POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CXLIX

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Continua and their non-separating subcontinua

WARSZAWA 1977
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

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Introduction

The theme of this paper is the study of the internal structure of continua. In particular, we are interested in a continuum M and in certain subcontinua of M which "lie at an end" of M. This basic idea has interested many previous writers, notably Bing, Miller, Rosen, Sorgenfrey, Thomas and Whyburn. They have formulated various notions of "lying at an end" of a continuum; e. g., "E-continuum", "end continuum", and "terminal continuum." One property which all of these ideas share is that if K is a subcontinuum which "lies at an end" of a continuum M, then $M-K$ is connected; i. e., K is non-separating in M. This is the weakest notion of "lying at an end" which we shall study.

In Chapter I, we develop the implications between various ideas of "lying at an end" of a continuum and examples are presented to show that these implications cannot be reversed. We also show that combining several properties, each of which is weaker than one we are interested in, will give us some characterizations.

Given that a continuum M has a subcontinuum K which "lies at an end" of M, there is often a "large" subcontinuum of M which is irreducible between a pair of points. We begin the study of irreducibility in Chapter I and devote the whole of Chapter II to it. In addition to new characterizations of irreducible continua, this chapter presents alternative proofs for some well-known theorems of Sorgenfrey.

In Chapter III we study subcontinua of M which are both terminal and non-cutting. Such continua are generalizations of E-continua and many known theorems on E-continua can be extended.

We have tried to keep our terminology in agreement with that of previous authors. One place where we were unable to do so is the idea of terminal continua as used in [2] and [5]. However, a way out of our difficulty can be found by distinguishing between those subcontinua K of M which "lie at an end" of M and those which "lie at an end" of each subcontinuum of M properly containing K; e. g., between te minal and absolutely terminal continua. Thus in Chapter IV, we study those continua K of M which are absolutely non-separating.

Finally, in Chapter V we investigate those subcontinua of M which are minimal with respect to "lying at an end" of M. In some cases, these continua must be degenerate or indecomposable.

Preliminaries

In this paper a *continuum* is a compact connected metric space. A continuum M is said to be *decomposable* if it is the union of two proper subcontinua. Otherwise the continuum is *indecomposable*. If A and B are proper subcontinua of M and the union of A and B is M, then we will write $M = A \oplus B$, and say that $A \oplus B$ is a *decomposition* of M. (This idea of decomposition is of course different from the usual decomposition into mutually exclusive subsets. It should always be clear from the context which kind of decomposition is meant.)

Let M be a continuum and K be a proper subcontinuum of M. K is *non-separating* (*non-cutting*) in M if and only if $M - K$ is connected (continuum-wise connected). Clearly every non-cutting subcontinuum is non-separating.

A continuum M is *irreducible* or *irreducible between a pair of points*, if there are two points p and q in M such that no proper subcontinuum of M contains both p and q. Continua which fail to be irreducible are said to be *reducible*.

If A is a subset of the continuum M, then the closure of A in M will be denoted by $\overline{A}$, the interior of A in M by $\operatorname{int}(A)$, and the boundary of A in M by $\operatorname{Bd}(A)$.

For terms and notation used but not defined herein, the reader is referred to [9] and [15].

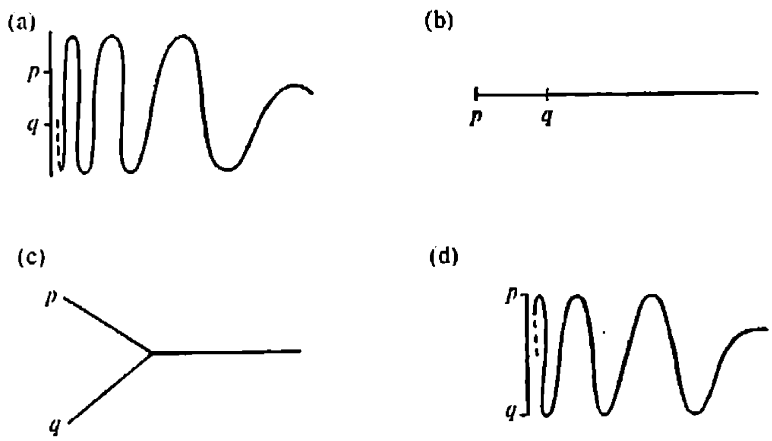
Chapter I

Basic types and properties of non-separating subcontinua

Terminal and end continua

DEFINITION 1.1. A proper subcontinuum K of M is said to be a *terminal continuum* of M provided that if $M = A \oplus B$ and $A \cap K \neq \emptyset \neq B \cap K$ then $M = A \oplus K$ or $M = B \oplus K$.

EXAMPLE 1.2. In each example below, the arc $K = [p, q]$ is a terminal continuum.



THEOREM 1.3. *Each terminal continuum of M is non-separating in M.*

Proof. Let K be a subcontinuum of M and suppose that U and V are disjoint open sets such that $M - K = U \cup V$. Then $K \cup U$ and $K \cup V$ are proper subcontinua of M and $M = (K \cup U) \oplus (K \cup V)$. Thus K is not terminal. ■

THEOREM 1.4. *A subcontinuum K of M is not a terminal continuum of M if and only if there is a decomposition of M, $M = A \oplus B$, such that $K \subset A \cap B$.*

Proof. Suppose K is not terminal. There are subcontinua C and D such that $M = C \oplus D$, $C \cap K \neq \emptyset \neq D \cap K$, and $C \cup K \neq M \neq D \cup K$. Let $A = C \cup K$ and $B = D \cup K$. Then $M = A \oplus B$ and $K \subset A \cap B$. ■

It follows from the definition of terminal continuum that if the condition holds, then K is not terminal.

THEOREM 1.5. *If K is a terminal continuum of M and L is a proper subcontinuum of M which contains K, then L is a terminal continuum of M.*

Proof. If L fails to be terminal, then the preceding theorem shows that there is a decomposition of M, $M = A \oplus B$, such that $L \subset A \cap B$. Clearly $K \subset A \cap B$, so the preceding theorem shows that K is not terminal. ∎

COROLLARY 1.6. *If K is a terminal continuum of M and L is a subcontinuum of M which contains K, then L does not separate M.*

A terminal continuum of M may or may not cut M as shown by Example 1.2 (a) and (b). Conversely, a non-cutting continuum may fail to be terminal (e. g., a subarc of a simple closed curve). However if, M is irreducible this last situation cannot arise.

THEOREM 1.7. *Suppose that M is irreducible. Then every non-cutting continuum is terminal.*

Proof. If M is irreducible from p to q and K is a non-cutting continuum of M, then K contains either p or q. In either case, Theorem 1.5 shows that K is a terminal continuum because $\{p\}$ and $\{q\}$ are terminal continua. ∎

THEOREM 1.8. *Suppose that M is irreducible and K is a subcontinuum of M. Then K is terminal if and only if M is irreducible between a pair of points, one of which belongs to K.*

Proof. Let $\{p, q\} \subset M$ such that M is irreducible from p to q. Assume K is terminal. If $K \cap \{p, q\} = \emptyset$, we wish to show that M is irreducible from p to some point of K or from q to some point of K. If this fails, choose a point $z \in K$. Then there are proper subcontinua A and B of M such that $\{p, z\} \subset A$ and $\{q, z\} \subset B$. Since M is irreducible from p to q, $M = A \oplus B$. Now $p \notin B \cup K$ and $q \notin A \cup K$ which is contrary to K being terminal.

For the converse assume $p \in K$. By Theorem 1.4, $\{p\}$ is terminal. Thus K is terminal by Theorem 1.5. ∎

DEFINITION 1.9. A proper subcontinuum K of M is said to be an *end continuum* of M provided M is not the union of two proper subcontinua each intersecting K.

In Example 1.2 (a) and (d) K is an end continuum, while in (b) and (c) K is not an end continuum. An immediate consequence of the definitions is the following theorem.

THEOREM 1.10 *Each end continuum of M has void interior in M.*

The relationship between terminal and end continua is given by the following theorem.

THEOREM 1.11. *Let K be a subcontinuum of M. Then K is an end continuum if and only if K is a terminal continuum with void interior.*

Proof. The necessity of the condition is an immediate consequence of the definitions and Theorem 1.10.

Suppose that K is terminal, $\mathrm{int}(K) = \emptyset$, but K is not an end continuum. Thus there is a decomposition of M, $M = A \oplus B$, and $A \cap K \neq \emptyset \neq B \cap K$. Since K is terminal, $A \subset K \cup B$ or $B \subset K \cup A$. Assume that $A \subset K \cup B$. Now $M = K \cup B$ and since $M = \overline{M - K}$, then $B = M$. This is a contradiction, thus K is an end continuum of M. ■

COROLLARY 1.12. *A point $p \in M$ is a terminal point of M if and only if p is an end point of M.*

Terminal continua and irreducibility

Examining the examples of 1.2 suggests that while a continuum M having a terminal continuum K need not itself be irreducible, $\overline{M - K}$ must be an irreducible continuum. The main result of this section, Theorem 1.15, establishes that this is indeed the case. We then use this theorem to further study terminal and end continua and consider some interesting examples.

In order to prove the principal result of this section, we need the notion of a monotone upper-semi-continuous decomposition and the lemma below.

Let K be a proper subcontinuum of M and $\mathscr{G}$ be the monotone upper-semi-continuous decomposition of M whose only non-degenerate element is K. Let $M/\mathscr{G}$ be the hyperspace of this decomposition. Then $M/\mathscr{G}$ is a continuum.

LEMMA 1.13. *K is a terminal subcontinuum of M if and only if K is an end point of $M/\mathscr{G}$.*

Proof. Let $\pi: M \to M/\mathscr{G}$ denote the projection map. Suppose that K is an end point of $M/\mathscr{G}$. Then by Theorem 4, page 192 of [9], there is a point $p \in M/\mathscr{G}$ such that $M/\mathscr{G}$ is irreducible from p to K. If K fails to be terminal in M, then there are proper subcontinua A and B of M such that $K \subset A \cap B$ and $M = A \oplus B$. Suppose that $\pi^{-1}(p) \in A$ and let $b \in B - A$. Then $\pi(b) \in [\pi(B) - \pi(A)]$ so that $\pi(A)$ is a proper subcontinuum of $M/\mathscr{G}$ containing p and $\pi(K) = K$. This contradicts the irreducibility of $M/\mathscr{G}$ thus K is terminal in M.

Conversely if K is not an end point of $M/\mathscr{G}$, then there are proper subcontinua $\mathscr{A}$ and $\mathscr{B}$ of $M/\mathscr{G}$ so that $K \in \mathscr{A} \cap \mathscr{B}$ and $M/\mathscr{G} = \mathscr{A} \oplus \mathscr{B}$. Then $\pi^{-1}(\mathscr{A})$ and $\pi^{-1}(\mathscr{B})$ are proper subcontinua of M, each containing K. Since $M = \pi^{-1}(\mathscr{A}) \oplus \pi^{-1}(\mathscr{B})$, K is not terminal in M. ■

COROLLARY 1.14. *K is a terminal subcontinuum of M if and only if $M/\mathscr{G}$ is irreducible from K to some point.*

Proof. The theorem cited above in [9] essentially says that a point p of a continuum M is an end point of M if and only if M is irreducible from p to some point q. ■

Note that the preceding results give a sufficient condition that a continuum have a monotone mapping onto an irreducible continuum. Continua satisfying this condition (i. e., having a terminal subcontinuum) are thus generalizations of the "Continua of type A" of Thomas [14].

The following theorem will be the principal tool in the proofs of the irreducibility theorems of Chapter II.

THEOREM 1.15. *Suppose that M is a continuum and K is a non-separating subcontinuum of M. Then K is terminal if and only if there is a point $p \in M-K$ such that $\overline{M-K}$ is irreducible from p to each point of $\overline{M-K} \cap K$.*

Proof. Suppose that the condition holds, A and B are subcontinua of M such that $M = A \oplus B$, $A \cap K \neq \emptyset \neq B \cap K$, and $p \in A$. Let L be the component of $(A \cup K)-K$ which contains p. Since $L \subset M-K$, then $\bar{L} \subset \overline{M-K}$. Also since $\bar{L}$ meets K, $p \in \bar{L}$, and $\overline{M-K}$ is irreducible from p to $K \cap \overline{M-K}$, it follows that $\bar{L} = \overline{M-K}$. Moreover $L \subset A$, hence $\bar{L} \subset A$ which implies that $\overline{M-K} \subset A$. Therefore $M = A \oplus K$, so K is terminal.

