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SOME PROBLEMS CONCERNING ROOTS OF POLYNOMIALS WITH DIRICHLET CHARACTERS AS COEFFICIENTS

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For a given Dirichlet character $\chi \pmod q$ let us denote

$$\Phi_\chi(z) = \sum_{m=1}^{q-1} \chi(m) z^m.$$

Let the field k_χ be the extension of Q generated by the numbers $\chi(m)$, $m = 1, 2, \ldots, q-1$. It is evident that $k_\chi \subset Q(\zeta_q)$ and that $\Phi_\chi(z)$ is a polynomial over the ring of integers of the field k_χ. Let us denote by K_χ the splitting field of $\Phi_\chi(z)$. Obviously $k_\chi \subset K_\chi$. The most interesting case for us will be $k_\chi = Q$.

The polynomials $\Phi_\chi(z)$ appear in a natural way in the theory of Dirichlet L-series. In fact it is easy to see that for $s = \sigma + i\tau$, $\sigma > 0$

$$L(s, \chi)\Gamma(s) = \int_0^\infty \frac{\Phi_\chi(e^{-t})}{1-e^{-qt}} t^{s-1}\, dt.$$

From this formula it follows that there should exist close relations between the zeros of $\Phi_\chi(z)$ and the behaviour of $L(s, \chi)$. One of them can be seen at once: if $\chi = \bar{\chi}$ and $\Phi_\chi(z)$ has no roots in the interval $(0, 1)$ then $L(s, \chi)$ has no real roots in the critical strip, in particular Siegel's zero does not exist. Thus we have

PROBLEM 1. Are there infinitely many primitive real characters χ such that $\Phi_\chi(z)$ has no roots in the interval $(0, 1)$?

From the theory of Gaussian sums it follows that many zeros of $\Phi_\chi(z)$ are roots of unity.

Some information about the roots of $\Phi_\chi(z)$ can be deduced from the structure of K_χ. We have the following

THEOREM. (i) *If the extension K_χ/Q is abelian, $\chi = \bar{\chi}$, $2|q$ then all zeros of $\Phi_\chi(z)$ are roots of unity.*

(ii) *Let K_χ^0 denote the maximal real subfield of K_χ. If the extension K_χ^0/Q is totally real then $\Phi_\chi(z)$ has no roots in the interval* (0, 1).

The proof of this theorem follows from the well-known Kronecker–Weber theorem, the Kronecker theorem ([1], th. 2.1) and a result of R. Robinson [2].

PROBLEM 2. For a given Dirichlet character χ give an explicit construction of the field K_χ.

It is easily seen that K_χ cannot be quite arbitrary. For example, if χ is a primitive character mod q, $q = 2^a q_1$, $2 \nmid q_1$, $q_1 > 1$, $a \geqslant 2$, then

$$Q(\zeta_q) \subset K_\chi.$$

It can also be proved that there exists a function ψ such that

$$|\Delta_\chi| < \psi(n_\chi)$$

where n_χ and Δ_χ denote the degree and the discriminant of K_χ resp.

Note that in general the discriminant of an algebraic number field cannot be estimated from above in terms of degree only. This means that the last inequality gives a strong restriction upon the field K_χ.

Let us give some examples of K_χ. Let us look at the fields K_χ where χ are the characters of the first six fields $Q(\sqrt{-D})$ (note that there exists one-to-one correspondence between primitive real Dirichlet characters and quadratic extensions of rationals). If $D > 0$ is any square-free number and χ_D is the character of the field $Q(\sqrt{-D})$, then we shall write K_D instead of K_{χ_D}. We have:

D	K_D
1	Q
2	$Q(\zeta_4)$
3	Q
5	$Q(\zeta_{60})$
6	$Q(\zeta_{24})$
7	$Q(\sqrt{2}, \sqrt{2\sqrt{2}-1}, i\sqrt{2\sqrt{2}+1})$

Thus in the first five cases the corresponding L-functions have no real roots in the half-plane $\sigma > 0$.

Here the case $D = 7$ is interesting. Namely K_7 is a non-abelian extension of rationals of degree 8. But if χ' (mod 14) denotes the Dirichlet character induced by χ_7, then

$$K_{\chi'} = Q(\zeta_{12}),$$

and again we have $L(\sigma, \chi) \neq 0$ for $\sigma > 0$.

PROBLEM 3. Let χ' (mod q') be the Dirichlet character induced by χ (mod q), $q \mid q'$. Find the relations between K_χ and $K_{\chi'}$.

PROBLEM 4. Is it true that if $L(s, \chi) \neq 0$, χ (mod q) for $0 < s < 1$ then there exists a natural number q', $q \mid q'$ such that, for the induced character χ' (mod q'), the extension $K_{\chi'}/Q$ is abelian?

References

[1] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, Warszawa 1974.
[2] R. Robinson, *Some conjectures about cyclotomic fields*, Math. Comput. 19 (1965), pp. 210–217.

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