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On an initial-boundary value problem for the parabolic system which appears in free boundary problems for compressible Navier–Stokes equations

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CONTENTS

Abstract

The existence and uniqueness of solutions of an initial-boundary value problem for the second order linear parabolic system which appears in free boundary problems for compressible Navier–Stokes equations are proved. Moreover, an estimate in anisotropic Sobolev–Slobodetskii spaces with noninteger derivatives is found. The existence and regularity of solutions are shown by means of a regularizer in the same way as in the case of general parabolic systems. However, the problem must be considered separately because the boundary conditions are noncoercive. Moreover, Sobolev–Slobodetskii spaces with noninteger derivatives are used because both first and second order operators appear in the boundary conditions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$, with sufficiently smooth boundary S, $\Omega^T = \Omega \times (0,T)$, $S^T = S \times (0,T)$. In this paper we consider the following problem:

$$(1.1)\qquad \begin{cases} u_t - \mu\Delta u - \nu\nabla \operatorname{div} u = f & \text{in } \Omega^T, \\ T\bar{n} - \sigma\bar{n}\Big(\bar{n}\cdot\Delta_S \int\limits_0^t u(\tau)\,d\tau\Big) = g + \bar{n}\sigma \int\limits_0^t h(\tau)\,d\tau \equiv b & \text{on } S^T, \\ u|_{t=0} = u_0(x) & \text{in } \Omega, \end{cases}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$, $\sigma > 0$, $\mu > 0$, $\nu > 0$, $\bar{n}(x)$ is the unit outward vector normal to the boundary at x, Δ_S is the Laplace–Beltrami operator on S, $T = \{T_{ij}\} = \{\mu\,(\partial u_i/\partial x_j + \partial u_j/\partial x_i) + (\nu - \mu)\delta_{ij}\operatorname{div} u\}$ and the dot denotes the scalar product in $\mathbb{R}^3$. Projecting $(1.1)_2$ on the tangent and normal planes to S we write the boundary condition in the form

$$(1.2)\qquad \begin{cases} \Pi T(u)\bar{n} = b_1 & \text{on } S^T, \\ \bar{n}T(u)\bar{n} - \sigma\bar{n}\cdot\Delta_S \int\limits_0^t u(\tau)\,d\tau = b_2 & \text{on } S^T, \end{cases}$$

where $b_1 = \Pi b \equiv b - \bar{n}(\bar{n}\cdot b)$, $b_2 = b\cdot\bar{n} = g\cdot\bar{n} + \sigma\int_0^t h(\tau)\,d\tau$.

Let s_1, s_2 be local coordinates on a surface $S' \subset S$, so that S' is determined by $x = x(s_1, s_2)$, where $x = (x_1, x_2, x_3)$. Then

$$(1.3)\qquad \Delta_S = \frac{1}{\sqrt{g}}\frac{\partial}{\partial s_\alpha}\sqrt{g}g^{\alpha\beta}\frac{\partial}{\partial s_\beta} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial s_\alpha}\frac{\widehat{g}^{\alpha\beta}}{\sqrt{g}}\frac{\partial}{\partial s_\beta},$$

where the summation convention over repeated indices is used, $g_{\alpha\beta} = (\partial x_k/\partial s_\alpha)(\partial x_k/\partial s_\beta)$, $g = \det\{g_{\alpha\beta}\}$, $\{g^{\alpha\beta}\}$ is the inverse matrix to $\{g_{\alpha\beta}\}$ and $\{\widehat{g}^{\alpha\beta}\}$ is the matrix of algebraic complements for $\{g_{\alpha\beta}\}$.

Problem (1.1) appears in examination of free boundary problems for capillary compressible viscous fluids [3, 4]. The aim of this paper is to prove the existence and uniqueness of solutions of problem (1.1) and to find an a priori estimate which is used in [3, 4].

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Since the Laplace–Beltrami operator appears in the boundary condition $(1.1)_2$ we have to use the anisotropic Sobolev–Slobodetskii spaces with non-integer derivatives.

The main result of this paper is formulated in Theorem 4.4. The case with vanishing initial conditions is presented in Theorem 3.5. Comparing the cases with vanishing and nonvanishing initial conditions we see that in the latter the compatibility conditions $(4.16)_3$ must be added.

This paper is similar to Solonnikov's paper [1], where an analogous problem concerning Navier–Stokes equations is considered. However, many details make our paper essentially different from [1].

The author is greatly indebted to Professor V. A. Solonnikov for showing him the manuscript of [1], prior to publication.

Let us introduce some function spaces (see [1]). The anisotropic Sobolev–Slobodetskii space $W_2^{l,l/2}(\Omega^T)$ is equipped with the norm

$$\|u\|^2_{W_2^{l,l/2}(\Omega^T)} \equiv \int\limits_0^T \|u\|^2_{W_2^l(\Omega)}\,dt + \int\limits_\Omega \|u\|^2_{W_2^{l/2}((0,T))}\,dx\,,$$

where

$$\|u\|^2_{W_2^l(\Omega)} \equiv \sum_{|\alpha|\le[l]} \|D_x^\alpha u\|^2_{L_2(\Omega)} + \langle\!\langle u\rangle\!\rangle^2_{l,x,\Omega}$$

$$\equiv \sum_{|\alpha|\le[l]} \|D_x^\alpha u\|^2_{L_2(\Omega)} + \int\limits_\Omega\int\limits_\Omega \frac{|D_x^{[l]}u(x) - D_y^{[l]}u(y)|^2}{|x-y|^{3+2(l-[l])}}\,dx\,dy$$

(the last term is omitted for $l\in\mathbb{Z}$), $\Omega\subset\mathbb{R}^3$, $D_x^\alpha = \partial_{x_1}^{\alpha_1}\ldots\partial_{x_n}^{\alpha_n}$, $\partial_{x_i}^{\alpha_i} = \partial^{\alpha_i}/\partial x_i^{\alpha_i}$, $i=1,\ldots,n$, $|\alpha| = \alpha_1+\ldots+\alpha_n$, $[l]$ is the integer part of l,

$$\|u\|^2_{W_2^{l/2}((0,T))} \equiv \sum_{j\le[l/2]} \|\partial_t^j u\|^2_{L_2((0,T))} + \langle\!\langle u\rangle\!\rangle^2_{l/2,t,(0,T)}$$

$$= \sum_{j\le[l/2]} \|\partial_t^j u\|^2_{L_2((0,T))} + \int\limits_0^T\int\limits_0^T \frac{|\partial_t^{[l/2]}u(t) - \partial_{t'}^{[l/2]}u(t')|^2}{|t-t'|^{1+2(l/2-[l/2])}}\,dt\,dt'$$

and the last expression is omitted for $l/2\in\mathbb{Z}$. Moreover, we assume

$$\|u\|_{l,\Omega^T} = \|u\|_{W_2^{l,l/2}(\Omega^T)}\,,\quad \|u\|_{l,\Omega} = \|u\|_{W_2^l(\Omega)}\,.$$

To consider problem (1.1) with vanishing initial conditions we need a space of functions which admit a zero extension to $t<0$. Therefore for every $\gamma\ge 0$ we introduce the space $H_\gamma^{l,l/2}(\Omega^T)$ with norm

$$\|u\|^2_{H^{l,l/2}_\gamma(\Omega^T)} = \int_0^T e^{-2\gamma t}\|u\|^2_{l,\Omega}\,dt + \|u\|^2_{H^{0,l/2}_\gamma(\Omega^T)}\,.$$

Here for $l/2 \notin \mathbf{Z}$,

$$\|u\|^2_{H^{0,l/2}_\gamma(\Omega^T)} = \gamma^l \int_0^T e^{-2\gamma t}\|u\|^2_{0,\Omega}\,dt + \int_0^T e^{-2\gamma t}\,dt \int_0^\infty \frac{\|\partial_t^k u_0(\cdot,t-\tau) - \partial_t^k u_0(\cdot,t)\|^2_{0,\Omega}}{\tau^{1+2(l/2-k)}}\,d\tau$$

and $k = [l/2] < l/2$, $u_0(x,t) = u(x,t)$ for $t > 0$, $u_0(x,t) = 0$ for $t < 0$. For $l/2 \in \mathbf{Z}$

$$\|u\|^2_{H^{0,l/2}_\gamma(\Omega^T)} = \int_0^T e^{-2\gamma t}(\gamma^l\|u\|^2_{0,\Omega} + \|\partial_t^{l/2}u\|^2_{0,\Omega})\,dt\,,$$

and we assume that $\partial_t^j u|_{t=0} = 0$, $j = 0,\dots,l/2-1$, so $u_0(x,t)$ has a generalized derivative $\partial_t^{l/2}u_0$ in $\Omega \times (-\infty,T)$. Moreover, we write $\|u\|_{l,\gamma,\Omega^T} = \|u\|_{H^{l,l/2}_\gamma(\Omega^T)}$.

Finally, we need an interpolation inequality (see [1]). Let $u \in H^{r,r/2}_\gamma(\Omega^T)$, $r \le l+2$, $|\alpha| \le r$. Then for every $\varepsilon \in (0,1)$ and $0 \le q < r - |\alpha|$

$$\|D_x^\alpha u\|_{q,\gamma,\Omega^T} \le \varepsilon^{r-|\alpha|-q}\|u\|_{r,\Omega^T} + c\varepsilon^{-q-|\alpha|}\|e^{-\gamma t}u\|_{0,\Omega^T} \le (\varepsilon^{r-|\alpha|-q} + c\gamma^{-r/2}\varepsilon^{-q-|\alpha|})\|u\|_{r,\gamma,\Omega^T}\,. \tag{1.4}$$

We also write $\|u\|_{L_p(\Omega)} = |u|_{p,\Omega}$, $1 \le p \le \infty$.

2. Problem (1.1) in a half-space

In this section Ω is a half-space. Set $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n :\ x^n > 0\}$, $\mathbb{R}^{n+1}_T = \mathbb{R}^n \times (0,T)$, $\mathbb{D}^{n+1}_T = \mathbb{R}^n_+ \times (0,T)$, $n = 2,3$. First we introduce some spaces and show relations among them.

Similarly to Section 1 we define norms of $H^{l,l/2}_\gamma(\mathbb{R}^{n+1}_T)$, $H^{l,l/2}_\gamma(\mathbb{D}^{n+1}_T)$.

From [1] we recall the following definitions and results.

Any function $u \in H^{l,l/2}_\gamma(\mathbb{R}^{n+1}_T)$, $T < \infty$, can be extended to $\mathbb{R}^{n+1}_\infty$ in such a way that $u \in H^{l,l/2}_\gamma(\mathbb{R}^{n+1}_\infty)$ and

$$\|u\|_{l,\gamma,\mathbb{R}^{n+1}_\infty} \le c\|u\|_{l,\gamma,\mathbb{R}^{n+1}_T}\,. \tag{2.1}$$

Any function $u \in H_\gamma^{l,l/2}(\mathrm{D}_T^{n+1})$, $T \leq \infty$, can be extended to R_T^{n+1} in such a way that $u \in H_\gamma^{l,l/2}(\mathrm{R}_T^{n+1})$ and

$$\|u\|_{l,\gamma,\mathrm{R}_T^{n+1}} \leq c\|u\|_{l,\gamma,\mathrm{D}_T^{n+1}} . \tag{2.2}$$

For functions defined in R_∞^{n+1} and vanishing sufficiently fast at infinity we define the Fourier transform with respect to x and the Laplace transform with respect to t by the formula

$$\widetilde{f}(\xi, s) = \int_0^\infty e^{-st}\, dt \int_{\mathrm{R}^n} f(x,t) e^{-ix\cdot\xi}\, dx , \tag{2.3}$$

where $\operatorname{Re} s \geq 0$. For any $f \in H_\gamma^{l,l/2}(\mathrm{R}_\infty^{n+1})$ the Fourier–Laplace transform is defined for $\operatorname{Re} s \geq \gamma$ and by the Paley–Wiener theorem it is a holomorphic function of s in the half-space $\operatorname{Re} s > \gamma$.

