

PREDICTOR-CORRECTOR METHODS FOR NONLINEAR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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1. Introduction

In this paper we investigate predictor-corrector methods for the general nonlinear Volterra integral equation of the second kind

$$(1.1) \quad y(x) = g(x) + F\left(\int_a^x k(x, u, y(u)) du\right), \quad x \in [a, b].$$

For the functions $F(t)$, $k(x, u, z)$ and $g(x)$ we state the following properties:

(i) The kernel function $k(x, u, z)$ is defined and continuous for $(x, u, z) \in D$ where

$$D = \{(x, u, z): a \leq u \leq x \leq b, -\infty < z < +\infty\}.$$

(ii) The kernel function $k(x, u, z)$ satisfies a uniform Lipschitz condition in z , that is for all $(x, u, z_1) \in D$ and $(x, u, z_2) \in D$

$$(1.2) \quad |k(x, u, z_2) - k(x, u, z_1)| \leq q |z_2 - z_1|$$

where q is a constant.

(iii) The function $F(t)$ is continuous for all $t \in \mathbf{R} = \{t: -\infty < t < +\infty\}$ and satisfies a uniform Lipschitz condition in t , that is for all t_1, t_2

$$(1.3) \quad |F(t_2) - F(t_1)| \leq M |t_2 - t_1|$$

where M is a constant.

(iv) The function $g(x)$ is continuous for $x \in [a, b]$. Quadrature methods for the nonlinear Volterra integral equation

$$(1.4) \quad y(x) = g(x) + \int_a^x k(x, u, y(u)) du, \quad x \in [a, b]$$

and corresponding convergence properties has been studied by several authors, see C. T. H. Baker [1] (Ch. 6), L. M. Delves and J. Walsh [3] (Ch. 11), L. Garey [4] and B. Noble [6].

There is a similarity between Volterra integral equations and initial value problems for ordinary differential equations of the first order in the integrated form, therefore predictor-corrector methods can be constructed for Volterra integral equations. First of all L. Garey [4] studied this method for the equation (1.4). In the present paper such methods are derived for the equation (1.1). The used closed and open quadrature formulas are composed formulas. For the corresponding remainders we get a closed form, using the composition method for the remainders of quadrature formulas. The remainders of the used formulas are of fourth and sixth order.

For a continuous solution $y(x)$, $x \in [a, b]$ the quadrature methods yield discrete approximate solutions. To the step size $h = (1/n)(b - a)$, $n = 1, 2, \dots$ corresponds the system of quadrature points

$$\{x_i: x_i = a + ih, i = 0, \dots, n\}.$$

For the system of function values $y(x_1), \dots, y(x_n)$ we obtain a system of approximate values y_1, \dots, y_n .

At the initial point $x_0 = a$ we have $y(x_0) = g(x_0) + F(0)$, and y_0 denotes an approximate value for $y(x_0)$. The errors e_i are the quantities

$$(1.5) \quad e_i = y(x_i) - y_i, \quad i = 0, \dots, n.$$

A qualitative error control provides estimates for the absolute errors $|e_i|$. We have the property of discrete convergence, if

$$(1.6) \quad \lim_{h \rightarrow 0} \max_{i=0, \dots, n} |y(x_i) - y_i| = 0.$$

For equation (1.1) we get the following existence theorem.

THEOREM 1. *Suppose that the assumptions (i)–(iv) hold. Then there exists a unique continuous solution $y(x)$, $x \in [a, b]$ of equation (1.1).*

Proof. Let C denote the space of continuous functions on $[a, b]$ with the general norm

$$\|y\|_* = \exp(-\alpha x) |y(x)|$$

$a \leq x \leq b$

where α is a fixed positive real number. We define the nonlinear operator

$$T(y) = g(x) + F\left(\int_a^x k(x, u, y(u)) du\right)$$

which associates with each element $y \in C$ an element $T(y)$ of C . We have

$$T(y_2) - T(y_1) = F\left(\int_a^x k(x, u, y_2(u)) du\right) - F\left(\int_a^x k(x, u, y_1(u)) du\right),$$

and from (1.2) and (1.3) it follows that

$$\begin{aligned} |T(y_2) - T(y_1)| &\leq M \left| \int_a^x k(x, u, y_2(u)) du - \int_a^x k(x, u, y_1(u)) du \right| \\ &\leq Mq \int_a^x |y_2(u) - y_1(u)| du. \end{aligned}$$

