CONSEQUENCES OF THE LEIBNIZ CONDITION

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Let X be a linear space over a field $\mathcal{F}$ of scalars of characteristic zero. Denote by $L(X)$ the set of all linear operators with domains and ranges in X. Let $L_0(X) = \{A \in L(X) : \operatorname{dom} A = X\}$.

If X is a commutative algebra over $\mathcal{F}$ with an operator $D \in L(X)$ such that the domain of D is a subalgebra of X, i.e. such that $x, y \in \operatorname{dom} D$ implies $xy \in \operatorname{dom}$, then we shall write : $D \in \mathsf{A}(X)$. If $D \in \mathsf{A}(X)$ satisfies the classical *Leibniz condition* for the product:

$$D(xy) = xDy + yDx \quad \text{for } x, y \in \operatorname{dom} D \tag{1}$$

then X is said to be a *Leibniz algebra*.

In the present paper we shall study particular properties which distinguish Leibniz algebras among algebras with operators having a product rule of a more complicated form. Several examples of such conditions are given in [1]–[6]. Some of the presented results have been published before, sometimes with a slightly different formulation, sometimes with a slightly different proof.

In general, if $D \in \mathsf{A}(X)$ then we may write

$$f_D(x, y) = D(xy) - c_D(xDy + yDx) \quad \text{for } x, y \in \operatorname{dom} D,$$

where c_D is a scalar depending on D only. Clearly, f_D is a bilinear (i.e. linear in each variable) symmetric form. This form is called the *non-Leibniz component*. Recall that in the case of a right invertible operator $D \in \mathsf{A}(X)$ the algebra X is said to be a *D-algebra* (cf. [1], Section 6.1, also [6]).

By the definition of the non-Leibniz component, the product rule for every $D \in \mathsf{A}(X)$ can be written as follows:

$$D(xy) = c_D(xDy + yDx) + f_D(x, y) \quad \text{for } x, y \in \operatorname{dom} D.$$

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By definition, X is a Leibniz algebra if and only if $c_D = 1$ and $f_D = 0$.

Suppose that $D, D' \in \mathsf{A}(X)$, $\operatorname{dom} D \cap \operatorname{dom} D' \neq \emptyset$, and D, D' satisfy the Leibniz condition. Clearly, for every $a, b \in X$ the operator $aD + bD' \in \mathsf{A}(X)$ also satisfies the Leibniz condition.

The non-Leibniz components for powers of D are determined by recursive (equivalent) formulae. Namely, for all $k \in \mathbf{N}$, $x, y \in \operatorname{dom} D^k$ such that $f_D(x, y) \in \operatorname{dom} D^k$, we have $xy \in \operatorname{dom} D^k$ and

$$D^k(xy) = c_D^k(xD^k y + yD^k x) + f_D^{(k)}(x, y), \tag{2}$$

where $f_D^{(1)} = f_D$ and for $k = 2, 3, \ldots$,

$$\begin{aligned} f_D^{(k)}(x, y) = {} & c_D^k[(Dx)D^{k-1}y + (D^{k-1}x)Dy] \\ & + c_D^{k-1}[f_D(x, D^{k-1}y) + f_D(D^{k-1}x, y)] + Df_D^{(k-1)}(x, y) \end{aligned} \tag{3}$$

or

$$\begin{aligned} f_D^{(k)}(x, y) = {} & c_D^k[(Dx)D^{k-1}y + (D^{k-1}x)Dy] \\ & + c_D[f_D^{(k-1)}(x, Dy) + f_D^{(k-1)}(Dx, y)] + D^{k-1}f_D(x, y). \end{aligned} \tag{3'}$$

Thus we have

$$c_{D^k} = c_D^k \quad \text{and} \quad f_{D^k} = f_D^{(k)} \quad \text{for } k \in \mathbf{N}. \tag{4}$$

We also have

$$c_{pD} = c_D, \quad f_{pD}^{(k)} = p^k f_D^{(k)} \quad \text{for } p \in \mathcal{F},\ k \in \mathbf{N}.$$

Clearly, if X is a Leibniz algebra then X is not a Leibniz algebra for all D^n $(n \geq 2)$. Indeed, we have $c_{D^n} = c_D^n = 1$ and, by the Leibniz Formula

$$D^n(xy) = \sum_{k=0}^{n} \binom{n}{k}(D^k x)(D^{n-k}y) \quad \text{for } x, y \in \operatorname{dom} D, \tag{5}$$

we conclude that

$$f_D(x, y) = \sum_{k=1}^{n-1} \binom{n}{k}(D^k x)(D^{n-k}y) \neq 0 \quad \text{on } \operatorname{dom} D.$$

In order to give a characterization of those algebras which are Leibniz algebras we have to define some mappings (cf. [5] in particular cases and [6] in the general case).

