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GAUSS TYPE QUADRATURE FORMULAS FOR SINGULAR INTEGRALS

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On the finite or infinite interval (a, b) let $S(g; x)$ denote the Cauchy principal value integral

$$S(g; x) = \int_a^b g(y) \frac{p(y)}{y-x} dy. \tag{1}$$

The weight function $p(y)$ is nonnegative on (a, b), Hölder continuous at the point x, and we shall assume that all moments $\int_a^b y^N p(y) dy$ $(N = 0, 1, 2, \ldots)$ exist. Then there exists a sequence of polynomials $\{P_N(y)\}_{N=0}^{\infty}$ with $P_N(y) = k_N y^N + \ldots$ and $\int_a^b P_N(y) P_M(y) p(y) dy = h_N \delta_{NM}$. Further, let $Q_N(x)$ denote the "function of the second kind"

$$Q_N(x) = -S(P_N; x). \tag{2}$$

The Gauss quadrature formula with respect to the weight function $p(y)$

$$\int_a^b g(y) p(y) dy \approx \sum_{i=1}^{N} A_i^{(N)} g(y_i^{(N)}) \tag{3}$$

is exact whenever $g(y)$ is a polynomial of degree $\leqslant 2N-1$.

A Gauss type formula for $S(g; x)$ was given for special cases by Sanikidze [13] and Hunter [8]. Later it was generalized by Chawla and Ramakrishnan [1] and Theocaris [14]. This formula can be derived easily. Since

$$Q_N(x) = -P_N(x) \int_a^b \frac{p(y)}{y-x} dy - \int_a^b \frac{P_N(y) - P_N(x)}{y-x} p(y) dy,$$

by quadrature formula (3), we have

$$\int_a^b \frac{p(y)}{y-x}\,dy - \sum_{i=1}^{N} \frac{A_i^{(N)}}{y_i^{(N)}-x} = -\frac{Q_N(x)}{P_N(x)}. \tag{4}$$

Let $g_{2N}(y)$ denote a polynomial of degree $\leqslant 2N$. We can write

$$S(g_{2N};x) = \int_a^b \frac{g_{2N}(y)-g_{2N}(x)}{y-x}\,p(y)\,dy + g_{2N}(x)\int_a^b \frac{p(y)}{y-x}\,dy,$$

and further, by (3) and (4),

$$S(g_{2N};x) = \sum_{i=1}^{N} \frac{A_i^{(N)} g_{2N}(y_i^{(N)})}{y_i^{(N)}-x} - g_{2N}(x)\frac{Q_N(x)}{P_N(x)}.$$

Thus, the quadrature formula

$$S(g;x) \approx J_N(g;x) = \sum_{i=1}^{N} A_i^{(N)} \frac{g(y_i^{(N)})}{y_i^{(N)}-x} - g(x)\frac{Q_N(x)}{P_N(x)}, \tag{5}$$

where $x \neq y_i^{(N)}$ $(i = 1, \ldots, N)$, is exact for polynomials of degree $\leqslant 2N$ (comp. [5], [6], [7]).

Now we shall estimate the error of the quadrature formula (5) $R_N(g;x) = S(g;x) - J_N(g;x)$ in the case of a finite interval $[a, b]$ and of a continuously differentiable function $g(y)$. Let $L_{2N-1}^1(y)$ denote the polynomial of degree $2N-1$, for which

$$E_{2N-1}^{(1)}(g) = \max_{y\in[a,b]} |g'(y) - L_{2N-1}^1(y)|$$

is minimal. Further, let $R(x)$ denote the function $R(x) = g(x) - \int_a^x L_{2N-1}^1(y)\,dy$. Thus, $\max_{x\in[a,b]} |R'(x)| = E_{2N-1}^{(1)}(g)$, from which

$$\left|\frac{R(y)-R(x)}{y-x}\right| = \left|\frac{\int_x^y R'(t)\,dt}{y-x}\right| \leqslant E_{2N-1}^{(1)}(g). \tag{6}$$

Since formula (5) is exact for the polynomial $\int_a^x L_{2N-1}^1(y)\,dy$ of degree $2N$, we obtain

$$R_N(g;x) = \int_a^b \frac{R(y)-R(x)}{y-x}\,p(y)\,dy - \sum_{i=1}^{N} A_i^{(N)} \frac{R(y_i^{(N)})-R(x)}{y_i^{(N)}-x} +$$

$$+R(x)\left(\int_a^b \frac{p(y)}{y-x}\,dy - \sum_{i=1}^{N} \frac{A_i^{(N)}}{y_i^{(N)}-x} + \frac{Q_N(x)}{P_N(x)}\right).$$

