ON THE ORDER REDUCTION OF OPTIMAL CONTROL SYSTEMS

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1. Introduction

We consider a control system which is modelled by the equation

(1.1)
$$\dot{x}(t) = A_1(t)x(t) + A_2(t)y(t) + B_1(t)u(t), \quad x(0) = x^0,$$
$$\lambda\dot{y}(t) = A_3(t)x(t) + A_4(t)y(t) + B_2(t)u(t), \quad y(0) = y^0,$$

where $x(t) \in R^m$, $y(t) \in R^n$, $u(t) \in R^r$, the time t belongs to the interval $[0, 1]$ and λ is a positive small parameter. The states $x(t)$ and $y(t)$ represent slow and fast phenomena, respectively. For $\lambda = 0$ the order $m+n$ of system (1.1) reduces to m, i.e., (1.1) becomes

(1.2)
$$\dot{x}(t) = A_1(t)x(t) + A_2(t)y(t) + B_1(t)u(t), \quad x(0) = x^0,$$
$$0 = A_3(t)x(t) + A_4(t)y(t) + B_2(t)u(t)$$

for $t \in [0, 1]$, where the initial condition y^0 is dropped. Assuming that $A_4^{-1}(t)$ exists and solving the second equation in (1.2) with respect to $y(t)$, we obtain a "reduced" model

$$\dot{x}(t) = A_0(t)x(t) + B_0(t)u(t), \quad x(0) = x^0,$$

where

$$A_0(t) = A_1(t) - A_2(t)A_4^{-1}(t)A_3(t), \quad B_0(t) = B_1(t) - A_2(t)A_4^{-1}(t)B_2(t).$$

It is obvious that this order reduction procedure may lead to an essential simplification of the original model.

In this paper the following optimal control problem (P_λ) is considered: For fixed λ, find a function $\hat{u}_\lambda \in L_2^{(r)}(0, 1)$ which minimizes the following performance index:

(1.3)
$$J_\lambda(u) = g(z(1)) + \int_0^1 (f(z(t), t) + h(u(t), t)) dt,$$

where $z = (x, y)$, determined by (1.1), is an absolutely continuous function of the time.

The order reduction of singularly perturbed optimal control systems has been investigated in a number of papers; see the survey [7]. In the present form, problem (P_λ) has not been treated in the literature although similar problems have been studied under various conditions for the functions g, f and h (see, e.g., [4]–[6], [9]). In [9] the function g has the following special form:

$$g(x, y) = \pi_1(x) + \lambda\pi_2(x)y + \lambda y^T \pi_3(x)y$$

and the functions f and h are quadratic with respect to y and u. An asymptotic solution of the problem is constructed. Papers [4] and [6] deal with linear quadratic problems and use a Riccati equation technique. In [4] it is assumed that the matrix B_2 is equal to 0. In [6] the function f does not depend on y. Work [5] proves that the result in [6] is valid for the corresponding problem with nonquadratic but convex f and h. The present paper extends the analysis of [5] for problem (P_λ), in which both the terminal and the integral parts of the performance index are nonquadratic and depend explicitly on the fast trajectories. For the case $B_2 = 0$ we generalize the corresponding result in [4]. Our approach uses some properties of the solutions of singularly perturbed linear differential equations and estimates of the solutions of perturbed strongly convex extremal problems [2].

We shall denote by $|\cdot|$ the Euclidean norm and by an upper T the transposition. The norm of a real vector space E of functions defined on $[0, 1]$ will be denoted by $\|\cdot\|_E$. All constants which are independent of the time t and the perturbation parameter λ are denoted by c.

2. Preliminary lemmas

We first start from some auxiliary results. Denote by (x_λ, y_λ) the solution of the equation

$$(2.1) \quad \begin{aligned} \dot{x}(t) &= A_1(t)x(t) + A_2(t)y(t) + \varphi_0(t) + \Delta\varphi_k(t), & x(0) &= v_k, \\ \lambda_k\dot{y}(t) &= A_3(t)x(t) + A_4(t)y(t) + \psi_0(t) + \Delta\psi_k(t), & y(0) &= w_k, \end{aligned}$$

where $\lim_{k\to+\infty} \lambda_k = 0$, $\varphi_0 \in L_2^{(m)}(0, 1)$, $\psi_0 \in L_2^{(n)}(0, 1)$ and $\{\Delta\varphi_k\}$, $\{\Delta\psi_k\}$, $\{v_k\}$, $\{w_k\}$ are given sequences. The solution of the equation

$$(2.2) \quad \begin{aligned} \dot{x}(t) &= A_1(t)x(t) + A_2(t)y(t) + \varphi_0(t), & x(0) &= v_0, \\ 0 &= A_3(t)x(t) + A_4(t)y(t) + \psi_0(t) \end{aligned}$$

will be denoted by (x_0, y_0). We assume that

A1. The matrices $A_i(t)$ are continuous on $[0, 1]$. All the eigenvalues of the matrix $A_4(t)$ have negative real parts for $t \in [0, 1]$.

LEMMA 2.1. (i) *Let* $\lim\limits_{k \to +\infty} v_k = v_0$, $w_k = \omega_k/\lambda_k$, $\lim\limits_{k \to +\infty} \omega_k = \omega_0$, *and let the sequences* $\{\Delta\varphi_k\}$, $\{\Delta\psi_k\}$ *be weakly convergent to zero in* $L_2^{(m)}(0, 1)$ *and* $L_2^{(n)}(0, 1)$, *respectively. Let* x_0 *be determined by* (2.2) *with the initial condition* $x(0) = v_0 - A_2(0)A_4^{-1}(0)\omega_0$. *Then the sequence* $\{x_k\}$ *is uniformly bounded in* $[0, 1]$ *and for every* $\theta \in (0, 1)$,

(2.3) $$\lim_{k \to +\infty} \max_{\theta \leqslant t \leqslant 1} |x_k(t) - x_0(t)| = 0.$$

(ii) *If, additionally,* $\omega_0 = 0$, *then*

(2.4) $$\lim_{k \to +\infty} \|x_k - x_0\|_C = 0.$$

(iii) *Let* $\lim\limits_{k \to +\infty} \sqrt{\lambda_k} w_k = 0$ *and let all the above conditions hold. Then the sequence* $\{y_k\}$ *is* L_2-*weakly convergent to* y_0.

