SEQUENTIAL METHODS IN STATISTICS
BANACH CENTER PUBLICATIONS, VOLUME 16
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

A DIRECT METHOD OF COMPUTATION OF THE CHARACTERISTICS FOR SPRTs IN THE CASE OF DISCRETE RANDOM VARIABLES

KARL-HEINZ EGER

Department of Mathematics, Technological University, Karl-Marx-Stadt, G.D.R.

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. integer-valued random variables with a distribution depending on a parameter $\theta \in \Theta$, let N be the corresponding stopping time of an SPRT for H_0: $\theta = \theta_0$ against H_1: $\theta = \theta_1$ and let S be the randomly stopped sum $\sum_{i=1}^{N} X_i$. Under the condition that the slope of the straight line of acceptance of this test is a rational number we reduce the problem of computation of expectation values $E_\theta w(N, S)$ to that of solving of a system of simultaneous linear equations. As special cases we obtain a system of simultaneous linear equations for the OC and the moments of the sample number.

1. Introduction

Direct methods of the computation of the operating characteristic function (OC) and the average sample number function of a sequential probability ratio test (SPRT) in the case of discrete random variables were investigated in [1], [2], [3], [4], [9], [12], [15], [16], [17], [18] and [19]. As a rule these direct methods are characterized by very special assumptions – e.g., about the underlying distribution – and the amount of numerical calculations can be very large in practical situations. For a survey of these methods, see [8], [10], [11], [13], and [14]. In [5] a method was developed which allows us to compute the OC, the average sample number function and moreover the higher moments of the sample number by solving systems of simultaneous linear equations which only differ in their right-hand sides. Besides the comparatively small amount of numerical calculations, this method has the additional advantage that we can also compute the above characteristics for truncated SPRTs.

Here we consider the more general problem of the computation of those characteristics of SPRTs which may be represented by expectation values of certain functions of the sample number N and the randomly stopped sum $\sum_{i=1}^{N} X_i$. As a generalization of [5] we reduce also this problem to that of solving a system of simultaneous linear equations under the assumption that the slope of the straight line of acceptance is rational.

2. Preliminary considerations

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with density $f(x, \theta)$ with respect to some measure on the set of integer numbers Γ and let θ be a parameter with values in a parameter space Θ. We consider an SPRT for discriminating between the hypotheses H_0: $\theta = \theta_0$ and H_1: $\theta = \theta_1$, $\theta \neq \theta_1$, θ_0, $\theta_1 \in \Theta$. Then, by the definition of the SPRT we have the test variable

$$Z_n = \sum_{i=1}^{n} \ln\left(\frac{f(X_i, \theta_1)}{f(X_i, \theta_0)}\right) \qquad \text{for} \quad n = 1, 2, \ldots$$

and to given stopping bounds b and a, $-\infty < b < 0 < a < +\infty$, the stopping time

$$N = \min\{n \geqslant 1\colon\ Z_n \notin (b, a)\}. \tag{2.1}$$

We accept the hypothesis H_0 iff $Z_N \leqslant b$ and reject H_0 iff $Z_N \geqslant a$.

For the following investigations we assume that the critical inequalities of our test, which are defined by $b < Z_N < a$ for $n = 1, 2, \ldots$, may be written as

$$\hat{b} + g \cdot n < \sum_{i=1}^{n} X_i < \hat{a} + g \cdot n \qquad \text{for} \quad n = 1, 2, \ldots \tag{2.2}$$

where $\hat{b}$, $\hat{a}$ and g denote real numbers such that $-\infty < \hat{b} < 0 < \hat{a} < +\infty$ and $0 < g < +\infty$ holds. Other cases can be treated in an analogous manner.

We now investigate the computation of those characteristics of our test which are representable as expectation values in the following manner. Let Γ^+ be the set of positive integer numbers and $\Gamma_0^+ = \{0\} \cup \Gamma^+$. Let w be a measurable function defined on $\Gamma_0^+ \times \Gamma$. Then, supposing that the corresponding expectation value exists, we consider the computation of the expectation value $E_\theta w(N, \sum_{i=1}^{N} X_i)$. For this reason we introduce the following notation: Let $h^a(m)$ and $h^r(m)$ be, for $m \in \Gamma_0^+$, integer numbers defined by

$$h^a(m) := \min\{h \in \Gamma:\ h > \hat{b} + g \cdot m\}$$

and

$$h^r(m) := \max\{h \in \Gamma:\ h < \hat{a} + g \cdot m\}.$$

Let M be a set of lattice points, characterizing the continuation region of our test, defined by

$$M := \{m, k) \in \Gamma_0^+ \times \Gamma:\ h^a(m) \leqslant k \leqslant h^r(m)\}.$$

Then we obtain for the stopping time (2.1)

$$N = \min\{n \geqslant 1:\ (n, \sum_{i=1}^{n} X_i) \notin M\}$$

and we may say that the test starts at the lattice point (0, 0).