Using the introduced norm we obtain the inequality

$$|y_2(x) - y_1(x)| \leq \exp ax \|y_2 - y_1\|_*.$$

We conclude that

$$|T(y_2) - T(y_1)| \leq Mq \|y_2 - y_1\|_* \int_a^x \exp au du \leq Mq \|y_2 - y_1\|_* \frac{1}{a} \exp ax.$$

Consequently

$$\exp(-ax) |T(y_2) - T(y_1)| \leq \frac{Mq}{a} \|y_2 - y_1\|_*,$$

therefore

$$\|T(y_2) - T(y_1)\|_* \leq \frac{Mq}{a} \|y_2 - y_1\|_*.$$

Under the condition

$$Mq/a < 1$$

for α Banach's contractive mapping principle, see [9], Ch. 1, § 5, asserts a unique continuous solution.

Remark. The solution $y(x)$ of equation (1.1) is also differentiable, if the functions $g(x)$, $k(x, u, z)$ and $F(t)$ are differentiable.

2. Quadrature formulas

For the predictor-corrector methods we use special quadrature formulas for the integral

$$\int_a^{x_i} f(u) du, \quad i \geq m+1, \dots, n,$$

where x_m is the endpoint of a starting method. The employed nodes for the quadrature formulas are

$$\begin{aligned} x_{v/2}, \quad v = 0, 1, \dots, 2m, \\ x_v, \quad v = m+1, \dots, n. \end{aligned}$$

Type 1. Right-side closed formulas

$$(2.1) \quad \int_a^{x_i} f(u) du = h \sum_{v=0}^{2m} c_{i,v/2} f(x_{v/2}) + h \sum_{v=m+1}^i c_{iv} f(x_v) + r_{1i}(f(x)).$$

$r_{1i}(f(x))$ denotes the remainder. For the coefficients the following conditions hold

$$\begin{aligned} \sum_{v=0}^{2m} c_{i,v/2} + \sum_{v=m+1}^i c_{iv} &= i, \quad i \geq m+1, \\ c_{i,v/2} &\geq 0, \quad v = 0, \dots, 2m, \\ c_{iv} &\geq 0, \quad v = m+1, \dots, i. \end{aligned}$$

Type 2. Right-side open formulas

$$(2.2) \quad \int_a^{x_i} f(u) du = h \sum_{v=0}^{2m} d_{i,v/2} f(x_{v/2}) + h \sum_{v=m+1}^{i-1} d_{iv} f(x_v) + r_{2i}(f(x)).$$

$r_{2i}(f(x))$ denotes the remainder. For the coefficients the following conditions hold

$$\sum_{v=0}^{2m} d_{i,v/2} + \sum_{v=m+1}^{i-1} d_{iv} = i.$$

We introduce the following notations

$$\begin{aligned} \max\{c_{i,v/2} : i = m+1, \dots, n; v = 0, 1, \dots, 2m\} &= c_0, \\ \max\{c_{iv} : i = m+1, \dots, n; v = m+1, \dots, i\} &= c_1, \\ \max\{|d_{i,v/2}| : i = m+1, \dots, n; v = 0, 1, \dots, 2m\} &= d_0, \\ \max\{|d_{iv}| : i = m+1, \dots, n; v = m+1, \dots, i-1\} &= d_1. \end{aligned} \quad (2.3)$$

The quadrature formulas of type 1 and type 2 are composed quadrature formulas. The elementary formulas in the composition are closed and open Newton-Cotes formulas. We denote the closed formula for the interval $[a, x_i]$ by NFC i and the open formula by NFO i . These formulas and the corresponding remainders can be found in a table of E. Isaacson and H. B. Keller [5] (Ch. 7.1). For one special interval we need the Mac-

Laurin formula (notation LF)

$$(2.4) \quad \int_a^{a+3h} f(u) du = \frac{3}{8}h(3f(x_{1/2}) + 2f(x_{3/2}) + 3f(x_{5/2})) + \frac{21}{640}h^5 f^{(4)}(\xi),$$

see R. Zurmühl [10] (Ch. 3, § 3.5). The behaviour of the remainder is the same as that of the remainder of the open formula NFO 5. The composition of the quadrature formulas is constructed in such a way, that the remainder has a closed form. These calculations are founded on Lemma 1, see E. Isaacson and H. B. Keller [5] (Ch. 7.1).

The tables 1 and 2 contain the composed formulas for the predictor-corrector method.