(iv) *Let* $\lim\limits_{k \to +\infty} v_k = v_0$ *and* $\lim\limits_{k \to +\infty} \sqrt{\lambda_k} w_k = 0$. *Then the following estimation holds for* k *sufficiently large:*

(2.5) $$\|x_k - x_0\|_C + \|y_k - y_0\|_{L_2} \leqslant c(\|\Delta\varphi_k\|_{L_2} + \|\Delta\psi_k\|_{L_2} + \delta_k),$$

where $\lim\limits_{k \to +\infty} \delta_k = 0$.

(v) *Let the above conditions be satisfied and, additionally,* $\psi_0 \in C^{(n)}[0, 1]$, *let the sequence* $\{w_k\}$ *be bounded, and for every* $\theta_1 \in (0, 1)$,

$$\lim_{k \to +\infty} \max_{0 \leqslant t \leqslant \theta_1} |\Delta\psi_k(t)| = 0.$$

Then for every $\theta \in (0, 1/2)$,

(2.6) $$\lim_{k \to +\infty} \max_{\theta \leqslant t \leqslant 1-\theta} |y_k(t) - y_0(t)| = 0.$$

Proof. Let $Y(t, \tau, \lambda_k)$ be the fundamental matrix solution of the equation $\lambda_k \dot{z}(t) = A_4(t)z(t)$. By [10], there exist constants σ_0, $\sigma > 0$, such that

(2.7) $$|Y(t, \tau, \lambda_k)| \leqslant \sigma_0 \exp\left(-\sigma \frac{t - \tau}{\lambda_k}\right)$$

for all $t, \tau \in [0, 1]$, $t \geqslant \tau$. Writing $\Delta x_k = x_k - x_0$, $\Delta y_k = y_k - y_0$, we have

(2.8) $$\Delta x_k(t) = v_k - v_0 + A_2(0)A_4^{-1}(0)\omega_0 +$$
$$+ \int_0^t \left(A_1(\tau)\Delta x_k(\tau) + A_2(\tau)\Delta y_k(\tau) + \Delta\varphi_k(\tau)\right)d\tau,$$

(2.9) $$\Delta y_k(t) = Y(t, 0, \lambda_k)w_k + \frac{1}{\lambda_k}\int_0^t Y(t, \tau, \lambda_k)\left(A_3(\tau)\Delta x_k(\tau) +\right.$$
$$\left.+ \Delta\psi_k(\tau) - A_4(\tau)y_0(\tau)\right)d\tau - y_0(t).$$

In the sequel we use the following standard result: if $p \in L_1^{(1)}(0,1)$, $q \in L_2^{(1)}(0,1)$ and

$$r(t) = \int\limits_0^t p(t-\tau)q(\tau)d\tau,$$

then

(2.T) $$\|r\|_{L_2} \leqslant \|p\|_{L_1} \|q\|_{L_2}.$$

Let δ be an arbitrary positive number. Choose a function $y^\delta \in C_1^{(n)}[0,1]$ such that $\|y^\delta - y_0\|_{L_2} < \delta$. In view of (2.7), for every $t \in [0,1]$

(2.10) $$\left| \int\limits_0^t \frac{\partial}{\partial\tau} Y(t,\tau,\lambda_k)y^\delta(\tau)d\tau - y_0(t) \right|$$

$$\leqslant c \left(|y^\delta(t) - y_0(t)| + |y^\delta(0)| \exp\left(-\sigma\frac{t}{\lambda_k}\right) + \lambda_k \|\ddot{y}^\delta\|_C \right).$$

Let

$$\bar{y}_k(t) = \frac{1}{\lambda_k} \int\limits_0^t Y(t,\tau,\lambda_k)A_4(\tau)y_0(\tau)d\tau.$$

Since δ could be arbitrarily small, from (2.T) and (2.10), integrating by parts, we obtain

(2.11) $$\lim_{k\to+\infty} \|\bar{y}_k - y_0\|_{L_2} = 0.$$

For an arbitrary but fixed $\varepsilon > 0$ one can choose matrices $A_2^\varepsilon(t)$ and $A^\varepsilon(t)$, whose elements are smooth in the time variable, such that $\|A_2 - A_2^\varepsilon\|_C < \varepsilon$ and $\|A_4^{-1} - A^\varepsilon\|_C < \varepsilon$. From

$$\frac{1}{\lambda_k} \int\limits_0^t A_2(\tau) Y(\tau,0,\lambda_k)d\tau$$

$$= \int\limits_0^t A_2(\tau)A_4^{-1}(\tau)\frac{\partial}{\partial\tau} Y(\tau,0,\lambda_k)d\tau$$

$$= \frac{1}{\lambda_k} \int\limits_0^t \left(A_2(\tau)A_4^{-1}(\tau) - A_2^\varepsilon(\tau)A^\varepsilon(\tau) \right) A_4(\tau) Y(\tau,0,\lambda_k)d\tau +$$

$$+ \int\limits_0^t A_2^\varepsilon(\tau)A^\varepsilon(\tau)\frac{\partial}{\partial\tau} Y(\tau,0,\lambda_k)d\tau,$$

integrating by parts and taking advantage of (2.7) we get

$$(2.12) \quad \left| \frac{1}{\lambda_k} \int_0^t A_2(\tau) Y(\tau, 0, \lambda_k) \omega_k d\tau + A_2(0) A_4^{-1}(0) \omega_0 \right|$$

$$\leqslant c \left(\varepsilon + |\omega_k| \exp \left(-\sigma \frac{t}{\lambda_k} \right) + c_1(\varepsilon) \lambda_k + |\omega_k - \omega_0| \right),$$

where $c_1(\varepsilon) \geqslant \left\| \frac{d}{dt} (A_2^\varepsilon A^\varepsilon) \right\|_C$. Write

$$\xi_k(t) = A_3(t) \Delta x_k(t) + \Delta \psi_k(t)$$

and

$$\eta_k(t) = \frac{1}{\lambda_k} \int_0^t Y(t, \tau, \lambda_k) \xi_k(\tau) d\tau.$$

Applying (2.T), we get

$$(2.13) \qquad \|\eta_k\|_{L_2} \leqslant \frac{\sigma_0}{\sigma} \|\xi_k\|_{L_2}.$$