Definition 1. Suppose that $D \in \mathsf{A}(X)$. Let $\Omega : \operatorname{dom} D \to 2^{\operatorname{dom} D}$ be the multifunction defined as follows:

$$\Omega u = \{x \in \operatorname{dom} D : Du = uDx\} \quad \text{for } u \in \operatorname{dom} D. \tag{6}$$

Clearly,

$$\Omega^{-1}x = \{u \in \operatorname{dom} D : Du = uDx\} \quad \text{for } x \in \operatorname{dom} D.$$

The multifunction Ω is well-defined. Any invertible selector L of Ω is said to be a *logarithmic mapping* and its inverse $E = L^{-1}$ is said to be an *antilogarithmic mapping*. If $(u,x) \in \operatorname{graph}\Omega$ and L is an invertible selector of Ω then the element Lu is said to be a *logarithm* of u and Ex is said to be an *antilogarithm* of x. By $G[\Omega]$ we denote the set of all pairs (L,E), where L is an invertible selector of Ω and $E = L^{-1}$.

Clearly, by definition, we have

$$ELu = u, \quad LEx = x \quad \text{for all } (L,E) \in G[\Omega],\ (u,x) \in \operatorname{graph}\Omega, \tag{7}$$

$$DEx = (Ex)Dx, \qquad Du = uDLu. \tag{8}$$

A logarithmic mapping is said to be of *exponential type* if $L(uv) = Lu + Lv$ for $u, v \in \operatorname{dom}\Omega$. If $(L,E) \in G[\Omega]$ and L is of exponential type then $E(x+y) = (Ex)(Ey)$ for all $x, y \in \operatorname{dom}\Omega^{-1}$ (cf. [5], [6]).

THEOREM 1 (cf. [6], Theorem 1.3 and Corollary 9.5). *Suppose that $D \in \mathsf{A}(X)$ and $(L,E) \in G[\Omega]$. Then L is of exponential type if and only if X is a Leibniz algebra.*

Proof. We shall prove this theorem for right invertible D. For left invertible operators the multifunction Ω is single-valued and the proof is similar. Let then $D \in \mathsf{A}(X)$ be right invertible.

Necessity. Suppose that L is of exponential type, i.e. $L(uv) = Lu + Lv$ for all $u, v \in \operatorname{dom}\Omega$. By definition, we have $uDLu = Du$, $vDLv = Dv$ and

$$D(uv) = uvDL(uv) = uvD(Lu + Lv) = v(uDLu) + u(vDLv) = vDu + uDv.$$

Sufficiency. Suppose that X is a Leibniz algebra, i.e. $c_D = 1$ and $f_D = 0$. Then, by our assumption and Theorem 1.2 in [6] there is a right inverse R of D such that for all invertible $u, v \in \operatorname{dom}\Omega$ we have

$$L(uv) = c_D(Lu + Lv) + R[u^{-1}v^{-1}f_D(u,v)] = Lu + Lv. \tag{9}$$

Hence L is of exponential type. Note that for a left invertible operator D equality (9) holds independently of the choice of a left inverse. ∎

THEOREM 2 (cf. [6], Theorem 1.9). *Suppose that $D \in \mathsf{A}(X)$. Consider the following conditions*:

(i) *X is a Leibniz algebra*;
(ii) *X has a unit e*;
(iii) $\ker D = \{0\}$.

Each pair of these conditions excludes the remaining condition, *i.e. we have*:

(a) *If* $\ker D = \{0\}$ *then either X is not a Leibniz algebra or X has no unit*;
(b) *If X is a Leibniz algebra then either X has no unit or* $\ker D \neq \{0\}$.

Proof. (a) Suppose that X is a Leibniz algebra with unit e and $\ker D = \{0\}$. Then $De = De^2 = 2eDe = 2De$, which implies $De = 0$. Thus $e = 0$, a contradiction.

(b) Suppose that X is a Leibniz algebra. Let X have the unit e. Arguing as before, we conclude that $De = 0$, i.e. $\ker D \neq \{0\}$.

Suppose now that $\ker D = \{0\}$. Hence a unique element satisfying the equation $Dx = 0$ is $x = 0$. Suppose, moreover, that there is an $e \in X \setminus \{0\}$ such that $xe = ex = x$ for every $x \in X$. Then $Dx = xDe + eDx = xDe + Dx$. Thus $xDe = 0$. The arbitrariness of x implies $De = 0$, i.e. $e \in \ker D = \{0\}$. Hence $e = 0$, a contradiction. ∎

In the sequel we shall consider Leibniz algebras X with unit e. By Theorem 2, if $D \in \mathsf{A}(X)$ then D is not left invertible, whence not invertible. We may therefore restrict ourselves to Leibniz algebras with unit and with a right invertible operator D, i.e. D-algebras with unit. Recall that

- $R(X)$ is the set of all right invertible operators belonging to $L(X)$;
- $\mathcal{R}_D = \{R \in L_0(X) : DR = I\}$ is the set of all right inverses to a $D \in R(X)$;
- $\mathcal{F}_D = \{F \in L_0(X) : F^2 = F,\ FX = \ker D \text{ and } \exists_{R \in \mathcal{R}_D} FR = 0\}$ is the set of all *initial* operators for a $D \in R(X)$.

