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ON THE EULER-LAGRANGE INEQUALITY OF A CONVEX VARIATIONAL INTEGRAL IN ORLICZ SPACES

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1. Introduction

In this paper we consider variational inequalities of a variational integral

$$I(u) = \int_X F(x, u, \ldots, D^m u) \tag{1.1}$$

defined by a convex function F. We find a solution for the inequality

$$\sum_{|\alpha| \leqslant m} \int_X N_\alpha(x, u, \ldots, D^m u) D^\alpha(v-u) \geqslant (f, v-u), \tag{1.2}$$

$N_\alpha(x, t) = \partial F(x, t)/\partial t_\alpha$, in an Orlicz–Sobolev space, in which the semilinear form in (1.2) is coercive. The method is to approximate the nonlinearities N_α in (1.2) in a suitable way. This method is related to the truncation of nonlinearities used in [S]. We first present a general stability result, Theorem 3.1, according to which limits of certain solvable variational inequalities are solvable. The existence of solutions of the approximating inequalities is derived from variational principles under the hypothesis that the Young function giving a lower bound for $\sum_{|\alpha| \leqslant m} N_\alpha(x, t) t_\alpha$ satisfies the Δ_2-condition. This additional hypothesis seems plausible, since the Δ_2-condition merely restricts the growth from above. To be able to approximate the functions F and N_α well enough we use a modified Yosida approximation; this is studied in Section 4.

This note generalizes some results in the author's dissertation [Le]. Related work has been done by Hempel [H], Landes [La], Mustonen and Simader [MS] and Simader [S]. Truncation of nonlinearities as used in [S] seems to need a more special structure for the N_α. The Yosida approxima-

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tion extends this method to the case of gradients of convex functions. In [H] the Euler–Lagrange equations are studied from the point of view of the calculus of variations; it is shown that a function minimizing a variational integral satisfies the corresponding Euler–Lagrange equation. Our choice for the test functions v in (1.2) guarantees that a solution of (1.2) minimizes the functional $I(u)-(f, u)$.

2. Preliminaries

2.1. *Notation.* We let X be a bounded domain in $\boldsymbol{R}^N$ and $m \geqslant 1$ an integer. The number of multi-indices $a \in N^N$ such that $|a| \leqslant m$ is denoted by d. We let Ψ be a *Young function*, i.e. Ψ: $\boldsymbol{R} \to \boldsymbol{R}$ is even, convex and satisfies

$$\lim_{r \to 0} \frac{\Psi(r)}{r} = 0, \qquad \lim_{r \to \infty} \frac{\Psi(r)}{r} = \infty .$$

The complementary Young function of Ψ is denoted by $\bar{\Psi}$. The *Orlicz class* consists of functions u such that

$$\int_X \Psi(u(x)) dx < \infty,$$

and the *Orlicz space* $L_\Psi(X)$ is the linear hull of the Orlicz class. The Luxemburg norm in $L_\Psi(X)$ will be denoted by $\|u\|_\Psi$. The *Orlicz–Sobolev space* of functions u such that u and its distributional derivatives up to order m lie in $L_\Psi(X)$ is denoted by $W^m L_\Psi(X)$. Ordinary Sobolev spaces are denoted by $W^{m,p}(X)$, $1 \leqslant p < \infty$. The norm in $W^m L_\Psi(X)$ is given by

$$\|u\|_{m,\Psi} := \Big(\sum_{|a| \leqslant m} \|D^a u\|_\Psi^2 \Big)^{1/2}.$$

The closure in $L_\Psi(X)$ of bounded functions with compact support in $\bar{X}$ is denoted by $E_\Psi(X)$. Similarly to $W^m L_\Psi(X)$ one defines the space $W^m E_\Psi(X)$. We denote by $W^{-m} E_{\bar{\Psi}}(X)$ the space of continuous linear functionals f on $W^m L_\Psi(X)$ which have a representation

$$(f, v) = \sum_{|a| \leqslant m} \int_X f_a D^a v$$

with $f_a \in E_{\bar{\Psi}}(X)$. For more information on Orlicz spaces see [KJF].

Functions f: $X \times \boldsymbol{R}^d \to \boldsymbol{R}$ satisfying the *Carathéodory condition*

$$x \mapsto f(x, t) \text{ is measurable for all } t \in \boldsymbol{R}^d,$$
$$t \mapsto f(x, t) \text{ is continuous for a.e. } x \in X,$$

are used to define differential operators on $W^{m,1}(X)$ by $f(u)(x) := f\big(x, D(u)(x)\big)$, where $D(u) := (D^a u)_{|a| \leqslant m}$.

By V we denote a subspace of $W^m L_\Psi(X)$. For $\int_X uv$ we use the notation (u, v).

2.2. *The variational kernels.* Functions F and N_a, which define the variational integral (1.1) and the nonlinearities in (1.2), are assumed to satisfy the Carathéodory condition and the following one.

(F) (i) For a.e. $x \in X$ the function $t \mapsto F(x, t)$ is strictly convex and differentiable with partial derivatives $N_a := \partial F/\partial t_a$, $|a| \leqslant m$.

(ii) There exists a Young function Ψ and $k_0 \in L^1(X)$ such that

$$\sum_{|a| \leqslant m} N_a(x, t) t_a \geqslant \Psi(|t|) - k_0(x) \tag{2.3}$$

for a.e. $x \in X$ and for all $t \in \boldsymbol{R}^d$.