We introduce the norm

$$|\!|\!| u |\!|\!|^2_{l,\gamma,\mathrm{R}_\infty^{n+1}} = \int_{\mathrm{R}^n} d\xi \int_{-\infty}^{\infty} |\widetilde{u}(\xi,s)|^2 (|s| + \xi^2)^l\, d\xi_0 , \quad s = \gamma + i\xi_0 . \tag{2.4}$$

LEMMA 2.1 (see [1]). *There exist constants c_1 and c_2, which do not depend on u and γ, such that*

$$c_1 |\!|\!| u |\!|\!|_{l,\gamma,\mathrm{R}_\infty^{n+1}} \leq \|u\|_{l,\gamma,\mathrm{R}_\infty^{n+1}} \leq c_2 |\!|\!| u |\!|\!|_{l,\gamma,\mathrm{R}_\infty^{n+1}} . \tag{2.5}$$

For functions defined in D_∞^{n+1} the partial Fourier–Laplace transform with respect to $x' = (x_1, \ldots, x_{n-1})$ and t is introduced in the following way:

$$\widetilde{f}(\xi', s, x_n) = \int_0^\infty e^{-st}\, dt \int_{\mathrm{R}^{n-1}} f(x,t) e^{-ix'\cdot\xi'}\, dx' , \tag{2.6}$$

where $x' \cdot \xi' = x_1\xi_1 + \ldots + x_{n-1}\xi_{n-1}$. Then we define the norm

$$\begin{aligned} |\!|\!| u |\!|\!|^2_{l,\gamma,\mathrm{D}_\infty^{n+1}} = \sum_{j<l} \int_{\mathrm{R}^{n-1}} d\xi' \int_{-\infty}^{\infty} \left\| \partial_{x_n}^j \widetilde{u}(\xi', s, \cdot) \right\|^2_{0,\mathrm{R}_+^1} (\xi'^2 + |s|)^{l-j}\, d\xi_0 \\ + \int_{\mathrm{R}^{n-1}} d\xi' \int_{-\infty}^{\infty} \|\widetilde{u}(\xi', s, \cdot)\|^2_{l,\mathrm{R}_+^1}\, d\xi_0 , \qquad s = \gamma + i\xi_0 . \end{aligned} \tag{2.7}$$

LEMMA 2.2 (see [1]). *There exist constants c_3, c_4, which do not depend on u and γ, such that*

$$c_3 |||u|||_{l,\gamma,\mathrm{D}^{n+1}_\infty} \leq \|u\|_{l,\gamma,\mathrm{D}^{n+1}_\infty} \leq c_4 |||u|||_{l,\gamma,\mathrm{D}^{n+1}_\infty} . \tag{2.8}$$

We also need

LEMMA 2.3 (see [1]). *Let $u \in H^{l,l/2}_\gamma(\mathrm{R}^{n+1}_T)$ and $0 < 2m + |\alpha| < l$. Then $\partial^m_t D^\alpha_x u \in H^{l_1,l_1/2}_\gamma(\mathrm{R}^{n+1}_T)$, where $l_1 = l - 2m - |\alpha|$ and*

$$\|\partial^m_t D^\alpha_x u\|_{l_1,\gamma,\mathrm{R}^{n+1}_T} \leq c\|u\|_{l,\gamma,\mathrm{R}^{n+1}_T} . \tag{2.9}$$

Moreover, for $\rho \in (0, l_1)$ and $\varepsilon > 0$

$$\|\partial^m_t D^\alpha_x u\|_{\rho,\gamma,\mathrm{R}^{n+1}_T} \leq \varepsilon^{l_1-\rho}\|u\|_{l,\gamma,\mathrm{R}^{n+1}_T} + c\varepsilon^{-h}\|e^{-\gamma t}u\|_{0,\mathrm{R}^{n+1}_T} , \tag{2.10}$$

where $h = \rho + 2m + |\alpha|$.

Let $u \in H^{l,l/2}_\gamma(\mathrm{D}^{n+1}_T)$ and $0 \leq 2m + |\alpha| < l - 1/2$. Then $\partial^m_t D^\alpha_x u|_{x_n=0} \in H^{l_2,l_2/2}_\gamma(\mathrm{R}^n_T)$, where $l_2 = l - 2m - |\alpha| - 1/2$, and

$$\|\partial^m_t D^\alpha_x u|_{x_n=0}\|_{l_2,\gamma,\mathrm{R}^n_T} \leq c\|u\|_{l,\gamma,\mathrm{D}^{n+1}_T} . \tag{2.11}$$

Now we consider problem (1.1) in D^4_∞ . Therefore we have

$$\begin{cases} u_t - \mu\Delta u - \nu\nabla \operatorname{div} u = 0\,, & x_3 > 0, \\ \mu\left(\dfrac{\partial u_i}{\partial x_3} + \dfrac{\partial u_3}{\partial x_i}\right) = b_i\,, \quad i = 1,2\,, & x_3 = 0, \\ (\mu+\nu)\dfrac{\partial u_3}{\partial x_3} + (\nu - \mu)\left(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2}\right) + \sigma\Delta' \displaystyle\int_0^t u_3(\tau)\,d\tau = b_3\,, & \\ & x_3 = 0, \\ u|_{t=0} = 0\,, & x_3 > 0, \end{cases} \tag{2.12}$$

where $\Delta' = \partial^2_{x_1} + \partial^2_{x_2}$. On applying the Laplace–Fourier transformation (2.6) problem (2.12) takes the form

$$\begin{cases} \mu\dfrac{d^2\widetilde{u}_k}{dx_3^2} + \nu i\xi_k\dfrac{d\widetilde{u}_3}{dx_3} - (s + \mu\xi^2)\widetilde{u}_k - \nu\xi_k\xi_s u_s = 0\,, & \\ \qquad\qquad k = 1,2, \quad x_3 > 0, & \\ (\mu+\nu)\dfrac{d^2\widetilde{u}_3}{dx_3^2} + \nu i\xi_s\dfrac{d\widetilde{u}_s}{dx_3} - \left(s + \mu\xi^2\right)\widetilde{u}_3 = 0\,, & x_3 > 0, \end{cases} \tag{2.13}$$

$$(2.14)\quad \begin{cases} \mu\left(\dfrac{d\widetilde{u}_k}{dx_3}+i\xi_k\widetilde{u}_3\right)=\widetilde{b}_k\,, \quad k=1,2, & x_3=0,\\[2mm] (\mu+\nu)\dfrac{d\widetilde{u}_3}{dx_3}+(\nu-\mu)(i\xi_j\widetilde{u}_j)-\dfrac{\sigma}{s}\xi^2\widetilde{u}_3=\widetilde{b}_3\,, & x_3=0,\\[2mm] \widetilde{u}\to 0 \quad \text{as } x_3\to\infty\,, \end{cases}$$

where $\xi=(\xi_1,\xi_2)$, $\xi^2=\xi_1^2+\xi_2^2$.

Every solution to (2.13) vanishing at infinity has the form

$$(2.15)\qquad \widetilde{u}=\Phi(\xi,s)e^{-\tau_1x_3}+\psi(\xi,s)(\xi_1,\xi_2,i\tau_2)e^{-\tau_2x_3}\,,$$

where $\Phi(\xi,s)=(\varphi_1,\varphi_2,(i/\tau_1)(\xi_1\varphi_1+\xi_2\varphi_2))$, $\varphi_i=\varphi_i(\xi,s)$, $i=1,2$, $\tau_1=\sqrt{s/\mu+\xi^2}$, $\tau_2=\sqrt{s/(\mu+\nu)+\xi^2}$ and $\arg\tau_i\in(-\pi/4,\pi/4)$, $i=1,2$.

Putting (2.15) into (2.14) we obtain

$$(2.16)\qquad \begin{aligned} &\xi_i\xi\cdot\varphi+\tau_1^2\varphi_i+2\xi_i\tau_1\tau_2\psi=-\mu^{-1}\widetilde{b}_i\tau_1\,, \quad i=1,2,\\ &\left(2\mu+\frac{\sigma}{s}\frac{\xi^2}{\tau_1}\right)\xi\cdot\varphi+\left((\mu+\nu)\tau_2^2+(\mu-\nu)\xi^2+\frac{\sigma}{s}\xi^2\tau_2\right)\psi=i\widetilde{b}_3\,, \end{aligned}$$

where $\varphi=(\varphi_1,\varphi_2)$, $\xi\cdot\varphi=\xi_1\varphi_1+\xi_2\varphi_2$.

Solving (2.16) and using (2.15) one obtains

$$(2.17)\qquad \begin{aligned} \widetilde{u}_k(\xi,s,x_3)=\;&-\frac{\widetilde{b}_k}{\mu\tau_1}e_1-\frac{i\xi_k}{\mu D}\frac{\tau_1^2+\xi^2}{\tau_1+\tau_2}\frac{\nu s}{\mu(\mu+\nu)}\widetilde{b}_3e_0\\ &+\frac{i\xi_k}{\mu D}\frac{s}{\mu+\nu}\left(\frac{\tau_1-\tau_2}{\tau_1+\tau_2}\frac{\nu}{\mu}-1\right)\widetilde{b}_3e_1\\ &+\frac{i\xi_k}{\mu D}\frac{\nu s}{\mu(\mu+\nu)}\left(2\tau_1+\frac{\sigma}{s}\frac{\xi^2}{\mu}\right)\frac{i\xi\cdot\widetilde{b}}{\tau_1+\tau_2}e_0\\ &-\frac{i\xi_k}{\mu D}\left[\frac{2\nu}{\mu(\mu+\nu)}\frac{s}{\tau_1+\tau_2}+\frac{s}{\tau_1\mu}\right.\\ &-\frac{2}{\mu(\mu+\nu)}\frac{s^2+(2\mu+\nu)s\xi^2}{\tau_1\left(\tau_1\tau_2+\xi^2\right)}\\ &\left.+\frac{\nu}{\mu^2(\mu+\nu)}\frac{\sigma\xi^2}{\tau_1(\tau_1+\tau_2)}\right]i\xi\cdot\widetilde{b}e_1\,,\\ \widetilde{u}_3(\xi,s,x_3)=\;&-\frac{i\xi\cdot\widetilde{b}}{\mu\tau_1^2}e_1+\frac{\tau_2\left(\tau_1^2+\xi^2\right)}{\mu D(\tau_1+\tau_2)}\frac{\nu s}{\mu(\mu+\nu)}\widetilde{b}_3e_0 \end{aligned}$$

$$
-\frac{\tau_2 s}{\mu^2 D}\widetilde{b}_3 e_1 - \frac{\tau_2\left(2\tau_1 + \frac{\sigma}{s}\frac{\xi^2}{\mu}\right)}{\mu D(\tau_1+\tau_2)} \cdot \frac{\nu s}{\mu(\mu+\nu)} i\xi\cdot\widetilde{b}e_0
$$

$$
+\frac{1}{\mu D \tau_1^2}\left[2\tau_1\tau_2\frac{s}{\mu} - \frac{2\xi^2 s}{\tau_1\tau_2+\xi^2}\frac{s+(2\mu+\nu)\xi^2}{\mu(\mu+\nu)} + \frac{s}{\mu}\xi^2 + \frac{\sigma\xi^2\tau_2}{\mu^2}\right] i\xi\cdot\widetilde{b}e_1\,,
$$

where $k = 1,2$, $\xi\cdot\widetilde{b} = \xi_1\widetilde{b}_1 + \xi_2\widetilde{b}_2$, $e_1 = e^{-\tau_1 x_3}$, $e_0 = (e^{-\tau_1 x_3} - e^{-\tau_2 x_3}) \times (\tau_1-\tau_2)^{-1}$ and

$$
D = (\tau_1^2+\xi^2)^2 + \frac{\sigma\xi^2\tau_2}{\mu^2} - 4\tau_1\tau_2\xi^2 = \frac{s^2}{\mu^2} + 4\left(\frac{s}{\mu}+\xi^2\right)\xi^2 + \frac{\sigma\xi^2\tau_2}{\mu^2} - 4\tau_1\tau_2\xi^2\,. \tag{2.18}
$$

LEMMA 2.4. *If* $\operatorname{Re} s = \gamma > 0$, $\xi\in\mathbb{R}^2$, *then*

$$
|D| \geq \frac{|s|}{\mu}\left(\frac{\gamma}{\mu} + \frac{\sqrt{2\nu}}{\mu+\nu}\xi^2\right), \tag{2.19}
$$

$$
|s|^2 \leq \mu c_1 |D|\,, \qquad |\xi|^3 \leq \frac{\mu^2}{\sigma}\left(3+\frac{c_1}{\mu}\right)|D|\,, \tag{2.20}
$$

where

$$
c_1 = \max\left\{\frac{\mu+\nu}{2}, \mu + \left(\frac{\sigma}{\gamma}\sqrt{\frac{\mu+2\nu}{\nu(\mu+\nu)}} + 4(1+\sqrt{2})\frac{\mu(\mu+\nu)}{\nu}\right)\frac{\mu+\nu}{2\nu}\right\}.
$$

Proof. From (2.18) we write D in the form

$$
D = \frac{s}{\mu}\left[\frac{s}{\mu} + \frac{\sigma}{\mu}\frac{\tau_2}{s}\xi^2 + \frac{4\nu}{\mu+\nu}\xi^2\frac{\tau_1}{q}\right], \tag{2.21}
$$

where $q = \tau_1+\tau_2$. We have $\arg\tau_1 \in (-\pi/4, \pi/4)$, $|\tau_i| \geq \sqrt{\gamma/\nu_i}$, $i = 1,2$, $\nu_1 = \mu$, $\nu_2 = \nu+\mu$. Then $|q| \leq |\tau_1| + |\tau_2|$ and $|q| \geq |\operatorname{Re} q| = \operatorname{Re}\tau_1 + \operatorname{Re}\tau_2 = |\tau_1|\cos\arg\tau_1 + |\tau_2|\cos\arg\tau_2 \geq (1/\sqrt{2})(|\tau_1|+|\tau_2|)$.