Using (2.7), (2.T), (2.13) the Hölder inequality and integrating by parts, we obtain

$$(2.14) \quad \left| \int_0^t A_2(\tau) \eta_k(\tau) d\tau \right| \leqslant \int_0^t |A_2(\tau) - A_2^\varepsilon(\tau)| \; |\eta_k(\tau)| \; d\tau +$$

$$+ \left| \int_0^t A_2^\varepsilon(\tau) A^\varepsilon(\tau) A_4(\tau) \eta_k(\tau) d\tau \right| + \int_0^t |A_2^\varepsilon(\tau)| \; |A_4^{-1}(\tau) - A^\varepsilon(\tau)| \; |A_4(\tau) \eta_k(\tau)| d\tau$$

$$\leqslant c\varepsilon \|\xi_k\|_{L_2} + \left| \int_0^t A_2^\varepsilon(\tau) A^\varepsilon(\tau) \frac{\partial}{\partial \tau} \int_0^\tau Y(\tau, s, \lambda_k) \xi_k(s) ds \, d\tau \right| +$$

$$+ \left| \int_0^t A_2^\varepsilon(\tau) A^\varepsilon(\tau) \xi_k(\tau) d\tau \right|$$

$$\leqslant c\varepsilon \|\xi_k\|_{L_2} + \lambda_k \left(|A_2^\varepsilon(t) A^\varepsilon(t) \eta_k(t)| + \left| \int_0^t \frac{d}{d\tau} \left(A_2^\varepsilon(\tau) A^\varepsilon(\tau) \right) \eta_k(\tau) d\tau \right| + \right.$$

$$\left. + \left| \int_0^t A_2^\varepsilon(\tau) A^\varepsilon(\tau) \xi_k(\tau) d\tau \right| \right.$$

$$\leqslant c \left((\varepsilon + \sqrt{\lambda_k} + c_2(\varepsilon) \lambda_k) \|\xi_k\|_{L_2} + \right.$$

$$\left. + \int_0^t |\Delta x_k(\tau)| d\tau) + \left| \int_0^t A_2^\varepsilon(\tau) A^\varepsilon(\tau) \Delta \psi_k(\tau) d\tau \right| \right),$$

where $c_2(\varepsilon) \geqslant \left\|\dfrac{d}{dt}(A_2^* A^*)\right\|_{L_2}$. Taking into account (2.8), (2.9), (2.12), (2.13) and (2.14), we have

$$(2.15) \quad |\Delta x_k(t)| \leqslant |v_k - v_0| + c\left(\varepsilon + c_1(\varepsilon)\lambda_k + |\omega_k|\exp\left(-\sigma\frac{t}{\lambda_k}\right) + \right.$$

$$+ |\omega_k - \omega_0| + \left(\varepsilon + \sqrt{\lambda_k} + c_2(\varepsilon)\lambda_k\right)\left(\|\Delta x_k\|_{L_2} + \|\Delta \psi_k\|_{L_2}\right) +$$

$$+ \|\bar{y}_k - y_0\|_{L_2} + \int\limits_0^t |\Delta x_k(\tau)|\,d\tau\right) + \left|\int\limits_0^t \Delta\varphi_k(\tau)\,d\tau\right| +$$

$$+ \left.\left|\int\limits_0^t A_2^*(\tau)A^*(\tau)\Delta\psi_k(\tau)\,d\tau\right|\right).$$

Let us recall that if a sequence $\{z_k\}$ is L_2-weakly convergent to zero, then

$$\lim_{k\to+\infty}\ \max_{0\leqslant t\leqslant 1}\left|\int\limits_0^t z_k(\tau)\,d\tau\right| = 0,$$

and the sequence $\{\|z_k\|_{L_2}\}$ is bounded. Applying the Gronwall inequality to (2.15), we get

$$\|\Delta x_k\|_{L_2} \leqslant \|\Delta x_k\|_C \leqslant c\left(\varepsilon + \sqrt{\lambda_k} + c_1(\varepsilon)\lambda_k\right)\|\Delta x_k\|_{L_2} + c.$$

Choosing $\varepsilon < 1/c$ and tending to zero with λ_k, we conclude that

$$(2.16) \qquad\qquad \limsup_{k\to+\infty}\|\Delta x_k\|_{L_2} < +\infty.$$

Using this result in (2.15), we finally obtain

$$(2.17) \qquad |\Delta x_k(t)| \leqslant c\left(\varepsilon + \int\limits_0^t |\Delta x_k(\tau)|\,d\tau + |\omega_k|\exp\left(-\sigma\frac{t}{\lambda_k}\right) + \delta_k\right),$$

where $\lim\limits_{k\to+\infty} \delta_k = 0$ uniformly in $[0, 1]$. From the Gronwall lemma we get that for every $\theta \in (0, 1)$,

$$\lim_{k\to+\infty}\ \max_{\theta\leqslant t\leqslant 1} |\Delta x_k(t)| \leqslant c\varepsilon.$$

Since ε is arbitrarily small and Δx_k does not depend on ε, this relation implies (2.3).

Part (ii) of the statement follows immediately from (2.17).

If $\lim\limits_{k\to+\infty} \sqrt{\lambda_k}\,w_k = 0$, then from (2.3), (2.7), (2.9), (2.12) and (2.T)

$$\limsup_{k\to+\infty}\|\Delta y_k\|_{L_2} < +\infty.$$

In order to prove (iii) it is sufficient to show that for every $t \in [0, 1]$,

$$(2.18) \qquad \lim_{k \to +\infty} \int_0^t \Delta y_k(\tau) d\tau = 0.$$

Using a sequence of inequalities similar to (2.14), we have

$$\left| \frac{1}{\lambda_k} \int_0^t \int_0^\tau Y(\tau, s, \lambda_k) \Delta \psi_k(s) ds \, d\tau \right| \leqslant c \left(\varepsilon + \sqrt{\lambda_k} + c_3(\varepsilon) \lambda_k \right) \| \Delta \psi_k \|_{L_2} +$$

$$+ \left| \int_0^t A^*(\tau) \Delta \psi_k(\tau) d\tau \right|.$$

In view of this relation and (2.4), (2.11) we obtain (2.18).

The estimate for Δx_k in (2.5) follows immediately from (2.11), (2.15) and (2.16). In order to obtain the estimate for Δy_k we use (2.T), (2.9) and (2.11).