Elements of the kernel of a $D \in R(X)$ are said to be *constants*.

In the sequel, whenever we shall consider Leibniz D-algebras with unit e, we shall assume that $e \in \operatorname{dom} D$. Clearly, by Theorem 2, in such algebras $\ker D \neq \{0\}$.

PROPOSITION 1. *Suppose that X is a Leibniz D-algebra with unit e. Then $e \in \ker D$ and $f_D^{(n)}(x, e) = 0$ for all $x \in \operatorname{dom} D$ and $n \in \mathbf{N}$.*

Proof. By the Leibniz condition, we have $De = De^2 = 2eDe = De$, which implies $De = 0$. Hence $e \in \ker D$. By definition, $f_D^{(1)} = f_D = 0$. Suppose that $f_D^{(n)}(x, e) = 0$ for all $x \in \operatorname{dom} D$ and for an arbitrarily fixed $n \in \mathbf{N}$. Since $c_D = 1$, $De = 0$, by (3) we get

$$f_D^{(n+1)}(x, e) = (Dx)D^n e + (D^n x)De + f_D(x, D^n e) + f_D(D^n x, e) + Df_D^{(n)}(x, e) = 0.$$

This proves that $f_D^{(n)}(x, e) = 0$ for all $x \in \operatorname{dom} D$, $n \in \mathbf{N}$. ∎

PROPOSITION 2. *Suppose that X is a D-algebra with unit e which is a Leibniz D^2-algebra. Then $c_D^2 = 1$ and $f_D \neq 0$.*

Proof. Since, by our assumptions, X is a Leibniz D^2-algebra, we have $c_D^2 = c_{D^2} = 1$ and for all $x, y \in \operatorname{dom} D^2$,

$$0 = f_{D^2}(x, y) = f_D^{(2)}(x, y) = 2(Dx)(Dy) + f_D(x, Dy) + f_D(Dx, y) + Df_D(x, y).$$

Suppose that $f_D = 0$. By (3), $f_D^{(2)}(x, y) = 2(Dx)(Dy)$ for all $x, y \in \operatorname{dom} D$. Hence X is not a Leibniz D^2-algebra even when $c_D = 1$. This contradicts our assumptions. We therefore conclude that $f_D \neq 0$. ∎

OPEN QUESTION. Does there exist a D-algebra X with unit which is a Leibniz D^n-algebra for an $n \geq 2$?

EXAMPLE 1. Let X be a D-algebra with unit e such that

$$D(xy) = c(xDy + yDx) + d(Dx)(Dy) + axy \quad \text{for } x, y \in \operatorname{dom} D,$$

where $a, c, d \in \mathcal{F}$ do not vanish simultaneously. Suppose, moreover, that X is a Leibniz D^2-algebra. Then $e \notin \ker D$.

Indeed, by Proposition 2, we have $c_D^2 = c^2 = 1$ and $f_D(x, y) = d(Dx)(Dy) + axy \not\equiv 0$, i.e. a, d do not vanish simultaneously. Since X is a Leibniz D^2-algebra, we have $e \in \ker D^2$, which implies $De \in \ker D$. On the other hand, we have

$$(1 - 2c)De = f_D(e, e) = d(De)^2 + ae^2 = d(De)^2 + ae.$$

Suppose that $e \in \ker D$, i.e. $De = 0$. Then we get $a = 0$ and $f_D(x, y) = d(Dx)(Dy)$. By (3) and our assumption, we find for arbitrary $x, y \in \operatorname{dom} D$,

$$\begin{aligned} 0 = f_{D^2}(x, y) &= 2(Dx)(Dy) + c[f_D(x, Dy) + f_D(Dx, y)] + Df_D(x, y) \\ &= 2(Dx)(Dy) + 2cd[(Dx)(D^2y) + (D^2x)(Dy)] + d^2(D^2x)(D^2y). \end{aligned}$$

Let $R \in \mathcal{R}_D$ be arbitrarily fixed and let $x = y = Re$. Clearly, $x, y \in \operatorname{dom} D$ and $D^2x = D^2y = D^2Re = De = 0$. Hence $e = e^2 = (Dx)(Dy) = \frac{1}{2} f_{D^2}(x, y) = 0$, a contradiction. Thus $e \notin \ker D$.