Now we calculate

$$
|D| \geq \frac{|s|}{\mu}\left|\operatorname{Re}\left(\frac{s}{\mu} + \frac{\sigma}{\mu}\frac{\tau_2}{s}\xi^2 + \frac{4\nu}{\mu+\nu}\xi^2\frac{\tau_1}{q}\right)\right| \geq \frac{|s|}{\mu}\left(\frac{\gamma}{\mu} + \frac{\sigma}{\mu}\xi^2\operatorname{Re}\frac{\tau_2}{s} + \frac{4\nu}{\mu+\nu}\xi^2\operatorname{Re}\frac{\tau_1}{q}\right).
$$

Since $\arg s \in (-\pi/2, \pi/2)$ we get $\arg \tau_2 - \arg s \in (-\pi/2, \pi/2)$, because the signs of $\arg \tau_2$ and $\arg s$ are the same, so $\operatorname{Re}(\tau_2/s) \geq 0$. Moreover, $\arg q \in (-\pi/4, \pi/4)$ and $|\arg q| \leq |\arg \tau_1|$, so $\arg \tau_1 - \arg q \in (-\pi/4, \pi/4)$ and $\cos(\arg \tau_1 - \arg q) \in (1/\sqrt{2}, 1)$. Therefore $\operatorname{Re}(\tau_1/q) = (|\tau_1|/|q|)\cos(\arg \tau_1 - \arg q) \geq (1/\sqrt{2})(|\tau_1|/|q|)$ and $1/2 \leq |\tau_1|/|q| \leq 1/(1 + \sqrt{\mu/(\mu+\nu)})$. Thus, we have proved (2.19).

We show (2.20). Let $|s|/\nu \leq |\xi|^2$. Then using (2.19) one has

$$|s|^2 \leq \nu |s| |\xi|^2 \leq \frac{\mu(\mu+\nu)}{\sqrt{2}} D .$$

Let $|s|/\nu \geq |\xi|^2$. Since $D = s^2/\mu^2 + (\sigma/\mu^2)\tau_2\xi^2 + 4\xi^2\tau_1 \cdot (\tau_1 - \tau_2)$ one has

$$\begin{aligned} |s|^2 &\leq \mu^2 |D| + \sigma|\tau_2|\xi^2 + 4\mu^2\xi^2|\tau_1||\tau_1 - \tau_2| \\ &\leq \mu^2|D| + \sigma|\tau_2|\xi^2 + 4\mu^2\xi^2|\tau_1|(|\tau_1| + |\tau_2|) \\ &\leq \mu^2|D| + \left(\frac{\sigma}{\gamma}\sqrt{\frac{\mu+2\nu}{\nu(\mu+\nu)}} + 4(1+\sqrt{2})\frac{\mu(\mu+\nu)}{\nu}\right)|s|\xi^2 , \end{aligned}$$

where

$$|\tau_1| \leq \sqrt{\frac{\mu+\nu}{\mu\nu}|s|} , \qquad |\tau_2| \leq \sqrt{\frac{\mu+2\nu}{\nu(\mu+\nu)}|s|}$$

have been used. Finally, applying (2.19) gives

$$|s|^2 \leq \left(\mu^2 + \left(\frac{\sigma}{\gamma}\sqrt{\frac{\mu+2\nu}{\nu(\mu+\nu)}} + 4(1+\sqrt{2})\frac{\mu(\mu+\nu)}{\nu}\right)\frac{\mu(\mu+\nu)}{2\nu}\right)|D| .$$

Hence $(2.20)_1$ has been proved.

To show $(2.20)_2$ we estimate as follows:

$$\frac{\sigma}{\mu^2}|\xi|^3 \leq \frac{\sigma}{\mu^2}|\tau_2|\xi^2 \leq |D| + \frac{|s|^2}{\mu^2} + \frac{4\nu}{\mu(\mu+\nu)}\xi^2|s| \cdot \frac{|\tau_1|}{|q|} \leq 3|D| + \frac{c_1}{\mu}|D| .$$

This concludes the proof.

By the result of Solonnikov [1] the functions

$$e_i(x_3) = e^{-\tau_i x_3} , \quad i = 1, 2, \qquad e_0(x_3) = \frac{e^{-\tau_1 x_3} - e^{-\tau_2 x_3}}{\tau_1 - \tau_2}$$

satisfy

LEMMA 2.5. *For $\xi \in \mathbb{R}^2$, $s = \gamma + i\xi_0$, $\gamma > 0$, and for every nonnegative integer j and $\kappa \in (0,1)$,*

$$(2.22)\qquad \begin{aligned} &\int_0^\infty \left| \frac{d^j e_i(x_3)}{dx_3^j} \right| dx_3 \le \frac{1}{\sqrt{2}} |\tau_i|^{2j-1}, \quad i = 1,2, \\ &\int_0^\infty \int_0^\infty \left| \frac{d^j e_i(x_3+z)}{dx_3^j} - \frac{d^j e_i(x_3)}{dx_3^j} \right|^2 \frac{dx_3 dz}{z^{1+2\kappa}} \\ &\qquad\qquad \le c_2 |\tau_i|^{2(j+\kappa)-1}, \quad i = 1,2, \end{aligned}$$

where c_2 does not depend on τ_i, $|\xi|$.

Moreover,

$$(2.23)\qquad \begin{aligned} &\int_0^\infty \left| \frac{d^j e_0(x_3)}{dx_3^j} \right|^2 dx_3 \le c_3 \frac{|\tau_1|^{2j-1} + |\tau_2|^{2j-1}}{|\tau_1|^2}, \\ &\int_0^\infty \int_0^\infty \left| \frac{d^j e_0(x_3+z)}{dx_3^j} - \frac{d^j e_0(x_3)}{dx_3^j} \right|^2 \frac{dx_3 dz}{z^{1+2\kappa}} \\ &\qquad\qquad \le c_4 \frac{|\tau_1|^{2(j+\kappa)-1} + |\tau_2|^{2(j+\kappa)-1}}{|\tau_1|^2}. \end{aligned}$$

Proof. Inequalities (2.22) can be proved in exactly the same way as in [1], Lemma 3.1. The proof of (2.23) is similar to that for the corresponding inequalities in [1]. To prove $(2.23)_1$ we write $e_0(x_3)$ as the convolution

$$e_0(x_3) = -\int_0^{x_3} e^{-\tau_2(x_3-y)} e^{-\tau_1 y}\, dy.$$

Using the Young inequality one obtains

$$\int_0^\infty |e_0(x_3)|^2\, dx_3 \le \left(\int_0^\infty |e^{-\tau_1 y}|\, dy \right)^2 \int_0^\infty |e^{-\tau_2 x_3}|^2\, dx_3$$

$$\le \left(\int_0^\infty e^{-\mathrm{Re}\,\tau_1 y}\, dy \right)^2 \int_0^\infty e^{-2\mathrm{Re}\,\tau_2 x_3}\, dx_3 \le \frac{1}{2(\mathrm{Re}\,\tau_1)^2 \mathrm{Re}\,\tau_2} \le \frac{\sqrt{2}}{|\tau_1|^2 |\tau_2|},$$

where $|\tau_i| \le \sqrt{2}\,\mathrm{Re}\,\tau_i$, $i = 1,2$, has been used. Hence $(2.23)_1$ is proved for $j = 0$.

Since $e_0(x_3+z) - e_0(x_3) = e^{-r_1 x_3} e_0(z) + e_0(x_3)(e^{-r_2 z} - 1)$ we have

$$(2.24)\quad \int_0^\infty \int_0^\infty |e_0(x_3+z) - e_0(x_3)|^2 \frac{dx_3\, dz}{z^{1+2\kappa}}$$

$$\leq \frac{\sqrt{2}}{|r_1|} \int_0^\infty |e_0(z)|^2 \frac{dz}{z^{1+2\kappa}} + \frac{2\sqrt{2}}{|r_1|^2 |r_2|} \int_0^\infty |e^{-r_2 z} - 1|^2 \frac{dz}{z^{1+2\kappa}}.$$

To estimate the first term on the right-hand side of (2.24) we use

$$|e_0(z)| \leq |e^{-r_2 z}||r_1 - r_2|^{-1}|1 - e^{-(r_1 - r_2)z}| \leq z e^{-\mathrm{Re}\, r_2 z}$$

to obtain

$$(2.25)\quad \frac{1}{|r_1|} \int_0^\infty |e_0(z)|^2 \frac{dz}{z^{1+2\kappa}} \leq \frac{1}{|r_1|} \int_0^\infty z^{1-2\kappa} e^{-2\mathrm{Re}\, r_2 z}\, dz$$

$$\leq c \frac{|\mathrm{Re}\, r_2|^{2\kappa-2}}{|r_1|} \leq c \frac{|r_2|^{2\kappa-2}}{|r_1|}.$$

To estimate the second term on the right-hand side of (2.24) we apply the inequality

$$(2.26)\quad \int_0^\infty |1 - e^{-r_2 z}| \frac{dz}{z^{1+2\kappa}} \leq c |r_2|^{2\kappa}$$

which implies that

$$(2.27)\quad \frac{1}{|r_1|^2 |r_2|} \int_0^\infty |1 - e^{-r_2 z}|^2 \frac{dz}{z^{1+2\kappa}} \leq c \frac{|r_2|^{2\kappa-1}}{|r_1|^2}.$$

Therefore (2.25) and (2.27) yield

$$(2.28)\quad \int_0^\infty \int_0^\infty |e_0(x_3+z) - e_0(x_3)|^2 \frac{dx_3\, dz}{z^{1+2\kappa}} \leq c \frac{|r_2|^{2\kappa-1}}{|r_1|^2} \left(\frac{|r_1|}{|r_2|} + 1 \right).$$

Since $1 \leq |r_1|/|r_2| \leq \sqrt{(\mu+\nu)/\mu}$ from (2.28) we obtain

$$(2.29)\qquad \int_0^\infty\int_0^\infty |e_0(x_3+z)-e_0(x_3)|^2\,\frac{dx_3\,dz}{z^{1+2\kappa}}\le c\,\frac{|\tau_1|^{2\kappa-1}+|\tau_2|^{2\kappa-1}}{|\tau_1|^2}\,.$$

In this way (2.23) has been shown for $j=0$. For $j>0$ we have

$$\frac{d^j e_i(x_3)}{dx_3^j}=(-\tau_i)^j e_i(x_3)\,,\quad i=1,2,$$

$$\frac{d^j e_0(x_3)}{dx_3^j}=(-1)^j(\tau_1^{j-1}+\tau_1^{j-2}\tau_2+\ldots+\tau_2^{j-1})e_1(x_3)+(-1)^j\tau_2^j e_0(x_3)\,.$$

Hence, the lemma is proved.