Conversely, suppose that K is terminal. Let $\mathcal{G}$ be the monotone upper-semi-continuous decomposition of M whose only non-degenerate element is K. Then by Lemma 1.13, K is an end point of $M/\mathcal{G}$, so there is a point $q \in M/\mathcal{G}-\{K\}$ such that $M/\mathcal{G}$ is irreducible from q to K. Let $p = \pi^{-1}(q)$.

Assertion. $\overline{M-K}$ is irreducible from p to each point of $\overline{M-K} \cap K$. For suppose that there is a proper subcontinuum N of $\overline{M-K}$ which contains p and intersects K. If $M-K \subset N$, then $\overline{M-K} \subset N$ and N is not proper. Thus $M-K \not\subset N$, hence $\pi(N)$ is a proper subcontinuum of $M/\mathcal{G}$ containing $\pi(p) = q$ and $\pi(K) = K$. This contradicts the irreducibility of $M/\mathcal{G}$ and establishes the theorem. ■

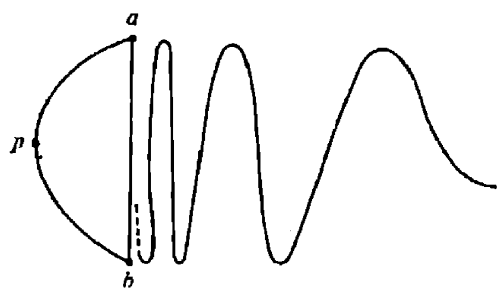
Using the preceding theorem and Theorem 1.11 we obtain immediately a charecterization of end continua, due to Rosen [11].

THEOREM 1.16. *A subcontinuum K is an end continuum of M if and only if there is a point $p \in M-K$ such that M is irreducible from p to each point of K.*

Further consideration of the examples presented thus far suggests that if K is a terminal continuum of M, then $\mathrm{Bd}(K) = \overline{M-K} \cap K$ is a continuum. However this is not so, as shown by the following simple example.

EXAMPLE 1.17. A terminal continuum K such that $\mathrm{Bd}(K)$ is not connected.

K is the arc from a to b which contains p, $\mathrm{Bd}(K) = \{a, b\}$.



We will return to the consideration of this example in Chapter III.

To say that a continuum K is terminal in M means that M cannot be decomposed in a specified way. With some restrictions, this is equivalent to saying that $\overline{M-K}$ cannot be decomposed in a specified way. Thus we establish the following result.

THEOREM 1.18. *Let K be a non-separating subcontinuum of M and suppose that $\overline{M-K}$ is decomposable. Then K is terminal if and only if $\overline{M-K}$ is not the union of two proper subcontinua each intersecting K.*

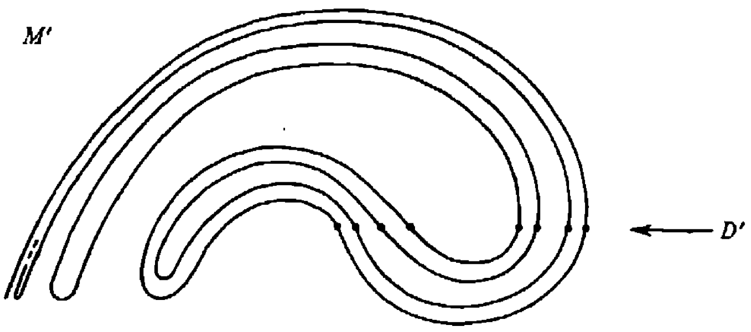
Proof. If K is terminal, then the condition follows from Theorem 1.15. Now suppose that the condition holds. Since $\overline{M-K}$ is decomposable, $\overline{M-K}$ is the union of two of its proper subcontinua A and B. Only one of these, say A, can meet K. Again applying this condition, it follows that each point of $\overline{M-K}\cap K$ is an end point (i. e., terminal point) of $\overline{M-K}$. Choosing $q\in\overline{M-K}\cap K$, and using Theorem 1.16, there is a point $p\in M-K$ such that $\overline{M-K}$ is irreducible from p to q. Note that $p\in B$.

If we assume that the implication fails and that K is not terminal, it follows from Theorem 1.15 that $\overline{M-K}$ is reducible from p to $K\cap\overline{M-K}$. Thus there is a proper subcontinuum N of $\overline{M-K}$ which contains p and meets $K\cap\overline{M-K}$. Clearly, $q\notin N$ and so $q\notin N\cup B$. It follows that $\overline{M-K}$ is the union of $N\cup B$ and A, each a proper subcontinuum of $\overline{M-K}$ meeting K. This contradicts the condition and shows that K must be a terminal continuum of M. ■

We wish to present an example to show that we cannot omit "$\overline{M-K}$ is decomposable" from the hypothesis of the preceding theorem. This example (1.20) is constructed by starting with an example of Cornette [3], which we now describe informally.

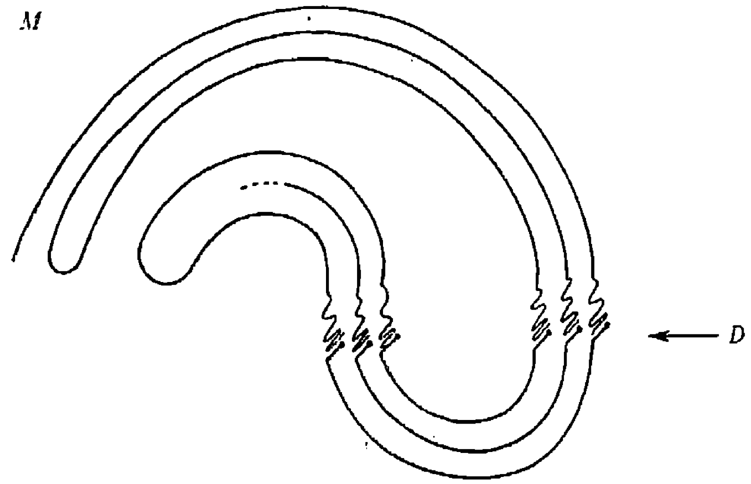
EXAMPLE 1.19. An indecomposable continuum M having a closed subset D such that (1) D meets each composant of M, (2) $M-D$ is connected, and (3) D is homeomorphic to the Cantor ternary set.

We commence with a well known indecomposable continuum M', formed by "winding on the Cantor set" in the plane (Fig. 4–5 p. 205 of [9]).



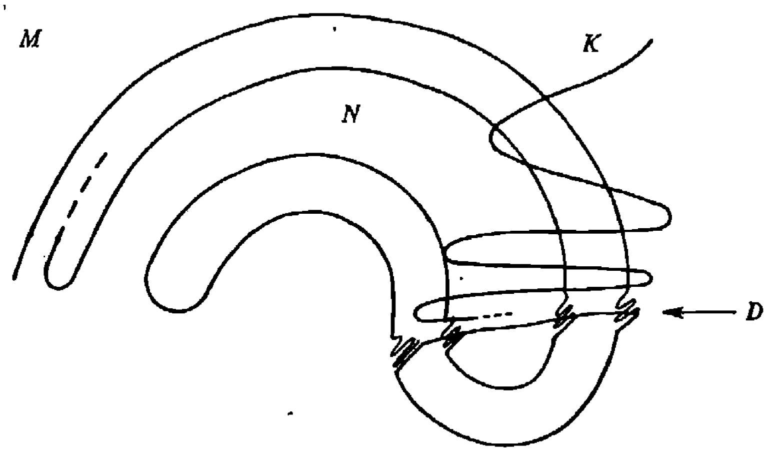
We let D' be the points of this continuum which lie on the ends of the right hand collection of semi-circles. Note that D' has properties (1) and (3) but not property (2). Thus to obtain the continuum M we remove the points of D' from M' and replace each point by a "graph of $y = \sin\frac{1}{x}$".

Let D be the set of upper end points of limit arcs of the graphs of $y = \sin\frac{1}{x}$.



EXAMPLE 1.20. Let M be the continuum described in Example 1.19 and let K be the "$\sin\frac{1}{x}$ continuum"; e.g., K is a homeomorphic copy of $\overline{\left\{\left(x, \sin\frac{1}{x}\right) : 0 < x \leqslant 1\right\}}$. Let N be the union of M and K, joined so that their intersection is the closed set D (satisfying (1), (2), and (3) of the preceding example) and contained in the "limit arc" of K.

It follows from Theorem 1.15 that K is not terminal in N, since $\overline{N-K} = M$ is not irreducible from any point to $\overline{N-K}\cap K = D$. However, $\overline{N-K}$ is not the union of any two of its proper subcontinua each meeting K.



Composants and E-continua

DEFINITION 1.21. Suppose M is a continuum and $p \in M$. Then the *composant of p in M*, denoted by C_p, is $\{x \in M\colon$ there is a proper subcontinuum of M containing p and $x\}$.

It is well known that C_p is connected and dense in M, and that $M-C_p$ is connected; moreover a continuum is irreducible if and only if M has a composant which is a proper subset of M ([9], p. 108). Using these facts, it follows from Theorem 1.16 that any subcontinuum of M which is contained in $M-C_p$ is an end continuum and that $M-C_p = \{x\colon M$ is irreducible from p to $x\}$.

THEOREM 1.22. *Suppose that M is a continuum and $p \in M$. The following are equivalent*:

(1) *$M-C_p$ is closed*;

(2) *$M-C_p$ is a continuum*;

(3) *$M-C_p$ is continuum-wise connected*;

(4) *If K is a subcontinuum of M such that $K\cap(M-C_p) \neq \emptyset$ and $K\cap C_p \neq \emptyset$, then K is decomposable.*

Proof. Since $M-C_p$ is always connected, (1) implies (2) and (2) implies (3).

We shall show that (3) implies (4). Suppose that (3) holds but (4) fails. Then there is an indecomposable subcontinuum K such that $K\cap(M-C_p) \neq \emptyset$ and $K \not\subset M-C_p$. There is a proper subcontinuum

L of M such that $p \in L$ and $L \cap K \neq \emptyset$. Since $L \cup K$ is a subcontinuum of M containing p and intersecting $M - C_p$, it follows that $L \cup K = M$. Since $L \subset C_p$, then $M - C_p \subset K$. Also since $M - C_p$ is a continuum-wise connected proper subset of K, there is a composant C of K such that $M - C_p \subset C$. There is a sequence N of proper subcontinua of K such that $C = \bigcup_{i=1}^{\infty} N_i$. Since K is indecomposable, each N_i is nowhere dense in K.

There is a sequence J of proper subcontinua of M such that $C_p = \bigcup_{i=1}^{\infty} J_i$. For each i, $(J_i \cap K)$ is a proper closed subset of K.

Assertion. $J_i \cap K$ does not contain a set V open relative to K. For suppose V is open relative to K and $V \subset J_i \cap K$. Then V, hence J_i, meets each composant of K. Let T be a proper subcontinuum of K meeting $M - C_p$ and meeting J_i. Since K is indecomposable, T contains no subset which is open relative to K. Thus $K - J_i \not\subset T$ which implies that $T \cup J_i$ is a proper subcontinuum of M containing p and meeting $M - C_p$. This contradiction establishes the assertion, hence for each positive integer i, $J_i \cap K$ is nowhere dense in K. Thus

$$K = (K \cap C_p) \cup [(M - C_p) \cap K] = \Big[K \cap \Big(\bigcup_{i=1}^{\infty} J_i\Big)\Big] \cup C = \Big[\bigcup_{i=1}^{\infty} (J_i \cap K)\Big] \cup \Big(\bigcup_{i=1}^{\infty} N_i\Big).$$

That is, K is the union of countably many closed nowhere dense sets which is a contradiction to the Baire Category Theorem. Thus (3) implies (4).