(iii) $F(\cdot, 0) \in L^1(X)$, $N_a(\cdot, 0) \in E_{\bar{\Psi}}(X)$, $|a| \leqslant m$.

(iv) K is a convex subset of V, closed in $W^{m,1}(X)$, and $0 \in K$.

We are concerned with the finding of a solution of the problem

(P) Given $f \in W^{-m} E_{\bar{\Psi}}(X)$ find $u \in K$ such that $\sum_{|a| \leqslant m} N_a(u) D^a u \in L^1(X)$ and

$$\sum_{|a| \leqslant m} \big(N_a(u), D^a(v-u)\big) \geqslant (f, v-u) \tag{2.4}$$

for all $v \in K$ such that $F(v) \in L^1(X)$.

2.5. *Remarks.* (i) If Ψ and $\bar{\Psi}$ both satisfy the global Δ_2-condition, i.e. if $W^m L_{\Psi}(X)$ is reflexive, we may let X be unbounded, too, and K need only be a closed convex subset of $W^m L_{\Psi}(X)$ with $0 \in K$.

(ii) Let condition (F) be satisfied, and assume that $u \in K$ is a solution of problem (P). Then by the strict convexity of F the mapping $t \mapsto \big(N_a(x, t)\big)_{|a| \leqslant m}$ is strictly monotone. Thus problem (P) can admit at most one solution. Moreover, by the convexity we also obtain $I(v) - (f, v) \geqslant I(u) - (f, u)$ for all $v \in K$ such that $F(v) \in L^1(X)$. If $F(v) \notin L^1(X)$, then by condition (F) (iii) we have $I(v) = +\infty$. Thus the function u minimizes the functional $I(v) - (f, v)$ on K.

(iii) Assume instead of (F) (ii) that $t \mapsto \big(N_a(x, t)\big)_{|a| \leqslant m}$ is strongly monotone in the sense that

$$\sum_{|a| \leqslant m} \big(N_a(x, t) - N_a(x, s)\big)(t_a - s_a) \geqslant \Psi(|t-s|)$$

for a.e. $x \in X$ and for all $t, s \in \boldsymbol{R}^d$. Then the solution $u(f)$ of problem (P) satisfies

$$\int_X \Psi\big(\big|D\big(u(f) - u(g)\big)\big|\big) \leqslant 2 \int \bar{\Psi}(2|\bar{f} - \bar{g}|),$$

where $\bar{f} = (f_a)_{|a| \leqslant m}$ and $\bar{g} = (g_a)_{|a| \leqslant m}$. Thus the mapping $f \mapsto u(f)$, $W^{-m} E_{\bar{\Psi}}(X) \to K$, is continuous if in K we use Ψ-mean convergence.

To solve problem (P) we use a method related to the duality principle (minmax-theorem) of the calculus of variations, cf. [B]. We approximate (P) by a sequence of solvable problems $(P)_j$ defined by a sequence of functions $F_j\colon X \times \boldsymbol{R}^d \to \boldsymbol{R}$ satisfying the Carathéodory condition and

(SF) (i) Each F_j satisfies condition (F), $N_a^j := \partial F_j/\partial t_a$, and (F) (ii) is satisfied by Ψ and k_0 independent of $j \in \boldsymbol{N}$.

(ii) The sequence (F_j) is increasing.

(iii) $F_j(x, t) \to F_0(x, t)$ for a.e. $x \in X$ and for all $t \in \boldsymbol{R}^d$.

(iv) $N_a^j(\cdot, 0) \to N_a^0(\cdot, 0)$ in $E_{\bar{\Psi}}(X)$.

(v) (K_j) is a decreasing sequence of convex subsets of V each containing the origin and closed in $W^{m,1}(X)$, and $\bigcap_{j>0} K_j = K_0$.

2.6. *Remark.* Conditions (SF) (i) and (iii) imply that $N_a^j(x, t) \to N_a^0(x, t)$ uniformly on $\{x\} \times E$ for a.e. $x \in X$ and for all compact sets $E \subset \boldsymbol{R}^d$; see [R, p. 248].

Corresponding to the functions F_j and N_a^j we have problem

$(P)_j$ Given $f \in W^{-m} E_{\bar{\Psi}}(X)$ find $u_j \in K_j$ such that $\sum_{|a| \leqslant m} N_a^j(u_j) D^a u_j \in L^1(X)$ and

$$\sum_{|a| \leqslant m} \left(N_a^j(u_j), D^a(v - u_j)\right) \geqslant (f, v - u_j) \tag{2.7}$$

for all $v \in K_j$ such that $F_j(v) \in L^1(X)$.

3. Stability

In this section we prove the following approximation result.

3.1. THEOREM. *Let condition* (SF) *be satisfied, and* $f \in W^{-m} E_{\bar{\Psi}}(X)$. *Assume that each problem* $(P)_j$, $j > 0$, *has a solution* $u_j \in K_j$. *Then* $(P)_0$ *is solvable.*

To prove Theorem 3.1 we need the following lemma.