PROPOSITION 3 (cf. [1], Section 6.1). *Suppose that X is a Leibniz D-algebra with unit e. Then*

(i) $D^n(zx) = zD^nx$ *for every* $z \in \ker D$, $x \in \operatorname{dom} D$ *and* $n \in \mathbf{N}$;

(ii) $zz' \in \ker D$ *whenever* $z, z' \in \ker D$, *i.e. a product of two constants is again a constant.*

Proof. Let $z \in \ker D$ and $x \in \operatorname{dom} D$ be arbitrarily fixed. Since $Dz = 0$, we get $D(zx) = zDx + xDz = zDx$. By an easy induction, this implies (i). Suppose now that $z, z' \in \ker D$. Then, by (i) we find $D(zz') = zDz' = 0$, which implies $zz' \in \ker D$. ∎

PROPOSITION 4 (cf. [1]). *Suppose that X is a Leibniz D-algebra with unit e and F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. If F is* almost averaging, *i.e.* $F(zx) = zFx$ *for all* $z \in \ker D$ *and* $x \in X$, *then* $R^n(zx) = zR^nx$ *for all* $z \in \ker D$, $x \in X$ *and* $n \in \mathbf{N}$.

Proof. Let $z \in \ker D$, $x \in X$ be arbitrarily fixed. Write $u = R(zx) - zRx$. Since F is almost averaging and $FR = 0$, we get $Fu = FR(zx) - F(zRx) = -zFRx = 0$. By Proposition 3(i), $Du = DR(zx) - D(zRx) = zx - zDRx = zx - zx = 0$. Hence $u \in \ker D$ and $u = Fu = 0$, i.e. $R(zx) = zRx$. By an easy induction, this implies the required formula. ∎

Again by induction we obtain the following *Integration of unit formula*:

PROPOSITION 5 (cf. [1]). *Suppose that X is a Leibniz D-algebra with unit e and F is an almost averaging initial operator for D corresponding to an $R \in \mathcal{R}_D$.*

Write $g = Re$. *Then*

$$R^n e = \frac{g^n}{n!} - \sum_{k=2}^{n} \frac{Fg^k}{k!} R^{n-k} e \qquad (n \in \mathbf{N}) \tag{10}$$

COROLLARY 1 (cf. Przeworska-Rolewicz and von Trotha [7]). *Suppose that all assumptions of Proposition* 5 *are satisfied. If* F *is multiplicative then*

$$R^n e = \frac{g^n}{n!} \qquad (n \in \mathbf{N}). \tag{11}$$

Proof. Clearly, if F is multiplicative then it is almost averaging, but not conversely. We will show by induction that $Fg^n = 0$ for all $n \in \mathbf{N}$. By definitions, $Fg = FRe = 0$. Suppose that $Fg^n = 0$ for an arbitrarily fixed $n \in \mathbf{N}$. Since F is multiplicative, we get $Fg^{n+1} = (Fg^n)(Fg) = 0$. ∎

Proposition 4 and Corollary 1 immediately imply

COROLLARY 2 (cf. [1]). *Suppose that all assumptions of Proposition* 5 *are satisfied and* F *is multiplicative. Then*

$$R^n z = z\frac{g^n}{n!} \qquad \text{for all } z \in \ker D \ (n \in \mathbf{N}). \tag{12}$$

THEOREM 3. *Suppose that either* $\mathcal{F} = \mathbf{R}$ *or* $\mathcal{F} = \mathbf{C}$, X *is a Leibniz* D*-algebra with unit* e *and a Banach space*, F *is a multiplicative initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$ *and* $g = Re$. *Then*

(i) *for every* $\lambda \in \mathcal{F}$ *and* $z \in \ker D$ *the series* $\sum_{n=0}^{\infty} \lambda^n R^n z$ *is convergent and*

$$(I - \lambda R)^{-1} z = \sum_{n=0}^{\infty} \lambda^n R^n z = z\mathrm{e}^{\lambda g};$$

(ii) *every* $\lambda \in \mathcal{F}$ *is an eigenvalue of* D *and the corresponding eigenvectors are*

$$e_\lambda z = (I - \lambda R)^{-1} z = z(I - \lambda R)^{-1} e = z\mathrm{e}^{\lambda g}, \qquad \text{where } z \in \ker D \text{ is arbitrary},$$

i.e.

$$\ker(D - \lambda I) = (I - \lambda R)^{-1} \ker D = \mathrm{e}^{\lambda g} \ker D \qquad \text{for all } \lambda \in \mathcal{F}. \tag{13}$$

Proof. By Corollary 2, for all $z \in \ker D$, $\lambda \in \mathcal{F}$ and for all $N \in \mathbf{N}$,

$$\Big\| \sum_{n=0}^{N} \lambda^n R^n z \Big\| = \Big\| z \sum_{n=0}^{N} \lambda^n \frac{g^n}{n!} \Big\| \le \|z\| \sum_{n=0}^{N} \frac{|\lambda|^n \|g\|^n}{n!}.$$

Hence the series in question is convergent to $(I - \lambda R)^{-1} z$. Writing $\mathrm{e}^{\lambda g} = \sum_{n=0}^{\infty} \frac{\lambda^n g^n}{n!}$ we get assertion (i) (cf. also [1]).