Now we show that (4) implies (1). Suppose that (4) holds but that (1) fails. Then $\overline{M - C_p}$ is a continuum properly containing $M - C_p$. Then by (4), $\overline{M - C_p}$ is decomposable so let $\overline{M - C_p} = A \oplus B$. If $M - C_p \subset A$, then $\overline{M - C_p} \subset A$ which is false. Thus $M - C_p \not\subset A$ and in like manner $M - C_p \not\subset B$. Since $\overline{M - C_p}$ properly contains $M - C_p$, either A or B intersects C_p. Suppose that $A \cap C_p \neq \emptyset$. There is a proper subcontinuum D of M containing p and intersecting A. Note that $D \subset C_p$. It follows that $D \cup A$ is a subcontinuum of M containing p and meeting $M - C_p$. Moreover $D \cup A$ is a proper subcontinuum of M because $(B - A) \cap (M - C_p) \neq \emptyset$ and $D \subset C_p$. This contradiction establishes the desired implication. ∎

COROLLARY 1.23 (Thomas [14]). *Suppose that M is a continuum such that each indecomposable subcontinuum has void interior. Then for each $p \in M$, $M - C_p$ is a continuum.*

Proof. Any subcontinuum K satisfying (4) has non-void interior, hence is decomposable. The corollary follows since (4) implies (2) in the preceding theorem. ∎

Following Miller ([10]), we shall call a continuum which is the complement of a composant an E-continuum.

DEFINITION 1.24. A subcontinuum K is an *E-continuum* of M if and only if $M-K$ is a composant of M.

Of the examples presented thus far only in Example 1.2 (d) is K an E-continuum. Clearly every E-continuum is an end continuum and is also non-cutting. Thus we have the following implications.

$$\begin{tikzcd}
E\text{-continua} \arrow[r] \arrow[dr] & \text{end continua} \arrow[r] & \text{terminal continua} \arrow[r] & \text{non-separating continua} \\
& \text{non-cutting continua} \arrow[urr] & &
\end{tikzcd}$$

The examples presented thus far show that none of these implications may be reversed.

In the next few theorems we characterize E-continua by using the non-separating continua previously studied.

THEOREM 1.25. *A continuum K is an E-continuum of M if and only if K is a maximal end continuum of M (i.e., K is not properly contained in any end continuum of M).*

Proof. Suppose that K is an E-continuum. There is a point $p \in M-K$ such that $K = M-C_p$. If L is a subcontinuum of M which properly contains K, then $L \cap C_p \neq \emptyset$. There is a proper subcontinuum N of M such that $p \in N$ and $L \cap N \neq \emptyset$. It follows that $M = L \oplus N$, so $\operatorname{int}(L) \neq \emptyset$. By Theorem 1.10, L is not an end continuum, hence K is a maximal end continuum.

Now suppose that K is a maximal end continuum. According to Theorem 1.16, there is a point p such that M is irreducible from p to each point of K. Let C_p be the composant of M determined by p; then $K \subset M-C_p$.

If K fails to be an E-continuum, then by Theorem 1.22 (4), there is an indecomposable subcontinuum L of M such that $L \cap C_p \neq \emptyset$ and $L \cap (M-C_p) \neq \emptyset$. It follows that $M-C_p \subset L$, so K is a proper subcontinuum of L. Thus K is contained in a composant of L. Since composants are continuum-wise connected dense sets, there is a proper subcontinuum J of L such that $K \subsetneq J$. From Theorem 1.5, it follows that J is a terminal continuum of M.

Since L is indecomposable, J has void interior in L, hence J has void interior in M. By Theorem 1.11, J is an end continuum of M and K is not a maximal end continuum. This contradiction establishes that K is an E-continuum. ■

THEOREM 1.26. *Suppose that K is an end continuum of M. Then the following are equivalent:*

(1) *K is an E-continuum;*

(2) *K is a maximal end continuum;*

(3) *K is a non-cutting end continuum.*

Proof. By the preceding theorem (1) and (2) are equivalent and it is clear that (1) implies (3). We show that (3) implies (1). Let K be a non-cutting end continuum of M. Then by Theorem 1.16 there is a composant C_p of M such that $K \subset M - C_p$. Let $q \in M - K$. Since $M - K$ is continuum-wise connected there is a subcontinuum L of $M - K$ such that $\{p, q\} \subset L$. Thus $q \in C_p$ and it follows that $M - C_p = K$. Therefore K is an E-continuum. ■

COROLLARY 1.27. *A subcontinuum K of M is an E-continuum if and only if K is terminal, non-cutting, and has void interior in M.*

Proof. According to Theorem 1.11, end continua are exactly those terminal continua which have void interior. ■

COROLLARY 1.28. *Suppose that M is an irreducible continuum. Then K is an E-continuum if and only if K is non-cutting and has void interior.*

Proof. If K is an E-continuum, then the preceding corollary shows that K is non-cutting and has void interior. Conversely, if K is non-cutting and has void interior, then Theorem 1.7 shows that K is a terminal continuum and hence an E-continuum. ■

DEFINITION 1.29. A continuum M is said to be *locally connected at a subcontinuum N* if for each open set U containing N there is an open connected set V such that $N \subset V \subset U$. (Notice that this is quite different from saying that M is locally connected at each point of N. The latter condition implies the former; if N is degenerate, they are equivalent.)

THEOREM 1.30. *The E-continua of a continuum M are exactly those end continua at which M is locally connected.*

Proof. First suppose that K is an end continuum of M and that M is locally connected at K. Let a and b be distinct points in $M - K$ and let U be an open set such that $K \subset U \subset \overline{U} \subset M - \{a, b\}$. There is an open connected set V such that $K \subset V \subset U$. Then by Theorem 1.5, $\overline{V}$ is a terminal continuum of M and by applying Theorem 1.3 we see that $M - \overline{V}$ is connected. Thus $\overline{M - \overline{V}}$ is a subcontinuum containing $\{a, b\}$ and $\overline{M - \overline{V}}$ lies in the complement of K. This implies that K is non-cutting, so by Theorem 1.26 (3), K is an E-continuum.

Now suppose that K is an E-continuum. Then it follows that K is an end continuum. Let U be an open set containing K and let $p \in M$ such that $K = M - C_p$. Let V be an open set such that $K \subset V \subset \overline{V} \subset U$ and $p \notin \overline{V}$. Let J be the component of V containing K. Then $\overline{J} \subset \overline{V}$, thus by Theorem 1.5, $\bar{J}$ is terminal. It follows that $\overline{M-\bar{J}}$ is a proper subcontinuum of M which contains $\{p\}$. Since $\{p\}$ is terminal, $\overline{M-\bar{J}}$ is also terminal. Therefore $M-(\overline{M-\bar{J}})$ is connected (Theorem 1.3). Thus we have $K \subset M-(\overline{M-\bar{J}}) \subset \bar{V} \subset U$, which implies that M is locally connected at K. ■

In Theorems 3.15 and 3.17 we will generalize and extend Theorems 1.26 and 1.30.

Chapter II

Necessary and sufficient conditions for irreducibility

In this chapter we continue the study of irreducibility begun in the second section of Chapter I. We begin by examining the role of terminal continua.

Decompositions into terminal continua

The following theorem is an immediate consequence of Theorem 1.16.

THEOREM 2.1. *A continuum M is irreducible if and only if M has an end continuum.*

Examples such as 1.2 (c) and 1.17 show that we cannot replace "end" by "terminal" in the above theorem. However, the next result does provide a characterization of irreducibility using terminal continua.

THEOREM 2.2. *Let M be a decomposable continuum. Then M is irreducible if and only if there is a terminal continuum K of M such that $\overline{M-K}$ is terminal.*

Proof. Suppose M is irreducible between the points p and q, and let A and B be subcontinua of M such that $M = A\oplus B$. We may assume that $p\in A-B$ and $q\in B-A$. Thus by Theorem 1.16 p and q are end points of M, hence terminal points of M. Now by Theorem 1.5, A is a terminal continuum, hence $\overline{M-A}$ is a subcontinuum containing q. Again applying Theorem 1.5, $\overline{M-A}$ is terminal.

Conversely, suppose that K is a terminal continuum of M such that $\overline{M-K}$ is terminal. It follows from Theorem 1.15 that there is a point $p\in M-K$ such that $\overline{M-K}$ is irreducible from p to each point of $\overline{M-K}\cap K$. Also since $\overline{M-K}$ is terminal, there is a point $q\in M-(\overline{M-K})$ such that $\overline{M-(\overline{M-K})}$ is irreducible from q to $[\overline{(M-\overline{M-K})}\cap\overline{M-K}]\subset K\cap\overline{M-K}$. Thus $M = \overline{M-\overline{M-K}}\oplus\overline{M-K}$. Now $\overline{M-\overline{M-K}}$ is irreducible from q to $[\overline{M-\overline{M-K}}\cap\overline{M-K}]$ and $\overline{M-K}$ is irreducible from p to $[\overline{M-\overline{M-K}}\cap$ $\cap\overline{M-K}]$.

Assertion. M is irreducible from p to q. For suppose that N is a proper subcontinuum of M containing p and q. Without loss of generality, we may suppose that $N \cap \overline{M-K}$ is a proper subset of $\overline{M-K}$. It follows that $N \cap (M-K)$ is a proper subset of $M-K$. Let C be the component of p in $N \cap (M-K)$. Since $\bar{C} \subset N \cap \overline{M-K}$ and $N \cap \overline{M-K}$ is a proper subset of $\overline{M-K}$, $\bar{C}$ is a proper subcontinuum of $\overline{M-K}$ containing p. Also by the "to the boundary theorem" ([9], Theorem 2, p. 172), $\bar{C} \cap K \neq \emptyset$. This contradicts the irreducibility of $\overline{M-K}$ and establishes the theorem. ∎

Corollary 2.3. *A continuum M is irreducible if and only if M contains a pair of disjoint terminal continua.*

Corollary 2.4. *Let M be a decomposable irreducible continuum such that each separating subcontinuum of M has void interior. Then M is the sum of two indecomposable terminal subcontinua.*

Proof. According to Theorem 2.2 there is a terminal subcontinuum K of M such that $\overline{M-K}$ is also terminal.

Assume $\overline{M-K}$ is decomposable, say $\overline{M-K} = A \oplus B$. Then by Theorem 1.18 one of these subcontinua, say A, does not intersect K. Thus $A \cup K$ is a proper closed subset of M and since $M-(A \cup K) \subset B$, it follows that $\operatorname{int}(B) \neq \emptyset$. Also $M-B$ is contained in the union of the two disjoint closed sets A and K but is contained in neither A nor K. Thus $M-B$ is not connected which implies that B separates M. This contradiction implies that $\overline{M-K}$ is indecomposable.

In a similar fashion, it follows that K is also indecomposable. ∎

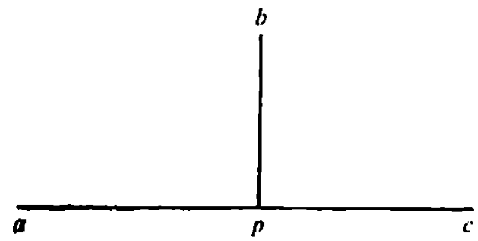
Theorem 2.5. *A necessary and sufficient condition that the continuum M be irreducible is that if $M = A \oplus B$, then both A and B are terminal continua.*

Proof. Suppose that M is irreducible from p to q and $M = A \oplus B$. We may assume that $p \in A$ and $q \in B$. It follows from Theorem 1.16 that p and q are terminal points; Theorem 1.5 guarantees that A and B are terminal continua.

Conversely if the condition holds, we may assume that M is decomposable. If $M = A \oplus B$, then A is terminal which implies that $\overline{M-A}$ is a continuum. Since $M = A \oplus \overline{M-A}$, $\overline{M-A}$ is terminal and by Theorem 2.2, M is irreducible. ∎

According to Theorem 2.5, M is irreducible provided each decomposition of M is a decomposition into terminal continua. We cannot replace this by "there exists a decomposition of M into terminal continua," as is shown by the following example.

