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A REMARK ON SYMBOLS OF AN OPERATOR

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1. Introduction

In the paper we shall give a unified definition of a few different symbols of an operator in $\boldsymbol{R}^n$ with tempered Schwartz kernel. A symbol will always be thought of as a function or distribution on the phase space. It will depend on a matrix parameter. Several important known symbols can be obtained by a proper choice of the parameter.

Let $\boldsymbol{R}^n$ denote the n-dimensional Euclidean space and let $\boldsymbol{R}_n$ be its dual. The value of a functional $\xi \in \boldsymbol{R}_n$ on a vector $x \in \boldsymbol{R}^n$ is denoted by $x\xi$.

The Lebesgue measures in $\boldsymbol{R}^n$ and $\boldsymbol{R}_n$ are denoted by dx and $d\xi$. We also put $đ\xi = (2\pi)^{-n} d\xi$.

The space $W = \boldsymbol{R}^n \times \boldsymbol{R}_n$ (called the *phase space*) is a symplectic space with the form

$$W \times W \ni ((x, \xi), (y, \eta)) \mapsto -x\eta + y\xi \in \boldsymbol{R}.$$

For any finite-dimensional vector space V with a scalar product, $\mathscr{S}(V)$ denotes the Schwartz space of rapidly decreasing functions and $\mathscr{S}'(V)$ (the dual of $\mathscr{S}(V)$) is the space of tempered distributions with the weak topology ([Sz]).

In the paper V will be $\boldsymbol{R}^n$, $\boldsymbol{R}_n$, $\boldsymbol{R}^n \times \boldsymbol{R}_n$ or $\boldsymbol{R}^n \times \boldsymbol{R}^n$.

The spaces $\mathscr{S}(\boldsymbol{R}^n)$ and $\mathscr{S}(\boldsymbol{R}_n)$ are connected by the Fourier transform isomorphisms:

$$\mathscr{F}\colon\ \mathscr{S}(\boldsymbol{R}^n) \to \mathscr{S}(\boldsymbol{R}_n), \qquad \mathscr{F}(f)(\xi) := \int e^{-ix\xi} f(x)\, dx,$$
$$\mathscr{F}^{-1}\colon\ \mathscr{S}(\boldsymbol{R}_n) \to \mathscr{S}(\boldsymbol{R}^n), \qquad \mathscr{F}^{-1}(g)(x) := \int e^{ix\xi} g(\xi)\, đ\xi.$$

We will also need the partial Fourier transform:

$$\mathscr{F}_{\mathrm{II}} = \mathscr{F}_{y\to\xi}\colon\ \mathscr{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n) \to \mathscr{S}(\boldsymbol{R}^n \times \boldsymbol{R}_n),$$
$$\mathscr{F}_{y\to\xi}(f)(x, \xi) := \int e^{-iy\xi} f(x, y)\, dy,$$

$$\mathscr{F}_{\mathrm{II}}^{-1} = \mathscr{F}_{\xi\to y}^{-1}\colon\ \mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n)\to\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n),$$
$$\mathscr{F}_{\xi\to y}^{-1}(g)(x, y) := \int e^{iy\xi} g(x, \xi)\,đ\xi.$$

The symplectic structure on $\boldsymbol{R}^n\times\boldsymbol{R}_n$ enables us to define the *symplectic Fourier transform*:

$$(1.1)\qquad {}^s\mathscr{F}\colon\ \mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n)\to\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n),$$
$${}^s\mathscr{F}(a)(x, \xi) := \int\int e^{-i(-x\eta+y\xi)} a(y, \eta)\,dy = \mathscr{F}_{y\to\xi}\otimes\mathscr{F}_{\eta\to x}^{-1}\big(a(y, \eta)\big) =: \hat{a}(x, \xi).$$

It is obvious that

$$(\hat{a})\hat{} = a,$$
$$\mathscr{F}_{\xi\to y}^{-1}\hat{a}(x, \xi) = \mathscr{F}_{\eta\to x}^{-1} a(y, \eta),$$
$$\mathscr{F}_{x\to\eta}\hat{a}(x, \xi) = \mathscr{F}_{y\to\xi} a(y, \eta).$$

An injection $\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n)\hookrightarrow\mathscr{S}'(\boldsymbol{R}^n\times\boldsymbol{R}_n)$ will be defined by

$$a[b] = \int\int a(x, \xi) b(x, \xi)\,dx\,đ\xi,\qquad a, b\in\mathscr{S}.$$

The Fourier transforms defined above for functions are extended in the standard way for tempered distributions.