More generally, we may use every other lattice point of M as a starting point for an SPRT for H_0 against H_1. With respect to this interpretation we introduce the following notation: Let $N(m, k)$ be for $(m, k) \in M$ a stopping time which is defined by

$$N(m, k) := \min\{n \geqslant 1:\ (m+n, k + \sum_{i=1}^{n} X_{m+i}) \notin M\}$$

and let $S(m, k)$ be a randomly stopped sum defined by

$$S(m, k) := k + \sum_{i=1}^{N(m,k)} X_{m+1}.$$

Definition 2.1. We shall say that an SPRT for H_0 against H_1 *starts at the lattice point* $(m, k) \in M$ if we use the stopping time $N(m, k)$ and accept or reject H_0 if

$$S(m, k) < h^a(m + N(m, k)) \qquad \text{or} \qquad S(m, k) > h^r(m + N(m, k))$$

respectively. We denote such a test by $T(m, k)$.

Remark 2.1. It is also possible to interpret the test $T(m, k)$ as a conditional SPRT for H_0: $\theta = \theta_0$ against H_1: $\theta = \theta_1$ under the condition that we have reached the point $(m, k) \in M$ after the m^{th} observation.

In preparation for the computation of the characteristics of the given test $T(0, 0)$ we now consider the more general problem of computation of the characteristics of the test $T(m, k)$ for $(m, k) \in M$. To solve this problem we need some geometrical properties of the set M.

Definition 2.2. The lattice points $(m, k) \in M$ and $(m', k') \in M$ are said to be equivalent (write: $(m, k) \sim (m', k')$) iff $k - g \cdot m = k' - g \cdot m'$ holds.

It follows from this definition that equivalent lattice points of M have the same distance to the straight line of acceptance taken in the direction of the ordinate. Furthermore, it is not difficult to see that the following lemma holds ([5]).

LEMMA 2.1. *Different equivalent lattice points of M exist iff g is a rational number.*

The meaning of the existence of equivalent lattice points will become clear in the following, where we shall see that, under certain assumptions, tests which are started at equivalent lattice points will have the same characteristics. Especially the following theorem holds:

THEOREM 2.1. *Suppose that $D_\theta^2 X_1 > 0$. Let g be a rational number with $g = g_2/g_1$, $g_1, g_2 \in \Gamma^+$. Let $(m, k) \in M$ and $(m', k') \in M$ be equivalent lattice points such that*

$$m' = m + rg_1 \quad \text{and} \quad k' = k + rg_2 \quad \text{for} \quad r \in \Gamma_0^+$$

holds. Then the random vectors $\big(N(m', k'), S(m', k') - rg_2\big)$ and $\big(N(m, k), S(m, k)\big)$ are identically distributed.

Proof. We consider the probabilities $P_\theta\big(N(m, k) = n,\ S(m, k) = s\big)$, $(n, s) \in \Gamma_0^+ \times \Gamma$. Since $D_\theta^2 X_1 > 0$, we have

$$P_\theta\big(N(m, k) < \infty\big) = P_\theta\big((N(m, k), S(m, k)) \notin M\big) = 1$$

and therefore

$$P_\theta\big(N(m, k) = n, S(m, k) = s\big) = 0 \quad \text{for} \quad (n, s) \in M.$$