3.2. LEMMA. *Let V be a normed space and (u_j) a sequence in V which converges weakly to u. Then there exists a sequence $(\bar{u}_j)$ of convex combinations of the u_j which converges strongly to u. Furthermore, if (I_j) is an increasing sequence of convex functions $I_j\colon V \to \boldsymbol{R} \cup \{\pm\infty\}$, then*

$$\limsup I_j(\bar{u}_j) \leqslant \limsup I_j(u_j). \tag{3.3}$$

Proof. The existence of $\bar{u}_j$ converging strongly to u follows from the lemma of Mazur, see e.g. [Y, p. 120]. For the latter statement we let $I := \limsup I_j(u_j) < +\infty$. Then, given $J > I$ there exists n_J such that $I_j(u_j) \leqslant J$ for $j \geqslant n_J$. If

$$\bar{u}_j = \sum_{k=j}^{k_j} \lambda_{j,k} u_k, \quad \lambda_{j,k} \geqslant 0, \quad \sum_{k=j}^{k_j} \lambda_{j,k} = 1, \tag{3.4}$$

then for $j \geqslant n_J$ the convexity yields

$$I_j(\bar{u}_j) \leqslant \sum_{k=j}^{k_j} \lambda_{j,k} I_j(u_k) \leqslant \sum_{k=j}^{k_j} \lambda_{j,k} I_k(u_k) \leqslant J.$$

Thus (3.3) follows. ∎

3.5. *Remark.* In the proof of Theorem 3.1 the assumption that (F_j) is increasing is used to obtain inequalities (3.3), (3.7) and (3.12). The contents of (3.7) is to guarantee that $\bar{u}_j$ is an admissible test function for (2.7). Inequality (3.12) gives the passage to the limit in $(\mathrm{P})_j$. Therefore if these three inequalities are separately verified, the statement of Theorem 3.1 remains true.

Proof of Theorem 3.1. We may assume that $F_j(x, 0) = N_a^j(x, 0) = 0$, $|a| \leqslant m$, since otherwise we consider $F_j(x, t) - F_j(x, 0) - \sum_{|a| \leqslant m} N_a^j(x, 0) t_a$ instead of $F_j(x, t)$ and $(f_j, v) := (f, v) - \sum_{|a| \leqslant m} \big(N_a^j(\cdot, 0), D^a v\big)$ instead of (f, v).

By (F) (iii) the function $v = 0$ is an admissible test function in (2.7). This substitution yields by (F) (ii)

$$\int_X \Psi\big(|D(u_j)|\big) - \int_X k_0 \leqslant \sum_{|a| \leqslant m} \big(N_a^j(u_j), D^a u_j\big) \leqslant (f_j, u_j)$$
$$\leqslant \int_X \bar{\Psi}(2|\bar{f}_j|) + \int_X \Psi\big(\tfrac{1}{2}|D(u_j)|\big),$$

where $\bar{f}_j = (f_{j,a})_{|a| \leqslant m}$ if $(f_j, v) = \sum_{|a| \leqslant m} (f_{j,a}, D^a v)$ with $f_{j,a} \in E_{\bar{\Psi}}(X)$. Therefore it follows that the integrals $\int_X \Psi\big(|D(u_j)|\big)$ remain bounded. Hence (u_j) is a bounded sequence in $W^m L_\Psi(X)$ and $\sum_{|a| \leqslant m} \big(N_a^j(u_j), D^a u_j\big)$ is bounded in $\mathbf{R}$. Since

$$0 \leqslant \int_X F_j(u_j) \leqslant \sum_{|a| \leqslant m} \big(N_a^j(u_j), D^a u_j\big) \leqslant \text{constant} \tag{3.6}$$

the sequence $\int_X F_j(u_j)$ is bounded. By the de la Vallée-Poussin theorem the sequence $\big(D(u_j)\big)$ is uniformly equi-integrable. Therefore the sequence (u_j) converges weakly in $W^{m,1}(X)$ and $D^a u_j \to D^a u$ in $L_\Psi(X)$ for the topology $\sigma\big(L_\Psi(X), E_{\bar{\Psi}}(X)\big)$ for each $|a| \leqslant m$.

Now, using Lemma 3.2 we find a sequence $(\bar{u}_j)$ of convex combinations of u_j converging to u strongly in $W^{m,1}(X)$. For some subsequence of (u_j) we may assume that $D^a \bar{u}_j \to D^a u$ a.e. in X for all $|a| \leqslant m$, and $\int_X F_j(u_j) \to I \in \mathbf{R}$. If $\bar{u}_j$ is given by (3.4), we obtain

$$0 \leqslant F_j(\bar{u}_j) \leqslant \sum_{k=j}^{k_j} \lambda_{j,k} F_j(u_k) \leqslant \sum_{k=j}^{k_j} \lambda_{j,k} F_k(u_k) \tag{3.7}$$
$$\leqslant \sum_{k=j}^{k_j} \lambda_{j,k} \sum_{|a| \leqslant m} N_a^k(u_k) D^a u_k.$$

Since by construction $\bar{u}_j \in K_j$, we may substitute $v = \bar{u}_j$ in (2.7) to obtain

$$(f_j, u_j - \bar{u}_j) \geqslant \sum_{|\alpha| \leqslant m} \big(N_\alpha^j(u_j), D^\alpha(u_j - \bar{u}_j)\big)$$
$$\geqslant \int_X F_j(u_j) - \int_X F_j(\bar{u}_j).$$

This yields $\liminf \int_X F_j(\bar{u}_j) \geqslant \liminf \int_X F_j(u_j)$. Therefore by the second part of Lemma 3.2 we have $\lim \int_X F_j(\bar{u}_j) = I$.