2. *T*-symbols

Let $T\in\mathrm{GL}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ be a linear automorphism of $\boldsymbol{R}^n\times\boldsymbol{R}^n$. Define

$$I_T = F_{\mathrm{II}}\circ T\colon\ \mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)\to\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n),$$
$$I_T(\mathscr{A})(x, \xi) := \int e^{-iy\xi}\mathscr{A}\big(T(x, y)\big)\,dy,\qquad \mathscr{A}\in\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n).$$

Obviously I_T is a continuous isomorphism and $I_T^{-1} = T^{-1}\circ\mathscr{F}_{\mathrm{II}}^{-1}$. Thus for $a\in\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n)$

$$I_T^{-1}(a)(x, y) = \int e^{i\pi_2 T^{-1}(x,y)\xi} a\big(\pi_1 T^{-1}(x, y), \xi\big)\,đ\xi$$

where $\pi_1(x, y) = x$, $\pi_2(x, y) = y$.

Define also

$$I_T^{\#}\colon\ \mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n)\to\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n),$$
$$I_T^{\#}(a)(x, y) = |\det T|^{-1}\int e^{-i\pi_2 T^{-1}(x,y)\xi} a\big(\pi_1 T^{-1}(x, y), \xi\big)\,đ\xi,$$
$$a\in\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n).$$

It is easy to show that

$$I_T(\mathscr{A})[b] = \mathscr{A}[I_T^{\#}(b)],\qquad \mathscr{A}\in\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n),\ b\in\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n).$$

This formula gives a continuous extension of I_T to tempered distributions,

$$I_T\colon\ \mathscr{S}'(\boldsymbol{R}^n\times\boldsymbol{R}^n)\to\mathscr{S}'(\boldsymbol{R}^n\times\boldsymbol{R}_n).$$

Let $A\colon \mathcal{S}(\boldsymbol{R}^n) \to \mathcal{S}'(\boldsymbol{R}^n)$ be a continuous linear operator and let $\mathscr{A} \in \mathcal{S}'(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ be the Schwartz kernel of A ([Ma]). In that situation we write $A = \mathrm{Op}(\mathscr{A})$, $\mathscr{A} = \mathrm{SK}(A)$.

DEFINITION. The distribution $I_T(\mathscr{A}) \in \mathcal{S}'(\boldsymbol{R}^n \times \boldsymbol{R}_n)$ is called the *T-symbol* of the operator A and will be denoted by $\sigma_T(A)$.

If $a = \sigma_T(A)$ we also write $A = \mathrm{Op}_T(a)$.

According to the definition we have the formulas

$$\mathscr{A}[\psi \otimes \varphi] = A\varphi[\psi], \qquad \varphi, \psi \in \mathcal{S}(\boldsymbol{R}^n),$$

$$\sigma_T(A)(x, \xi) = \mathscr{F}_{y \to \xi}\big(\mathscr{A}(T(x, y))\big), \tag{2.1}$$

$$\mathscr{A}(x, y) = \mathscr{F}^{-1}_{\xi \to \pi_2 T^{-1}(x,y)}\big(\sigma_T(A)(\pi_1 T^{-1}(x, y), \xi)\big), \tag{2.2}$$

$$A\varphi[\psi] = \sigma_T(A)[J_T(\psi \otimes \varphi)], \qquad \varphi, \psi \in \mathcal{S}(\boldsymbol{R}^n), \tag{2.3}$$

$$J_T(\chi)(x, \xi) = |\det T| \int e^{iy\xi} \chi(T(x, y))\, dy, \qquad \chi \in \mathcal{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n).$$

If $\mathscr{A} \in \mathcal{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ we obtain

$$A\varphi(x) = \int\int e^{i\pi_2 T^{-1}(x,y)\xi} \sigma_T(A)(\pi_1 T^{-1}(x, y), \xi)\, \varphi(y)\, dy\, d\!\!{}^-\xi. \tag{2.4}$$

For more general $\mathscr{A}$ it is possible to interpret the integral above as an oscillatory integral, but we shall not use this here.

From (1.1) and (2.2) we get an interesting relation between the T-symbol σ_T and its symplectic Fourier image $\hat{\sigma}_T$.