Next let the conditions in (v) hold. Since $y_0 \in C^{(n)}[0, 1]$, one can choose $y^\delta \in C_1^{(n)}[0, 1]$ such that $\| y^\delta - y_0 \| < \delta$. Then

$$(2.19) \qquad |\bar{y}_k(t) - y_0(t)| \leqslant \left| \frac{1}{\lambda_k} \int_0^t Y(t, \tau, \lambda_k) A_4(\tau) \big(y_0(\tau) - y^\delta(\tau) \big) d\tau \right| +$$

$$+ \left| \int_0^t \frac{\partial}{\partial \tau} Y(t, \tau, \lambda_k) y^\delta(\tau) d\tau - y_0(t) \right|$$

$$\leqslant c \left(\delta + |y^\delta(0)| \exp\left(-\sigma \frac{t}{\lambda_k} \right) + \lambda_k \| \ddot{y}^\delta \|_C \right).$$

Substituting (2.19) in (2.9), we obtain (2.10). This proves the lemma completely.

Consider now the control system (1.1) with the assumptions A1 and

A2. The matrices $B_i(t)$ are continuous and

$$(2.20) \qquad \text{rank}\, [B_2(1), A_4(1) B_2(1), \ldots, A_4(1)^{n-1} B_2(1)] = n.$$

Lemma 2.2. *Let* (x_0, y_0) *be the solution of* (1.2) *for a given* $u_0 \in C^{(r)}[0, 1]$. *For every point* $w \in R^n$ *and for every sequence* $\{\lambda_k\}$, $\lim_{k \to +\infty} \lambda_k = 0$, *there exists a uniformly bounded sequence of controls* $\{\delta u_k\}$ *so that if* (x_k, y_k) *corresponds to* $u_0 + \delta u_k$ *and* λ_k *according to* (1.1), *then for every* $\theta \in (0, 1/2)$,

$$(2.21) \qquad \lim_{k \to +\infty} (\| x_k - x_0 \|_C + \max_{\theta \leqslant t \leqslant 1-\theta} |y_k(t) - y_0(t)| +$$

$$+ |y_k(1) - w| + \max_{0 \leqslant t \leqslant \theta} |\delta u_k(t)|) = 0.$$

Proof. Let $(\tilde{x}_k, \tilde{y}_k)$ be the solution of (1.1) for λ_k and u_0. By Lemma 2.1 (ii) and (v), we obtain for every $\theta \in (0, 1)$,

$$(2.22) \qquad \lim_{k \to +\infty} \left(\|\tilde{x}_k - x_0\|_C + \max_{\theta \leqslant t \leqslant 1} |\tilde{y}_k(t) - y_0(t)| \right) = 0.$$

Let $t_k = 1 - \sqrt{\lambda_k}$. Introduce the matrices

$$M_k = \frac{1}{\lambda_k} \int_{t_k}^{1} Y(1, t, \lambda_k) B_2(t) B_2^T(t) \, Y^T(1, t, \lambda_k) \, dt,$$

$$\tilde{M}_k = \frac{1}{\lambda_k} \int_{t_k}^{1} \exp\left(A_4(1) \frac{1-t}{\lambda_k}\right) B_2(1) B_2^T(1) \exp\left(A_4^T(1) \frac{1-t}{\lambda_k}\right) dt,$$

$$M_0 = \int_{0}^{+\infty} \exp\left(A_4(1)t\right) B_2(1) B_2^T(1) \exp\left(A_4^T(1)t\right) dt.$$

Condition (2.20) implies that the matrix M_0 is nonsingular; see [8]. For $t \in [t_k, 1]$, write

$$P_k(t) = Y(1, t, \lambda_k) - \exp\left(A_4(1) \frac{1-t}{\lambda_k}\right).$$

Using (2.7), one can get the estimate

$$|P_k(t)| \leqslant c \max_{t_k \leqslant s \leqslant 1} |A_4(1) - A_4(s)|.$$

Hence

$$\lim_{k \to +\infty} |M_k - \tilde{M}_k| = 0.$$

Furthermore

$$|\tilde{M}_k - M_0| \leqslant c \int_{\lambda_k^{-1/2}}^{+\infty} \exp(-2\sigma s) \, ds \leqslant c \exp\left(-\frac{2\sigma}{\sqrt{\lambda_k}}\right),$$

which implies $\lim_{k \to +\infty} \tilde{M}_k = M_0$. Thus, for k sufficiently large the matrix M_k is nonsingular.

Let us define the sequence $\{\delta u_k\}$ as follows:

$$\delta u_k(t) = \begin{cases} 0 & \text{for} \quad t \in [0, t_k), \\ B_2^T(t) \, Y^T(1, t, \lambda_k) M_k^{-1}(w - y_0(1)) & \text{for} \quad t \in [t_k, 1], \end{cases}$$

where k is sufficiently large. Then

$$(2.23) \qquad |\delta u_k(t)| \leqslant c \exp\left(-\sigma \frac{1-t}{\lambda_k}\right),$$

and

$$y_k(1) - \tilde{y}_k(1) = \frac{1}{\lambda_k} \int\limits_0^1 Y(1, t, \lambda_k) A_s(t) \left(x_k(t) - \tilde{x}_k(t)\right) dt + w - y_0(1).$$

Lemma 2.1(ii) yields

$$\lim_{k \to +\infty} \|x_k - x_0\|_C = 0.$$

In view of (2.22) we obtain

$$\lim_{k \to +\infty} y_k(1) = w.$$

The convergence of y_k to y_0 follows from Lemma 2.1(v).

Remark 2.1. The above lemma implies that the controllability of the reduced system and condition A2 are sufficient conditions for controllability of the full-order system; see [3] for details.