For $(n, s) \notin M$ we obtain the following: Since g is a rational number we get

$$h^a(m' + l) = h^a(m + l) + rg_2 \quad \text{for} \quad l \in \Gamma_0^+$$

and

$$h^r(m' + l) = h^r(m + l) + rg_2 \quad \text{for} \quad l \in \Gamma_0^+.$$

Thus, since the $X_1, X_2, \ldots$ are assumed to be i.i.d. random variables, we have for $(n, s) \notin M$

$$P_\theta\big(N(m, k) = n, S(m, k) = s\big) = P_\theta\Big(h^a(m + l) \leqslant k + \sum_{i=1}^{l} X_{m+i} \leqslant h^r(m + l)$$

$$\text{for } l = 1, \ldots, n-1 \text{ and } k + \sum_{i=1}^{n} X_{m+i} = s\Big)$$

$$= P_\theta\Big(h^a(m + l) + rg_2 \leqslant k + rg_2 + \sum_{i=1}^{l} X_{m'+i} \leqslant h^r(m + l) + rg_2 \text{ for } l = 1, \ldots, n-1$$

$$\text{and } k+rg_2+\sum_{i=1}^{n} X_{m'+i} = s+rg_2\big)$$

$$= P_\theta\big(h^a(m'+l) \leqslant k'+\sum_{i=1}^{l} X_{m'+i} \leqslant h^r(m'+l) \quad \text{for } l = 1, \ldots, n-1$$

$$\text{and } k'+\sum_{i=1}^{n} X_{m'+i} = s+rg_2\big)$$

$$= P_\theta\big(N(m', k') = n,\ S(m', k')-rg_2 = s\big),$$

which completes the proof.

Some remarks:

Remark 2.2. The assertion of Theorem 2.1 may be formulated also in the following manner: Let w be a measurable function, defined on $\Gamma_0^+ \times \Gamma$. Suppose that $E_\theta w\big(N(m, k), S(m, k)\big)$ exists. Then we have under the assumptions of this theorem

$$E_\theta w\big(N(m, k), S(m, k)\big) = E_\theta w\big(N(m', k'), S(m', k')-rg_2\big)$$

and especially

$$E_\theta w\big(N(m, k)\big) = E_\theta w\big(N(m', k')\big).$$

Remark 2.3 It was supposed above that relation (2.2) holds, which is a comparatively weak assumption. For instance, we may write the critical inequalities in this form if $\ln\big(f(X_1, \theta_1)/f(X_1, \theta_0)\big) = uX_1 - v$ holds, where u, $u > 0$, and v denote given real numbers. Therefore, the class of probability distributions characterized by (2.2) contains, e.g., the binomial, Poisson, geometric and negative binomial distributions.

3. The computation of the characteristics

In this section we shall reduce the problem of the computation of $E_\theta w\big(N(m, k), S(m, k)\big)$ for $(m, k) \in M$ and $\theta \in \Theta$ to that of solving a system of simultaneous linear equations which only differ in their right-hand sides. We introduce the following notation:

$K(m) := \{k \in \Gamma\colon\ h^a(m) \leqslant k \leqslant h^r(m)\}, \quad m \in \Gamma_0^+;$

$\bar{K}(m) := \Gamma - K(m), \quad m \in \Gamma_0^+;$

$w_k^\theta(m) := E_\theta w\big(N(m, k), S(m, k)\big), \quad (m, k) \in M,\ \theta \in \Theta;$

$w^\theta(m) := \{w_k^\theta(m)\}_{k \in K(m)};$

$C_{kk'}(m, m') := \{h^a(m+l) \leqslant k+\sum_{i=1}^{l} X_{m+i} \leqslant h^r(m+l) \quad \text{for} \quad l = 1, \ldots, m'-m-$ -1 and $k+\sum_{i=1}^{m'-m} X_{m+i} = k'\}$ – the event of reaching the lattice point $(m', k') \in \Gamma^+ \times \Gamma$ by the test $T(m, k)$, $(m, k) \in M$, $m < m'$;

$c^{\theta}_{kk'}(m, m') := P_{\theta}(C_{kk'}(m, m'))$;
$C^{\theta}(m, m') := \{c^{\theta}_{kk'}(m, m')\}_{k \in K(m), k' \in K(m')}$.
$c^{\theta}_{k'}(m, m') := \{c_{kk'}(m, m')\}_{k \in K(m)}$ for $k' \in \bar{K}(m')$;
E – a unit matrix of the same type as $C^{\theta}(m, m+g_1)$
For $g_1, g_2 \in \Gamma^+$ let d be a function defined on $\Gamma_0^+ \times \Gamma$ by
$d(n, s) := w(n+g_1, s) - w(n, s-g_2)$;
$d^{\theta}_k(m) := E_{\theta} d(N(m, k), S(m, k)), (m, k) \in M$;
$d^{\theta}(m) := \{d^{\theta}_k(m)\}_{k \in K(m)}$.
Then we obtain the following assertion:

THEOREM 3.1. *Let g be a rational number such that $g = g_2/g_1$, $g_1, g_2 \in \Gamma^+$, holds. Suppose that the corresponding expectation values exist. Then we have for $m \in \Gamma_0^+$ and $\theta \in \Theta$*

$$(E - C^{\theta}(m, m+g_1)) \cdot w^{\theta}(m) = \sum_{n=1}^{g_1} v^{\theta}_n(m) + C^{\theta}(m, m+g_1) \cdot d^{\theta}(n+g_1), \quad (3.1)$$

where $v^{\theta}_n(m)$ is defined by

$$v^{\theta}_n(m) = \sum_{k' \in \bar{K}(m+n)} w(m+n, k') \cdot c^{\theta}_{k'}(m, m+n). \quad (3.2)$$

Proof. We consider for $(m, k) \in M$ and $k \in K(m)$ the system of events

$$\{\{C_{kk'}(m, m+1)\}_{k' \in \bar{K}(m+1)}, \ldots, \{C_{kk'}(m, m+g_1)\}_{k' \in \bar{K}(m+g_1)}, \{C_{kk'}(m, m+g_1)\}_{k' \in K(m+g_1)}\}.$$

This system forms a complete system of pairwise mutually exclusive events. Then, by the formula of total probability, it follows for $k \in K(m)$ that

$$w^{\theta}_k(m) = \sum_{n=1}^{g_1} \sum_{k' \in \bar{K}(m+n)} E_{\theta}(w(N(m, k), S(m, k)) \mid C_{kk'}(m, m+n)) \times \times c^{\theta}_{kk'}(m, m+n) + s', \quad (3.3)$$

where s' is defined by

$$s' := \sum_{k' \in K(m+g_1)} E_{\theta}(w(N(m, k), S(m, k)) \mid C_{kk'}(m, m+g_1)) \cdot c^{\theta}_{kk'}(m, m+g_1). \quad (3.4)$$

According to the definition of the events $C_{kk'}(m, m+g_1)$ we have for $n = 1, \ldots, g_1$ and $k' \in \bar{K}(m+n)$

$$E_{\theta}(w(N(m, k), S(m, k)) \mid C_{kk'}(m, m+n)) = w(m+n, k'). \quad (3.5)$$

For the conditional expectation values in the sum s' we obtain the following: By the definition of d, by Theorem 2.1 and because the $X_1, X_2, \ldots$ are assumed to be a sequence of i.i.d. random variables we have for $k' = h + g_2 \in K(m+g_1)$

$$\begin{aligned}
&E_\theta\big(w(N(m,k),S(m,k))\,|\,C_{kk'}(m,m+g_1)\big)\\
&\quad= E_\theta w\big(g_1+N(m+g_1,h+g_2),S(m+g_1,h+g_2)\big)\\
&\quad= E_\theta w\big(N(m+g_1,h+g_2),S(m+g_1,h+g_2)-g_2\big)\\
&\qquad+E_\theta d\big(N(m+g_1,h+g_2),S(m+g_1,h+g_2)\big)\\
&\quad= w_h^\theta(m)+d_{h+g_2}^\theta(m+g_1).
\end{aligned}\tag{3.6}$$

Putting together relations (3.3), (3.4), (3.5) and (3.6), we get for $k\in K(m)$

$$\begin{aligned}
w_k^\theta(m) &= \sum_{n=1}^{g_1}\sum_{k'\in\bar{K}(m+n)} w(m+n,k')c_{kk'}^\theta(m,m+n)+\\
&+\sum_{h\in K(m)} c_{kh+g_2}^\theta(m,m+g_1)w_h^\theta(m)+\sum_{h\in K(m)} c_{kh+g_2}^\theta(m,m+g_1)d_{h+g_2}^\theta(m+g_1),
\end{aligned}$$

and the proof is complete. ∎

It follows from this theorem that we can compute $w^\theta(m)$ by solving a system of linear equations if we can compute, besides the probabilities $P_\theta(C_{kk'}(m,m'))$, the vector $d^\theta(m+g_1)$. Here are some remarks:

Remark 3.1. Since $X_1, X_2, \ldots$ are assumed to be i.i.d. random variables, we can compute the matrix $C^\theta(m,m+g_1)$ and the vectors $c_{k'}^\theta(m,m+n)$ for $n=1,\ldots,g_1$ and $k'\in\bar{K}(m+n)$ in the following manner:

(i) For $n=1,\ldots,g_1$ we have

$$C^\theta(m,m+n)=\sum_{j=m}^{m+n-1} C^\theta(j,j+1)$$

where with respect to the elements of $C^\theta(j,j+1)$

$$c_{kk'}^\theta(j,j+1)=P_\theta(k+X_{j+1}=k')=P_\theta(X_1=k'-k)$$

for $k\in K(j)$ and $k'\in K(j+1)$ holds.