PROPOSITION 1.

$$\hat{\sigma}_T = \sigma_{T \circ v}$$

where $v\colon \boldsymbol{R}^n \times \boldsymbol{R}^n \ni (x, y) \mapsto (y, x) \in \boldsymbol{R}^n \times \boldsymbol{R}^n$.

Proof.

$$\begin{aligned}\hat{\sigma}_T(x, \xi) &= \mathscr{F}_{y \to \xi} \mathscr{F}^{-1}_{\eta \to x} \sigma_T(y, \eta) = \mathscr{F}_{y \to \xi} \mathscr{F}^{-1}_{\eta \to x} \mathscr{F}_{z \to \eta}\big(\mathscr{A}(T(y, z))\big) \\ &= \mathscr{F}_{y \to \xi}\big(\mathscr{A}(T(y, x))\big) = \mathscr{F}_{y \to \xi}\big(\mathscr{A}((T \circ v)(x, y))\big) = \sigma_{T \circ v}(x, \xi).\end{aligned}$$

Remark. It follows easily from (2.1) that $\sigma_{v \circ T}(A) = \sigma_T(A^+)$ where A^+ denotes the formal adjoint operator.

Now we shall show that various known symbols of operators are T-symbols for suitable T chosen in $\mathrm{GL}(2) \subset \mathrm{Aut}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$.

3. Weyl symbol

Let $T_W(x, y) = (x + y/2, x - y/2)$, $x, y \in \boldsymbol{R}^n$. The T_W-symbol of an operator is the well-known *Weyl symbol* (see: [Hö], [Sh], [Re]). We shall write $\sigma_W(A) := \sigma_{T_W}(A)$. We have $T_W^{-1}(x, y) = \big((x+y)/2, x - y\big)$ and formulas (2.1)–(2.4) take the following form:

$$\sigma_W(A)(x, \xi) = \mathscr{F}_{y \to \xi}\big(\mathscr{A}(x + y/2, x - y/2)\big), \tag{3.1}$$

$$\mathscr{A}(x, y) = \mathscr{F}^{-1}_{\xi \to x-y}\left(\sigma_W(A)\left(\frac{x+y}{2}, \xi\right)\right), \tag{3.2}$$

$$A\varphi[\psi] = \sigma_W(A)[J_W(\psi \otimes \varphi)], \tag{3.3}$$

$$J_W(\psi \otimes \varphi)(x, \xi) = \int e^{iy\xi} \psi(x+y/2)\, \varphi(x-y/2)\, dy,$$

$$A\varphi(x) = \int\int e^{i(x-y)\xi} \sigma_W(A)\left(\frac{x+y}{2}, \xi\right) \varphi(y)\, dy\, đ\xi. \tag{3.4}$$

4. Mikhlin–Giraud symbol

Let $T_{MG}(x, y) = (x, x-y)$. The T_{MG}-symbol of an operator A is the usual symbol of the (pseudodifferential) operator A (see: [Hö], [Sh], [Ta]). It is also called the polarized symbol ([Hw]). We shall denote it by $\sigma_{MG}(A)$ and call the *Mikhlin–Giraud* symbol of A.

We have $T_{MG}^{-1}(x, y) = (x, x-y)$ and

$$\sigma_{MG}(A)(x, \xi) = \mathscr{F}_{y \to \xi}(\mathscr{A}(x, x-y)), \tag{4.1}$$

$$\mathscr{A}(x, y) = \mathscr{F}^{-1}_{\xi \to x-y}(\sigma_{MG}(A)(x, \xi)), \tag{4.2}$$

$$A\varphi[\psi] = \sigma_{MG}(A)[J_{MG}(\psi \otimes \varphi)], \tag{4.3}$$

$$J_{MG}(\psi \otimes \varphi)(x, \xi) = \int e^{iy\xi} \psi(x)\, \varphi(x-y)\, dy = e^{ix\xi} \psi(x)\, \hat{\varphi}(\xi).$$

If $\mathscr{A} \in \mathscr{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ we have the well-known formula

$$\begin{aligned} A\varphi(x) &= \int\int e^{i(x-y)\xi} \sigma_{MG}(A)(x, \xi)\, \varphi(y)\, dy\, đ\xi \\ &= \int e^{ix\xi} \sigma_{MG}(A)(x, \xi)\, \hat{\varphi}(\xi)\, đ\xi, \qquad \varphi \in \mathscr{S}. \end{aligned} \tag{4.4}$$

5. Kohn–Nirenberg symbol

Let $T_{KN}(x, y) = (x+y, x)$. The T_{KN}-symbol of an operator A will be called the *Kohn–Nirenberg symbol* and denoted by $\sigma_{KN}(A)$. It is also called the dual symbol ([Sh]).