Example 2.6. Let M be the union of three arcs, $[a, p]$, $[b, p]$, and $[c, p]$ such that each pair intersect only in $\{p\}$. Then each of the arcs $[a, b]$ and $[a, c]$ is a terminal continuum of M.



The reason that this example fails to be irreducible is that M is a triod. Because of the importance of triods in relation to irreducibility, we denote much of the next section to them.

Triods and Sorgenfrey's theorems

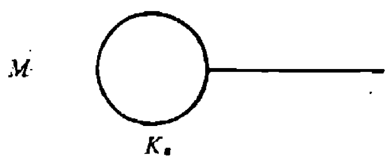
DEFINITION 2.7 (Sorgenfrey [12]). A continuum M is a *type* 1 *triod* provided that M is the essential sum of three proper subcontinua such that all three have a point in common.

M is a *type* 3 *triod* if M is the essential sum of three proper subcontinua such that all three intersect, the common part of each pair is the common part of all three, and this common part is a proper subcontinuum of each of them.

A type 3 triod is usually just called a *triod*. If a continuum contains no (type 3) triods, then it is said to be *a-triodic*.

Clearly every type 3 triod is a type 1 triod. Moreover, Sorgenfrey has shown ([12]) that each type 1 triod contains a type 3 triod. It follows that in the definition of "a-triodic" we may omit any reference to "type."

EXAMPLE 2.8. A type 1 triod which is not a type 3 triod. This continuum is the union of an arc and a simple closed curve, intersecting only in an end point of the arc.



Let K be the simple closed curve.

Note that the simple closed curve, K, is a terminal continuum.

THEOREM 2.9. *Let M be a continuum which is not a type* 1 *triod. If $M = A \oplus B$, then each of A and B is non-separating.*

Proof. Suppose to the contrary that there is a decomposition of M, $M = A \oplus B$, such that $M - A$ is not connected. Let S and T be mutually separated sets such that $M - A = S \cup T$. Then both S and T are connected. For if S is the union of mutually separated sets R and Q, then $R \cup A, Q \cup A$, and $T \cup A$ are all continua with a common point, which contradicts our hypothesis.

Clearly, $\bar{S} \cap \bar{T} = \emptyset$, otherwise $\bar{S}, \bar{T}$, and A would be a forbidden decomposition.

Assertion. No subcontinuum in $A \cap B$ intersects both $\bar{S}$ and $\bar{T}$. For if L is a subcontinuum in $A \cap B$ such that $L \cap \bar{S} \neq \emptyset \neq L \cap \bar{T}$, then $\bar{S} \cup L$, $\bar{T} \cup L$, and A would be a forbidden decomposition. This establishes the assertion.

It follows that $A \cap B$ is the union of the disjoint closed sets D and E with $(A \cap B \cap \bar{S}) \subset D$ and $(A \cap B \cap \bar{T}) \subset E$. Thus $B = (D \cup \bar{S}) \cup (E \cup \bar{T})$. Since $(D \cup \bar{S})$ and $(E \cup \bar{T})$ are disjoint closed sets, this implies that B is not connected which is a contradiction.

Therefore B, and likewise A, is non-separating. ∎

THEOREM 2.10 (Sorgenfrey [13]). *A necessary and sufficient condition that the continuum M be irreducible is that if M is the essential sum of three proper subcontinua, then some pair fail to intersect.*

Proof. It is clear that the condition is necessary for the irreducibility of M. We shall assume that the condition holds and show that M is irreducible.

Clearly we may assume that M is decomposable, and that M is not the essential sum of three subcontinua such that all three intersect. Applying Theorem 2.9 we conclude that each decomposition is a decomposition into non-separating continua. By Theorem 2.5 we see that if M is reducible, there is a decomposition of M, $M = A \oplus B$, such that one of the continua, say A, fails to be terminal. It follows that there are continua X and Y such that $M = X \oplus Y$ and $A \subset X \cap Y$. Since X is non-separating, $M - X$ is a proper open connected subset of $M - A$. It follows that the continuum $\overline{M - A}$ is decomposable. Applying Theorem 1.18 we have proper subcontinua Q and R of $\overline{M - A}$ such that both Q and R intersect A and $\overline{M - A} = Q \cup R$. However, Q, R, and A now form a decomposition of M of the type contrary to the condition. Therefore M is irreducible. ∎

DEFINITION 2.11. A continuum M is said to be *unicoherent* provided if $M = A \oplus B$, then $A \cap B$ is a continuum. A continuum is *hereditarily unicoherent* if every subcontinuum is unicoherent.

THEOREM 2.12 (Sorgenfrey [12]). *If M is a unicoherent continuum which is not a triod, then M is irreducible.*

Proof. Suppose that the theorem fails and M is reducible. Then by the preceding theorem, M is the essential sum of subcontinua A, B, and C such that $A \cap B \neq \emptyset, B \cap C \neq \emptyset$, and $A \cap C \neq \emptyset$. Since M is unicoherent, $A \cap (B \cup C)$ is connected which implies that $A \cap B \cap C \neq \emptyset$. Then

$$L = [A \cap (B \cup C)] \cup [B \cap (A \cup C)] \cup [C \cap (A \cup B)]$$

is a continuum. Let $A' = L \cup A$, $B' = L \cup B$, and $C' = L \cup C$. Then $M = A' \oplus B' \oplus C'$. Now L is a proper subcontinuum of A', B' and C' and $A' \cap B' \cap C' = A' \cap B' = A' \cap C' = B' \cap C' = L$. This implies that M is a triod which is impossible. Thus M is irreducible. ∎

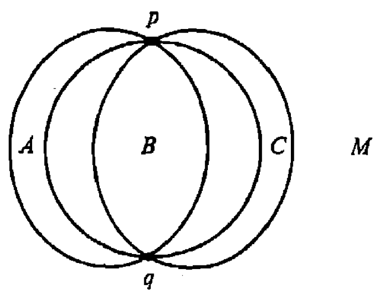
THEOREM 2.13. *A continuum M is irreducible if and only if M is not a type* 1 *triod and each non-separating subcontinuum with non-void interior is a terminal continuum.*

Proof. Assume first that M is irreducible. Then by Theorem 2.10, M is not a type 1 triod. Let K be a non-separating subcontinuum with non-void interior. Then $\overline{M-K}$ is a proper subcontinuum of M and $M = K \oplus \overline{M-K}$. Thus by Theorem 2.5 K is a terminal continuum.

Now assume that M is a decomposable continuum which is not a type 1 triod and each non-separating subcontinuum with non-void interior is terminal. Let A and B be subcontinua of M such that $M = A \oplus B$. Applying Theorem 2.9 we see that neither A nor B separates M. Clearly both A and B have non-void interior, so by hypothesis A and B are terminal continua. It follows from Theorem 2.4 that M is irreducible. ∎

The following example shows that one cannot replace "M is not a type 1 triod" by "M is not a type 3 triod" in the preceding theorem.

EXAMPLE 2.14. Let A, B, and C be indecomposable continua such that each of A, B, and C is irreducible between the points p and q, and $A \cap B = A \cap C = B \cap C = \{p, q\}$. Let $M = A \oplus B \oplus C$.



The only non-separating subcontinua of M with non-void interior are those continua containing some pair of the subcontinua A, B, and C.

So M is not a type 3 triod, each non-separating subcontinuum with non-void interior is terminal, but clearly M is reducible.

THEOREM 2.15. *Suppose that M a is continuum which is not a type* 1 *triod, K is terminal in M, and $\overline{M-K}$ is decomposable. Then M is irreducible.*

Proof. Let A and B be subcontinua of M such that $\overline{M-K} = A \oplus B$ Then according to Theorem 1.18 one of these subcontinua, say A, does not intersect K.

Assume that A is not terminal. Now $M = (K \cup B) \oplus A$, so Theorem 2.9 implies that $\overline{M-A}$ is a continuum. Also since $\operatorname{int}(K) \neq \emptyset$ (Theorem 1.11) and $K \subset \overline{M-A}$, it follows that $\overline{M-A}$ is decomposable. According to Theorem 1.18, there is a decomposition of $\overline{M-A}$, say $\overline{M-A} = S \oplus T$, so that $A \cap S \neq \emptyset \neq A \cap T$.

Now $M = (B \cup S) \cup (B \cup T) \cup (A \cup B)$ and $(B \cup S) \cap (B \cup T) \cap (A \cup B) \neq \emptyset$, so this sum of subcontinua is not an essential sum.

Case 1. Suppose that $M = (B \cup S) \cup (B \cup T) = B \cup S \cup T$ and let $x \in M - (B \cup K)$. Clearly $x \in \operatorname{int}(A)$. But since $M = B \cup S \cup T$ and $x \notin B$, then $x \in S \cup T = \overline{M-A}$. So this case leads to a contradiction.

Case 2. If $M = (B \cup S) \cup (A \cup B) = S \cup A \cup B$, then S, A, and B is an essential sum of subcontinua. Let $x \in S \cap A$. Then $x \in \overline{M-A} \cap A \subset B$, so $S \cap A \cap B \neq \emptyset$. This implies that M is a type 1 triod which is a contradiction.

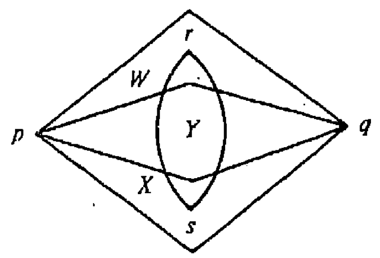
Case 3. If $M = (B \cup T) \cup (A \cup B) = T \cup A \cup B$, then as in case 2, the subcontinua T, A, and B form a type 1 triodic decomposition of M which is contrary to the hypothesis.

Hence, in any case, a contradiction is reached so it follows that A is a terminal continuum. Thus according to Corollary 2.3, M is irreducible. ■

The following example shows that the theorem above does not hold if $\overline{M-K}$ is indecomposable.

EXAMPLE 2.16.

$M = W \cup X \cup Y$, where W, X, and Y are indecomposable continua.



W is irreducible between any pair of $\{p, q, r\}$;
X is irreducible between any pair of $\{p, q, s\}$;
Y is irreducible between r and s.

Let $K = W \cup X$. Then K is terminal and $\overline{M-K}$ is indecomposable. Clearly M fails to be irreducible between any pair of points.

Note that Example 2.8 shows that the preceding theorem fails if we replace "not a type 1 triod" by "not a type 3 triod."

COROLLARY 2.17. *Suppose that M is a continuum such that each indecomposable subcontinuum of M has void interior. M is irreducible if and only if* (i) *M is not a type* 1 *triod, and* (ii) *M contains a terminal continuum.*

Proof. If M is irreducible, then (i) and (ii) follow from Theorems 2.10 and 2.2. If (i) and (ii) hold, then M is irreducible according to the preceding theorem. ∎

COROLLARY 2.18. *Suppose that M is hereditarily decomposable. Then M is irreducible if and only if* (i) *M is not a type* 1 *triod, and* (ii) *M contains a terminal continuum.*

COROLLARY 2.19. *Suppose that M is hereditarily decomposable. Then M is irreducible if and only if* (i) *M is not a type* 1 *triod, and* (ii) *M has a decomposition into terminal continua.*

DEFINITION 2.20. A continuum M is said to be *irreducible about a subset S of M* provided no proper subcontinuum of M contains S.

Many of the results in this chapter can be extended in an obvious way to a continuum which is irreducible about a finite subset.

According to Theorem 2.5, if M is a decomposable continuum which is irreducible between a pair of points, then every decomposition of M is a decomposition into terminal continua. We now show the existence of terminal decompositions for continua which are irreducible about certain types of subsets.

THEOREM 2.21. *Suppose that the decomposable continuum M is irreducible about a subset S, s_1 and s_2 are distinct points of S, and M is not irreducible about $S-\{s_1\}$ or $S-\{s_2\}$. Then M can be decomposed into terminal continua.*

Proof. Let N be a proper subcontinuum of M such that $s_1 \notin N$ and $S-\{s_1\} \subset N$. Clearly N does not separate M. For if U and V are mutually separated sets such that $M-N = U \cup V$ and $s_1 \in U$, then $N \cup U$ would be a proper subcontinuum of M containing S.