Let $\varepsilon > 0$ and choose j_ε such that for $j \geqslant j_\varepsilon$

$$(f_j, u_j - \bar{u}_j) \leqslant \varepsilon, \qquad \int_X F_j(u_j) \geqslant I - \varepsilon, \qquad \int_X F_j(\bar{u}_j) \leqslant I + \varepsilon.$$

Then for $j \geqslant j_\varepsilon$

$$\int_X \big(F_j(u_j) + F_j(\bar{u}_j) - 2F_j\big((u_j + \bar{u}_j)/2\big)\big) \tag{3.8}$$
$$\leqslant 2 \int_X \big(F_j(u_j) - F_j\big((u_j + \bar{u}_j)/2\big)\big) + 2\varepsilon$$
$$\leqslant 2 \sum_{|\alpha| \leqslant m} \big(N_\alpha^j(u_j), D^\alpha\big(u_j - (u_j + \bar{u}_j)/2\big)\big) + 2\varepsilon$$
$$\leqslant (f_j, u_j - \bar{u}_j) + 2\varepsilon \leqslant 3\varepsilon.$$

Since the integrand in the first line of (3.8) is nonnegative, we obtain

$$F_j(u_j) + F_j(\bar{u}_j) - 2F_j\big((u_j + \bar{u}_j)/2\big) \to 0 \quad \text{a.e. in } X. \tag{3.9}$$

Now, we show that for some subsequence

$$D^\alpha u_j \to D^\alpha u \quad \text{a.e. in } X \text{ for all } |\alpha| \leqslant m. \tag{3.10}$$

For this let ξ be a limit point of $\big(D(u_j)(x)\big)_{j \in N}$. Since $D^\alpha \bar{u}_j(x) \to D^\alpha u(x)$ and $F_j(x, t) \to F_0(x, t)$ uniformly in t on compact sets, we obtain from (3.9)

$$F_0(x, \xi) + F_0\big(x, D(u)(x)\big) - 2F_0\big(x, (\xi + D(u)(x))/2\big) = 0.$$

The strict convexity yields $\xi = D(u)(x)$. Thus (3.10) follows.

Since $N_\alpha^j(u_j) \to N_\alpha^0(u)$ a.e. in X, inequality (3.6) and Fatou's lemma yield $\sum_{|\alpha| \leqslant m} N_\alpha^0(u) D^\alpha u \in L^1(X)$. Because $\bar{u}_j \in K_j$, $\bar{u}_j \to u$ in $W^{m,1}(X)$, and $\bigcap_{j>0} K_j = K_0$, we have $u \in K_0$. Now, for $v \in K_0$ such that $F_0(v) \in L^1(X)$ we have $v \in K_j$ and $F_j(v) \in L^1(X)$. Since

$$\sum_{|\alpha| \leqslant m} N_\alpha^j(u_j) D^\alpha(v - u_j) \leqslant F_j(v) - F_j(u_j) \leqslant F_j(v) \leqslant F_0(v), \tag{3.11}$$

we obtain by Fatou's lemma

$$\limsup \sum_{|\alpha| \leqslant m} \big(N_\alpha^j(u_j), D^\alpha(v - u_j)\big) \leqslant \sum_{|\alpha| \leqslant m} \big(N_\alpha^0(u), D^\alpha(v - u)\big). \tag{3.12}$$

Using (3.12) in (2.7) gives

$$\sum_{|\alpha|\leqslant m} (N_\alpha^0(u), D^\alpha(v-u)) \geqslant (f_0, v-u).$$

Thus u solves $(\mathrm{P})_0$. ∎

3.13. *Remark*. The statements made in Remark 2.5 (i) follow easily from the above proof. Namely, if $W^m L_\Psi(X)$ is reflexive we do not need to use the weak compactness in $W^{m,1}(X)$. Instead of this we can work in $W^m L_\Psi(X)$ all the time.

4. The Yosida approximation

In this section we shall extend a well-known method of approximating convex functions and their derivatives, cf. [B, Ch. 2]. Let $F\colon \boldsymbol{R}^d \to \boldsymbol{R}$ be a convex function and $\Phi\colon \boldsymbol{R}\to\boldsymbol{R}$ a continuous strictly increasing odd function satisfying $\Phi(0)=0$ and $\Phi(r)\to+\infty$ as $r\to+\infty$. We assume that F is differentiable, $F(0)=0$ and for $N:=DF$, $N(0)=0$, and that Φ is locally Lipschitz on $]0,\infty[$. Set

$$K(x) := \Phi(|x|)\frac{x}{|x|} \quad \text{for } x\in\boldsymbol{R}^d, \qquad \text{and} \qquad \Psi(r) := \int_0^r \Phi(s)\,ds.$$

For $\varepsilon>0$ define

$$F_\varepsilon(x) := \inf\{\varepsilon^{-1}\Psi(|x-y|)+F(y)\colon\ y\in\boldsymbol{R}^d\}. \tag{4.1}$$

Since the function $y\mapsto\varepsilon^{-1}\Psi(|x-y|)+F(y)$ is strictly convex and tends to $+\infty$ as $|y|\to+\infty$, it attains its minimum at a unique point $y=:J_\varepsilon(x)$, where

$$K(y-x)+\varepsilon N(y) = 0. \tag{4.2}$$

This shows that $x\mapsto x+K^{-1}(\varepsilon N(x))$ is surjective. Injectivity follows easily from the monotonicity of N and the strict monotonicity of K. Thus J_ε is a homeomorphism. We set

$$N_\varepsilon(x) := \varepsilon^{-1}K(x-J_\varepsilon(x)) = N(J_\varepsilon(x)). \tag{4.3}$$

We collect some properties of N_ε in

4.4. Proposition. (i) *N_ε is monotone.*

(ii) $|x-J_\varepsilon(x)| \leqslant \Phi^{-1}(\varepsilon|N(x)|)$.