We have $T_{KN}^{-1}(x, y) = (y, x-y)$ and

$$\sigma_{KN}(A)(x, \xi) = \mathscr{F}_{y \to \xi}(\mathscr{A}(x+y, x)), \tag{5.1}$$

$$\mathscr{A}(x, y) = \mathscr{F}^{-1}_{\xi \to x-y}(\sigma_{KN}(A)(y, \xi)), \tag{5.2}$$

$$A\varphi[\psi] = \sigma_{KN}(A)[J_{KN}(\psi \otimes \varphi)], \tag{5.3}$$

$$J_{KN}(\psi \otimes \varphi)(x, \xi) = \int e^{iy\xi} \psi(x+y)\, \varphi(x)\, dy = e^{-ix\xi} \varphi(x)\, \hat{\psi}(-\xi).$$

If $\mathscr{A} \in \mathscr{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ and $\varphi \in \mathscr{S}$ we get

$$\begin{aligned} A\varphi(x) &= \int\int e^{i(x-y)\xi} \sigma_{KN}(A)(y, \xi)\, \varphi(y)\, dy\, đ\xi \\ &= \mathscr{F}^{-1}_{\xi \to x} \mathscr{F}_{y \to \xi}(\sigma_{KN}(A)(y, \xi)\, \varphi(y)). \end{aligned} \tag{5.4}$$

6. Shubin τ-symbol

Let $\tau \in [0, 1]$ and $T_\tau(x, y) = (x+\tau y,\ x-(1-\tau)y)$, $x, y \in \boldsymbol{R}^n$. The T_τ-symbol was considered by M. A. Shubin (e.g. [Sh]). We shall call it the *Shubin τ-symbol* and denote by $\sigma_\tau(A)$.

Since $T_\tau^{-1}(x, y) = ((1-\tau)x+\tau y,\ x-y)$ we have, as above, the formulas:

$$\sigma_\tau(A)(x, \xi) = \mathscr{F}_{y\to\xi}(\mathscr{A}(x+\tau y,\ x-(1-\tau)y)), \tag{6.1}$$

$$\mathscr{A}(x, y) = \mathscr{F}^{-1}_{\xi\to x-y}(\sigma_\tau(A)((1-\tau)x+\tau y,\ \xi)), \tag{6.2}$$

$$A\varphi[\psi] = \sigma_\tau(A)[J_\tau(\psi\otimes\varphi)], \tag{6.3}$$

$$J_\tau(\psi\otimes\varphi)(x, \xi) = \int e^{iy\xi}\psi(x+\tau y)\,\varphi(x+(\tau-1)y)\,dy.$$

For $\mathscr{A} \in \mathscr{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ and $\varphi \in \mathscr{S}(\boldsymbol{R}^n)$ we have

$$A\varphi(x) = \iint e^{i(x-y)\xi}\sigma_\tau(A)((1-\tau)x+\tau y,\ \xi)\,\varphi(y)\,dy\,d\!\!\bar{}\,\xi. \tag{6.4}$$

Let us observe that $\sigma_1 = \sigma_{\mathrm{KN}}$, $\sigma_0 = \sigma_{\mathrm{MG}}$, $\sigma_{1/2} = \sigma_{\mathrm{W}}$.

7. Segal symbol

Let $T_{\mathrm{S}}(x, y) = (x+y,\ y)$; the resulting T_{S}-symbol will be called the *Segal symbol*. It is known in quantum mechanics as the Weyl transform ([Se], [Hw]).