(ii) If $(D - \lambda I)x = 0$ then we have $0 = (D - \lambda I)x = D(I - \lambda R)x$. Hence $(I - \lambda R)x = z \in \ker D$. Together with (i) this implies that every scalar λ is an eigenvalue of D, the corresponding eigenvectors are of the form $x = e_\lambda z = z(I - \lambda R)^{-1} e = z\mathrm{e}^{\lambda g}$ and that formula (13) holds. ∎

Proposition 6 (cf. [1]) *Suppose that X is a Leibniz D-algebra with unit e. Then there is an operator $T \in L_0(X)$ such that*

$$DT - TD = I \quad \text{on } \operatorname{dom} D. \tag{14}$$

Namely, $Tx = gx$, *where $g = Re$ and $R \in \mathcal{R}_D$ is arbitrarily fixed.*

Proof. Let $R \in \mathcal{R}_D$ be arbitrarily fixed and let $g = Re$. Since $Dg = DRe = e$, we have for every $x \in \operatorname{dom} D$,

$$(DT - TD)x = D(gx) - gDx = xDg + gDx - gDx = xe = x.$$

The arbitrariness of x implies (14). ■

It is not known if the converse statement is also true. However, we have

Proposition 7. *Suppose that X is a D-algebra with unit e for which there exists an $R \in \mathcal{R}_D$ such that*

$$D(gx) - gDx = x \quad \text{for all } x \in \operatorname{dom} D, \text{ where } g = Re. \tag{15}$$

Then $c_D = 1$ and $f_D(g, x) = 0$ for all $x \in \operatorname{dom} D$.

Proof. Suppose that D satisfies (15). Then for every $x \in \operatorname{dom} D$ we have

$$x = D(gx) - gDx = c_D(gDx + xDg) + f_D(g, x) - gDx = (c_D - 1)gDx + x + f_D(g, x).$$

The arbitrariness of x implies $c_D = 1$ and $f_D(g, x) = 0$. ■

Proposition 8. *If all assumptions of Proposition 7 are satisfied and g is not a zero divisor (in particular, if g is invertible), then $e \in \ker D$.*

Proof. Put $x = e$ in (14). Then we get $e = D(ge) - gDe = e - gDe$, which implies $gDe = 0$. Since g is not a zero divisor, we have $De = 0$. Hence $e \in \ker D$. ■

Theorem 4. *Suppose that all assumptions of Proposition 7 are satisfied and g is not a zero divisor. Then the set*

$$P_0(R) = \operatorname{lin}\{zg^n : z \in \ker D;\ n \in \mathbf{N}_0\} \tag{16}$$

is a Leibniz D-algebra with unit e.

Proof. By Proposition 7, $c_D = 1$. We shall prove by induction that

$$Dg^n = ng^{n-1} \quad \text{for all } n \in \mathbf{N}. \tag{17}$$

Indeed, for $n = 1$ we have $Dg = DRe = e = g^0$. Suppose formula (17) to be true for an arbitrarily fixed $n \in \mathbf{N}$. Then, by (15),

$$D(g^{n+1}) = D(gg^n) = gDg^n + g^n = ngg^{n-1} + g^n = (n+1)g^n.$$

This proves (17) for all $n \in \mathbf{N}$. Proposition 3(i) and (17) imply that

$$D(zg^n) = zDg^n = nzg^{n-1} \quad \text{for all } z \in \ker D,\ n \in \mathbf{N}. \tag{18}$$

We shall now prove that

$$f_D(g^m, g^n) = 0 \quad \text{for all } m, n \in \mathbf{N}_0. \tag{19}$$

Indeed, if $m = n = 0$ then, by Proposition 8, $f_D(e,e) = D(e^2) - 2eDe = De - 2De = -De = 0$. Again by Proposition 7, for $x = g^n$ ($n \in \mathbf{N}$) we have $f_D(g,g^n) = 0$. We shall apply induction with respect to m. Suppose (19) to be true for every $n \in \mathbf{N}_0$ and for an arbitrarily fixed $m \in \mathbf{N}_0$. Then, by (17), we find

$$\begin{aligned} f_D(g^{m+1},g^n) &= D(g^{m+1}g^n) - g^{m+1}Dg^n - g^nDg^{m+1} \\ &= Dg^{m+n+1} - ng^{m+1}g^{n-1} - (m+1)g^ng^m \\ &= (m+n+1)g^{m+n} - (m+n+1)g^{m+n} = 0. \end{aligned}$$