Assertion. The continuum $\overline{M-N}$ is irreducible from s_1 to each point of $\overline{M-N} \cap N$. If this is not the case, there is a proper subcontinuum B of $\overline{M-N}$ such that $s_1 \in B$ and $B \cap N \neq \emptyset$. Then $B \cup N$ is a proper subcontinuum of M containing S which is contrary to the hypothesis.

Therefore N is a non-separating continuum and $\overline{M-N}$ is irreducible from s_1 to each point of $\overline{M-N} \cap N$. According to Theorem 1.15, N is a terminal continuum.

In a similar manner construct a terminal continuum L such that $s_2 \notin L$ and $S-\{s_2\} \subset L$. Then the continuum $N \cup L$ contains S which implies that $M = N \oplus L$. ∎

Corollary 2.22. *Let M be a decomposable continuum. If M is irreducible about a finite subset, then there is a decomposition, $M = A \oplus B$, such that both A and B are terminal continua.*

Corollary 2.23. *If the continuum M is irreducible about a finite subset, then every point of M belongs to a terminal continuum.*

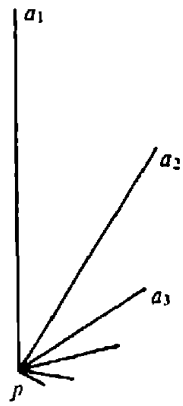
Consider the following example of a continuum which is irreducible about a countable closed subset.

Example 2.24.

M is the union of a countable collection of arcs $[p, a_i]$, $i \in [1, \infty)$, with a common end point p and $\lim_{i\to\infty}(\operatorname{dia}[p, a_i]) = 0$;

M is irreducible about the countable closed set $S = \{a_i\}_{i=1}^{\infty} \cup \{p\}$.

M



Note that M can be decomposed into terminal continua.

Question 2.25. Do the preceding two corollaries hold if the word "finite" is replaced with the words "countable closed"?

Chapter III

Terminal and non-cutting continua

Properties of terminal non-cutting continua

In Chapter I (Corollary 1.27) it was established that a necessary and sufficient condition for a subcontinuum to be an E-continuum is that it be terminal, non-cutting, and have void interior. Thus "terminal and non-cutting" is a reasonable substitute for "E-continuum."

Considering Example 1.17, we see that a terminal continuum K of M may have boundary which is not connected. However, if K is also a non-cutting continuum of M, then this cannot happen. Indeed, the following theorem shows that a terminal and non-cutting continuum occurs by intersecting two continua along an E-continuum of one of them.

THEOREM 3.1. *A subcontinuum K of M is terminal and non-cutting if and only if the boundary of K is an E-continuum of $\overline{M-K}$.*

Proof. Suppose that K is terminal and non-cutting in M. By Theorem 1.15, there is a $p \in M-K$ such that $\overline{M-K}$ is irreducible from p to $\mathrm{Bd}(K) = K \cap \overline{M-K}$. Let C_p be the composant of p in $\overline{M-K}$. Clearly $K \cap \overline{M-K} \subset (\overline{M-K}) - C_p$. Since K is non-cutting, it follows that $M-K \subset C_p$. Also if $x \in (\overline{M-K}) - C_{p'}$, then $x \notin M-K$. Therefore $K \cap \overline{M-K} = (\overline{M-K}) - C_p$. Since $\overline{M-K} - C_p$ is connected, this implies that $K \cap \overline{M-K}$ is an E-continuum of $\overline{M-K}$.

Suppose now that $\overline{M-K} \cap K$ is an E-continuum of $\overline{M-K}$. There is a $q \in M-K$ such that $\overline{M-K} \cap K = \overline{M-K} - C_q$. It follows from Theorem 1.15 that K is terminal while it is clear that K is non-cutting in M. ∎

THEOREM 3.2. *Let K be a terminal non-cutting continuum of M. Then one and only one of the following holds*:

(1) *M is irreducible*;

(2) *K is decomposable and* $\mathrm{Bd}(K)$ *is not a terminal continuum of K.*

Proof. Suppose that (1) fails. Then by Theorem 2.2, $\overline{M-K}$ fails to be a terminal continuum of M. Applying Theorem 1.4, we obtain a decomposition of M, $M = A \oplus B$, such that $\overline{M-K} \subset A \cap B$. Each component of $A - \overline{M-K}$ has a limit point in $\overline{M-K}$ and each such limit point is a point of $\mathrm{Bd}(K)$. Since $\mathrm{Bd}(K)$ is a continuum by the preceding theorem, $\overline{A-\overline{M-K}} \cup \mathrm{Bd}(K)$ is a proper subcontinuum of K with interior. In like fashion, $\overline{B-\overline{M-K}} \cup \mathrm{Bd}(K)$ is a proper subcontinuum of K with interior. It follows that (2) holds.

Suppose now that (2) holds. By Theorem 1.4, K is the union of two proper subcontinua D and E of K, each containing $\mathrm{Bd}(K)$. Then D, E, and $\overline{M-K}$ form a type 1 triod, so M cannot be irreducible. ∎

COROLLARY 3.3. *If K is a non-cutting terminal subcontinuum of M and K is indecomposable, then M is irreducible.*

We note that in the preceding two theorems (Theorem 3.2 and Corollary 3.3), we could have weakened our hypothesis by replacing "K is noncutting" with "$\mathrm{Bd}(K)$ is connected."

THEOREM 3.4. *Suppose that K is a terminal non-cutting continuum of M and L is a subcontinuum of M properly containing K. Then $K \subset \mathrm{int}(L)$.*

Proof. Assume that $K \not\subset \mathrm{int}(L)$. Then $K \cap \overline{M-L} \neq \emptyset$. Since L is terminal, $M-L$ is connected and $K \cup \overline{M-L}$ is a subcontinuum of M. Now K is terminal and $M = L \cup (K \cup \overline{M-L})$ which implies that $K \cup \overline{M-L} = M$. Thus $\mathrm{int}(L) \subset K$.

Let $p \in M-L$ such that $\overline{M-L}$ is irreducible from p to $\overline{M-L} \cap L$. Let $q \in L-K$. Then $q \notin \mathrm{int}(L)$ which implies that $q \in L \cap \overline{M-L}$. Since K does not cut p from q, there is a subcontinuum $N \subset M-K$ such that $\{p, q\} \subset N$. Now $M = K \cup \overline{M-L}$, thus $N \subset \overline{M-L}$. However, $\overline{M-L}$ is irreducible from p to q so $N = \overline{M-L}$. This says that $K \cap \overline{M-L} = K \cap N = \emptyset$ which implies that M is not connected. This contradiction establishes the theorem. ∎

COROLLARY 3.5. *Suppose that K is an E-continuum of M and L is a subcontinuum of M properly containing K. Then $K \subset \mathrm{int}(L)$.*

Proof. By Corollary 1.27 an E-continuum is terminal and non-cutting so it follows from the preceding theorem that $K \subset \mathrm{int}(L)$. ∎

Local connectivity at K and K-aposyndesis

The concept of a continuum being locally connected at a subcontinuum K was introduced in the third section of Chapter I. If K is a terminal continuum of M, the following theorem relates this idea to that of being a non-cutting continuum.

THEOREM 3.6. *Let K be a terminal continuum in M. Then K is non-cutting if and only if M is locally connected at K.*

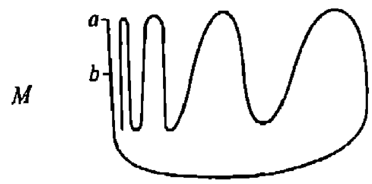
Proof. First assume that M is locally connected at K. Let p and q be distinct points in $M-K$ and let U be an open set such that $K \subset U \subset M-\{p, q\}$. There is an open connected set V such that $K \subset V \subset \overline{V} \subset U$. Thus by Theorem 1.5, $\overline{V}$ is terminal and it follows from Theorem 1.3 that $M-\overline{V}$ is connected. Therefore $\{p, q\} \subset \overline{M-V} \subset M-K$ which implies that K is non-cutting in M.

Now suppose that K is non-cutting and let U be an open set containing K. By applying Theorem 3.1 we see that $\overline{M-K}\cap K$ is an E-continuum of $\overline{M-K}$. Thus by Theorem 1.30, $\overline{M-K}$ is locally connected at $\overline{M-K}\cap K$.

There is a set W, open in M, such that $K\cap\overline{M-K} \subset W\cap\overline{M-K} \subset U\cap\overline{M-K}$ and $W\cap\overline{M-K}$ is a connected set which is open relative to $\overline{M-K}$. Let $V = K\cup W$. Clearly V is connected and $K \subset V \subset U$. Since $M-V = \overline{M-K}\cap(M-W)$, it follows that V is open in M and M is locally connected at K. ∎

The above theorem is not true if K is not terminal. Consider the following example.

EXAMPLE 3.7.



K is the arc $[a, b]$ and is not terminal. Although K is non-cutting in M, M is not locally connected at K.

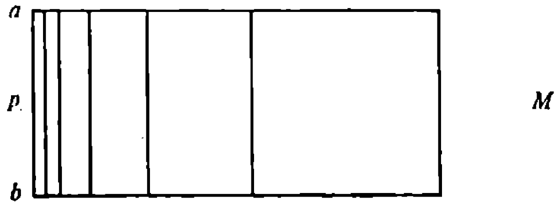
DEFINITION 3.8. Let M be a continuum and K be a proper subcontinuum of M. Then M is said to be *K-aposyndetic* provided that for each $x \in M-K$ there is a subcontinuum N of M such that $x \in \operatorname{int}(N) \subset N \subset M-K$.

If for each $m \in M$, M is $\{m\}$-aposyndetic, then M is just said to be *aposyndetic* [8].

It is easy to verify that if K is a subcontinuum of M then M is K-aposyndetic if and only if M is semi-locally-connected at K. (i.e., if V is an open subset of M containing K, then there is an open set U such that $K \subset U \subset V$ and $M-U$ has finitely many components).

The following examples show that neither "K-aposyndetic" nor "locally connected at K" implies the other.

EXAMPLE 3.9. An aposyndetic continuum M with a non-separating point p such that M is not locally connected at p. M is the union of a square and a countable collection of intervals.



A non-aposyndetic example may be found on page 113 of [7].

EXAMPLE 3.10. A continuum which is locally connected at a non-separating point p, but not $\{p\}$-aposyndetic. This continuum, called the Cantor swastika, is the union of four copies of $C\times[0,1]$, where C is the Cantor ternary set. The four copies are joined in an alternating manner with the point p as the common part of the four copies.

THEOREM 3.11. *Let K be a terminal non-cutting subcontinuum of M and suppose that L is a subcontinuum of M which properly contains K. Then L is K-aposyndetic.*

Proof. Let $x\in L-K$. By Theorem 3.4, $K\subset \operatorname{int}(L)$ and by Theorem 3.6, M is locally connected at K. So there is a connected open set V such that $K\subset V\subset\overline{V}\subset\operatorname{int}(L)$ and $x\notin\overline{V}$. We wish to show that $L-\overline{V}$ is connected. Suppose to the contrary, that $\overline{V}$ separates L. Since K is a terminal continuum of M, Corollary 1.6 shows that $\overline{V}$ cannot separate M. Thus $M\neq L$. Using this corollary again, we see that $M-L$ is connected. Thus $\overline{M-L}$ is a non-void continuum.

Now $L-\overline{V}$ is the union of two separated sets A and B. According to Theorem 2, page 172 of [9], each component of $L-\overline{V}$ has a limit point in $\overline{V}$, so $A\cup\overline{V}$ and $B\cup\overline{V}$ are continua. Moreover, since $\operatorname{int}(L)$ is an open set containing $\overline{V}$, $A\cap\operatorname{int}(L)\neq\emptyset$ and since $\operatorname{int}(L)$ misses $\overline{M-L}$ it follows that $A\not\subset\overline{M-L}$. Similarly, $B\not\subset\overline{M-L}$. However, $\overline{M-L}$ meets L so $\overline{M-L}$ meets A or B. Say $\overline{M-L}\cap A\neq\emptyset$. Then each of $\overline{M-L}\cup A\cup\overline{V}$ and $B\cup\overline{V}$ is a continuum containing K. Moreover, each is a proper subcontinuum of M, since $B\not\subset\overline{M-L}\cup(A\cup\overline{V})$. It follows that $M=\overline{M-L}\cup(A\cup\overline{V})\oplus(B\cup\overline{V})$, and K is not terminal. This contradiction shows that $L-\overline{V}$ is connected.