*Method 1: Starting method up to x_4 ,
used closed quadrature formulas*

$[a, x_{2i+1}], \quad i \geq 2$	NFC 2	$i-1$ times
	NFC 3	1 time
$[a, x_{2i}], \quad i \geq 3$	NFC 2	i times

used open quadrature formulas

$[a, x_{4i}], \quad i \geq 2$	NFO 4	i times
$[a, x_{4,i+1}], \quad i \geq 1$	NFO 4	$i-1$ times
	NFO 5	1 time
$[a, x_{4,i+2}], \quad i \geq 1$	NFO 4	for the interval $[a, x_2]$ with the nodes $x_{1/2}, x_1, x_{3/2}$
	NFO 4	i times
$[a, x_{4,i+3}], \quad i \geq 1$	NFO 4	for the interval $[a, x_2]$ with the nodes $x_{1/2}, x_1, x_{3/2}$
	NFO 4	$i-1$ times
	NFO 5	1 time

*Method 2: Starting method up to x_8
used closed quadrature formulas*

$[a, x_{4i}], \quad i \geq 3$	NFC 4	i times
$[a, x_{4,i+1}], \quad i \geq 2$	NFC 4	$i-1$ times
	NFC 5	1 time
$[a, x_{4,i+2}], \quad i \geq 2$	NFC 4	for the interval $[a, x_2]$ with the nodes $x_0, x_{1/2}, \dots, x_2$
	NFC 4	i times
$[a, x_{4,i+3}], \quad i \geq 2$	NFO 4	for the interval $[a, x_2]$ with the nodes $x_0, x_{1/2}, \dots, x_2$
	NFC 4	$i-1$ times
	NFC 5	1 time

used open quadrature formulas

$[a, x_{6i}], \quad i \geq 2$	NFO 6	i times
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$[a, x_{6,i+1}], i \geq 2$	NFO 6	$i-1$ times
	NFO 7	1 time
$[a, x_{6,i+2}], i \geq 2$	NFO 6	$i-2$ times
	NFO 7	2 times
$[a, x_{6,i+3}], i \geq 1$	* NFO 6	for the interval $[a, x_3]$ with the nodes $x_{1/2}, x_1, \dots, x_{5/2}$
	NFO 6	i times
$[a, x_{6,i+4}], i \geq 1$	NFO 6	for the interval $[a, x_3]$ with the nodes $x_{1/2}, x_1, \dots, x_{5/2}$
	NFO 6	$i-1$ times
$[a, x_{6,i+5}], i \geq 2$	NFO 7	1 time
	NFO 6	for the interval $[a, x_3]$ with the nodes $x_{1/2}, x_1, \dots, x_{5/2}$
	NFO 6	$i-2$ times
	NFO 7	2 times

Remark. In the method 2 we choose for the interval $[a, x_{11}]$ a special composition

LF for the intervals $[a, x_3]$ and $[x_3, x_6]$ with the nodes $x_{1/2}, x_1, \dots, x_{11/2}$,
NFO 5 1 time.

EXAMPLE. For the remainder of the open quadrature formula for the interval $[a, x_{4i}]$ of method 1 we get

$$(x_{4i} - a)^{\frac{28}{90}} h^4 f^{(4)}(\xi), \quad a < \xi < x_{4i}.$$

3. The predictor-corrector method

A predictor-corrector method consists of two parts: a starting method and a quadrature method. The starting method up to the node x_m is a special method for the determination of approximate values $y_{v/2}$ for the values $y(x_{v/2})$, $v = 1, 2, \dots, 2m$. The construction of J. T. Day [2] for starting values of equation (1.4) can be modified, so that we obtain starting values for equation (1.1) in the described manner. The predictor-corrector method yields a system of predictor values $\{\bar{y}_i: i = m+1, \dots, n\}$ and a system of corrector values $\{y_i: i = m+1, \dots, n\}$. For every node x_i , $i \geq m+1$, we consider equation (1.1) and calculate the integral value using the quadrature formulas (2.1) and (2.2). For the values $g(x_i)$, $i \geq m+1$, and the remainders we introduce the following notations

$$g_i = g(x_i), \quad i = m+1, \dots, n,$$

$$\varrho_{1i} = r_{1i} \left(k(x_i, x, y(x)) \right),$$

$$\varrho_{2i} = r_{2i} \left(k(x_i, x, y(x)) \right).$$

Then we have

$$(3.1) \quad y(x_i) = g_i + F \left(h \sum_{v=0}^{2m} c_{i,v/2} k(x_i, x_{v/2}, y(x_{v/2})) + \right. \\ \left. + h \sum_{v=m+1}^i c_{iv} k(x_i, x_v, y(x_v)) + \varrho_{1i} \right),$$