This proves (19) for all $m, n \in \mathbf{N}_0$. By (15), for all $z, z' \in \ker D$, $m, n \in \mathbf{N}_0$,

$$\begin{aligned} f_D(zg^m, z'g^n) &= D(zg^mz'g^n) - zg^mD(z'g^n) - z'g^nD(zg^m) \\ &= zz'D(g^{m+n}) - zz'g^mDg^n - z'zg^nD(g^m) \\ &= zz'(Dg^{m+n} - g^mDg^n - g^nDg^m) = zz'f_D(g^m,g^n) = 0. \end{aligned}$$

We therefore conclude that $f_D(x,y) = 0$ for all $x, y \in P_0(R)$, i.e. $f_D|_{P_0(R)} = 0$. Hence $P_0(R)$ is a Leibniz D-algebra. ∎

An immediate consequence of Theorem 4 is

COROLLARY 3. *Suppose that either* $\mathcal{F} = \mathbf{R}$ *or* $\mathcal{F} = \mathbf{C}$, X *is a* D*-algebra with unit* e *and a complete linear metric space*, f_D *is continuous* (*in both variables*), $R \in \mathcal{R}_D$, $g = Re$ *is not a zero divisor and* (15) *holds. Let the set* $P_0(R)$ *be defined by* (16). *If* $\overline{P_0(R)} = X$ *then* X *is a Leibniz* D*-algebra with unit* e.

PROPOSITION 9. *Suppose that all assumptions of Proposition* 7 *are satisfied and* g *is not a zero divisor. Then the elements* g^n ($n \in \mathbf{N}_0$) *are linearly independent.*

Proof (by induction). By Proposition 8, $Dg = DRe = e \in \ker D$. Suppose that e and g are linearly dependent. Then there are scalars c_0, c_1 not vanishing simultaneously such that $c_0e + c_1g = 0$. Hence $c_1e = c_1Dg = D(c_0e + c_1g) = 0$, which implies $c_1 = 0$. We get $c_0e = 0$, which implies $c_0 = 0$, a contradiction with our assumption. Hence e and g are linearly independent.

Let now the elements $e, g, \ldots, g^n$ be linearly independent for an arbitrarily fixed $n \geq 1$. Suppose that the elements $e, g, \ldots, g^{n+1}$ are linearly dependent. This means that there are scalars $c_0, \ldots, c_{n+1}$ not vanishing simultaneously and such that $c_0e + \ldots + c_{n+1}g^{n+1} = 0$. By (17),

$$c_1e + \ldots + c_{n+1}(n+1)g^n = D(c_0e + c_1g + \ldots + c_{n+1}g^{n+1}) = 0,$$

which contradicts our inductive assumption. We therefore conclude that the elements $e, g, \ldots, g^n$ are linearly independent for all $n \in \mathbf{N}_0$. ∎

Note. Write

$$P(R) = \operatorname{lin}\{R^nz : z \in \ker D;\ n \in \mathbf{N}_0\}. \tag{16'}$$

The set $P(R)$ is independent of the choice of R and the elements R^nz are linearly independent (cf. [1]). Let the initial operator F for D corresponding to R be multiplicative. Suppose that all assumptions of Proposition 7 are satisfied and g is not a zero divisor. By our assumptions and Theorem 4, $P_0(R)$ is a Leibniz D-algebra. Hence (12) holds. Hence $zg^n = n!zR^n e = n!R^n z$ whenever $z \in \ker D$, $n \in \mathbf{N}_0$. We therefore conclude that $P(R) = P_0(R)$. Clearly, in the case when dim ker $D = 1$, i.e. $\ker D = \operatorname{lin}\{e\}$, we have $P(R) = \operatorname{lin}\{g^n : n \in \mathbf{N}_0\}$.