Hence $\overline{L-\overline{V}}$ is a continuum containing x in its interior. Since $K\subset V\subset\overline{V}$, $L-\overline{V}\subset L-V\subset L-K$ and $\overline{L-\overline{V}}\subset L-V\subset L-K$. Thus $K\cap\overline{L-\overline{V}}=\emptyset$ and L is K-aposyndetic. ■

COROLLARY 3.12. *Let K be an E-continuum of M and L be a subcontinuum of M properly containing K. Then L is K-aposyndetic.*

Note that example 3.7 shows that "terminal" cannot be dropped in the hypothesis of Theorem 3.11.

COROLLARY 3.13. *Suppose that K is terminal and non-cutting and L is a subcontinuum of M properly containing K. Then $\overline{L-K}$ is decomposable.*

Proof. Let $x \in L-K$. There is a set V, open in L, and a subcontinuum H of L such that $x \in V \subset H \subset L-K$. Thus $\overline{L-K}$ contains a proper subcontinuum with interior. Hence $\overline{L-K}$ is decomposable. ∎

We will conclude this section by proving two theorems which summarize much of our work on terminal and non-cutting continua. We require the following result, which is similar to a theorem of Whyburn (Theorem 4.1 page 50 of [15]).

THEOREM 3.14. *Suppose K is a subcontinuum of M and M is K-aposyndetic. Then K does not cut M if and only if K does not separate M.*

Proof. Using the definition of K-aposyndetic, it is easy to see that each subset of $M-K$ which is maximal with respect to being continuum-wise connected is both open and closed in $M-K$. ∎

THEOREM 3.15. *Suppose that K is a terminal continuum of M. Then the following are equivalent.*

(1) *K is a non-cutting continuum*;

(2) *M is locally connected at K*;

(3) *M is K-aposyndetic.*

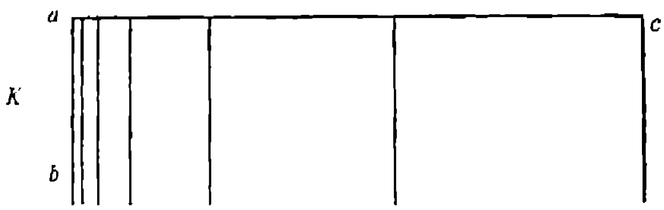
Proof. By Theorem 3.6, (1) and (2) are equivalent and Theorem 3.11 shows that (1) implies (3). To see that (3) implies (1) we note that since K is terminal, K does not separate M. It follows from the preceding theorem that K is non-cutting. ∎

In the previous theorem if "non-separating" is substituted for "terminal", then neither one of (2) and (3) implies the other.

To see that (3) does not imply (2), in example 3.9 let K be the arc from a to b which contains the point p. Then K is non-separating, M is K-aposyndetic, but M fails to be locally connected at K.

To see that (2) does not imply (3), consider the following example.

EXAMPLE 3.16.



M is the interval $[a, c]$ together with the union of a countable collection of intervals each with an end point on $[a, c]$.

Note that $K = [a, b]$ is a non-separating subcontinuum of M, M is locally connected at K, but M fails to be K-aposyndetic.

The following result extends Theorems 1.26 and 1.30.

THEOREM 3.17. *Suppose that K is an end continuum of M. Then the following are equivalent:*

(1) *K is a non-cutting continuum;*

(2) *K is an E-continuum;*

(3) *K is a maximal end continuum;*

(4) *M is locally connected at K;*

(5) *M is K-aposyndetic.*

Proof. Using Theorem 3.15, we see that (1), (4), and (5) are equivalent. By Theorem 1.26, (1), (2), and (3) are equivalent. ■

Arcs and monotone decompositions

We begin with a simple characterization of arcs.

THEOREM 3.18. *A non-degenerate continuum M is an arc if and only if each non-separating point of M is a non-cutting terminal point.*

Proof. If M is an arc, it is clear that the condition is satisfied.

To prove that the condition implies that M is an arc, it suffices to show that M has exactly two non-separating points. There exists a non-separating point of M, say p (see [15], p. 54). Since p is a terminal point, there is a point q such that M is irreducible from p to q. Clearly q is a terminal point so q is also a non-separating point.

Let $r \in M - \{p, q\}$. Since M is irreducible from p to q, r cuts p from q. By hypothesis, r must separate M. Hence M is an arc. ■

As we pointed out in Chapter I, if M is a continuum which has a terminal continuum, there is a monotone upper-semi-continuous decomposition of M such that the hyperspace is an irreducible continuum. We will conclude this chapter by proving two theorems giving sufficient conditions for the hyperspace to be an arc.

LEMMA 3.19 [1]. *Suppose that M is a continuum with a decomposition, $M = I \oplus N$, such that I is irreducible from a point p to $I \cap N$. If M is aposyndetic, then I is an arc and N is aposyndetic.*

Using this lemma we obtain the following theorem.

THEOREM 3.20. *Suppose that M is an aposyndetic continuum and K is a terminal continuum of M.* Then $\overline{M-K}$ *is an arc.*

Proof. If K has void interior in M, then by Theorem 1.11, K is an end continuum of M. Applying Theorem 1.16, we see that M is irreducible from some point $p\in M-K$ to each point of K. Choosing $q\in K$, M is irreducible from p to q. Any point of M different from p and q cuts M, and since M is aposyndetic, any such point separates M. Thus M is an arc.

Suppose that $\mathrm{int}(K)\neq\emptyset$. Since $M=K\oplus\overline{M-K}$ and K is terminal, it follows from Theorem 1.15 that there is a point $p\in M-K$ such that $\overline{M-K}$ is irreducible from p to $K\cap\overline{M-K}$. Thus by Lemma 3.19, $\overline{M-K}$ is an arc. ■

THEOREM 3.21. *Suppose that M is an aposyndetic continuum and K is terminal in M. There is a monotone upper-semi-continuous decomposition $\mathscr{G}$ of M such that the hyperspace, $M/\mathscr{G}$, is an arc.*

Proof. Let $\mathscr{G}$ be the monotone upper-semi-continuous decomposition of M whose only non-degenerate element is K. Let π be the projection of M onto $M/\mathscr{G}$.

Now by Theorem 3.20, $\overline{M-K}$ is an arc. Let $\{q\}=K\cap\overline{M-K}$ and let $p\in\overline{M-K}$ such that $\overline{M-K}$ is irreducible between p and q.

Suppose $y\in M/\mathscr{G}-\{p,K\}$. Then $y\in\overline{M-K}-\{p,q\}$. Thus y separates $\overline{M-K}$ so let S and T be mutually separated sets such that $\overline{M-K}-\{y\}=S\cup T$. Then $M/\mathscr{G}-\{y\}=\pi(S)\cup\pi(T)$ and it follows that $\pi(S)$ and $\pi(T)$ are mutually separated in $M/\mathscr{G}$. Therefore p and K are the only non-separating points of $M/\mathscr{G}$. Thus $M/\mathscr{G}$ is an arc. ■

We next show that in the preceding theorem "aposyndetic" can be replaced with "hereditarily decomposable" and the conclusion still holds. (The decomposition, $\mathscr{D}$, in this case is not necessarily the same as $\mathscr{G}$, in Theorem 3.21).

The following lemma by Miller is needed. The continuum, T, described in the lemma is called a *C-continuum.*

LEMMA 3.22 [10]. *Let N be an irreducible hereditarily decomposable continuum, H and K be E-continua of N, and $p\in N-(H\cup K)$. There is a continuum T in N such that* (i) $p\in T$, (ii) $T\subset N-(H\cup K)$, (iii) *there are mutually separated sets U and V such that $N-T=U\cup V$, and* (iv) $N=\overline{U}\cup\overline{V}$.

THEOREM 3.23. *Suppose the hereditarily decomposable continuum M contains a terminal continuum. Then there is a monotone upper-semi-continuous decomposition of M onto an arc.*

Proof. Let K be a terminal continuum in M. By Theorem 1.15, there is a point $q\in M-K$ such that $\overline{M-K}$ is irreducible from q to $\overline{M-K}\cap K$. Let A be the E-continuum of $\overline{M-K}$ containing q and B be the E-continuum of $\overline{M-K}$ which contains $\overline{M-K}\cap K$. Note that $B\cap K\neq\emptyset$.

Let $\mathscr{D}$ be the monotone upper-semi-continuous decomposition of M whose elements are K, A, and all the C-continua of $M-K$.

Then by Theorem 2.2 of [10], the hyperspace $M/\mathscr{D}$ is an arc. ■

COROLLARY 3.24. *Suppose that M is a hereditarily decomposable (aposyndetic) continuum and K is a terminal continuum of M. There is a terminal non-cutting continuum L of M such that $K \subset L$.*

Proof. By Theorem 3.23 (Theorem 3.21), M can be monotonially decomposed onto an arc $M/\mathscr{D}$ $(M/\mathscr{G})$. It follows that K is an E-continuum of $M/\mathscr{D}$. Let J be a proper subcontinuum of $M/\mathscr{D}$ which properly contains K and let π be the projection of M onto $M/\mathscr{D}$. Then $L = \pi^{-1}(J)$ has the desired properties. ■

Chapter IV

Absolutely non-separating continua

Basic types and properties

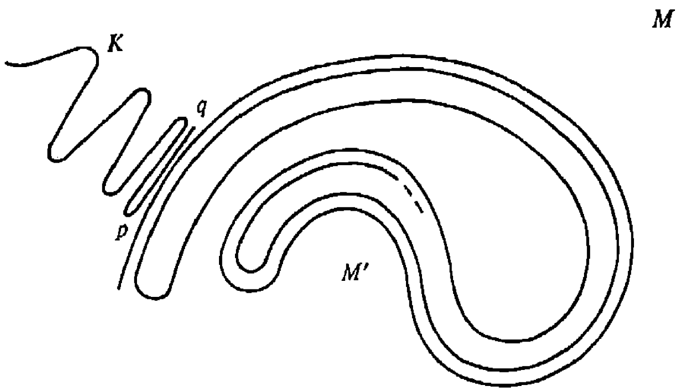
In this chapter we extend the notion of a subcontinuum K "lying at an end of M" to K "lying at an end of every subcontinuum of M which properly contains K." For terminal continua (points) this notion is equivalent to that used in [2] and [5].

DEFINITION 4.1. A proper subcontinuum K of the continuum M is said to be an *absolutely non-separating* (*terminal*, *end*, *E*, *non-cutting*, *respectively*) *continuum* if K is a non-separating (terminal, end, E, non-cutting, respectively) continuum of each subcontinuum L of M which properly contains K.

The subcontinuum K of M in Example 1.2 (b), (c), and (d) is an absolutely terminal continuum while in 1.2 (a), K is not such a continuum.

EXAMPLE 4.2.

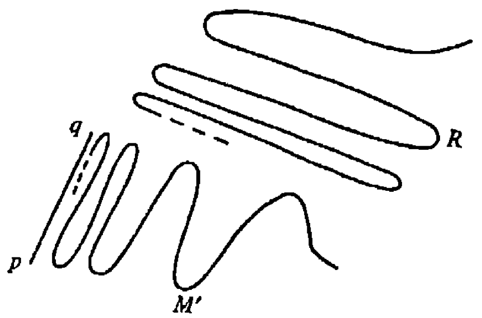
(a)



M is the union of the indecomposable continuum M' and K (a "$\sin\frac{1}{x}$ continuum") such that $K \cap M' = [p, q]$.

K is a terminal continuum but is neither an end continuum nor an absolutely non-separating continuum.

(b)



M is the union of M' (a "$\sin\frac{1}{x}$ continuum") and the ray R in E_3 which limits on M'. K is the arc $[p, q]$.

K is an absolute end continuum which fails to be non-cutting in M.