(iii) $|N_\varepsilon(x)| \leqslant |N(x)|$.

(iv) $(J_\varepsilon(x), x)\geqslant 0, \quad |J_\varepsilon(x)|\leqslant|x|, \quad |x-J_\varepsilon(x)|\leqslant|x|$.

(v) $\varepsilon|N_\varepsilon(x)| \leqslant |K(x)|$.

(vi) *If $\Phi(r)=r^\alpha$ with $\alpha>0$, then $(N_\varepsilon)_\delta = N_\gamma$, where $\Phi(\gamma)=\Phi(\varepsilon)+\Phi(\delta)$. Especially $|N_\varepsilon(x)|\to|N(x)|$.*

Proof. Since

$$\begin{aligned}&\big(J_\varepsilon^{-1}(x)-J_\varepsilon^{-1}(y),\ N(x)-N(y)\big)\\&\quad=\big(x-y,\ N(x)-N(y)\big)+\big(K^{-1}(\varepsilon N(x))-K^{-1}(\varepsilon N(y)),\ N(x)-N(y)\big)\geqslant 0,\end{aligned}$$

the monotonicity of N_ε follows. Part (ii) follows from

$$\begin{aligned}0&\leqslant\big(x-J_\varepsilon(x),\ N(x)-N_\varepsilon(x)\big)\\&=\big(x-J_\varepsilon(x),\ N(x)\big)-\varepsilon^{-1}\Phi\big(|x-J_\varepsilon(x)|\big)|x-J_\varepsilon(x)|.\end{aligned}$$

Thus we also have $|N_\varepsilon(x)|=\varepsilon^{-1}\big|K\big(x-J_\varepsilon(x)\big)\big|=\varepsilon^{-1}\Phi\big(|x-J_\varepsilon(x)|\big)\leqslant|N(x)|$. For (iv) we use

$$\begin{aligned}\big(y,\ J_\varepsilon^{-1}(y)\big)&=|y|^2+\big(y,\ K^{-1}(\varepsilon N(y))\big)\\&=|y|^2+\Phi\big(|\varepsilon N(y)|\big)|N(y)|^{-1}\big(y,\ N(y)\big)\geqslant|y|^2.\end{aligned}$$

Therefore (iv) obtains. Inequality (v) follows by

$$\varepsilon|N_\varepsilon(x)|=\big|K\big(x-J_\varepsilon(x)\big)\big|=\Phi\big(|x-J_\varepsilon(x)|\big)\leqslant\Phi(|x|)=|K(x)|.$$

Part (vi) follows by using the defining equation (4.2) twice and using the multiplicative property $\Phi^{-1}(rs)=\Phi^{-1}(r)\Phi^{-1}(s)$.

4.5. *Remark.* If $\Phi(r)=r$, then N_ε is the Yosida approximation studied for example in [B, Ch. 2]. In this case J_ε is a monotone contraction and N_ε satisfies $\varepsilon|N_\varepsilon(x)-N_\varepsilon(y)|\leqslant|K(x)-K(y)|$ (where $K=$ identity). It remains open which of these properties hold in the above case.

Next we collect some properties of F_ε in

4.6. Proposition. (i) *F_ε is convex.*

(ii) $F_\varepsilon(x)=\varepsilon^{-1}\Phi\big(|x-J_\varepsilon(x)|\big)+F\big(J_\varepsilon(x)\big)$.

(iii) $F\geqslant F_\delta\geqslant F_\varepsilon$ *for* $0<\delta<\varepsilon$.

(iv) $F_\varepsilon\to F$ *uniformly on compact sets as* $\varepsilon\to 0$.

(v) $F_\varepsilon(x)\leqslant\varepsilon^{-1}\Phi(|x|)$.

(vi) *F_ε is differentiable and* $F'_\varepsilon=N_\varepsilon$.

Proof. To prove (i) let $x_0,\ x_1\in\mathbf{R}^d$, $0\leqslant r\leqslant 1$ and let $y_i=J_\varepsilon(x_i)$. Set $y=(1-r)y_0+ry_1$. Then

$$\begin{aligned}F_\varepsilon\big((1-r)x_0+rx_1\big)&\leqslant\varepsilon^{-1}\Psi\big(|(1-r)x_0+rx_1-y|\big)+F(y)\\&\leqslant(1-r)F_\varepsilon(x_0)+rF_\varepsilon(x_1)\end{aligned}$$

because of the convexity of Ψ and F. Part (ii) follows from (4.2); (iii) and (v) are obvious. Since by Proposition 4.4(ii) we have $J_\varepsilon(x)\to x$ uniformly on compact sets, claim (iv) follows by using the inequality