$$(3.2) \quad y(x_i) = g_i + F \left(h \sum_{v=0}^{2m} d_{i,v/2} k(x_i, x_{v/2}, y(x_{v/2})) + \right. \\ \left. + h \sum_{v=m+1}^{i-1} d_{iv} k(x_i, x_v, y(x_v)) + \varrho_{2i} \right)$$

for $i \geq m+1$. In equation (3.2) the remainder ϱ_{2i} is neglected and the values $y(x_{v/2})$, $v = 0, \dots, 2m$, and $y(x_v)$, $v = m+1, \dots, i-1$, are replaced by the starting values $y_{v/2}$ and calculated corrector values y_v . Now we obtain the new predictor value

$$(3.3) \quad \bar{y}_i = g_i + F \left(h \sum_{v=0}^{2m} d_{i,v/2} k(x_i, x_{v/2}, y_{v/2}) + h \sum_{v=m+1}^{i-1} d_{iv} k(x_i, x_v, y_v) \right).$$

In the next step the remainder ϱ_{1i} in equation (3.1) is neglected. We use the starting values, the known corrector values y_v , $v = m+1, \dots, i-1$, and the predictor value \bar{y}_i instead of the functional values $y(x_{v/2})$ and $y(x_v)$. Then we get the new corrector value

$$(3.4) \quad y_i = g_i + F \left(h \sum_{v=0}^{2m} c_{i,v/2} k(x_i, x_{v/2}, y_{v/2}) + h \sum_{v=m+1}^{i-1} c_{iv} k(x_i, x_v, y_v) + \right. \\ \left. + h c_{ii} k(x_i, x_i, \bar{y}_i) \right).$$

Equations (3.3) and (3.4) define the predictor-corrector method. The order of magnitude of the errors depends on the errors of the starting values and the quadratur errors.

THEOREM 2. Suppose that the values $y_{v/2}$, $v = 0, \dots, 2m$, are constructed by using a starting procedure. The values y_{m+1}, \dots, y_n are calculated using equations (3.3) and (3.4). We assume

$$(3.5) \quad M |\varrho_{1i}| + c_1 M^2 q h |\varrho_{2i}| \leq s(h), \quad i = m+1, \dots, n.$$

We introduce

$$(3.6) \quad E(h) = s(h) + M q h (c_0 + M q h c_1 d_0) \sum_{v=0}^{2m} |y(x_{v/2}) - y_{v/2}|.$$

Then we have the following estimation

$$(3.7) \quad |y(x_i) - y_i| \leq E(h) [1 + Mqhc_1(1 + Mqhd_1)]^{-(m+1)} \times \\ \times \exp iMqhc_1(1 + Mqhd_1)$$

for $i \geq m+1$.

For the proof we use the

LEMMA. If the members of the sequence (w_n) , $n \geq m+1$, satisfy the inequalities

$$(3.8) \quad w_{n+1} \leq \sigma_1, \\ w_i \leq \sigma_1 + \beta \sum_{v=m+1}^{i-1} w_v, \quad i > m+1,$$

where $\sigma_1 > 0$ and $\beta > 0$, then the following inequalities are valid

$$(3.9) \quad w_i \leq \sigma_1(1 + \beta)^{i-m-1}, \quad i \geq m+1,$$

$$(3.10) \quad w_i \leq \frac{\sigma_1}{(1 + \beta)^{m+1}} \exp i\beta, \quad i \geq m+1.$$

The inequalities (3.9) follow immediately by induction. For the validity of inequality (3.10) we notice that

$$1 + \beta \leq \exp \beta.$$

Proof of Theorem 2. For convenience we introduce the notations

$$(3.11) \quad z_{v/2} = y(x_{v/2}), \quad e_{v/2} = z_{v/2} - y_{v/2}, \quad v = 0, \dots, 2m, \\ z_v = y(x_v), \quad e_v = z_v - y_v, \quad v = m+1, \dots, 2m.$$

Using (3.1) and (3.4) we get

$$z_i - y_i = F \left(h \sum_{v=0}^{2m} c_{i,v/2} k(x_i, x_{v/2}, z_{v/2}) + h \sum_{v=m+1}^i c_{iv} k(x_i, x_v, z_v) + \varrho_{1i} \right) - \\ - F \left(h \sum_{v=0}^{2m} c_{i,v/2} k(x_i, x_{v/2}, y_{v/2}) + h \sum_{v=m+1}^{i-1} c_{iv} k(x_i, x_v, y_v) + \right. \\ \left. + h c_{ii} k(x_i, x_i, \bar{y}_i) \right),$$