THEOREM 2.6. *Let $l>1/2$, $\gamma>0$, $b_1,b_2\in H_\gamma^{l+1/2,l/2+1/4}(\mathbf{R}^3_\infty)$, $b_3=d_1+\sigma\int_0^t d_2(\tau)d\tau$, $d_1\in H_\gamma^{l+1/2,l/2+1/4}(\mathbf{R}^3_\infty)$ and $d_2\in H_\gamma^{l-1/2,l/2-1/4}(\mathbf{R}^3_\infty)$. Then for solutions* (2.17) *of the problem* (2.12) *the following estimate holds*:

$$(2.30)\qquad \sum_{i=1}^3\|u_i\|_{l+2,\gamma,\mathbf{D}^4_\infty}$$
$$\le c(\gamma)\Big(\sum_{\alpha=1}^2\|b_\alpha\|_{l+1/2,\gamma,\mathbf{R}^3_\infty}+\|d_1\|_{l+1/2,\gamma,\mathbf{R}^3_\infty}+\|d_2\|_{l-1/2,\gamma,\mathbf{R}^3_\infty}\Big)\,,$$

where $c(\gamma)$ remains bounded for $\gamma\ge\gamma_0>0$.

Proof. We write $\widetilde{u}_i$, $i=1,2,3$, given by (2.17) in the form

$$(2.31)\qquad \widetilde{u}_j=-\frac{1}{\mu\tau_1}A_{jk}\widetilde{b}_k e_1(x_3)+\frac{1}{\mu D}(U_{jk}\widetilde{b}_k e_1(x_3)-V_{jk}\widetilde{b}_k e_0(x_3))\,;$$

we recall that the summation convention is used and A_{jk}, U_{jk}, V_{jk}, $j,k=1,2,3$, are entries of the matrices

$$A=\begin{pmatrix}-1&0&0\\0&-1&0\\i\xi_1/\tau_2&i\xi_2/\tau_2&0\end{pmatrix},\quad U=\begin{pmatrix}\xi_1^2a_1&\xi_1\xi_2a_1&i\xi_1a_2\\\xi_1\xi_2a_1&\xi_2^2a_1&i\xi_2a_2\\i\xi_1a_3&i\xi_2a_3&-\tau_2s/\mu\end{pmatrix},$$

$$V=\begin{pmatrix}\xi_1^2a_4&\xi_1\xi_2a_4&i\xi_1a_5\\\xi_1\xi_2a_4&\xi_2^2a_4&i\xi_2a_5\\i\xi_1a_6&i\xi_2a_6&-a_7\end{pmatrix},$$

where

$$a_1 = \frac{2\nu}{\mu(\mu+\nu)}\frac{s}{\tau_1+\tau_2} + \frac{s}{\tau_1\mu} - \frac{2s}{\mu(\mu+\nu)\tau_1}\frac{s+(2\mu+\nu)\xi^2}{\tau_1\tau_2+\xi^2}$$

$$+ \frac{\nu}{\mu^2(\mu+\nu)}\frac{\sigma\xi^2}{\tau_1(\tau_1+\tau_2)},$$

$$a_2 = \frac{\nu}{\mu}\frac{\tau_1-\tau_2}{\tau_1+\tau_2} - 1,$$

$$a_3 = \frac{1}{\mu\tau_1^2}\left(2\tau_1\tau_2 s - \frac{2\xi^2 s(s+(2\mu+\nu)\xi^2)}{(\mu+\nu)(\tau_1\tau_2+\xi^2)} + s\xi^2 + \frac{\sigma\xi^2\tau_2}{\mu}\right),$$

$$a_4 = \frac{\nu}{\mu(\mu+\nu)}\frac{s}{\tau_1+\tau_2}\left(2\tau_1 + \frac{\sigma}{s}\frac{\xi^2}{\mu}\right),$$

$$a_5 = \frac{\nu}{\mu(\mu+\nu)}\frac{s(\tau_1^2+\xi^2)}{\tau_1+\tau_2},$$

$$a_6 = \frac{\nu s}{\mu(\mu+\nu)}\frac{\tau_2\left(2\tau_1 + \frac{\sigma}{s}\frac{\xi^2}{\mu}\right)}{\tau_1+\tau_2},$$

$$a_7 = \frac{\nu s}{\mu(\mu+\nu)}\frac{\tau_2(\tau_1^2+\tau_2^2)}{\tau_1+\tau_2}.$$

The remaining part of the proof is similar to the proof of the estimate of $\widetilde{v}$ in the proof of Theorem 3.1 in [1]. Applying the norm (2.7) to the vector $\widetilde{u}_j$ given by (2.31) we obtain

$$\text{(2.32)} \quad \|u_j\|^2_{l+2,\gamma,\mathbb{D}^4_\infty} \le c\int_{\mathbb{R}^2}\int_{-\infty}^{\infty}\left(\frac{|A_{jk}\widetilde{b}_k|^2}{|\tau_1|^2} + \frac{|U_{jk}\widetilde{b}_k|^2}{|D|^2}\right)$$

$$\times\left(E_{1,[l]+2,l-[l]} + \sum_{i=0}^{[l]+2}(|s|+\xi^2)^{l+2-i}E_{1,i}\right)d\xi\,d\xi_0$$

$$+ c\int_{\mathbb{R}^2}\int_{-\infty}^{\infty}\frac{|V_{jk}\widetilde{b}_k|^2}{|D|^2}\Big(E_{0,[l]+2,l-[l]}$$

$$+ \sum_{i=0}^{[l]+2}(|s|+\xi^2)^{l+2-i}E_{0,i}\Big)\,d\xi\,d\xi_0 \equiv I_1,$$

where

$$E_{\mu,i} = \int_0^\infty \left| \frac{d^i e_\mu(x_3)}{dx_3^i} \right| dx_3 ,$$

$$E_{\mu,i,\kappa} = \int_0^\infty \int_0^\infty \left| \frac{d^i e_\mu(x_3+z)}{dx_3^i} - \frac{d^i e_\mu(x_3)}{dx_3^i} \right|^2 \frac{dx_3\, dz}{z^{2+2\kappa}} ,$$

$\mu = 0, 1$. Using Lemma 2.5 in (2.32) one gets

$$(2.33) \qquad I_1 \le c \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \left(\frac{|A_{jk}\widetilde{b}_k|^2}{|r_1|^2} + \frac{|U_{jk}\widetilde{b}_k|^2}{|D|^2} + \frac{|V_{jk}\widetilde{b}_k|^2}{|r_1|^2 |D|^2} \right) \times \left(|s| + \xi^2\right)^{l+2-1/2} d\xi\, d\xi_0 .$$

Applying the form of the matrices A, U, V, the relation $|\widetilde{b}_3| \le c(|\widetilde{d}_1|)$, the inequalities

$$|r_1 + r_2| \ge \frac{1}{\sqrt{2}}(|r_1| + |r_2|) , \quad c_1 \le \frac{|r_1|}{|r_2|} \le c_2 ,$$

$$c_{1i}(|s| + \xi^2) \le r_i^2 \le c_{2i}(|s| + \xi^2) , \quad i = 1, 2,$$

and finally Lemma 2.4, we deduce that (2.32) and (2.33) yield

$$\sum_{j=1}^3 |||u_j|||^2_{l+2,\gamma,\mathrm{D}^4_\infty} \le c(\gamma) \Big(\sum_{\alpha=1}^2 |||b_\alpha|||_{l+1/2,\gamma,\mathbb{R}^3_\infty} + |||d_1|||_{l+1/2,\gamma,\mathbb{R}^3_\infty} + |||d_2|||_{l-1/2,\gamma,\mathbb{R}^3_\infty} \Big) .$$

Now Lemmas 2.1, 2.2 give (2.30). This concludes the proof.

Now we consider the nonhomogeneous problem

$$(2.34) \quad \begin{cases} u_t - \mu\Delta u - \nu\nabla \operatorname{div} u = f , \\ u|_{t=0} = 0 , \\ \mu \left(\dfrac{\partial u_i}{\partial x_3} + \dfrac{\partial u_3}{\partial x_i} \right) = b_i , \quad i = 1, 2, \\ (\mu+\nu)\dfrac{\partial u_3}{\partial x_3} + (\nu - \mu)\left(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} \right) + \sigma\Delta' \displaystyle\int_0^t u_3(\tau)\, d\tau = b_3 , \end{cases}$$

where f is prescribed in D^4_T and $b = (b_1, b_2, b_3)$ in $\mathbb{R}^3_T$.

THEOREM 2.7. *Let* $l > 1/2$, $\gamma > 0$. *Assume that* $f \in H_\gamma^{l,l/2}(\mathbb{D}_T^4)$, $b_i \in H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T^3)$, $i = 1,2$, $b_3 = d_1 + \sigma\int_0^t d_2(\tau)\,d\tau$, $d_1 \in H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T^3)$, $d_2 \in H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T^3)$. *Then problem* (2.34) *has a unique solution for* $\nu \geq (1/3)\mu$ *such that* $u \in H_\gamma^{l+2,l/2+1}(\mathbb{D}_T^4)$ *and*

$$\|u\|_{l+2,\gamma,\mathbb{D}_T^4} \leq c(\gamma)\Big(\|f\|_{l,\gamma,\mathbb{D}_T^4} + \sum_{i=1}^{2}\|b_i\|_{l+1/2,\gamma,\mathbb{R}_T^3} + \|d_1\|_{l+1/2,\gamma,\mathbb{R}_T^3} + \sigma\|d_2\|_{l-1/2,\gamma,\mathbb{R}_T^3}\Big)\,. \tag{2.35}$$

P r o o f (see the proof of Theorem 3.2 from [1]). We extend f to a function f' on $\mathbb{R}_\infty^4$ in such a way that $f' \in H_\gamma^{l,l/2}(\mathbb{R}_\infty^4)$ and (see (2.1) and (2.2))

$$\|f'\|_{l,\gamma,\mathbb{R}_\infty^4} \leq c\|f\|_{l,\gamma,\mathbb{D}_\infty^4}\,.$$

Let ω be a solution of the problem

$$L(\partial_x,\partial_t)\omega = f'\,, \tag{2.36}$$

where $L(\partial_x,\partial_t)$ is the differential operator of $(1.1)_1$, $\partial_x = (\partial_{x_1},\dots,\partial_{x_n})$. Applying the Fourier–Laplace transformation (2.3) to (2.36) yields

$$L(i\xi,s)\widetilde{\omega} = \widetilde{f}'\,, \tag{2.37}$$

so $\widetilde{\omega} = L^{-1}(i\xi,s)\widetilde{f}'$. Since $\det L(i\xi,s) = (s+\mu\xi^2)^2(s+(\mu+\nu)\xi^2)$ and $s = \gamma + i\xi_0$ with $\gamma \geq \gamma_0 > 0$, one obtains easily

$$\|\omega\|_{l+2,\gamma,\mathbb{R}_\infty^4} \leq c\|\widetilde{f}'\|_{l,\gamma,\mathbb{R}_\infty^4}\,,$$

and so

$$\|\omega\|_{l+2,\gamma,\mathbb{D}_T^4} \leq c\|f\|_{l,\gamma,\mathbb{D}_T^4}\,. \tag{2.38}$$

Now $v = u - \omega$ is a solution to the problem

$$\begin{cases} v_t - \mu\Delta v - \nu\nabla\operatorname{div} v = 0\,, \\ v|_{t=0} = 0\,, \\ \mu\left(\dfrac{\partial v_i}{\partial x_3} + \dfrac{\partial v_3}{\partial x_i}\right) = h_i\,, \quad i = 1,2, \\ (\mu+\nu)\dfrac{\partial v_3}{\partial x_3} + (\nu-\mu)\left(\dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2}\right) + \sigma\Delta'\displaystyle\int_0^t v_3(\tau)\,d\tau = h_3\,, \end{cases} \tag{2.39}$$

where

$$(2.40)\qquad \begin{aligned} h_i &= b_i - \mu\left(\frac{\partial\omega_i}{\partial x_3} + \frac{\partial\omega_3}{\partial x_i}\right), \quad i = 1,2, \\ h_3 &= b_3 - (\mu+\nu)\frac{\partial\omega_3}{\partial x_3} - (\nu-\mu)\left(\frac{\partial\omega_1}{\partial x_1} + \frac{\partial\omega_2}{\partial x_2}\right) - \sigma\Delta'\int_0^t \omega_3(\tau)\,d\tau \\ &\equiv h_3' + \int_0^t h_3''(\tau)\,d\tau\,. \end{aligned}$$

Using (2.40) we have

$$\begin{aligned} &\|h_i\|_{l+1/2,\gamma,\mathbb{R}_T^3} \le c(\|b_i\|_{l+1/2,\gamma,\mathbb{R}_T^3} + \|\omega\|_{l+2,\gamma,\mathbb{D}_T^4}), \quad i = 1,2, \\ &\|h_3'\|_{l+1/2,\gamma,\mathbb{R}_T^3} \le c'(\|d_1\|_{l+1/2,\gamma,\mathbb{R}_T^3} + \|\omega\|_{l+2,\gamma,\mathbb{D}_T^4}), \\ &\|h_3''\|_{l-1/2,\gamma,\mathbb{R}_T^3} \le c''(\|d_2\|_{l-1/2,\gamma,\mathbb{R}_T^3} + \|\omega\|_{l+2,\gamma,\mathbb{D}_T^4}), \end{aligned}$$

so applying Theorem 2.6 to problem (2.39) and using Lemmas 2.1, 2.2 we see that $u = v + \omega$ is a solution of (2.34) and estimate (2.35) holds.