$\varepsilon^{-1}\Psi(\Phi^{-1}(\varepsilon r)) \leqslant r\Phi^{-1}(\varepsilon r)$ for $\varepsilon > 0$, $r \geqslant 0$. If J_ε were differentiable, we could calculate as follows:

$$
\begin{aligned}
(4.7) \qquad F'_\varepsilon(x)h &= \varepsilon^{-1}\big(K(x-J_\varepsilon(x)), (1-J'_\varepsilon(x))h\big)+F'(J_\varepsilon(x))J'_\varepsilon(x)h \\
&= \varepsilon^{-1}\big(\varepsilon N(J_\varepsilon(x)), (1-J'_\varepsilon(x))h\big)+\big(N(J_\varepsilon(x)), J'_\varepsilon(x)h\big) \\
&= \big(N_\varepsilon(x), h\big).
\end{aligned}
$$

This calculation can be carried through by using finite differences and appropriate remainders if J_ε is locally Lipschitz. We now show that J_ε can be approximated by locally Lipschitzian mappings of proper type.

Let $\delta > 0$ and let η be an increasing continuous even function satisfying $\eta(r) = 1$ for $r > 2\delta$, $\eta(r) = 0$ for $|r| < \delta$ and linear on $[\delta, 2\delta]$. Since Φ is locally Lipschitz on $]0, \infty[$, the function $x \mapsto \delta x + \eta(|x|)K(x)$ is locally Lipschitz on $\mathbf{R}^d$ and also strongly monotone. Let

$$A(x, y) := \delta(y-x)+\eta(|x-y|)K(y-x)+\varepsilon N(y).$$

Then for each $x \in \mathbf{R}^d$ there is a unique $J^\delta(x)$ such that $A(x, J^\delta(x)) = 0$. Moreover, $x \mapsto J^\delta(x)$ is locally Lipschitz. Let $\Phi_\delta(r) := \delta r + \eta(r)\Phi(r)$, and define $F_{\varepsilon,\delta}$ and $N_{\varepsilon,\delta}$ as in (4.1) and (4.2). Then $J^\delta(x)$ is the solution of (4.2). By (4.7), $F_{\varepsilon,\delta}$ is differentiable, and $F'_{\varepsilon,\delta} = N_{\varepsilon,\delta}$. Since Φ is strictly increasing, $J^\delta(x)$ tends to the solution of the equation $K(y-x)+\varepsilon N(y) = 0$ as $\delta \to 0$. Thus $J^\delta(x) \to J_\varepsilon(x)$ and also $F_{\varepsilon,\delta}(x) \to F_\varepsilon(x)$ and $N_{\varepsilon,\delta}(x) \to N_\varepsilon(x)$ as $\delta \to 0$. Therefore the theorem follows from the next lemma.

4.8. LEMMA. *Let (F_j) be a sequence of differentiable convex functions on $\mathbf{R}^d$, and assume that $F_j(x) \to F(x)$ and $F'_j(x) =: N_j(x) \to N(x)$ for each $x \in \mathbf{R}^d$. If N is continuous, then F is differentiable, and $F' = N$.*

Proof. Let $x, y \in \mathbf{R}^d$, and define $f_j(r) := F_j(x+ry)$ and $h_j(r) := \big(N_j(x+ry), y\big)$, and similarly for f and n. Then $f'_j =: n_j$ is increasing, and $f_j(r) \to f(r)$, $n_j(r) \to n(r)$ for all $r \in \mathbf{R}$. Since n is continuous, it follows that $n_j \to n$ uniformly on compact sets. Therefore f is differentiable, and $f' = n$. Hence the function F has directional derivatives $F'(x)y = f'(0) = n(0) = \big(N(x), y\big)$, and the lemma follows. ∎

4.9. *Remark.* The above propositions and their proofs hold also even if F is not assumed to be differentiable. In this case N is the subdifferential of F, and equation (4.2) should read $K(y-x)+\varepsilon N(y) \ni 0$. The mapping J_ε is no more bijective, but it is surjective and constant on the sets $x+K^{-1}(\varepsilon N(x))$. In Proposition 4.4 the norms $|N(x)|$ should be replaced by $|N^0(x)|$, where $N^0(x) \in N(x)$ is the element with minimal norm. The above propositions also apply if $\mathbf{R}^d$ is replaced by a Hilbert space H and F is a convex lower semicontinuous function on H. In the finite-dimensional case Φ need not be locally Lipschitz on $]0, \infty[$, since in the approximation we may take $\Phi_\delta(r) = \delta r + \eta(r)\Phi * \tau_\delta(r)$ for a suitable τ.

5. The theorem

In this section we shall show that problem (P) is solvable under mild regularity properties of Ψ. Let Φ be a continuous strictly increasing odd function on $\boldsymbol{R}$ with $\Phi(r) \to +\infty$ as $r \to +\infty$. We assume that $\Psi(r) := \int_0^r \Phi(s)\,ds$ satisfies the Δ_2-condition, i.e. $\Psi(2r) \leqslant k\Psi(r)$ for some $k > 0$ and for all $r \leqslant r_0$. This is equivalent to

$$r\Phi(r) \leqslant c_1 \Psi(r) + c_2 \quad \text{for all } r \geqslant 0 \tag{5.1}$$

where $c_1, c_2 \geqslant 0$ are constants.