LEMMA 2.3. *Let z_λ be the solution of the equation*

$$(2.24) \quad z(t) = w(\lambda) + a \exp\left(-\sigma \frac{t}{\lambda}\right) + \frac{d}{\lambda} \exp\left(-\sigma \frac{1-t}{\lambda}\right) +$$

$$+ \frac{b}{\lambda} \int\limits_0^1 \exp\left(-\sigma \frac{|t - \tau|}{\lambda}\right) z(\tau) d\tau,$$

where $w(\lambda) \to c$ as $\lambda \to 0$; a, d, b, σ are positive constants and $\sigma > 2b$. Then for every $\theta \in (0, 1/2)$,

$$(2.25) \qquad \lim_{\lambda \to 0} \max_{\theta \leqslant t \leqslant 1 - \theta} |z_\lambda(t)| = 0.$$

Proof. Equation (2.24) is equivalent to the following boundary value problem:

$$\lambda \ddot{z} = r^2 z - \sigma^2 w(\lambda),$$

$$\lambda \dot{z}(0) - \sigma z(0) = -w(\lambda)\sigma - 2a\sigma,$$

$$\lambda \dot{z}(1) + \sigma z(1) = w(\lambda)\sigma + \frac{2d\sigma}{\lambda},$$

where $r^2 = \sigma(\sigma - 2b)$. Let $D_1 = \sigma - r$, $D_2 = \sigma + r$. This problem has the

following exact solution:

$$z_\lambda(t) = \frac{w(\lambda)\sigma}{\sigma - 2b} + \left(-\frac{2bw(\lambda)\sigma}{\sigma - 2b} \left(D_2 \exp\left(-r\frac{1-t}{\lambda} \right) - \right.\right.$$

$$-\exp\left(-r\frac{t}{\lambda} \right) \right) - D_1 \left(\exp\left(-r\frac{2-t}{\lambda} \right) - \exp\left(-r\frac{t+1}{\lambda} \right) \right) +$$

$$+2a\sigma\left(D_2\exp\left(-r\frac{t}{\lambda} \right) - D_1\exp\left(-r\frac{t-2}{\lambda} \right) \right) + \frac{2d\sigma}{\lambda}\left(D_2\exp\left(-r\frac{1-t}{\lambda} \right) - \right.$$

$$\left.\left. -D_1\exp\left(-r\frac{t+1}{\lambda} \right) \right)\right) \Big/ \left(D_2^2 - D_1^2\exp\left(-\frac{2r}{\lambda} \right) \right),$$

which satisfies (2.25). ∎

3. L_2-convergence of the optimal control

In the sequel we concern ourselves with the optimal control problem (P_λ), defined in Introduction. We assume that A1, A2 and the following two conditions hold:

A3. The function $g(x, y)$ is continuous, convex and bounded below in R^{m+n}. For a fixed bounded set $X \subset R^m$ there exists a continuous function $Q: X \to R^n$ such that

$$g\big(x, Q(x)\big) \leqslant g(x, y)$$

for all $(x, y) \in X \times R^n$ and the function $g\big(x, Q(x)\big)$ is convex.

A4. The function f is continuous and there exists an integrable function ψ such that $f(x, y, t) \geqslant \psi(t)$ for all $(x, y) \in R^{m+n}$ and $t \in [0, 1]$; $f(\cdot, \cdot, t)$ is convex and differentiable for all $t \in [0, 1]$ and the derivatives f'_x, f'_y are continuous. The function h is continuous, it is strongly convex with respect to u in R^r with a constant $\varkappa$ uniformly in $[0, 1]$, i.e., for each $u, v \in R^r$, $a \in [0, 1]$, and $t \in [0, 1]$,

$$h\big(au + (1-a)v, t\big) \leqslant ah(u, t) + (1-a)h(v, t) - a(1-a)\varkappa |u - v|^2.$$

By a standard argument [8], there exists a unique optimal solution $(\hat{x}_\lambda, \hat{y}_\lambda, \hat{u}_\lambda)$ of problem (P_λ) for each $\lambda > 0$ such that the maximum principle holds. Moreover, the optimal control $\hat{u}_\lambda$ can be considered as a function which is continuous with respect to the time in $[0, 1]$; see [2].

Let us introduce the following optimal control problem (P_0) for the reduced system (1.2): minimize the functional

$$(3.1) \qquad J_0^*(u) = g\big(x(1), Q\big(x(1)\big)\big) + \int_0^1 \big(f\big(x(t), y(t), t\big) + h\big(u(t), t\big)\big) dt$$

over $L_2^{(r)}(0,1)$, where x, y are determined by (1.2). We shall denote the continuous function which represents the optimal control for (P_0) by $\hat{u}_0$, and the optimal trajectory by $(\hat{x}_0, \hat{y}_0)$ (where $\hat{y}_0(t) = -A_4^{-1}(t)\,(A_3(t)\hat{x}_0(t) + B_2(t)\hat{u}_0(t)))$.

THEOREM 3.1. *The following relation holds*:

$$\lim_{\lambda \to 0}(|J_\lambda(\hat{u}_\lambda) - J_0^*(\hat{u}_0)| + \|\hat{u}_\lambda - \hat{u}_0\|_{L_2} + \|\hat{x}_\lambda - \hat{x}_0\|_C + \|\hat{y}_\lambda - \hat{y}_0\|_{L_2}) = 0.$$

Proof. Let the function $\bar{u}$ be determined as follows:

$$\bar{u}(t) = \arg\min h(u,t), \quad u \in R^r.$$

for all $t \in [0,1]$. Then $\bar{u}(\cdot)$ is a continuous function [2], and

$$\varkappa |\hat{u}_\lambda(t) - \bar{u}(t)|^2 \leqslant h(\hat{u}_\lambda(t), t) - h(\bar{u}(t), t).$$

By the boundedness of g and f we get

$$J_\lambda(0) \geqslant J_\lambda(\hat{u}_\lambda) \geqslant \varkappa \|\hat{u}_\lambda - \bar{u}\|_{L_2}^2 + c.$$

Applying Lemma 2.1(v) we conclude that $\limsup_{\lambda \to 0} J_\lambda(0) < +\infty$; hence $\limsup_{\lambda \to 0} \|\hat{u}_\lambda\|_{L_2} < +\infty$.