To prove uniqueness we put $f = 0$, $b_i = 0$, $i = 1,2,3$, in (2.34). Writing $(2.34)_1$ in the form

$$(2.41)\qquad u_{jt} - \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + (\nu-\mu)\delta_{ij}\operatorname{div} u\right] = 0\,,$$

then multiplying (2.41) by u_j, summing over j and integrating over $\mathbb{D}_T^4$ gives

$$\frac{d}{dt}\left(\int_{\mathbb{R}_t^3} u^2\,dx\,d\tau + \sigma\int_{\mathbb{R}^2}\left|\int_0^t \nabla' u_3\,d\tau\right|^2\Big|_{x_3=0}\,dx'\right) + \mu\int_{\mathbb{R}_t^3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 dx\,d\tau$$
$$+(\nu-\mu)\int_{\mathbb{R}_t^3} |\operatorname{div} u|^2\,dx\,d\tau = 0\,.$$

Hence for $\nu \ge (1/3)\mu$ we have uniqueness. This concludes the proof.

Now we formulate the similar result for the Cauchy problem:

THEOREM 2.8. *Let $l > 1/2$, $\gamma > 0$, $f \in H_\gamma^{l,l/2}(\mathbb{R}_T^4)$. Then the Cauchy problem*

$$(2.42)\qquad u_t - \mu\Delta u - \nu\nabla\operatorname{div} u = f\,, \quad u|_{t=0} = 0\,,$$

has a unique solution for $\nu \geq (1/3)\mu$ *such that* $u \in H_\gamma^{l+2,l/2+1}(\mathbb{R}_T^4)$ *and*

$$\|u\|_{l+2,\gamma,\mathbb{R}_T^4} \leq c(\gamma)\|f\|_{l,\gamma,\mathbb{R}_T^4}. \tag{2.43}$$

In this case a proof is simpler than the proof of Theorem 2.7 because the problem without boundary is considered only.

3. A priori estimate and existence of solutions of problem (1.1) with vanishing initial conditions

We consider the following problem in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} L(\partial_x, \partial_t)u \equiv u_t - \mu\Delta u - \nu\nabla \operatorname{div} u = f & \text{in } \Omega^T, \\ u\big|_{t=0} = 0 & \text{in } \Omega, \\ B_1(x, \partial_x)u \equiv \Pi T(u)\bar{n} = \Pi b, & \text{on } S^T, \\ B_2(x, \partial_x)u \equiv \bar{n}T(u)\bar{n} - \sigma\bar{n}\Delta_S \int_0^t u(\tau)\, d\tau = b \cdot \bar{n} & \text{on } S^T, \end{cases} \tag{3.1}$$

where $\Pi b = b - (b \cdot \bar{n})\bar{n}$, $b \cdot \bar{n} = d_1 + \int_0^t d_2(\tau)\, d\tau$. We write $B(x, \partial_x)u = (B_1(x, \partial_x)u,\ B_2(x, \partial_x)u)$.

Let $f \in H_\gamma^{l,l/2}(\Omega^T)$, $b \in H_\gamma^{l+1/2,l/2+1/4}(S^T)$, $d_1 \in H_\gamma^{l+1/2,l/2+1/4}(S^T)$, $d_2 \in H_\gamma^{l-1/2,l/2-1/4}(S^T)$.

To find an a priori estimate and to prove the existence of solutions of (3.1) we introduce a partition of unity. We introduce two collections of open subsets $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$, $k \in \mathfrak{M} \cup \mathfrak{N}$, such that $\bar{\omega}^{(k)} \subset \Omega^{(k)} \subset \Omega$, $\bigcup_k \omega^{(k)} = \bigcup_k \Omega^{(k)} = \Omega$, $\bar{\Omega}^{(k)} \cap S = \emptyset$ for $k \in \mathfrak{M}$ and $\bar{\Omega}^{(k)} \cap S \neq \emptyset$ for $k \in \mathfrak{N}$. Assume that at most N_0 of the $\Omega^{(k)}$ have nonempty intersection. Suppose $\sup_k \operatorname{diam} \Omega^{(k)} \leq 2\lambda$ for some $\lambda > 0$. Let $\varsigma^{(k)}(x)$ be a smooth function such that $0 \leq \varsigma^{(k)}(x) \leq 1$, $\varsigma^{(k)}(x) = 1$ for $x \in \omega^{(k)}$, $\varsigma^{(k)}(x) = 0$ for $\Omega \setminus \Omega^{(k)}$ and $|D_x^\nu \varsigma^{(k)}(x)| \leq c/\lambda^{|\nu|}$. Then $1 \leq \sum_k \left(\varsigma^{(k)}(x)\right)^2 \leq N_0$. Introduce the function

$$\eta^{(k)}(x) = \frac{\varsigma^{(k)}(x)}{\sum_l \left(\varsigma^{(l)}(x)\right)^2}.$$

We have $\eta^{(k)}(x) = 0$ for $x \in \Omega \setminus \Omega^{(k)}$, $\sum_k \eta^{(k)}(x)\varsigma^{(k)}(x) = 1$ and $|D^\nu \eta^{(k)}(x)| \leq c/\lambda^{|\nu|}$. By $\xi^{(k)}$ we denote a centre of $\omega^{(k)}$ and $\Omega^{(k)}$ for $k \in \mathfrak{M}$ and a centre of $\bar{\omega}^{(k)} \cap S$ and $\bar{\Omega}^{(k)} \cap S$ for $k \in \mathfrak{N}$. To find an a priori estimate we consider problem (3.1) locally. We restrict our considerations to a neighbourhood of the boundary only. Take a point $\xi^{(k)} \in S$, $k \in \mathfrak{N}$. Since problem (3.1) is invariant with respect to translations and rotations we can introduce a local coordinate system $y = (y_1, y_2, y_3)$ with centre at $\xi^{(k)}$ such that the part $\widetilde{S}^{(k)} = S \cap \bar{\Omega}^{(k)}$ of the boundary is described by $y_3 = F(y_1, y_2)$, and then introduce new coordinates by

$$(3.2)\qquad \begin{aligned} z_i &= y_i\,, \quad i = 1,2,\\ z_3 &= y_3 - F(y_1, y_2)\,. \end{aligned}$$

We will write $z = \Phi_k(y)$, where $y' = (y_1, y_2) \in K_\lambda = \{y' \in \mathbb{R}^2 : \ |y'| \le \lambda\}$.

We assume that $S \in W_2^{l+3/2}$ and $\|F\|_{l+3/2,K_\lambda} \le M$, where M is a positive constant which does not depend on $\xi \in S$. We extend F to a function $\widetilde{F}$ on $\mathbb{R}^3_+$ in such a way that $\|\widetilde{F}\|_{l+2,\mathbb{R}^3_+} \le cM$. Moreover, $\widetilde{F}$ satisfies $\widetilde{F}(0) = 0$, $\nabla\widetilde{F}(0) = 0$. Therefore the following inequalities hold:

$$(3.3)\qquad |\widetilde{F}(z)| \le c\lambda M\,, \quad |\nabla\widetilde{F}(z)| \le c\lambda^\beta M\,,$$

where $\beta = \min\{l - 1/2, 1\}$ for $l \ne 3/2$ and $\beta < 1$ for $l = 3/2$.

Multiplying (3.1) by $\varsigma^{(k)}(x)$ and introducing the variables (3.2) with the extended function $\widetilde{F}$ one obtains

$$(3.4)\qquad \begin{aligned} L(\partial_z, \partial_t)\widetilde{u}^{(k)} &= (L(\partial_z, \partial_t) - L(\partial_z - \nabla\widetilde{F}\partial_{z_3}, \partial_t))\widetilde{u}^{(k)}\\ &\quad + L(\partial_z - \nabla\widetilde{F}\partial_{z_3}, \partial_t)\widetilde{u}^{(k)}\\ &\quad - \widehat{\varsigma}^{(k)}(z)L(\partial_z - \nabla\widetilde{F}\partial_{z_3}, \partial_t)\widehat{u}^{(k)} + \widetilde{f}^{(k)}\\ &\equiv \widetilde{f}^{(k)} + l_1(\widehat{u}^{(k)})\,,\\ B_i(\xi^{(k)}, \partial_z)\widetilde{u}^{(k)} &= (B_i(\xi^{(k)}, \partial_z) - B_i(z, \partial_z))\widetilde{u}^{(k)}\\ &\quad + (B_i(z, \partial_z) - B_i(z, \partial_z - \nabla\widetilde{F}\partial_{z_3}))\widetilde{u}^{(k)}\\ &\quad + B_i(z, \partial_z - \nabla\widetilde{F}\partial_{z_3})\widetilde{u}^{(k)}\\ &\quad - \widehat{\varsigma}^{(k)}(z)B_i(z, \partial_z - \nabla\widetilde{F}\partial_{z_3})\widehat{u}^{(k)} + \widetilde{b}_i^{(k)}\\ &\equiv \widetilde{b}_i^{(k)} + l_{2i}(\widehat{u}^{(k)}) + \delta_{i2}\sigma\int_0^t l_3(\widehat{u}^{(k)})\,d\tau\,, \qquad i = 1,2, \end{aligned}$$

where $\widehat{g}^{(k)}(z) = g(\Phi_k^{-1}(z))$, $\widetilde{g}^{(k)}(z) = \widehat{g}^{(k)}(z)\widehat{\varsigma}^{(k)}(z)$, $k \in \mathfrak{N}$, l_1, l_{21} are vectors, l_{22}, l_3 are scalars, $\widetilde{b}_1^{(k)} = \Pi\widetilde{b}^{(k)}$, $\widetilde{b}_2^{(k)} = \bar{n} \cdot \widetilde{b}^{(k)}$ and δ_{ij} is the Kronecker delta. Finally, $\widetilde{g}^{(k)}(x) = g(x)\varsigma^{(k)}(x)$, $k \in \mathfrak{M}$. Moreover, the operators $L(\partial_z, \partial_t)$, $B(\xi^{(k)}, \partial_z)$ are equal to the operators from the left-hand sides of (2.12).

Extending all functions in (3.4) by zero (which does not spoil the regularity) we obtain problem (3.4) in the half-space $z_3 > 0$. Hence Theorem 2.7 yields

$$(3.5)\quad \begin{aligned} \|\widetilde{u}^{(k)}\|_{l+2,\gamma,D_T^4} &\le c_1(\|\widetilde{f}^{(k)}\|_{l,\gamma,D_T^4} + \|\widetilde{b}_1^{(k)}\|_{l+1/2,\gamma,R_T^3} \\ &\quad + \|\widetilde{d}_1^{(k)}\|_{l+1/2,\gamma,R_T^3} + \|\widetilde{d}_2^{(k)}\|_{l-1/2,\gamma,R_T^3}) \\ &\quad + c_2\Big(\|l_1\|_{l,\gamma,D_T^4} + \sum_{i=1}^{2}\|l_{2i}\|_{l+1/2,\gamma,R_T^3} + \|l_3\|_{l-1/2,\gamma,R_T^3}\Big). \end{aligned}$$

To estimate the right-hand side of (3.5) we need

LEMMA 3.1 (see [1]). *Let* $a \in W_2^{l+1}(\Omega)$, $f \in W_2^l(\Omega)$, $g \in W_2^{l+1}(\Omega)$, $l > 1/2$, $\Omega \subset R^3$. *Then*

$$(3.6)\quad \begin{aligned} &\|af\|_{l,\Omega} \le c_1|a|_{\infty,\Omega}\|f\|_{l,\Omega} + \|a\|_{l+1,\Omega}(\varepsilon\|f\|_{l,\Omega} + c_2(\varepsilon)\|f\|_{0,\Omega}), \\ &\|af\|_{l,\Omega} \le c_3\|f\|_{l,\Omega}(\varepsilon\|a\|_{l+1,\Omega} + c_4(\varepsilon)\|a\|_{0,\Omega}), \\ &\|ag\|_{l+1,\Omega} \le c_5|a|_{\infty,\Omega}\|g\|_{l+1,\Omega} \\ &\qquad + \|a\|_{l+1,\Omega}(\varepsilon\|g\|_{l+1,\Omega} + c_6(\varepsilon)\|g\|_{0,\Omega}), \end{aligned}$$

where $\varepsilon > 0$.