5.2. THEOREM. *Let condition* (F) *be satisfied and assume that* Ψ *is as above. Then problem* (P) *is solvable.*

Set $M_c(x) := \max\{F(x, s)\colon |s| \leqslant c\}$ for $c > 0$.

5.3. COROLLARY. *Let the hypothesis of Theorem* 5.2 *be satisfied, and assume that each* $M_c \in L^1_{\mathrm{loc}}(X)$. *Then there exists* $u \in W^m L_\Psi(X)$ *such that* $N_a(u) \in L^1_{\mathrm{loc}}(X)$, $|a| \leqslant m$, *and*

$$\sum_{|a| \leqslant m} (N_a(u), D^a v) = (f, v) \quad \textit{for all } v \in C_0^\infty(X). \tag{5.4}$$

Proof. Let $u \in V$ be a solution of problem (P). Since $M_c \in L^1_{\mathrm{loc}}(X)$ for $c > 0$ and $F(\cdot, 0) \in L^1(X)$, each $v \in C_0^\infty(X)$ is an admissible test function for (2.4). The convexity of $t \mapsto F(x, t)$ yields the inequality

$$\sum_{|a| \leqslant m} N_a(x, t) s_a \leqslant \sum_{|a| \leqslant m} N_a(x, t) t_a + F(x, s) - F(x, t). \tag{5.5}$$

Fix an index $a \in \boldsymbol{N}^N$, and let $s_a := \operatorname{sgn} N_a(x, t)$ and $s_b := 0$ for $b \neq a$. Then (5.5) gives

$$|N_a(x, t)| \leqslant \sum_{|a| \leqslant m} N_a(x, t) t_a + M_1(x) - F(x, t). \tag{5.6}$$

We use (5.6) for $t_a = D^a u$, $|a| \leqslant m$. Since u solves (P), $\sum_{|a| \leqslant m} N_a(u) D^a u \in L^1(X)$. Then by (F) (iii) we also have $F(u) \in L^1(X)$. Therefore the assumption $M_1 \in L^1_{\mathrm{loc}}(X)$ yields a locally integrable bound for $N_a(u)$.

Fix $v \in C_0^\infty(X)$. We use (2.4) for rv, $r \in \boldsymbol{R}$, to obtain

$$r\Big(\sum_{|a| \leqslant m} (N_a(u), D^a v) - (f, v)\Big) \geqslant \sum_{|a| \leqslant m} (N_a(u), D^a u) - (f, u). \tag{5.7}$$

Now divide both sides of inequality (5.7) by r, and then let $r \to \pm\infty$ to obtain (5.4). ■

To prove Theorem 5.2 we apply Theorem 3.1 twice. We may assume that $F(x, 0) = N_a(x, 0) = 0$, since otherwise we consider $F(x, t) - F(x, 0) - \sum_{|a| \leqslant m} N_a(x, 0) t_a$ instead of $F(x, t)$ and $(f, v) - \sum_{|a| \leqslant m} (N_a(\cdot, 0), D^a v)$ instead of (f, v). It is easily seen that condition (F) remains true, and that the new problem (P) is equivalent to the original one.

For $\eta > 0$ and $\varepsilon > 0$ let

$$F_\varepsilon^\eta(x, t) := \eta\Psi(|t|) + F_\varepsilon(x, t), \tag{5.8}$$

where $t \mapsto F_\varepsilon(x, t)$ is defined by (4.1) for a.e. $x \in X$. We also denote $N_a^{\eta,\varepsilon} := \partial F_\varepsilon^\eta / \partial t_a$.

In the following three lemmas we show that problem (P) is solvable for F_ε^η.

5.9. LEMMA. *F_ε^η satisfies condition* (F).

Proof. First we consider the Carathéodory condition. Let $\{s_j : j \in N\}$ be a dense subset of R^d, and $f_j(x) := \varepsilon^{-1}\Psi(|s_j - t|) + F(x, s_j)$. Then each f_j is measurable and $F_\varepsilon(x, t) = \inf\{f_j(x) : j \in N\}$. The partial derivatives $\partial F_\varepsilon / \partial t_a$ are then measurable, too. From the result of Section 4 it follows that the mappings $t \mapsto F_\varepsilon(x, t)$ and $t \mapsto \partial F_\varepsilon(x, t)/\partial t_a$ are continuous. Therefore F_ε^η and $N_a^{\eta,\varepsilon}$ satisfy the Carathéodory condition.

Since $\sum_{|a| \leqslant m} (\partial F_\varepsilon(x, t)/\partial t_a) t_a \geqslant 0$, we get

$$\sum_{|a| \leqslant m} N_a^{\eta,\varepsilon}(x, t) t_a \geqslant \eta \sum_{|a| \leqslant m} \Phi(|t|) \frac{t_a}{|t|} t_a = \eta\Phi(|t|)|t| \geqslant \eta\Psi(|t|). \tag{5.10}$$

Hence condition (F) (ii) is satisfied. Condition (F) (iii) holds trivially, since $F_\varepsilon^\eta(x, 0) = N_a^{\eta,\varepsilon}(x, 0) = 0$. ∎

In the next lemma we show that the functional $I(u) := \int_X F_\varepsilon^\eta(u) - (f, u)$ attains its minimum on K.