$i \geq m+1$. Applying the Lipschitz condition (1.3) we obtain

$$|z_i - y_i| \leq M \left| h \sum_{v=0}^{2m} c_{i,v/2} [k(x_i, x_{v/2}, z_{v/2}) - k(x_i, x_{v/2}, y_{v/2})] + \right. \\ \left. + h \sum_{v=m+1}^{i-1} c_{iv} [k(x_i, x_v, z_v) - k(x_i, x_v, y_v)] + \right. \\ \left. + h c_{ii} [k(x_i, x_i, z_i) - k(x_i, x_i, \bar{y}_i)] + \varrho_{1i} \right|.$$

Applying the Lipschitz condition (1.2), the notations (2.3) and (3.11), we have

$$(3.12) \quad |z_i - y_i| \leq Mqh c_0 \sum_{v=0}^{2m} |e_{v/2}| + Mqh c_1 \sum_{v=m+1}^{i-1} |e_v| + \\ + Mqh c_1 |z_i - \bar{y}_i| + M |e_{1i}|.$$

For $z_i - \bar{y}_i$ we obtain the expression

$$z_i - \bar{y}_i = F \left(h \sum_{v=0}^{2m} d_{i,v/2} k(x_i, x_{v/2}, z_{v/2}) + h \sum_{v=m+1}^{i-1} d_{iv} k(x_i, x_v, z_v) + e_{2i} \right) - \\ - F \left(h \sum_{v=0}^{2m} d_{i,v/2} k(x_i, x_{v/2}, y_{v/2}) + h \sum_{v=m+1}^{i-1} d_{iv} k(x_i, x_v, y_v) \right).$$

Using the Lipschitz conditions (1.3) and (1.2) we have the estimates

$$(3.13) \quad |z_i - \bar{y}_i| \leq M \left| h \sum_{v=0}^{2m} d_{i,v/2} [k(x_i, x_{v/2}, z_{v/2}) - k(x_i, x_{v/2}, y_{v/2})] + \right. \\ \left. + h \sum_{v=m+1}^{i-1} d_{iv} [k(x_i, x_v, z_v) - k(x_i, x_v, y_v)] + e_{2i} \right|, \\ |z_i - \bar{y}_i| \leq Mqh d_0 \sum_{v=0}^{2m} |e_{v/2}| + Mqh d_1 \sum_{v=m+1}^{i-1} |e_v| + M |e_{2i}|.$$

Substituting this inequality in (3.12) we obtain

$$(3.14) \quad |z_i - y_i| \leq Mqh(c_0 + Mqh c_1 d_0) \sum_{v=0}^{2m} |e_{v/2}| + \\ + Mqh c_1 (1 + Mqh d_1) \sum_{v=m+1}^{i-1} |e_v| + \\ + M |e_{1i}| + c_1 M^2 qh |e_{2i}|, \quad i \geq m+1.$$

From the inequality (3.5) and the notation (3.6) it follows, that

$$(3.15) \quad |z_i - y_i| \leq E(h) + Mqh c_1 (1 + Mqh d_1) \sum_{v=m+1}^{i-1} |e_v|, \quad i \geq m+1.$$

For the special case $i = m+1$ this estimation has the form

$$|z_{m+1} - y_{m+1}| \leq E(h).$$

Applying the lemma gives the estimation (3.7).

Using the remainders of the quadrature formulas, we obtain the order of the predictor-corrector methods. The following table contains the conditions for methods 1 and 2 and the resulting order of the approximate method.

	method 1	method 2
The functions $g(x)$, $F(x)$ are continuous differentiable up to the order	4	6
For the function $k(x, u, z)$ exist partial derivatives up to the order	4	6
For the solutions exist derivatives up to the order	4	6
Starting method for the starting values	$y_{v/2}, v = 1, \dots, 8$	$y_{v/2}, v = 1, \dots, 16$
Errors of the starting values	$O(h^4)$	$O(h^6)$
Error of the predictor-corrector method	$O(h^4)$	$O(h^6)$

For the methods 1 and 2 we have the property of discrete convergence (1.6).

The described predictor-corrector methods are part of a library program for Volterra integral equations of the second kind. Moreover the library program contains various methods for the solution of linear Volterra integral equations, see for example [7] and [8], and methods for error control. This library has been developed in cooperation with the computer centre Schwerin, DDR. The detailed programs for the computer have been elaborated by E. Tabbert. Calculations of special examples for the stepsize $h = 10^{-2}$ for steps from $x_0 = 1$ to $x = 3$ gave results with a maximum error of 10^{-8} .

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