Using $(3.6)_{1,2}$ implies

$$(3.7)\quad \begin{aligned} &\|(L(\partial_z,\partial_t) - L(\partial_z - \nabla\widetilde{F}\partial_{z_3},\partial_t))\widetilde{u}^{(k)}\|_{l,R_+^3} \\ &\quad \le c(1 + \|\nabla\widetilde{F}\|_{l+1,R_+^3})[|\nabla\widetilde{F}|_{\infty,R_+^3}\|\widetilde{u}^{(k)}\|_{l+2,R_+^3} \\ &\qquad + \|\nabla\widetilde{F}\|_{l+1,R_+^3}(\varepsilon\|\widetilde{u}^{(k)}\|_{l+2,R_+^3} + c(\varepsilon)\|\widetilde{u}^{(k)}\|_{0,R_+^3})]. \end{aligned}$$

By (3.3) the right-hand side of (3.7) is bounded by

$$\varphi(\|\nabla\widetilde{F}\|_{l+1,R_+^3})((\lambda+\varepsilon)\|\widetilde{u}^{(k)}\|_{l+2,R_+^3} + c(\varepsilon)\|\widetilde{u}^{(k)}\|_{0,R_+^3}),$$

where φ is a second order positive polynomial. Since $\widetilde{F}$ does not depend on time we obtain

$$(3.8)\quad \begin{aligned} &\|(L(\partial_z,\partial_t) - L(\partial_z - \nabla\widetilde{F}\partial_{z_3},\partial_t))\widetilde{u}^{(k)}\|_{l,\gamma,D_T^4} \\ &\quad \le \varphi(\|\nabla\widetilde{F}\|_{l+1,R_+^3})\cdot((\lambda+\varepsilon)\|\widetilde{u}^{(k)}\|_{l+2,\gamma,D_T^4} + c(\varepsilon)\|\widetilde{u}^{(k)}\|_{0,\gamma,D_T^4}). \end{aligned}$$

In a similar way we estimate l_1, l_{21}, l_{22} and l_3.

Finally, we get

$$(3.9)\quad \|l_1\|_{l,\gamma,D_T^4} + \|l_{21}\|_{l+1/2,\gamma,R_T^3} + \|l_{22}\|_{l+1/2,\gamma,R_T^3} + \|l_3\|_{l-1/2,\gamma,R_T^3}$$

$$\leq \varepsilon \|\widehat{u}^{(k)}\|_{l+2,\gamma,\mathbf{D}_T^4} + c(\varepsilon)\|e^{-\gamma t}\widehat{u}^{(k)}\|_{0,\mathbf{D}_T^4},$$

provided $\lambda \leq \varepsilon$.

By the interpolation inequality (1.4), the last expression in (3.9) is estimated by

$$c(\varepsilon)(\varepsilon_1^{l+2} + \gamma^{-l/2-1})\|\widehat{u}^{(k)}\|_{l+2,\gamma,\mathbf{D}_T^4}. \tag{3.10}$$

Applying (3.9) and (3.10) in (3.5), summing the results over all $k \in \mathfrak{M} \cup \mathfrak{N}$ and going back to the old variables in the case $k \in \mathfrak{N}$ we obtain for sufficiently small ε, ε_1 and sufficiently large γ

$$\|u\|_{l+2,\gamma,\Omega^T} \leq c\Big[\|f\|_{l,\gamma,\Omega^T} + \sum_{i=1}^{2} \|b_i\|_{l+1/2,\gamma,S^T} + \|d_1\|_{l+1/2,\gamma,S^T} + \|d_2\|_{l-1/2,\gamma,S^T}\Big]. \tag{3.11}$$

Now we prove the existence of solutions to problem (3.1) by applying the notion of regularizer (see [2]). To construct the regularizer we define $u^{(k)}(x,t)$ and $\widehat{u}^{(k)}(z,t)$ to be the solutions to the following Cauchy and boundary value problems:

$$\begin{cases} L(\partial_x,\partial_t)u^{(k)}(x,t) = \widetilde{f}^{(k)}(x,t), \\ u^{(k)}(x,t)|_{t=0} = 0, \quad k \in \mathfrak{M}, \end{cases}$$
$$\begin{cases} L(\partial_z,\partial_t)\widehat{u}^{(k)}(z,t) = \widetilde{f}^{(k)}(z,t), \\ B(\xi^{(k)},\partial_z)\widehat{u}^{(k)}(z,t) = \widetilde{b}^{(k)}(z,t), \\ \widehat{u}^{(k)}(z,t)|_{t=0} = 0, \quad k \in \mathfrak{N}, \end{cases} \tag{3.12}$$

respectively. Solutions to problems (3.12) are described by Theorem 2.8 and constructed in (2.17). Since

$$\widehat{u}^{(k)}(z,t) = \widehat{u}^{(k)}(\Phi_k(y),t) = u^{(k)}(\Phi_k^{-1}\Phi_k(y),t) = u^{(k)}(y,t),$$

we define

$$u(x,t) = \sum_{k\in\mathfrak{M}\cup\mathfrak{N}} \eta^{(k)}(x,t)u^{(k)}(x,t). \tag{3.13}$$

Let $h = (f,b_1,d)$, $b_1 = \Pi b$, $d = (d_1,d_2)$ and $f \in H_\gamma^{l,l/2}(\Omega^T)$, $b_1 \in H_\gamma^{l+1/2,l/2+1/4}(S^T)$, $d_1 \in H_\gamma^{l+1/2,l/2+1/4}(S^T)$, $d_2 \in H_\gamma^{l-1/2,l/2-1/4}(S^T)$. Moreover, let $u \in H_\gamma^{l+2,l/2+1}(\Omega^T)$. Also, set $V_\gamma^l = H_\gamma^{l+2,l/2+1}(\Omega^T)$, $H_\gamma^l = H_\gamma^{l,l/2}(\Omega^T)\times H_\gamma^{l+1/2,l/2+1/4}(S^T)\times H_\gamma^{l+1/2,l/2+1/4}(S^T)\times H_\gamma^{l-1/2,l/2-1/4}(S^T)$. Then (3.13) implies the existence of a linear operator R known as regularizer, such that $u = Rh$ and $R : H_\gamma^l \to V_\gamma^l$. The estimate (3.11) yields

LEMMA 3.2. *Let $S \in W_2^{l+3/2}$, $h \in H_\gamma^l$, $l > 1/2$, with γ sufficiently large. Then R is a bounded linear operator from H_γ^l into V_γ^l:*

$$\|Rh\|_{V_\gamma^l} \le c\|h\|_{H_\gamma^l}\,, \tag{3.14}$$

where c does not depend on h.

Now we write problem (3.1) in the following short form:

$$Au = h\,. \tag{3.15}$$

THEOREM 3.3 (see [2]). *Let $S \in W_2^{l+3/2}$, $h \in H_\gamma^l$, $l > 1/2$, with γ sufficiently large. Then*

$$ARh = h + Th\,, \tag{3.16}$$

where T is a bounded operator in H_γ^l with small norm for small λ and large γ.

Proof. By definition $ARh = (LRh, BRh)$. Using the form (3.13) of u we have

$$\begin{aligned} LRh = &\sum_{k\in\mathfrak{M}\cup\mathfrak{N}} \eta^{(k)} L(\partial_x,\partial_t)u^{(k)}(x,t) \\ &+ \sum_{k\in\mathfrak{M}\cup\mathfrak{N}} (L(\partial_x,\partial_t)\eta^{(k)}u^{(k)}(x,t) - \eta^{(k)}L(\partial_x,\partial_t)u^{(k)}(x,t)) \equiv h + T_1 h\,. \end{aligned}$$

The first sum above is equal to f because for $k \in M$, $x \in \Omega^{(k)}$ we have $L(\partial_x,\partial_t)u^{(k)}(x,t) = \widetilde{f}^{(k)}(x,t)$ and for $k \in \mathfrak{N}$, $x \in \Omega^{(k)}$ we obtain

$$\begin{aligned} L(\partial_x,\partial_t)u^{(k)}(x,t) &= L(\partial_y,\partial_t)u^{(k)}(y,t)|_{y=y(x)} \\ &= L(\partial_z - \nabla\widetilde{F}\partial_{z_3},\partial_t)\widehat{u}^{(k)}(z,t)|_{z=\Phi_k(y(x))} \\ &= L(\partial_z,\partial_t)\widehat{u}^{(k)}(z,t)|_{z=\Phi_k(y(x))} \\ &\quad + (L(\partial_z - \nabla\widetilde{F}\partial_{z_3},\partial_t) - L(\partial_z,\partial_t))\widehat{u}^{(k)}(z,t)|_{z=\Phi_k(y(x))}\,, \end{aligned}$$

where the first term on the right-hand side is equal to $\widetilde{f}^{(k)}(z,t)|_{z=\Phi_k(y(x))}$. Therefore we express the operator T_1 in the form

$$\begin{aligned} T_1 h = &\sum_{k\in\mathfrak{M}\cup\mathfrak{N}} (L(\partial_x,\partial_t)\eta^{(k)}u^{(k)}(x,t) - \eta^{(k)}L(\partial_x,\partial_t)u^{(k)}(x,t)) \\ &+ \sum_{k\in\mathfrak{N}} \eta^{(k)}(L(\partial_z - \nabla\widetilde{F}\partial_{z_3},\partial_t) - L(\partial_z,\partial_t))\widehat{u}^{(k)}(z,t)|_{z=\Phi_k(y(x))}\,. \end{aligned}$$

Similarly we have

$$(3.17)\quad BRh = B\sum_{k\in\mathfrak{N}}\eta^{(k)}(x)u^{(k)}(x,t)|_S = \sum_{k\in\mathfrak{N}}\eta^{(k)}(x)B(x,\partial_x)u^{(k)}(x,t)|_S$$

$$+\sum_{k\in\mathfrak{N}}(B(x,\partial_x)\eta^{(k)}(x)u^{(k)}(x,t) - \eta^{(k)}(x)B(x,\partial_x)u^{(k)}(x,t))|_S\,,$$

where we define T_2 by $BRh = h + T_2h$. The first term on the right-hand side can be written in the form

$$(3.18)\quad \sum_{k\in\mathfrak{N}}\eta^{(k)}(x)(B(x,\partial_x) - B(\xi^{(k)},\partial_x))u^{(k)}(x,t)|_S$$

$$+\sum_{k\in\mathfrak{N}}\eta^{(k)}(x)(B(\xi^{(k)},\partial_z - \nabla\widetilde{F}\partial_{z_3}) - B(\xi^{(k)},\partial_z))\widehat{u}^{(k)}(z,t)|_{z=\Phi_k(y(x))}|_S$$

$$+\sum_{k\in\mathfrak{N}}\eta^{(k)}(x)B(\xi^{(k)},\partial_z)\widehat{u}^{(k)}(z,t)|_{z=\Phi_k(y(x))}|_S\,,$$

where the last term is equal to $\sum_{k\in\mathfrak{N}}\eta^{(k)}(x)\widetilde{b}^{(k)}(z,t)|_{z=\Phi_k(y(x))} = b(x,t)$, $x\in S$. Hence T_2h is the sum of the second term on the right-hand side of (3.17) and of the first two terms in (3.18).

First we estimate the norm of the operator T_1. By using Lemma 2.3 and the interpolation inequality (1.4) the first term in T_1h is estimated in the following way:

$$\Big\|\sum_{k\in\mathfrak{M}\cup\mathfrak{N}}(L\eta^{(k)}u^{(k)} - \eta^{(k)}Lu^{(k)})\Big\|_{l,\gamma,\Omega^T} \le c\sum_k\|u^{(k)}\|_{l+1,\gamma,Q^{(k)}}$$

$$\le c(\varepsilon^{\delta_1}+\gamma^{-\delta_2})\sum_k\|u^{(k)}\|_{l+2,\gamma,Q^{(k)}} \le c(\varepsilon^{\delta_1}+\gamma^{-\delta_2})\sum_k\|u^{(k)}\|_{l+2,\gamma,Q^{(k)}}\,,$$

where $\delta_i>0$, $i=1,2$, $Q^{(k)}=\Omega^{(k)}\times(0,T)$, $k\in\mathfrak{M}\cup\mathfrak{N}$.