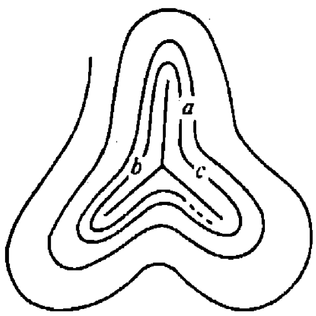
Other authors have generalized the notion of "terminal" points, as defined in [10], to "terminal continua."

For example, in [6] Gordh studied "terminal subcontinua" of hereditarily unicoherent compact connected Hausdorff spaces in which every indecomposable "subcontinuum" (closed connected subspace) is irreducible. His main result is that such a space is irreducible about the union of its indecomposable "terminal" subcontinua.

In [6] a "subcontinuum" (closed connected set) H is said to be "terminal" in the compact connected Hausdorff space M if, (i) H is contained in an irreducible "subcontinuum" of M, and (ii) every irreducible "subcontinuum" containing H is irreducible about $H \cup \{x\}$ for some $x \in M$.

If M is a chainable metric continuum, then our notion of an absolutely terminal continuum of M is equivalent to the definition of a "terminal continuum" in [6]. The definitions need not be equivalent in a continuum which fails to be chainable as shown by the following example.

EXAMPLE 4.3.



M Let K be triod a, b, c in M.

K is "terminal" in the sense of [6] but K is not an absolutely terminal continuum.

Note in Example 4.3 that not only is K an end continuum of M but also every subcontinuum L of M which K separates is an end continuum of M. We now show that this is always the case.

LEMMA 4.4 *Let K be an end continuum of M, L be a subcontinuum of M containing K. If K separates L, then L is an end continuum of M.*

Proof. Suppose that K separates L, and let C_p be a composant of M such that $K \subset M - C_p$. Let A and B be separated sets such that $L - K = A \cup B$.

Assertion. $L \subset M - C_p$. For suppose this is not the case. Then there is a $q \in (L \cap C_p)$ and we may assume that $q \in A$. There is a proper subcontinuum N such that $\{p, q\} \subset N$. Since $A \cup K$ is a continuum, then $M = N \cup (A \cup K)$. Thus $B \subset N$ which implies that $\bar{B} \subset N$ and $N \cap K \neq \emptyset$. Therefore $M = N \cup K$ and K is not an end continuum. This contradiction establishes the assertion. Thus $L \subset M - C_p$ which implies that L is an end continuum of M. ∎

THEOREM 4.5. *Let K be a non-cutting subcontinuum of M. If K is terminal, then K is an absolutely non-cutting continuum.*

Proof. Suppose that K is a terminal continuum and let L be a subcontinuum of M which properly contains K. Let $\mathscr{G}$ be the monotone upper-semi-continuous decomposition of M whose only non-degenerate element is K. Let $M/\mathscr{G}$ be the hyperspace of this decomposition and let π be the projection from M onto $M/\mathscr{G}$.

According to Lemma 1.13, K is an end continuum of $M/\mathscr{G}$. Since K is non-cutting in M, hence in $M/\mathscr{G}$, it follows from Theorem 1.26 that K is an E-continuum in $M/\mathscr{G}$. Clearly $\pi[L] = L'$ is a subcontinuum of $M/\mathscr{G}$ which contains K. Now K is a maximal end continuum of $M/\mathscr{G}$ (Theorem 1.25) thus by the contrapositive of Lemma 4.4, K does not separate L'.

It follows that K does not separate L, hence K is an absolutely non-separating continuum. Since L is K-aposyndetic (Theorem 3.11), it follows from Theorem 3.14 that K is an absolutely non-cutting continuum. ∎

COROLLARY 4.6. *A subcontinuum K of M is non-cutting and absolutely terminal if and only if K is an absolutely non-cutting and absolutely terminal continuum.*

COROLLARY 4.7. *Suppose that the continuum M is irreducible and K is a subcontinuum of M. Then K is non-cutting in M if and only if K is absolutely non-cutting.*

Proof. Suppose M is irreducible between the points p and q and K is non-cutting in M. Clearly one of the terminal points p or q must belong to K. Thus by Theorem 1.5, K is a terminal continuum. It follows from the preceding theorem that K is an absolutely non-cutting continuum. ∎

COROLLARY 4.8. *If K is an E-continuum of M, then K is an absolutely non-cutting continuum.*

Proof. It follows from Corollary 1.27 that E-continua are terminal and non-cutting, hence absolutely non-cutting by the previous theorem. ■

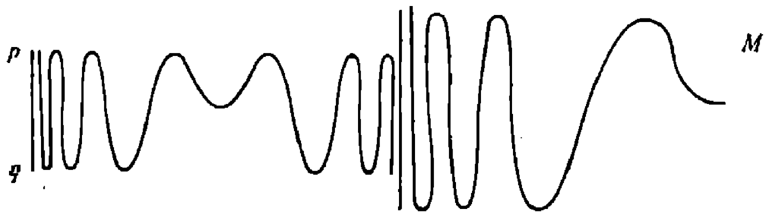
THEOREM 4.9. *Let K be an E-continuum of M and L be a subcontinuum of M which properly contains K. Then K is an E-continuum of L if and only if K is a terminal continuum of L.*

Proof. If K is an E-continuum of L, then clearly it is also terminal in L.

Suppose that K is a terminal continuum of L. According to Corollary 4.8, K is non-cutting in L. Let $p \in M$ such that $K = M - C_p$ and let $q \in L - K$. Then $q \in C_p$, so let N be a proper subcontinuum of M containing p and q. Clearly $N \subset C_p$. Since $L-K$ is connected, $\overline{L-K}$ is a subcontinuum of L which contains q, and $M = N \oplus \overline{L-K}$. Therefore, $K \subset \overline{L-K}$ which implies that K has void interior relative to L. Thus by Corollary 1.27, K is an E-continuum of L. ■

EXAMPLE 4.10.

Let $K = [p, q]$.



K is an E-continuum of M and is also an absolutely non-cutting continuum. Note that K is not an absolutely terminal continuum.

COROLLARY 4.11. *Let K be an E-continuum of M. Then K is an absolute E-continuum if and only if K is absolutely terminal.*

COROLLARY 4.12. *Let K be an E-continuum of M and L be a subcontinuum of M which properly contains K. Then K is an E-continuum of L if and only if L is irreducible.*

Proof. If K is an E-continuum of L, then clearly L is irreducible.

Assume that L is irreducible between the points p and q. Since K is an E-continuum of M, Corollary 4.8 implies that K is non-cutting in L. Clearly one of the terminal points p or q of L belongs to K. Thus by Theorem 1.5, K is terminal in L and it follows from the preceding theorem that K is an E-continuum of L. ■

COROLLARY 4.13. *Let K be an E-continuum of M. Then K is an absolute E-continuum if and only if every subcontinuum of M, which properly contains K, is irreducible.*

THEOREM 4.14. *Let K be a non-cutting terminal continuum in M and L be a subcontinuum of M which properly contains K. If K is an end continuum of L, then K is an E-continuum of both L and M.*

Proof. Suppose that K is an end continuum of L. Then K has void interior in L and hence in M. So by Corollary 1.27, K is an E-continuum of M.

Since K is an end continuum, hence a terminal continuum of L, it follows from Theorem 4.9 that K is an E-continuum of L. ∎

Hereditarily irreducible continua

Definition 4.15. The continuum M is said to be *hereditarily irreducible* provided every subcontinuum of M is irreducible.

In [2] Bing defined a snake-like (chainable) continuum and proved that such a continuum is hereditarily irreducible.

Theorem 4.16. *Let M be a continuum which is hereditarily irreducible and K be a non-cutting subcontinuum of M. Then K is an absolutely non-cutting continuum if and only if K is an absolutely terminal continuum.*

Proof. First suppose that K is absolutely terminal. Then, in particular, K is a terminal continuum of M so by Theorem 4.5 K is an absolutely non-cutting continuum.

Now assume that K is absolutely non-cutting and let L be a subcontinuum of M which properly contains K. Since L is irreducible and K is non-cutting in L, Theorem 1.7 implies that K is a terminal continuum of L. It follows that K is an absolutely terminal continuum. ∎

Corollary 4.17. *Let M be a continuum which is hereditarily irreducible. Then K is a non-cutting subcontinuum of M if and only if K is absolutely non-cutting and is an absolutely terminal continuum.*

Theorem 4.18. *Suppose that M is an a-triodic hereditarily unicoherent continuum and K is a terminal continuum of M with* $\mathrm{int}(K) \neq \emptyset$. *Then K is absolutely terminal.*

Proof. Suppose to the contrary that there is a subcontinuum L of M such that $K \subset L$ and K is not terminal in L. Since L is unicoherent and not a triod, it follows from Theorem 2.12 that L is irreducible. Then Theorem 2.13 guarantees that K must separate L. So $L-K$ is the union of two separated sets A and B. Moreover, $A \cup K$ and $B \cup K$ are continua.

Claim: $\overline{M-L} \cap L \cap K \neq \emptyset$.

If the claim fails, then $\overline{M-L} \cap L \subset A \cup B$ and since A and B are separated, we may assume that $\overline{M-L} \cap L \subset A$. Then $B \cup K$ and $A \cup K \cup \overline{M-L}$ are proper subcontinua of M containing K. Since K is terminal in M and $M = (B \cup K) \oplus (A \cup K \cup \overline{M-L})$ this contradicts Theorem 1.4. Thus the claim is established.

It follows that $\overline{M-L}\cap(K\cup A)$, $\overline{M-L}\cap(K\cup B)$, and K are all three continua which intersect. Since $\mathrm{int}(K)\neq\emptyset$, it follows that no one of these continua is contained in the union of the other two. Thus, the union of these three continua is a triod. This contradiction concludes the proof. ∎

COROLLARY 4.19. *If M is a chainable continuum, K is a terminal continuum, and* $\mathrm{int}(K)\neq\emptyset$, *then K is absolutely terminal.*

Proof. Chainable continua are a-triodic and hereditarily unicoherent, as is shown in [2]. ∎

The proof of the following theorem may be found in [4]. (Note: what we call "absolutely terminal continua" here are called "terminal continua" in [4].)

THEOREM 4.20. *If M is an a-triodic hereditarily unicoherent continuum, K is an absolutely terminal continuum of M, and $\mathscr{E}$ is a chain of open sets covering M, then there is a chain $\mathscr{F}$ of open sets covering M such that $\mathscr{F}$ refines $\mathscr{E}$ and the last link of $\mathscr{F}$ meets K.*

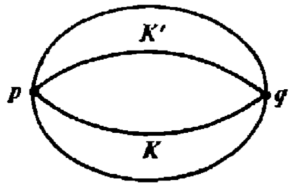
It follows from [2] and Theorem 2.12 that each of the following classes of continua is a subclass of the next: chainable continua, a-triodic and hereditarily unicoherent continua, hereditarily irreducible continua. It follows from [10] that, in the case of hereditarily decomposable continua, the three classes coincide. Hence the following result follows immediately from Theorem 4.18.

THEOREM 4.21. *If M is a hereditarily decomposable and hereditarily irreducible continuum and K is a terminal continuum of M with* $\mathrm{int}(K)\neq$ $\emptyset$, *then K is absolutely terminal.*

If we delete "hereditarily decomposable" from the hypothesis of the above theorem, then the statement is no longer true, as is shown by the following example.

EXAMPLE 4.22.

Both K and K' are pseudo-arcs with only a pair of points $\{p, q\}$, in common. Let $M = K\cup K'$.



M is hereditarily irreducible, K is a terminal continuum of M with $\mathrm{int}(K)\neq\emptyset$, but K is not absolutely terminal in M.