We have $T_{\mathrm{S}}^{-1}(x, y) = (x-y,\ y)$ and

$$\sigma_{\mathrm{S}}(A)(x, \xi) = \mathscr{F}_{y\to\xi}(\mathscr{A}(x+y,\ y)), \tag{7.1}$$

$$\mathscr{A}(x, y) = \mathscr{F}^{-1}_{\xi\to y}(\sigma_{\mathrm{S}}(A)(x-y,\ \xi)), \tag{7.2}$$

$$A\varphi[\psi] = \sigma_{\mathrm{S}}(A)[J_{\mathrm{S}}(\psi\otimes\varphi)], \tag{7.3}$$

$$J_{\mathrm{S}}(\psi\otimes\varphi)(x, \xi) = \int e^{iy\xi}\psi(x+y)\,\varphi(y)\,dy, \qquad \varphi, \psi \in \mathscr{S}(\boldsymbol{R}^n).$$

For $\mathscr{A}$ tempered we have

$$A\varphi(x) = \iint e^{iy\xi}\sigma_{\mathrm{S}}(A)(x-y,\ \xi)\,\varphi(y)\,dy\,d\!\!\bar{}\,\xi.$$

From Proposition 2.1 it follows that $\sigma_{\mathrm{S}}(A) = \hat{\sigma}_{\mathrm{KN}}(A)$.

8. Howe symbol

Let $T_{\mathrm{H}}(x, y) = (x/2+y,\ -x/2+y)$. The T_{H}-symbol was considered by R. Howe in [Hw]. We shall denote it by $\sigma_{\mathrm{H}}(A)$.

The Howe symbol is related via the symplectic Fourier transform to the Weyl symbol: $\sigma_{\mathrm{H}}(A) = \hat{\sigma}_{\mathrm{W}}(A)$. This follows at once from Prop. 2.1.

Formulas (2.1)–(2.4) now take the form

$$\sigma_{\mathrm{H}}(A)(x, \xi) = \mathscr{F}_{y\to\xi}(\mathscr{A}(x/2+y,\ -x/2+y)), \tag{8.1}$$

$$\mathscr{A}(x, y) = \mathscr{F}^{-1}_{\xi \to (x+y)/2}\big(\sigma_{\mathrm{H}}(A)(x-y, \xi)\big), \tag{8.2}$$

$$A\varphi[\psi] = \sigma_{\mathrm{H}}(A)[J_{\mathrm{H}}(\psi \otimes \varphi)], \tag{8.3}$$

$$J_{\mathrm{H}}(\psi \otimes \varphi)(x, \xi) = \int e^{iy\xi} \psi(x/2+y)\, \varphi(-x/2+y)\, dy.$$

For an operator A with tempered Schwartz kernel $\mathscr{A}$ we have

$$A\varphi(x) = \int\int e^{(i/2)(x+y)\xi} \sigma_{\mathrm{H}}(A)(x-y, \xi)\, \varphi(y)\, dy\, d\!\!{}^{-}\xi. \tag{8.4}$$

9. Relation between different T-symbols

It is easy to give a formula relating the T-symbols of an operator for different T.

PROPOSITION 2.

$$\begin{aligned}\sigma_{T_2}(A)(x, \xi) &= \mathscr{F}_{y \to \xi}(T_1^{-1} T_2).\mathscr{F}^{-1}_{\eta \to y}(\sigma_{T_1}(A)(x, \eta)) \\ &= \mathscr{F}_{y \to \xi}.\mathscr{F}^{-1}_{\eta \to \pi_2 T_1^{-1} T_2(x,y)}\big(\sigma_{T_1}(A)(\pi_1 T_1^{-1} T_2(x, y), \eta)\big).\end{aligned}$$

Proof. We use (2.1) and obvious calculations:

$$\begin{aligned}\sigma_{T_2}(A)(x, \xi) &= \mathscr{F}_{y \to \xi}\big(\mathscr{A}(T_2(x, y))\big), \\ \mathscr{A}(T_1(x, y)) &= \mathscr{F}^{-1}_{\eta \to y}(\sigma_{T_1}(A)(x, \eta)), \\ \mathscr{A}(x, y) &= \mathscr{F}^{-1}_{\eta \to \pi_2 T_1^{-1}(x,y)}\big(\sigma_{T_1}(A)(\pi_1 T_1^{-1}(x, y), \eta)\big), \\ \mathscr{A}(T_2(x, y)) &= \mathscr{F}^{-1}_{\eta \to \pi_2 T_1^{-1} T_2(x,y)}\big(\sigma_{T_1}(A)(\pi_1 T_1^{-1} T_2(x, y), \eta)\big), \\ \sigma_{T_2}(A)(x, \xi) &= \mathscr{F}_{y \to \xi}\, \mathscr{F}^{-1}_{\eta \to \pi_2 T_1^{-1} T_2(x,y)}\big(\sigma_{T_1}(A)(\pi_1 T_1^{-1} T_2(x, y), \eta)\big).\end{aligned}$$

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