5.11. LEMMA. *There exists $u = u_{\eta,\varepsilon} \in K$ such that*

$$I(u) = \inf_{v \in K} I(v).$$

Proof. Let $(u_j) \subset K$ be a minimizing sequence. Since $F_\varepsilon^\eta(x, t) \geqslant \eta\Psi(|t|)$, it follows that the integrals $\int_X \Psi(|D(u_j)|)$ remain bounded. Now, as in the proof of Theorem 3.1 we may assume that $D^a u_j \to D^a u$ in $\sigma(L_\Psi(X), E_{\bar\Psi}(X))$ and weakly in $L^1(X)$ for $|a| \leqslant m$. If $\bar u_j$ denote the convex combinations, we obtain by Fatou's lemma and Lemma 3.2

$$I(u) \leqslant \limsup I(\bar u_j) \leqslant \limsup I(u_j) = \inf_{v \in K} I(v).$$

Since $u \in K$, the lemma is proved. ∎

5.12. LEMMA. *The function $u = u_{\eta,\varepsilon} \in K$ solves the variational inequality*

$$\sum_{|a| \leqslant m} (N_a^{\eta,\varepsilon}(u), D^a(v - u)) \geqslant (f, v - u) \tag{5.13}$$

for all $v \in K$.

Proof. Since u minimizes the functional I on K, inequality (5.13) follows if we show that $v \mapsto \int_X F_\varepsilon^\eta(v)$ has directional derivatives at u equal to $\sum_{|\alpha| \leqslant m} (N_\alpha^{\eta,\varepsilon}(u), D^\alpha v)$. For this we use the Δ_2-property of Ψ.

Let $u, v \in W^m L_\Psi(X)$ and $t \in \mathbf{R}$, $|t| \leqslant 1$. A simple calculation using Proposition 4.4(v) yields

$$\left|\frac{\partial}{\partial t} F_\varepsilon^\eta(x, D(u)+tD(v))\right| \leqslant (\eta+\varepsilon^{-1})\,\Phi(|D(u)+tD(v)|)\,|D(v)| \tag{5.14}$$

$$\leqslant (\eta+\varepsilon^{-1})\,\Phi(|D(u)|+|D(v)|)\,|D(v)|.$$

For $r, s \geqslant 0$ we have by Young's inequality

$$\begin{aligned} \Phi(r+s)s &\leqslant \bar{\Psi}(\Phi(r+s)) + \Psi(s) \\ &= (r+s)\Phi(r+s) - \Psi(r+s) + \Psi(s) \\ &\leqslant (c_1-1)\Psi(r+s) + c_2 + \Psi(s) \\ &\leqslant (c_1-1)\frac{k}{2}(\Psi(r)+\Psi(s)) + c_2 + \Psi(s), \end{aligned} \tag{5.15}$$

where we have used (5.1).

Since $|D(u)|$, $|D(v)| \in L_\Psi(X)$, we conclude by Lebesgue's dominated convergence theorem that $\int_X F_\varepsilon^\eta(u)$ has directional derivative

$$\int_X \sum_{|\alpha| \leqslant m} N_\alpha^{\eta,\varepsilon}(u)\, D^\alpha v. \quad \blacksquare$$

Proof of Theorem 5.2. Let (ε_j) be a decreasing sequence of positive numbers with $\varepsilon_j \to 0$, and let $F_j := F_{\varepsilon_j}^\eta$ for a fixed $\eta > 0$. By Propositions 4.4 and 4.6 Theorem 3.1 can be applied to inequality (5.13). Hence we obtain a function $u = u_\eta \in K$ such that $\sum_{|\alpha| \leqslant m} N_\alpha(u) D^\alpha u \in L^1(X)$ and

$$\sum_{|\alpha| \leqslant m} (\eta K_\alpha(u) + N_\alpha(u), D^\alpha(v-u)) \geqslant (f, v-u) \tag{5.16}$$

for all $v \in K$ such that $F(v) \in L^1(X)$, where $K_\alpha(t) := \Phi(|t|)\, t_\alpha/|t|$.

To prove that u_η tends to a solution of problem (P) as $\eta \to 0$, we let (η_j) be a decreasing sequence of positive numbers with $\eta_j \to 0$. Let $F_j(x, t) := \eta_j \Psi(|t|) + F(x, t)$. Theorem 3.1 does not directly apply to the sequence (F_j), since (F_j) is not increasing. However, we can use Remark 3.5 to overcome this. First, inequality (3.3) remains valid for $I_j(v) = \int_X F_j(v)$, since now

$$\limsup I_j(\bar{u}_j) = \limsup \int_X F(\bar{u}_j) \leqslant \limsup \int_X F(u_j)$$

$$= \limsup I_j(u_j).$$

Inequality (3.7) was used to guarantee that the functions $\bar{u}_j$ are admissible test functions for (2.7), too. Here this condition reduces to verifying (3.7) with F_j and N_α^j replaced by F and N_α, respectively. Hence the functions $\bar{u}_j$ are admissible test functions for (5.16).

To obtain (3.12) we note that

$$\sum_{|\alpha| \leqslant m} N_\alpha^j(u_j) D^\alpha(v-u_j) \leqslant F_j(v) = \eta_j \Psi(|D(v)|) + F(v). \tag{5.17}$$

Hence the left-hand side of inequality (5.17) has an integrable bound (independent of j) for each $v \in K$ such that $F(v) \in L^1(X)$. Therefore inequality (3.12) follows by Fatou's lemma, and Theorem 5.2 is proved. ■

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