Another sufficient condition for a D-algebra to be a Leibniz D-algebra can be obtained by means of properties of shifts induced by D. We shall give here without proof the following

Proposition 10 (cf. [4], Theorem 2.3 and Corollary 2.1). *Suppose that X is a complete linear metric locally convex space* (*over* $\mathbf{C}$ *or* $\mathbf{R}$), $D \in R(X)$ *is closed*, $\ker D \neq \{0\}$, F *is a continuous initial operator corresponding to a right inverse* R almost quasinilpotent on $\ker D$, *i.e. such that* $\lim_{n\to\infty} \lambda^n R^n z = 0$ *whenever* $z \in \ker D$ *and* $1/\lambda$ *is a regular value of* R. *Let* $A(\mathbf{R})$ *be an additive subgroup of the reals. Let* $S_{A(\mathbf{R})} = \{S_h\}_{h\in A(\mathbf{R})} \subset L_0(X)$ *be a strongly continuous semigroup* (*group*) *of* true shifts, *i.e. continuous operators satisfying the condition*

$$S_0 = I, \quad S_h R^k F = \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} R^j F \quad \text{for } h \in A(\mathbf{R}),\ k \in \mathbf{N}_0.$$

Let $P(R)$ *be defined by formula* (16′). *Suppose that* $\overline{P(R)} = X$. *Then* D *is an infinitesimal generator for* $S_{A(\mathbf{R})}$. *Hence* $\overline{\operatorname{dom} D} = X$ *and* $S_h D = DS_h$ *on* $\operatorname{dom} D$. *Moreover*, *the canonical mapping defined by*

$$\kappa x = \{\hat{x}(h)\}_{h\in A(\mathbf{R})}, \quad \text{where} \quad \hat{x}(h) = FS_h x \quad \text{for } x \in X, \tag{20}$$

is an isomorphism (*hence separates points*) *and*

$$\kappa D = \frac{\mathrm{d}}{\mathrm{d}t}\kappa; \quad \kappa R = \int_0^t \kappa; \quad \kappa F x = \kappa x|_{t=0}; \\ (\kappa S_h x)(t) = (\kappa x)(t+h) \quad (t, h \in A(\mathbf{R})). \tag{21}$$

Theorem 5. *Suppose that all assumptions of Proposition* 10 *are satisfied*, X *is a* D*-algebra with unit* e *and* F, S_h ($h \in A(\mathbf{R})$ *are multiplicative. Then* X *is a Leibniz* D*-algebra.*

P r o o f. By our assumptions, for all $x, y \in X$, $h \in A(\mathbf{R})$ we have

$$\widehat{(xy)}(h) = FS_h(xy) = (FS_h x)(FS_h y) = \hat{x}(h)\hat{x}(h).$$

Hence the canonical mapping κ is multiplicative. The first formula of (21) now implies that for all $x, y \in \operatorname{dom} D$,

$$\begin{aligned} \kappa D(xy) &= \frac{\mathrm{d}}{\mathrm{d}t}\kappa(xy) = \frac{\mathrm{d}}{\mathrm{d}t}[(\kappa x)(\kappa y)] \\ &= (\kappa x)\frac{\mathrm{d}}{\mathrm{d}t}(\kappa y) + (\kappa y)\frac{\mathrm{d}}{\mathrm{d}t}(\kappa x) \\ &= (\kappa x)(\kappa D y) + (\kappa y)(\kappa D x) = \kappa(xDy + yDx). \end{aligned}$$

Since κ separates points, we conclude that $D(xy) = xDy + yDx$ for $x, y \in \operatorname{dom} D$. Hence X is a Leibniz D-algebra. ■

Clearly, there are Leibniz D-algebras such that the canonical mapping is not multiplicative. For instance, this is so when the initial operator under consideration is not multiplicative. Several such examples are given in [1].

COROLLARY 5. *Suppose that all assumptions of Proposition* 10 *are satisfied*, X *is a* D*-algebra with unit* e *and* F *and* S_h *for* $h \in A(\mathbf{R})$ *are multiplicative. If* X *is not a Leibniz* D*-algebra then* $\dim\ker D \neq 1$.

Proof. Suppose that $\dim\ker D = 1$. By Corollary 6.1 in [3], X is a Leibniz D-algebra. This is a particular case of Theorem 5, which yields our conclusion. ■

In order to examine the *Trigonometric Identity* in D-algebras, we shall use again Definition 1 (cf. also Theorem 1).

THEOREM 6 (cf. [6], Corollary 7.2). *Suppose that* $\mathcal{F} = \mathbf{C}$, X *is a* D*-algebra with unit* e, *the domain of the multifunction* Ω^{-1} *is symmetric* (*i.e.* $-x \in \operatorname{dom}\Omega^{-1}$ *whenever* $x \in \operatorname{dom}\Omega^{-1}$) *and* $(L, E) \in G[\Omega]$. *Write*

$$Cx = \frac{1}{2}[E(ix) + E(-ix)], \quad Sx = \frac{1}{2i}[E(ix) - E(-ix)] \quad \textit{if } ix \in \operatorname{dom}\Omega^{-1}.$$

Then

(i) *for all* $ix \in \operatorname{dom}\Omega^{-1}$ *the mappings* C *and* S *are even and odd*, *respectively*, *i.e.* $C(-x) = Cx$, $S(-x) = -Sx$. *Moreover*,

$$C(0) = z \in \ker D \setminus \{0\}, \quad S(0) = 0, \quad DCx = -(Sx)Dx, \quad DSx = (Cx)Dx;$$