Choose a sequence $\{\lambda_k\}$, $\lim_{k \to +\infty} \lambda_k = 0$, such that the corresponding sequence of controls $\{\hat{u}_{\lambda_k}\}$ is L_2-weakly convergent to $\tilde{u}$. By Lemma 2.1(ii) the corresponding trajectory $\hat{x}_{\lambda_k}$ is C-convergent to $\tilde{x}$ and $\hat{y}_{\lambda_k}$ is L_2-weakly convergent to $\tilde{y}$, where $(\tilde{x}, \tilde{y})$ solves (1.2) for $u = \tilde{u}$. Since the integral part of the performance index (1.3) is L_2-weakly lower semicontinuous [1], we have

(3.2) $$J_0^*(\hat{u}_0) \leqslant J_0^*(\tilde{u}) \leqslant \liminf_{k \to +\infty} J_{\lambda_k}^*(\hat{u}_{\lambda_k}),$$

where $J_{\lambda_k}^*(\hat{u}_{\lambda_k})$ is determined by (3.1) for $(\hat{x}_{\lambda_k}, \hat{y}_{\lambda_k})$. On the other hand,

(3.3) $$J_{\lambda_k}^*(\hat{u}_{\lambda_k}) \leqslant J_{\lambda_k}(\hat{u}_{\lambda_k}).$$

Applying Lemma 2.2, we choose a sequence of controls $\{u_k\}$, $u_k(t) \to \hat{u}_0(t)$ for all $t \in [0,1)$, $\sup_k \|u_k\|_C < +\infty$, such that the trajectory (x_k, y_k), corresponding to u_k and λ_k according to (1.1) satisfies

$$\lim_{k \to +\infty} (\|x_k - \hat{x}_0\|_C + |y_k(1) - Q(\hat{x}_0(1))| + |y_k(t) - \hat{y}_0(t)|) = 0$$

for all $t \in (0,1)$. Hence

(3.4) $$\lim_{k \to +\infty} J_{\lambda_k}(\hat{u}_{\lambda_k}) \leqslant \lim_{k \to +\infty} J_{\lambda_k}(u_k) = J_0^*(\hat{u}_0).$$

Combining (3.2)–(3.4), we obtain that $J_\lambda(\hat{u}_\lambda)$ is convergent to $J_0^*(\hat{u}_0)$. Using the strong convexity of $h(\cdot, t)$, we obtain L_2-strong convergence of $\hat{u}_\lambda$ to $\hat{u}_0$. The convergences of $\hat{x}_\lambda$ and $\hat{y}_\lambda$ follow from Lemma 2.1.

Remark 3.1. The following example shows that assumption A3 is essential for the obtained result. Consider problem (P_λ) with

$$g(x, y) = g_1(x) + c^T y,$$

where $c \in R^n$, $c \neq 0$. From Lemma 2.2 it follows that for every $\mu > 0$ and for every sequence $\{\lambda_k\}$, $\lim\limits_{k \to +\infty} \lambda_k = 0$, there exists a sequence of controls $\{u_k\}$ such that $\lim\limits_{k \to +\infty} y_k(1) = -\mu c$, and the remaining part of the performance index $J_{\lambda_k}(u_k)$ is tending to a constant a, which does not depend on μ. Hence

$$\lim_{k \to +\infty} J_{\lambda_k}(\hat{u}_{\lambda_k}) \leqslant \lim_{k \to +\infty} J_{\lambda_k}(u_k) = -\mu |c|^2 + a.$$

Since μ is arbitrarily chosen,

$$\lim_{\lambda \to 0} J_\lambda(\hat{u}_\lambda) = -\infty.$$

Remark 3.2. The boundedness below of g in A3 can be replaced by the following condition: the functions $g, Q, h(\cdot, t)$ are C_1 and h'_u is continuous. In order to prove Theorem 3.1 we use the relations:

$$\varkappa \|\hat{u}_\lambda\|_{L_2}^2 \leqslant J_\lambda(0) - J_\lambda(\hat{u}_\lambda) \leqslant J_\lambda(0) - J_\lambda^*(\hat{u}_\lambda)$$

$$\leqslant \left(g'_x(z_\lambda) + g'_y(z_\lambda) Q'\big(\bar{x}_\lambda(1)\big)\right)^T \big(\bar{x}_\lambda(1) - \hat{x}_\lambda(1)\big) +$$

$$+ \int_0^1 \left(f'_x\big(\bar{v}_\lambda(t), t\big)^T\big(\bar{x}_\lambda(t) - \hat{x}_\lambda(t)\big) + \right.$$

$$\left. + f'_y\big(\bar{v}_\lambda(t), t\big)^T\big(\bar{y}_\lambda(t) - \hat{y}_\lambda(t)\big) - h'_u(0, t)^T \hat{u}_\lambda(t)\right) dt,$$

where $z_\lambda = \big(\bar{x}_\lambda(1), Q\big(\bar{x}_\lambda(1)\big)\big)$ and $\bar{v}_\lambda = (\bar{x}_\lambda, \bar{y}_\lambda)$, where $(\bar{x}_\lambda, \bar{y}_\lambda)$ is determined by (1.1) for $u = 0$. Applying Lemma 2.1(v), we obtain that $\lim\limits_{\lambda \to 0} \sup \|\hat{u}_\lambda\|_{L_2} < +\infty$.

4. Uniform convergence of the optimal control

We consider problem (P_λ) under conditions A1–A4 and

A5. The functions g, Q are differentiable and the derivatives g'_x, g'_y, Q' are continuous. The function g is strongly convex with respect to y uniformly in x, belonging to a bounded set in R^m, or the function g does not depend on y. The derivatives f'_x, f'_y are Lipschitz continuous with respect

to y with a constant L_y for $x \in X$, $y \in R^n$, $t \in [0, 1]$, where X is a bounded set in R^m. The function h is differentiable with respect to u and the derivative h'_u is continuous. (In this case the boundedness below of g in A3 can be dropped; see Remark 3.2.)

Using the maximum principle [8] we obtain the relations

(4.1) $h'_u(\hat{u}_\lambda(t), t) = B_1^T(t)p_\lambda(t) + B_2^T(t)q_\lambda(t)$

for all $t \in [0, 1]$, where (p_λ, q_λ) is the solution of the adjoint equation

$$\dot{p}(t) = -A_1^T(t)p(t) - A_3^T(t)q(t) + f'_x(\hat{x}_\lambda(t), \hat{y}_\lambda(t), t),$$

(4.2) $\lambda\dot{q}(t) = -A_2^T(t)p(t) - A_4^T(t)q(t) + f'_y(\hat{x}_\lambda(t), \hat{y}_\lambda(t), t),$

$$p(1) = -g'_x(\hat{x}_\lambda(1), \hat{y}_\lambda(1)), \quad q(1) = -\frac{1}{\lambda}g'_y(\hat{x}_\lambda(1), \hat{y}_\lambda(1)).$$