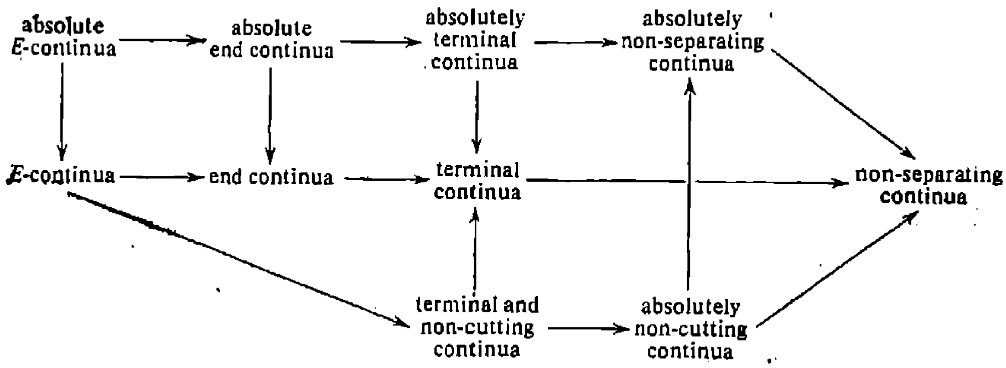
The following theorem appeared in [2] and the proof follows from our basic definition and Theorem 1.4 of Chapter I.

THEOREM 4.23 (Bing). *A necessary and sufficient condition for the continuum M to be indecomposable is for each point of M to be a terminal point of M.*

The notion of an absolutely terminal continuum (point) gives the following corollary.

COROLLARY 4.24. *The continuum M is hereditarily indecomposable if and only if each point of M is an absolutely terminal point.*

The following diagram shows the implications between the different kinds of continua that we have considered. Examples presented throughout this paper illustrate that none of the implications can be reversed.



Chapter V

Minimal continua

Existence of certain minimal continua

According to Theorem 1.16 every subcontinuum of an end continuum is also an end continuum. Of course, this is not the case for terminal continua as is shown by several examples in this paper. This suggests the study of "minimal" terminal continua and other subcontinua of M which are "minimal" with respect to certain properties.

DEFINITION 5.1. Suppose $\mathscr{P}$ is a property and K is a subcontinuum of M. Then K is said to be a *minimal* $\mathscr{P}$ continuum provided K has property $\mathscr{P}$ and no proper subcontinuum of K has property $\mathscr{P}$.

Suppose, for example, that $\mathscr{P}$ is the property of being terminal. Then the continuum K in Example 1.2(c) and K in Example 1.17 are both minimal terminal continua.

The notion of an inducible property and the Brouwer Reduction Theorem, stated below from [15], are useful tools in showing the existence of certain minimal $\mathscr{P}$ continua.

A property $\mathscr{Q}$ is said to be *inducible* provided that when each set of a monotone decreasing sequence, $A_1, A_2, A_3, \ldots$, of compact sets has property $\mathscr{Q}$ so does $\bigcap_{j=1}^{\infty} A_j$.

THEOREM 5.2 (Brouwer Reduction). *Suppose $\mathscr{Q}$ is an inducible property and K is a non-void compact set having $\mathscr{Q}$. Then K contains a non-void closed set, K', such that K' has property $\mathscr{Q}$ and no proper closed subset of K' has property $\mathscr{Q}$.*

THEOREM 5.3. *Each terminal continuum of M contains a minimal terminal continuum.*

Proof. (We show that the property of being terminal is inducible) Let A is a decreasing sequence of terminal continua of M and let $L = \bigcap_{j=1}^{\infty} A_j$. Suppose that the continuum L is not terminal in M. Then by Theorem 1.4, there is a decomposition of M, $M = S \oplus T$, such that L

$\subset S \cap T$. Let $s \in S-(T \cup L)$ and $t \in T-(S \cup L)$. Then $M-\{s, t\}$ is open and $L \subset M-\{s, t\}$. There is a positive integer j so that $A_j \subset M-\{s, t\}$. Clearly, $s \in S-(T \cup A_j)$, $t \in T-(S \cup A_j)$, and A_j intersects both S and T. This implies that A_j is not terminal which is a contradiction. Thus L is terminal so the property of being terminal is inducible.

Therefore, by Theorem 5.2, each terminal continuum of M contains a minimal terminal continuum. ∎

Next we show the existence of a minimal continuum with respect to the absolutely non-separating property.

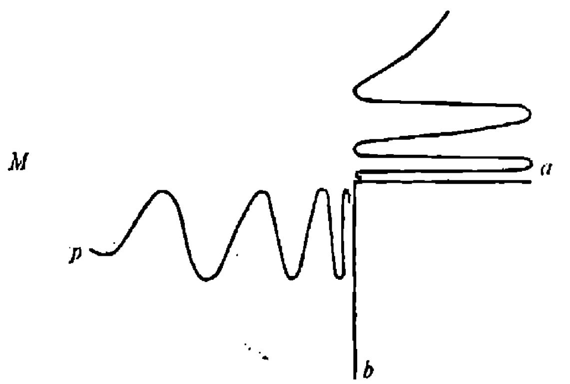
THEOREM 5.4. *Each absolutely non-separating subcontinuum of M contains a minimal absolutely non-separating continuum.*

Proof. Suppose K is a decreasing sequence of absolutely non-separating subcontinua of M and L is a subcontinuum such that $N = \bigcap_{j=1}^{\infty} K_j \subset L$. Now for each positive integer i, K_i does not separate $L \cup K_i$ which implies that $(L \cup K_i)-K_i = L-K_i$ is connected. Thus $L-\bigcap_{i=1}^{\infty} K_i = \bigcup_{i=1}^{\infty} (L-K_i)$ is an increasing sequence of connected sets and hence is connected so N does not separate L. It follows that N is absolutely non-separating. Therefore the property of being absolutely non-separating is inducible. The conclusion follows from the Brouwer Reduction Theorem. ∎

With the obvious modifications to the proof of the preceding two theorems it is easily seen that all the subcontinua which "lie at an end of M" contain a minimal such continuum.

THEOREM 5.5. *Each non-separating (non-cutting, terminal, terminal and non-cutting, end, E, absolutely non-separating, absolutely non-cutting, absolutely terminal, absolute end, absolute E, respectively) continuum contains a minimal such continuum.*

EXAMPLE 5.6.



Let K be the union of the arc from a to b with the "$\sin \frac{1}{x}$ continuum" which contains p.

Then K is neither minimal terminal nor minimal non-cutting in M, but K is minimal with respect to both properties simultaneously.

The structure of minimal continua

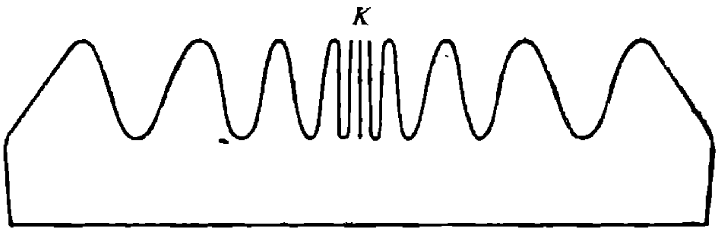
Recall in Chapter I (Theorem 1.25) that E-continua were characterized as maximal end continua. It is necessary that E-continua are also minimal non-cutting.

THEOREM 5.7. *Let K be an E-continuum of M. Then K is minimal non-cutting in M.*

Proof. Let $p \in M$ and C_p be the composant of p in M such that $K = M - C_p$. Suppose $J \subsetneq K$ and J is non-cutting in M. Let $q \in K - J$. Since J does not cut M between p and q, there is a proper subcontinuum of M containing $\{p, q\}$. This implies that $q \in C_p$ which is a contradicion. Thus J cuts M and it follows that K is a minimal non-cutting subcontinum of M. ■

In the following example K, the limit arc of the $\sin \frac{1}{x}$ curves, is a minimal non-cutting continuum but obviously is not an E-continuum. So the converse of the above theorem is false.

EXAMPLE 5.8.



Since every subcontinuum of an end continuum is also an end continuum, it follows that minimal end continua are degenerate. While the example above demonstrates that this is not the case for minimal non-cutting continua, some of the other minimal continua are also degenerate.

THEOREM 5.9. *Let K be a minimal non-separating continuum of M. Then K is degenerate.*

Proof. Suppose that K is non-degenerate. Every non-degenerate continuum contains at least two non-separating points (see [15], p.54) so let $\{p, q\} \subset K$ such that neither p nor q separates K. Since K is minimal non-separating in M, it follows that both the continua $\{p\}$ and $\{q\}$ separate M.

Let S and T be disjoint open sets such that $M - \{p\} = S \cup T$ and assume that $q \in S$. Now $K - \{p\}$ is connected and $(K - \{p\}) \cap S \neq \emptyset$ which implies that $K - \{p\} \subset S$.

Case 1. If $S-K=\emptyset$, then it follows that $S\subset K-\{p\}$. Therefore we have $S=K-\{p\}$ which says that $K=S\cup\{p\}$. Then $(S\cup\{p\})-\{q\}=K-\{q\}$ is connected. Since $\overline{T}=T\cup\{p\}$ is also connected and intersects $K-\{q\}$, then $M-\{q\}=(K-\{q\})\cup\overline{T}$ is connected. This is a contradiction since $\{q\}$ separates M.

Case 2. If $S-K\neq\emptyset$, then the non-empty open sets T and $S-K$ are disjoint and $M-K=T\cup(S-K)$. Since K does not separate M, this is also a contradiction.

Thus in either case a contradiction is reached so K must be degenerate. ■

THEOREM 5.10. *If M is irreducible, then the minimal terminal continua of M are degenerate.*

Proof. Suppose that M is irreducible and K is a minimal terminal continuum in M. According to Theorem 1.8, M is irreducible between a pair of points one of which, say p, belongs to K. Then by Theorem 1.16 $\{p\}$ is an end continuum, hence terminal continuum of M. Since K is minimal terminal it follows that $K=\{p\}$. ■

Note that if M does contain a terminal continuum, then the converse of the preceding theorem holds.

COROLLARY 5.11. *Suppose M is not a type 1 triod, K is a minimal terminal continuum in M and $\overline{M-K}$ is decomposable. Then K is degenerate.*

Proof. According to Theorem 2.15, M is irreducible so the conclusion follows from the preceding theorem. ■

The continuum K in Example 2.16 is a non-degenerate minimal terminal continuum, hence the corollary fails if $\overline{M-K}$ is indecomposable.

Note also that Example 2.8 demonstrates the preceding corollary fails if we replace the words "not a type 1 triod" with "not a type 3 triod."

The proof of the following theorem may be found in [5].

THEOREM 5.12 (Fugate). *Suppose that M is an a-triodic hereditarily unicoherent continuum and K is an absolutely terminal continuum in M. If $K=A\oplus B$, then either A or B is an absolutely terminal continuum in M.*

Applying Theorem 5.12 to minimal absolutely terminal continua of M we obtain the following corollary.

COROLLARY 5.13. *If M is a-triodic and hereditarily unicoherent, then the non-degenerate minimal absolutely terminal continua of M are indecomposable.*

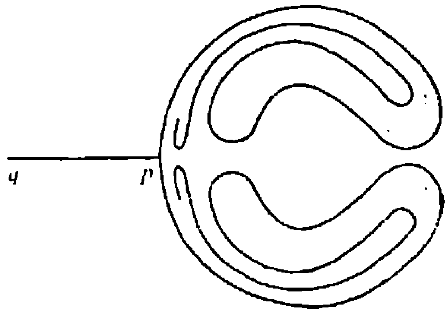
Note that Theorem 5.10 and Corollary 5.11 demonstrate situations where minimal terminal continua are (degenerate) end continua. This suggest the following question.

QUESTION 5.14. Suppose that M is hereditarily decomposable and K is a terminal continuum in M. If K is minimal non-cutting in M, does it follow that K is an end continuum?

The answer to the question above is negative if M fails to be herditarily decomposable. In the following example K is terminal and minimal non-cutting, but of course is not an end continuum of M.

EXAMPLE 5.15.

K is the union of two indecomposable continua which have only one point, p, in common.



M is the union of K and the arc $[q, p]$.

QUESTION 5.16. Is there a hereditarily decomposable continuum M with the property that every proper subcontinuum cuts?

If the answer to Question 5.16 is yes, let K be such a continuum and let $M = K \cup [a, b]$ where $[a, b]$ is an arc and $K \cap [a, b] = \{a\}$. Then K is terminal and minimal non-cutting in M but fails to be an end continuum which implies the answer to Question 5.14 is negative.

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