By Theorems 2.7 and 2.8 the above expression is estimated by

$$c(\varepsilon^{\delta_1}+\gamma^{-\delta_2})\|h\|_{H^l_\gamma}\,.$$

Now we estimate the second term in T_1h. By Lemma 3.1 we have

$$\Big\|\sum_{k\in\mathfrak{N}}\eta^{(k)}(L(\partial_z - \nabla\widetilde{F}(z)\partial_{z_3},\partial_t) - L(\partial_z,\partial_t))\widehat{u}^{(k)}(z,t)|_{z=\Phi_k(y(x))}\Big\|_{l,\gamma,\Omega^T}$$

$$\le c\sum_{k\in\mathfrak{N}}(\|(\nabla\widetilde{F}\nabla^2\widetilde{F}\nabla\widehat{u}^{(k)})|_{z=\Phi_k(y(x))}\|_{l,\gamma,Q^{(k)}}$$

$$+\|(\nabla\widetilde{F})^2\nabla^2\widehat{u}^{(k)}|_{z=\Phi_k(y(x))}\|_{l,\gamma,Q^{(k)}})$$

$$\le cp(\|\nabla\widetilde{F}\|_{l+1,\Omega})\sum_{k\in\mathfrak{N}}(\|\nabla\widehat{u}^{(k)}|_{z=\Phi_k(y(x))}\|_{l,\gamma,Q^{(k)}} + \|\widehat{u}^{(k)}|_{z=\Phi_k(y(x))}\|_{[l+2],\gamma,Q^{(k)}})$$

$$+ c|\nabla F|^2_{\infty,\Omega}\sum_{k\in\mathfrak{N}}\|\widehat{u}^{(k)}|_{z=\Phi_k(y(x))}\|_{l+2,\gamma,Q^{(k)}} ,$$

where p is a second order polynomial. Finally, by the interpolation inequality (1.4) and (3.3) the above expression is estimated by

$$c(\varepsilon^{\delta_1}+\lambda^{\delta_2}+\gamma^{-\delta_3})\sum_{k\in\mathfrak{N}}\|\widehat{u}^{(k)}|_{z=\Phi_k(y(x))}\|_{l+2,\gamma,Q^{(k)}}$$
$$\le c(\varepsilon^{\delta_1}+\lambda^{\delta_2}+\gamma^{-\delta_3})\|h\|_{H^l_\gamma} ,$$

where $\delta_i > 0$, $i = 1,2,3$, and the last inequality follows from Theorem 2.7. Similar considerations apply to the norm of T_2 . Summarizing we have

$$\|Th\|_{H^l_\gamma} \le c(\varepsilon^{\delta_1}+\lambda^{\delta_2}+\gamma^{-\delta_3})\|h\|_{H^l_\gamma} .$$

This concludes the proof.

THEOREM 3.4. *For every $v \in V^l_\gamma$*

$$RAv = v + Wv , \tag{3.19}$$

where W is a bounded operator in V^l_γ whose norm can be made small for small λ and large γ.

Proof. Let $R^{(k)}$ be the operator which solves problems (3.12). Hence $u^{(k)}(x,t) = R^{(k)}\widetilde{f}^{(k)}(x,t)$, $k \in \mathfrak{M}$ and $\widehat{u}^{(k)}(z,t) = R^{(k)}[\widetilde{f}^{(k)}(z,t),\widetilde{b}^{(k)}(z,t)]$, $k \in \mathfrak{N}$. Let Z_k, $k \in \mathfrak{N}$, be the operator which transforms functions depending on z into functions depending on x. Then we have

$$\begin{aligned} RAv = &\sum_{k\in\mathfrak{N}}\eta^{(k)}Z_kR^{(k)}[Z_k^{-1}\varsigma^{(k)}L(\partial_x,\partial_t)v, Z_k^{-1}B(x,\partial_x)v|_S] \\ &+\sum_{k\in\mathfrak{M}}\eta^{(k)}R^{(k)}\varsigma^{(k)}L(\partial_x,\partial_t)v \\ = &\sum_{k\in\mathfrak{N}}\eta^{(k)}Z_kR^{(k)}[Z_k^{-1}(\varsigma^{(k)}L(\partial_x,\partial_t)v - L(\partial_x,\partial_t)\varsigma^{(k)}v), \\ &\qquad Z_k^{-1}(\varsigma^{(k)}B(x,\partial_x)v - B(x,\partial_x)\varsigma^{(k)}v)|_S] \\ &+\sum_{k\in\mathfrak{N}}\eta^{(k)}Z_kR^{(k)}[0, Z_k^{-1}(B(x,\partial_x) - B(\xi^{(k)},\partial_x))\varsigma^{(k)}v|_S] \end{aligned} \tag{3.20}$$

$$+\sum_{k\in\mathfrak{N}}\eta^{(k)}Z_kR^{(k)}[Z_k^{-1}L(\partial_x,\partial_t)\varsigma^{(k)}v,$$

$$Z_k^{-1}B(\xi^{(k)},\partial_x)\varsigma^{(k)}v|_S]$$

$$+\sum_{k\in\mathfrak{M}}\eta^{(k)}R^{(k)}(\varsigma^{(k)}L(\partial_x,\partial_t)v-L(\partial_x,\partial_t)\varsigma^{(k)}v)$$

$$+\sum_{k\in\mathfrak{M}}\eta^{(k)}R^{(k)}L(\partial_x,\partial_t)\varsigma^{(k)}v\,.$$

By uniqueness for the Cauchy problem $(3.12)_1$ we have

$$R^{(k)}L(\partial_x,\partial_t)\varsigma^{(k)}v=\varsigma^{(k)}v\,,\quad k\in\mathfrak{M}\,,\tag{3.21}$$

so the last term on the right-hand side in (3.20) is equal to v. The third term on the right-hand side of (3.20) has the form

$$\begin{aligned}(3.22)\quad &\sum_{k\in\mathfrak{N}}Z_kR^{(k)}[L(\partial_z-\nabla\widetilde{F}(z)\partial_{z_3},\partial_t)\widetilde{v}^{(k)},\\ &\qquad B^{(k)}(\xi^{(k)},\partial_z-\nabla\widetilde{F}(z)\partial_{z_3})\widetilde{v}^{(k)}|_{z_3=0}]\\ &=\sum_{k\in\mathfrak{N}}(Z_kR^{(k)}[(L(\partial_z-\nabla\widetilde{F}(z)\partial_{z_3},\partial_t)-L(\partial_z,\partial_t))\widetilde{v}^{(k)},\\ &\qquad(B^{(k)}(\xi^{(k)},\partial_z-\nabla\widetilde{F}(z)\partial_{z_3})-B^{(k)}(\xi^{(k)},\partial_k))\widetilde{v}^{(k)}|_{z_3=0}]+Z_k\widetilde{v}^{(k)})\,,\end{aligned}$$

where $\widetilde{v}^{(k)}$ is the solution to problem $(3.12)_{2,3}$, so

$$\begin{aligned}Z_kR^{(k)}[L(\partial_z,\partial_t)\widetilde{v}^{(k)},B^{(k)}(\xi^{(k)},\partial_z)\widetilde{v}^{(k)}|_{z_3=0}]&=Z_k\widetilde{v}^{(k)}\\ &=\varsigma^{(k)}(x)v(x,t)\,,\quad k\in\mathfrak{N}\,,\end{aligned}$$

and $B^{(k)}$ is obtained from B by applying Z_k^{-1}.

Therefore the operator W is determined by the first, second and fourth expressions on the right-hand side of (3.20) and the first term on the right-hand side of (3.22).

The fourth term on the right-hand side of (3.20) is estimated in the following way:

$$\begin{aligned}(3.23)\quad &\|\eta^{(k)}R^{(k)}(\varsigma^{(k)}L(\partial_x,\partial_t)v-L(\partial_x,\partial_t)\varsigma^{(k)}v)\|_{l+2,\gamma,\Omega^T}\\ &\le c\|\nabla\varsigma^{(k)}\nabla v\|_{l,\gamma,\Omega^T}+\|\nabla^2\varsigma^{(k)}v\|_{l,\gamma,\Omega^T}\le c\|v\|_{l+1,\gamma,\Omega^T}\\ &\le c(\varepsilon^{\delta_1}+\gamma^{-\delta_2})\|v\|_{l+2,\gamma,\Omega^T}\,,\end{aligned}$$

where the interpolation inequality (1.4) has been used and $\delta_i>0$, $i=$

1, 2. Similarly the V_γ^l norm of the remaining part of Wv is estimated by $cp(\|\nabla\widetilde{F}\|_{l+1,\Omega})(\varepsilon^{\delta_1}+\gamma^{-\delta_2}+\lambda^{\delta_3})\|v\|_{l+2,\gamma,\Omega^T}$, $\delta_3>0$. Therefore we have shown that

$$\|Wv\|_{V_\gamma^l}\le c(\varepsilon^{\delta_1}+\gamma^{-\delta_2}+\lambda^{\delta_3})\|v\|_{V_\gamma^l}\,.$$

Hence the norm of W can be made small for small ε, λ and large γ. This concludes the proof.

For γ sufficiently large and ε, λ sufficiently small the norms of W and T are less than one. Therefore Theorems 3.3 and 3.4 imply

THEOREM 3.5. *Let* $f\in H_\gamma^{l,l/2}(\Omega^t)$, $b\in H_\gamma^{l+1/2,l/2+1/4}(S^t)$, $d_1\in H_\gamma^{l+1/2,l/2+1/4}(S^t)$, $d_2\in H_\gamma^{l-1/2,l/2-1/4}(S^t)$, $S\in W_2^{l+3/2}$, $l>1/2$, $t\le T$. *Then for sufficiently large* γ *there exists a unique solution of problem* (3.1) *such that* $u\in H_\gamma^{l+2,l/2+1}(\Omega^t)$ *and*

$$\|u\|_{l+2,\gamma,\Omega^t}\le c(\|f\|_{l,\gamma,\Omega^t}+\|b\|_{l+1/2,\gamma,S^t}+\|d_1\|_{l+1/2,\gamma,S^t}+\|d_2\|_{l-1/2,\gamma,S^t}),\quad t\le T\,. \tag{3.24}$$

4. Existence of solutions of problem (1.1)

In this section we prove the existence of solutions of problem (1.1) with nonvanishing initial conditions. We need the following result.

LEMMA 4.1 (see [1]). *Suppose l is not an odd integer. Then every function* $u\in W_2^{l,l/2}(\Omega^T)$, $T<\infty$, *satisfying*

$$\partial_t^i u|_{t=0}=0\,,\quad i=0,\dots,[(l-1)/2],$$

belongs to $H_0^{l,l/2}(\Omega^T)$ *and the norm* $\|u\|_{l,0,\Omega^T}$ *is equivalent to the norm*

$$(\langle\langle u\rangle\rangle_{l,\Omega^T}^2+T^{-l+2k}\|\partial_t^k u\|_{0,\Omega^T}^2)^{1/2}\,,\quad k=[l/2]\,, \tag{4.1}$$

where

$$\langle\langle u\rangle\rangle_{l,\Omega^T}^2=\int_0^T\langle\langle u\rangle\rangle_{l,x,\Omega}^2\,dt+\int_\Omega\langle\langle u\rangle\rangle_{l/2,t,(0,T)}^2\,dx\,.$$

Moreover,

$$\langle\langle u\rangle\rangle_{l,x,\Omega} = \Big(\sum_{|\alpha|=l}\int_\Omega |D_x^\alpha u|^2\,dx\Big)^{1/2} \quad \textit{for } l\in \mathbb{N}\cup\{0\},$$

$$\langle\langle u\rangle\rangle_{l/2,t,(0,T)} = \Big(\int_0^T |\partial_t^{l/2}u|^2\,dt\Big)^{1/2} \quad \textit{for } l/2\in \mathbb{N}\cup\{0\}.$$

The last term in (4.1) *can be omitted if* $l/2 > k+1/2$. *Moreover,*

$$\|u\|^2_{l,0,\Omega^T} \le c(\langle\langle u\rangle\rangle^2_{l,\Omega^T} + T^{-l+2k}\|\partial_t^k u\|^2_{0,\Omega^T}), \tag{4.2}$$

where c *does not depend on* T. *For noninteger* $l/2$ (*see* [1])

$$\|u\|^2_{l,0,\Omega^T} = \langle\langle u\rangle\rangle^2_{l,\Omega^T} + \frac{1}{l-2k}\int_0^T \|\partial_t^k u\|^2_{0,\Omega}\frac{dt}{t^{l-2k}}. \tag{4.3}$$

Therefore Lemma 4.1 follows from the following result:

LEMMA 4.2 (see [1]). *Let* $l<1$. *Then for every* $u\in W_2^{0,l/2}(\Omega^T)$

$$\int_0^T \|u\|^2_{0,\Omega}\frac{dt}{t^l} \le c_1\int_0^T\int_0^T \|u(\cdot,t-\tau)-u(\cdot,t)\|^2_{0,\Omega}\frac{dt\,d\tau}{\tau^{1+l}} + \frac{c_2}{T^l}\int_0^T\|u\|^2_{0,\Omega}\,dt. \tag{4.4}$$

The inequality also holds for $l\in(1,2)$ *but if* $u(x,0)=0$ *the second term on the right-hand side of* (4.4) *does not appear. The constants* c_1, c_2 *do not depend on* T *and* u.