For the reduced problem (P_0) we have similarly

(4.3) $h'_u(\hat{u}_0(t), t) = B_1^T(t)p_0(t) + B_2^T(t)q_0(t),$

where p_0, q_0 satisfy

$$\dot{p}(t) = -A_1^T(t)p(t) - A_3^T(t)q(t) + f'_x(\hat{x}_0(t), \hat{y}_0(t), t),$$

(4.4) $0 = -A_2^T(t)p(t) - A_4^T(t)q(t) + f'_y(\hat{x}_0(t), \hat{y}_0(t), t),$

$$p(1) = -g'_x(\hat{x}_0(1), Q(\hat{x}_0(1)))$$

since

$$g'_y(\hat{x}_0(1), Q(\hat{x}_0(1))) = 0.$$

Let us recall that the constants σ_0 and σ are defined in (2.7), $\varkappa$ is the strong convexity parameter of the function $h(\cdot, t)$, and L_y is the Lipschitz constant of f'_y with respect to y.

THEOREM 4.1. *Suppose that*

A6. $$\max\left[\frac{1}{2\varkappa}\|B_2\|_C^2, L_y\right] \leqslant \frac{\sigma}{2\sigma_0}.$$

Then for every $\theta \in (0, 1/2)$,

$$\lim_{\lambda \to 0} \max_{\theta \leqslant t \leqslant 1-\theta} (|\hat{u}_\lambda(t) - \hat{u}_0(t)| + |\hat{y}_\lambda(t) - \hat{y}_0(t)|) = 0.$$

Proof. We first prove that

(4.5) $$\lim_{\lambda \to 0} g'_y(\hat{x}_\lambda(1), \hat{y}_\lambda(1)) = 0.$$

Since the integral part of $J_\lambda(u)$ is L_2-weakly lower semicontinuous with respect to (x, y, u) and $\lim_{\lambda \to 0} J_\lambda(\hat{u}_\lambda) = J_0^*(\hat{u}_0)$, we conclude that

(4.6) $\liminf_{\lambda \to 0} g\big(\hat{x}_\lambda(1), \hat{y}_\lambda(1)\big) \leqslant g\big(\hat{x}_0(1), Q\big(\hat{x}_0(1)\big)\big)$.

On the other hand,

(4.7) $g\big(\hat{x}_\lambda(1), \hat{y}_\lambda(1)\big) \geqslant \lim_{\lambda \to 0} g\big(\hat{x}_\lambda(1), Q\big(\hat{x}_\lambda(1)\big)\big) = g\big(\hat{x}_0(1), Q\big(\hat{x}_0(1)\big)\big)$.

Combining (4.6) and (4.7), we get

$$\lim_{\lambda \to 0} g\big(\hat{x}_\lambda(1), \hat{y}_\lambda(1)\big) = g\big(\hat{x}_0(1), Q\big(\hat{x}_0(1)\big)\big).$$

Using A5, we obtain that (4.5) holds. Applying Lemma 2.1(ii) to equation (4.2), we get

(4.8) $\lim_{\lambda \to 0} \|p_\lambda - p_0\|_C = 0$.

Let $\Delta y = \hat{y}_\lambda - \hat{y}_0$ and $\Delta q_\lambda = q_\lambda - q_0$. From Theorem 3.1, a relation similar to (2.10) and from (4.8) we have

(4.9)

$$|\Delta q_\lambda(t)| \leqslant \frac{a(\lambda)}{\lambda} \exp\left(-\sigma \frac{1-t}{\lambda}\right) + \frac{\sigma_0 L_y}{\lambda} \int_t^1 \exp\left(-\sigma \frac{\tau-t}{\lambda}\right) |\Delta y_\lambda(\tau)| \, d\tau + a_1(\lambda),$$

where $a(\lambda)$ and $a_1(\lambda)$ tend to zero as $\lambda \to 0$, uniformly in $[0, 1]$. The strong convexity of h implies (see [2]) that

(4.10) $2\varkappa |\hat{u}_\lambda(t) - \hat{u}_0(t)| \leqslant \|B_1\|_C \|p_\lambda - p_0\|_C + \|B_2\|_C |\Delta q_\lambda(t)|$

for all $t \in [0, 1]$. Then, using Theorem 3.1, (2.10) and (4.10), we get

(4.11)

$$|\Delta y_\lambda(t)| \leqslant c \exp\left(-\sigma \frac{t}{\lambda}\right) + \frac{\sigma_0 \|B_2\|_C^2}{2\lambda\varkappa} \int_t^0 \exp\left(-\sigma \frac{t-\tau}{\lambda}\right) |\Delta q_\lambda(\tau)| \, d\tau + a_2(\lambda),$$

where $a_2(\lambda) \to 0$ as $\lambda \to 0$, uniformly in $[0, 1]$.

Let $v_\lambda(t) = |\Delta y_\lambda(t)| + |\Delta q_\lambda(t)|$. From (4.9) and (4.11) we have

(4.12) $v_\lambda(t) \leqslant a_3(\lambda) + c \exp\left(-\sigma \frac{t}{\lambda}\right) + \frac{a(\lambda)}{\lambda} \exp\left(-\sigma \frac{1-t}{\lambda}\right) +$

$$+ \frac{b}{\lambda} \int_0^1 \exp\left(-\sigma \frac{|t-\tau|}{\lambda}\right) v_\lambda(\tau) \, d\tau,$$

where

$$a_3(\lambda) = a_1(\lambda) + a_2(\lambda), \quad b = \sigma_0 \max\left[\frac{1}{2\varkappa} \|B_2\|_C^2, L_y\right].$$

Since $2b/\sigma < 1$, the spectral radius of the integral operator in the right-hand side of (4.12) is less then 1. Then, for all $t \in [0, 1]$,

$$v_\lambda(t) \leqslant z_\lambda(t),$$

where z_λ satisfies equation (2.24). Applying Lemma 2.3 and (4.10), we complete the proof.

Remark 4.1. The above result could be interpreted in the following way. Consider a family of problems $(P_\lambda(\gamma))$ under conditions A1–A5, where

$$J_\lambda(u, \gamma) = g\big(x(1), y(1)\big) + \int_0^1 \big(\gamma f\big(x(t), y(t), t\big) + h\big(u(t), t\big)\big) dt.$$

Then there exists $\gamma_0 > 0$ such that Theorem 4.1 holds for the problem $P_\lambda(\gamma)$ for each $\gamma < \gamma_0$.