The constants $A_i^{(N)}$ are positive, and from (3), $\sum_{i=1}^{N} A_i^{(N)} = \int_a^b p(y)dy$. Thus,

$$|R_N(g;x)| \leqslant 2E_{2N-1}^{(1)}(g) \int_a^b p(y)\,dy. \tag{7}$$

Hence the convergence of $J_N(g;x)$ to $S(g;x)$ for continuously differentiable functions $g(y)$. This convergence statement was given in [5] and [6]. A convergence proof for Lipschitz continuous functions was given by Elliott [2]. In the case of the Jacobi weight function, the convergence for Hölder continuous functions was proved by Tsamasphyros and Theocaris [15]. In the general case Elliott [2] has demonstrated that for Hölder continuous functions $g(y)$, $J_{N_k}(g;x)$ for a certain subsequence $\{N_k\}$ of indices converges to $S(g;x)$.

Now we shall give an expression for the error $R_N(g;x)$ in the case of functions $g(y)$, which possess a continuous and bounded derivative of order $2N+1$ on the finite or infinite interval (a,b). We can construct a polynomial of degree $2N$, for which $g(y_i^{(N)}) = P(y_i^{(N)})$, $g'(y_i^{(N)}) = P'(y_i^{(N)})$ $(i = 1, \dots, N)$ and $g(x) = P(x)$. Then, for each point y, there is a point $\xi(y) \in (a,b)$ with

$$g(y) = P(y) + \frac{1}{(2N+1)!} \prod_{i=1}^{N} (y-y_i^{(N)})^2 (y-x) g^{(2N+1)}\big(\xi(y)\big)$$

(see, e.g. [11]). Since quadrature formula (5) gives $S(P;x)$ exactly, and in all quadrature nodes $g(y)$ is equal to $P(y)$, we have

$$R_N(g;x) = \frac{1}{(2N+1)!} \int_a^b \prod_{i=1}^{N} (y-y_i^{(N)})^2 g^{(2N+1)}\big(\xi(y)\big) p(y)\,dy.$$

Using the generalized mean value theorem of integral calculus, we obtain

$$R_N(g;x) = \frac{g^{(2N+1)}(\xi)}{(2N+1)!} \frac{h_N}{k_N^2} \quad (\xi \in (a,b)). \tag{8}$$

This error term was given in [5] and [6].

Another method of giving a Gauss type quadrature formula for Cauchy principal value integrals was reported by Korneichuk [10]. The formula

$$S(g;x) \approx I_N(g;x) = \sum_{i=1}^{N} A_i^{(N)} \frac{g(y_i^{(N)})}{y_i^{(N)}-x} \left(1 - \frac{Q_N(x)}{Q_N(y_i^{(N)})}\right), \tag{9}$$

where $x \neq y_i^{(N)}$ $(i = 1, \dots, N)$, however, is exact only for polynomials of degree $\leqslant N-1$. On the other hand, the value $g(x)$ is not needed in this formula. An estimation of the error is given in [10], convergence results can be found in [3], [4] and [5], and an expression of the error term is suggested in [5].

In the zeros $x_k^{(N)}$ of the function of the second kind $Q_N(x)$, the number and distribution of which was studied for instance in [6] and [7], both formula (5) and formula (9) are of the simple form

$$S(g; x_k^{(N)}) \approx \sum_{i=1}^{N} A_i^{(N)} \frac{g(y_i^{(N)})}{y_i^{(N)} - x_k^{(N)}} \quad (Q_N(x_k^{(N)}) = 0). \tag{10}$$

This formula has the structure of Gauss formula (3) for nonsingular integrals. It is exact, whenever $g(y)$ is a polynomial of degree $\leqslant 2N$, and for the remainder the formulas (7) and (8) also hold.

The formula (10) is especially suitable for the numerical solution of the singular integral equation

$$\int_a^b g(y) \frac{p(y)}{y - x} dy + \int_a^b g(y) k(x, y) p(y) dy = r(x). \tag{11}$$

Using formulas (10) and (3) at the points $x_k^{(N)}$, equation (11) can be approximated by the system of equations

$$\sum_{i=1}^{N} A_i^{(N)} g(y_i^{(N)}) \left[\frac{1}{y_i^{(N)} - x_k^{(N)}} + k(x_k^{(N)}, y_i^{(N)}) \right] = r(x_k^{(N)}) \qquad (Q_N(x_k^{(N)}) = 0). \tag{12}$$

This system has already been used by several authors (for references, see, e.g. [6], [7]). The convergence of the method was proved in the case of special weight functions in [6] and [7].

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