Now we consider problem (1.1) with nonvanishing initial conditions. Let $u_0\in W_2^{l+1}(\Omega)$, $l\in(1/2,1)$. Then there exists a function $v\in W_2^{l+2,l/2+1}(\Omega^T)$ such that

$$v|_{t=0} = u_0 \tag{4.5}$$

and

$$\|v\|_{l+2,\Omega^T} \le c\|u_0\|_{l+1,\Omega}. \tag{4.6}$$

Therefore by putting $\omega = u - v$ problem (1.1) can be written in the form

$$
(4.7)\quad \begin{cases}
\omega_t - \mu\Delta\omega - \nu\nabla\mathrm{div}\,\omega = f - v_t + \mu\Delta v + \nu\nabla\mathrm{div}\,v \equiv f', \\
\Pi T(\omega)\bar{n} = -\Pi T(v)\bar{n} + \Pi g \equiv \Pi g', \\
\bar{n}T(\omega)\bar{n} - \sigma\bar{n}\Big(\bar{n}\cdot\Delta_S \int\limits_0^t \omega(\tau)\,d\tau\Big) \\
\qquad = -\bar{n}T(v)\bar{n} + \sigma\bar{n}\Big(\bar{n}\cdot\Delta_S \int\limits_0^t v(\tau)\,d\tau\Big) + \bar{n}g + \sigma\int\limits_0^t h(\tau)\,d\tau \\
\qquad \equiv h_1' + \sigma\int\limits_0^t h_2'(\tau)\,d\tau \equiv h', \\
\omega|_{t=0} = 0\,.
\end{cases}
$$

We need the following result:

LEMMA 4.3 (see [1]). *Let* $u \in W_2^{l,l/2}(\Omega^T)$ *and* $l' = l - 2\alpha - |\beta| > 0$. *Then* $\partial_t^\alpha D_x^\beta u \in W_2^{l',l'/2}(\Omega^T)$ *and*

$$
\|\partial_t^\alpha D_x^\beta u\|_{l',\Omega^T} \le c\|u\|_{l,\Omega^T}\,. \tag{4.8}
$$

Moreover, if $l' > 1/2$, *then* $\partial_t^\alpha D_x^\beta u|_{S^T} \in W_2^{l'-1/2,l'/2-1/4}(S^T)$ *and*

$$
\|\partial_t^\alpha D_x^\beta u\|_{l',\Omega^T} \le c\|u\|_{l,\Omega^T}\,. \tag{4.9}
$$

Finally, if $l' > 1$, *then* $\partial_t^\alpha D_x^\beta u|_{t=\mathrm{const}} \in W_2^{l'-1,l'/2-1/2}(\Omega)$ *and*

$$
\|\partial_t^\alpha D_x^\beta u\|_{l'-1,\Omega} \le c\|u\|_{l,\Omega^T}\,. \tag{4.10}
$$

From now on, assume that $f \in W_2^{l,l/2}(\Omega^T)$, $g \in W_2^{l+1/2,l/2+1/4}(S^T)$, $h \in W_2^{l-1/2,l/2-1/4}(S^T)$. Using Lemma 4.3 we have $f' \in W_2^{l,l/2}(\Omega^T)$, $g' \in W_2^{l+1/2,l/2+1/4}(S^T)$, $h_1' = W_2^{l+1/2,l/2+1/4}(S^T)$, $h_2' \in W_2^{l-1/2,l/2-1/4}(S^T)$. By Lemma 4.1 and the fact that the $H_\gamma^{l,l/2}$ and $H_0^{l,l/2}$ norms are equivalent for $T < \infty$, we have

$$
(4.11)\quad \begin{aligned}
&\|f'\|_{l,\gamma,\Omega^T} \le c\|f'\|_{l,0,\Omega^T} \le c'\|f'\|_{l,\Omega^T}\,, \\
&\|g'\|_{l+1/2,\gamma,S^T} \le c\|g'\|_{l+1/2,0,S^T} \le c'\|g'\|_{l+1/2,S^T}\,, \\
&\|h_1'\|_{l+1/2,\gamma,S^T} \le c\|h_1'\|_{l+1/2,0,S^T} \le c'\|h'\|_{l+1/2,S^T}\,, \\
&\|h_2'\|_{l-1/2,\gamma,S^T} \le c\|h_2'\|_{l-1/2,0,S^T} \le c'\|h_2'\|_{l-1/2,S^T}\,.
\end{aligned}
$$

By Lemma 4.3 and the definition of the function v one obtains

$$(4.12) \quad \|f'\|_{l,\Omega^T} + \|g'\|_{l+1/2,S^T} + \|h_1'\|_{l+1/2,S^T} + \|h_2'\|_{l-1/2,S^T} \\ \le c(\|f\|_{l,\Omega^T} + \|g\|_{l+1/2,S^T} + \|h\|_{l-1/2,S^T} + \|u_0\|_{l+1,\Omega}) \equiv cX(l)\,.$$

Therefore by Theorem 3.5 there exists a unique solution ω to problem (4.7) such that $\omega \in H_2^{l+2,l/2+1}(\Omega^T)$ and

$$(4.13) \qquad \|\omega\|_{l+2,0,\Omega^T} \le e^{2\gamma T}\|\omega\|_{l+2,\gamma,\Omega^T} \le c(\gamma,T)X(l)\,,$$

where (4.11) has been used. By Lemma 4.1 it follows that $\omega \in W_2^{l+2,l/2+1}(\Omega^T)$ and

$$(4.14) \qquad \|\omega\|_{l+2,\Omega^T} \le c(\gamma,T)X(l)\,, \quad l \in (1/2,1)\,.$$

Since $u = \omega + v$, where v is determined by (4.5) and (4.6), we have proved that there exists a solution u to problem (1.1) such that $u \in W_2^{l+2,l/2+1}(\Omega^T)$ and

$$(4.15) \qquad \|u\|_{l+2,\Omega^T} \le c(\gamma,T)X(l)\,, \quad l \in (1/2,1)\,.$$

Now we prove the existence of solutions of problem (1.1) in $W_2^{l+2,l/2+1}(\Omega^T)$ for $l > 1$. But in this case some compatibility conditions are necessary. Let $g^{(j)}(x) = \partial_t^j g(x,t)|_{t=0}$. Differentiating system (1.1) with respect to time and setting $t = 0$ we get the following recurrent system:

$$(4.16) \qquad \begin{aligned} &u^{(j+1)} = \mu\Delta u^{(j)} + \nu\nabla \operatorname{div} u^{(j)} + f^{(j)}\,,\\ &u^{(0)} = u_0\,,\\ &T(u^{(j)})\bar n - \sigma\bar n(\bar n\Delta_S u^{(j-1)}) = b^{(j)}\,. \end{aligned}$$

From $(4.16)_1$ we have

$$(4.17) \quad \begin{aligned} \|u^{(j+1)}\|_{l+1-2(j+1),\Omega} &\le c(\|u^{(j)}\|_{l+1-2j,\Omega} + \|f^{(j)}\|_{l-1-2j,\Omega})\\ &\le c\Big(\|u_0\|_{l+1,\Omega} + \sum_{0\le j\le(l-1)/2}\|f^{(j)}\|_{l-1-2j,\Omega}\Big)\\ &\le c(\|u_0\|_{l+1,\Omega} + \|f\|_{l,\Omega^T})\,. \end{aligned}$$

Therefore we obtain

$$(4.18) \qquad \sum_{0\le j\le(l+1)/2}\|u^{(j)}\|_{l+1-2j,\Omega} \le (\|u_0\|_{l+1,\Omega} + \|f\|_{l,\Omega^T})\,.$$

Moreover, equation $(4.16)_3$ implies the compatibility conditions.

Now let $v \in W_2^{l+2,l/2+1}(\Omega^T)$, $l > 1$, be such that

$$D_t^j v|_{t=0} = u^{(j)}(x)\,, \quad 0 \le j \le (l+1)/2\,, \tag{4.19}$$

and

$$\|v\|_{l+2,\Omega^T} \le c \sum_{0 \le j \le (l+1)/2} \|u^{(j)}\|_{l+1-2j,\Omega}\,. \tag{4.20}$$

Next, let $\omega = u - v$ be the solution to problem (4.7). By the construction of v we get $\partial_t^i f'|_{t=0} = 0$, $0 \le i \le (l-1)/2$, $\partial_t^i g'|_{t=0} = 0$, $0 \le i \le l/2 - 1/4$, $\partial_t^i h_1'|_{t=0} = 0$, $0 \le i \le l/2 - 1/4$, $\partial_t^i h_2'|_{t=0} = 0$, $0 \le i \le l/2 - 1/2 - 1/4$. Hence, by Lemma 4.1, $f' \in H_0^{l,l/2}(\Omega^T)$, $g' \in H_0^{l+1/2,l/2+1/4}(S^T)$, $h_1' \in H_0^{l+1/2,l/2+1/4}(S^T)$, $h_2' \in H_0^{l-1/2,l/2-1/4}(S^T)$. Then by the equivalence of $H_0^{l,l/2}$ and $H_\gamma^{l,l/2}$ norms for $T < \infty$ we obtain

$$\|f'\|_{l,\gamma,\Omega^T} + \|g'\|_{l+1/2,\gamma,\Omega^T} + \|h_1'\|_{l+1/2,\gamma,\Omega^T} + \|h_2'\|_{l-1/2,\gamma,\Omega^T} \le cX(l)\,. \tag{4.21}$$

Then by Theorem 3.5 there exists a solution to problem (4.7) such that $\omega \in H_\gamma^{l+2,l/2+1}(\Omega^T)$ and

$$\|\omega\|_{l+2,\gamma,\Omega^T} \le cX(l)\,. \tag{4.22}$$

Using the fact that $\partial_t^i \omega|_{t=0} = 0$, $0 \le i \le (l+1)/2$, the equivalence between $H_\gamma^{l,l/2}$ and $H_0^{l,l/2}$ norms for $T < \infty$ and Lemma 4.1 we find that $\omega \in W_2^{l+2,l/2+1}(\Omega^T)$ and

$$\|\omega\|_{l+2,\Omega^T} \le c\|\omega\|_{l+2,\gamma,\Omega^T} \le cX(l)\,. \tag{4.23}$$

Hence by the definition of v we see that $u = \omega + v \in W_2^{l+2,l/2+1}(\Omega^T)$ and

$$\|u\|_{l+2,\Omega^T} \le cX(l)\,. \tag{4.24}$$

Therefore we have proved the main result of this paper:

THEOREM 4.4. *Assume that* $f \in W_2^{l,l/2}(\Omega^T)$, $g \in W_2^{l+1/2,l/2+1/4}(S^T)$, $h \in W_2^{l-1/2,l/2-1/4}(S^T)$, $u_0 \in W_2^{l+1}(\Omega)$, $S \in W_2^{l+3/2}$, $l > 1/2$, $T < \infty$ *and* l *is not an odd integer. Assume that the compatibility conditions* $(4.16)_3$ *for* $j \le [l/2 - 1/4]$ *are satisfied. Then there exists a unique solution to problem* (1.1) *such that* $u \in W_2^{l+2,l/2+1}(\Omega^T)$ *and*

$$\|u\|_{l+2,\Omega^T} \le c(\|f\|_{l,\Omega^T} + \|g\|_{l+1/2,S^T} + \|h\|_{l-1/2,S^T} + \|u_0\|_{l+1,\Omega})\,. \tag{4.25}$$

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