Assuming that the function f is independent of y, one can choose a problem, equivalent to (P_λ), for which A6 holds. Theorem 4.1 generalizes the corresponding result from [5].

Remark 4.2. Suppose that $B_2 = 0$, A1 and A4 hold, the function g is convex and C_1 and h is differentiable with respect to u with h'_u being continuous. From Lemma 2.1 (for $\psi_0 = \Delta\psi_k = 0$) we obtain that $\hat{y}_k(1) \to \hat{y}_0(1)$, when $\hat{u}_\lambda$ tends L_2-weakly to $\hat{u}_0$, and $\lim\limits_{\lambda \to 0} \sup \|\hat{y}_\lambda\|_C < +\infty$. Moreover,

(4.13)
$$\max_{0 \leqslant t \leqslant 1} |\hat{y}_\lambda(t) - \hat{y}_0(t)| \leqslant c \|\hat{x}_\lambda - \hat{x}_0\|_C + a_4(\lambda)$$

$$\leqslant c \|\hat{u}_\lambda - \hat{u}_0\|_{L_2} + a_4(\lambda),$$

where $\theta \in (0, 1)$, $\lim\limits_{\lambda \to 0} a_4(\lambda) = 0$. The performance index for the reduced problem will have the form

$$J_0(u) = g\big(x(1), -A_4^{-1}(1)A_3(1)x(1)\big) + \int_0^1 \big(f\big(x(t), y(t), t\big) + h\big(u(t), t\big)\big) dt,$$

where (x, y) is determined by (1.2) with $B_2 = 0$. By repeating the arguments of Remark 3.2 and using (4.13) we obtain that $\lim\limits_{\lambda \to 0} \sup \|\hat{u}_\lambda\|_{L_2} < +\infty$. Choosing a L_2-weakly convergent subsequence of $\{\hat{u}_\lambda\}$, we have

$$J_0(\hat{u}_0) \leqslant \liminf_{\lambda \to 0} J_\lambda(\hat{u}_\lambda) \leqslant \lim_{\lambda \to 0} J_\lambda(\hat{u}_0) = J_0(\hat{u}_0).$$

By the strong convexity of $J_\lambda(\cdot)$ and Lemma 2.1(v) we get

$$\lim_{\lambda \to 0} \Big(\|\hat{u}_\lambda - \hat{u}_0\|_{L_2} + \|\hat{x}_\lambda - \hat{x}_0\|_C + \max_{0 \leqslant t \leqslant 1} |\hat{y}_\lambda(t) - \hat{y}_0(t)|\Big) = 0$$

for all $\theta \in (0, 1)$, and $\{\hat{y}_\lambda\}$ is uniformly bounded. Applying Lemma 2.1(i) to equation (4.2), we obtain that for every $\theta \in (0, 1)$,

$$\lim_{\lambda \to 0} \max_{0 \leqslant t \leqslant \theta} |p_\lambda(t) - p_0(t)| = 0,$$

where p_0 satisfies (4.4) with the final condition

$$p_0(1) = -g'_x(z) + A_3^T(1) \left(A_4^{-1}(1)\right)^T g'_y(z),$$

with $z = \left(\hat{x}_0(1), -A_4^{-1}(1)A_3(1)\hat{x}_0(1)\right)$. From (4.10) we get

$$\lim_{\lambda \to 0} \max_{0 \leqslant t \leqslant \theta} |\hat{u}_\lambda(t) - \hat{u}_0(t)| = 0.$$

Note that if g does not depend on y, then $\lim_{\lambda \to 0} \|\hat{u}_\lambda - \hat{u}_0\|_C = 0$. A similar result is obtained in [4] for quadratic g, f and h.

Remark 4.3. If the function g depends only on x, the result obtained is valid for problem (P_λ) with additional control constraints, i.e., $u(t) \in U$ where U is a closed and convex set in R^r.

5. Final discussion

This paper studies the qualitative effects due to changes of the system order for optimal control problems. Our analysis differs from the related papers in the following:

(i) The fast trajectories are involved in both the terminal and the integral parts of the performance index.

(ii) The performance index is not quadratic.

(iii) We do not use asymptotic expansions of the optimal solution.

The results obtained provide a basis for a validation of static models for optimal control systems with a dynamics fast compared with the optimization horizon. Consider the problem

$$\lambda \dot{z}(t) = A_4(t)z(t) + B_2(t)u, \quad z(0) = z^0,$$

(5.1)

$$J_\lambda(u) = g(z(1)) + \int_0^1 \left(f(z(t), t) + h(u(t), t)\right) dt \to \min,$$

where $0 < \lambda \leqslant 1$. Suppose that system (5.1) is controlled by $\hat{u}_0$ which is the optimal control for the static problem

$$f\left(-A_4^{-1}(t)B_2(t)u, t\right) + h(u, t) \to \min, \quad t \in [0, 1].$$

Then this suboptimal (open-loop) control structure leads to performance losses given by

$$S(\lambda) = J_\lambda(\hat{u}_0) - J_\lambda(\hat{u}_\lambda).$$

The function $S(\lambda)$ is called the *sensitivity measure* of the considered control structure; see [11]. Our analysis implies that, on assumptions related to A1–A6,

$$\lim_{\lambda \to 0} S(\lambda) = 0,$$

that is, for small λ the performance losses will be small enough. Moreover, the optimal solution is tending to the limit solution, uniformly in each compact subset of $(0, 1)$. Note that the limit problem does not depend on the function g (when it is bounded below; see Remark 3.1).

Similar conclusion can be obtained for the case where the optimization horizon is very large. Consider the problem

$$\dot{z}(t) = A_4 z(t) + B_2 u(t), \qquad z(0) = z^0, \qquad t \in [0, a],$$

$$J(u) = g\big(z(a)\big) + \int_0^a \big(f\big(z(t)\big) + h\big(u(t)\big)\big) dt \to \min.$$

After a time transformation, applying Theorem 4.1, one can get

$$\lim_{\substack{a \to +\infty \\ t \to +\infty \\ t/a < 1}} \hat{u}_a(t) = \hat{u}_0,$$

where

$$\hat{u}_0 = \arg \min \big(f(-A_4^{-1} B_2 u) + h(u)\big).$$

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