(ii) *if* X *is a Leibniz* D*-algebra then the Trigonometric Identity holds*, *i.e.*

$$(Cx)^2 + (Sx)^2 = e \quad \textit{whenever} \quad ix \in \operatorname{dom}\Omega^{-1}. \tag{22}$$

PROPOSITION 11 (cf. [6], Proposition 7.2). *Suppose that all assumptions of Theorem* 6 *are satisfied and the Trigonometric Identity* (22) *holds. Then*

(i) $e \in \ker D$;

(ii) *if* X *is* an almost Leibniz D*-algebra*, *i.e.* $f_D(z, x) = 0$ *for* $z \in \ker D$, $x \in \operatorname{dom} D$, *then* $c_D = 1$;

(iii) *if* $c_D = 1$ *then* $f_D(u, e) = 0$ *for all invertible* $u \in \operatorname{dom}\Omega$ (cf. Propositions 7 and 8).

Proposition 11 shows that D-algebras with the Trigonometric Identity (22) are, in a sense, very similar to Leibniz algebras. However, we have still an

OPEN QUESTION. Do there exist non-Leibniz D-algebras with the Trigonometric Identity (22)?

COROLLARY 6 (cf. [6], Corollary 7.6). *Suppose that all assumptions of Theorem* 6 *are satisfied*, X *is a Leibniz* D*-algebra*, $R \in \mathcal{R}_D$, $g = Re$ *and* $\lambda \in \mathbf{C}$.

If $\lambda ig \in \operatorname{dom}\Omega^{-1}$ *then*

$$\ker(D^2+\lambda^2 I)=\{zC(\lambda g)+z'S(\lambda g): z,z'\in\ker D\}.$$

We shall now consider properties of *smooth* elements in Leibniz algebras, i.e. elements belonging to

$$D_\infty=\bigcap_{k\in\mathbf{N}}\operatorname{dom}D^k. \tag{23}$$

Let $D\in R(X)$ and let F be an initial operator for D corresponding to an $R\in\mathcal{R}_D$. Write

$$\begin{aligned} Q_R(D)&=\{x\in D_\infty: R^nD^nx=x;\ n\in\mathbf{N}\}\\ &=\{x\in D_\infty: FD^nx=0\ ;n\in\mathbf{N}_0\}. \end{aligned} \tag{24}$$

The set $Q_R(D)$ is said to be the space of *singular elements* for D (cf. [2]).

By an application of Leibniz Formula (5) and the Taylor Formula

$$I=\sum_{n=0}^{N}R^nFD^n+R^ND^N \quad \text{on } \operatorname{dom}D^N\ (N\in\mathbf{N}) \tag{25}$$

(cf. [1]), we get

Proposition 12 (cf. [2]). *Suppose that X is a Leibniz algebra with unit e and F is a multiplicative initial operator for D corresponding to an $R\in\mathcal{R}_D$. Then*

(i) *the sets $S=P(R)$, $Q_R(D)$, $Q_R(D)\oplus S$ and D_∞ are Leibniz subalgebras of X and*

$$S\subset Q_R(D)\oplus S\subset D_\infty\subset\operatorname{dom}D\subset X;$$

(ii) *$Q_R(D)$ is an ideal in $Q_R(D)\oplus S$ and in D_∞.*

Proposition 13 (cf. [2]). *Suppose that X is a complete linear metric space (over $\mathbf{R}$ or $\mathbf{C}$) and a Leibniz D-algebra with unit e and F is a continuous multiplicative initial operator for D corresponding to an $R\in\mathcal{R}_D$. Consider the space of D-*analytic *elements*:

$$\begin{aligned} A_R(D)&=\Big\{x\in D_\infty:\sum_{n=0}^{\infty}R^nFD^nx \text{ is convergent and } x=\sum_{n=0}^{\infty}R^nFD^nx\Big\}\\ &=\{x\in D_\infty:\lim_{n\to\infty}R^nD^nx\}. \end{aligned} \tag{26}$$

Then

(i) *if $x\in A_R(D)$ then*

$$x=\sum_{n=0}^{\infty}\frac{g^n}{n!}FD^nx,\quad \text{where}\quad g=Re;$$

(ii) *the sets $A_R(D)$ and $A_R(D)\oplus Q_R(D)$ are Leibniz subalgebras of X and*

$$S\subset A_R(D)\subset A_R(D)\oplus Q_R(D)\subset D_\infty\subset\operatorname{dom}D\subset X;$$

(iii) *$Q_R(D)$ is an ideal in $A_R(D)\oplus Q_R(D)$.*

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