(ii) For $n=1,\ldots,g_1$ and $k'\in\bar{K}(m+n)$ we have

$$c_{k'}^\theta(m,m+n)=C^\theta(m,m+n-1)c_{k'}^\theta(m+n-1,n)$$

where

$$c_{kk'}^\theta(m+n-1,n)=P_\theta(X_1=k'-k)$$

for $k\in K(m+n-1)$ and $k'\in\bar{K}(m+n)$ and $C^\theta(m,n):=E$ holds. This means that we can compute the vectors $c_{k'}^\theta(m,m+n)$ parallel to the computation of $C^\theta(m,m+g_1)$.

Remark 3.2. The application of Theorem 3.1 may lead to a new problem, namely the computation of the vector $d^\theta(m+g_1)$, which we may also interpret as a vector of the characteristics of our test. Depending on the properties of the function w under consideration it will not always be possible to reduce these characteristics to known characteristics. A simple example is the computation of negative moments of the sample number.

Remark 3.3. Assertions about the characteristics of truncated SPRTs may be obtained if we solve the given system of linear equations by the method of iteration, using special initial conditions. For the computation of the OC and the moments of the sample number of truncated SPRTs we refer to [5].

Remark 3.4. Numerical results for the Poisson distribution are contained in [7].

Remark 3.5. By means of the generalized SPRTs $T(m, k)$ considered here it is further possible to compute directly the operating characteristic functions and the moments of the sample number of the Sobel–Wald test for discriminating between k hypotheses, where $k \geqslant 2$ ([6]).

4. Example

4.1. The operating characteristic function. In calculating the probability of acceptance of H_0 by the test $T(m, k)$ by means of a system of linear equations we introduce the following notation:

$q_k^\theta(m)$ – the probability of acceptance of H_0 by the test $T(m, k)$, $(m, k) \in M$, $\theta \in \Theta$;

$q^\theta(m) := \{q_k^\theta(m)\}_{k \in K(m)}$;

$a_k^\theta(m, m+n)$ – the probability of acceptance of H_0 by the test $T(m, k)$ at the n^{th} sampling stage, $n = 1, \ldots, g_1$, $k \in K(m)$; $a^\theta(m, m+n) := \{a_k^\theta(m, m+n)\}_{k \in K(m)}$.

Then the following theorem holds:

THEOREM 4.1. *Under the conditions of Theorem* 3.1 *we have for* $m \in \Gamma_0^+$

$$(E - C^\theta(m, m+g_1))q^\theta(m) = \sum_{n=1}^{g_1} a^\theta(m, m+n) \tag{4.1}$$

with

$$a^\theta(m, m+n) = C^\theta(m, m+n-1)\, a^\theta(m+n-1, m+n) \tag{4.2}$$

and

$$a_k^\theta(m+n-1, m+n) = P_\theta(X_1 < h^a(m+n)-k) \tag{4.3}$$

for $n = 1, \ldots, g_1$ *and* $k \in K(m+n-1)$.

Proof. By the definition of $q_k^\theta(m)$ we have

$$q_k^\theta(m) := P_\theta(\text{acceptance of } H_0 \text{ by } T(m, k))$$

$$= E_\theta \chi_{\{S(m,k) < h^a(m+N(m,k))\}}$$

and

$$w(n, s) := \chi_{\{s < h^a(m+n)\}}$$

for $(n, s) \in \Gamma_0^+ \times \Gamma$. Under the conditions of Theorem 3.1 we have

$$h^a(m+n+g_1) = h^a(m+n)+g_2 \qquad \text{for} \quad n = 1, 2, \ldots,$$

so that we obtain

$$\begin{aligned} d(n, s) &:= w(n+g_1, s)-w(n, s-g_2) \\ &= \chi_{\{s < h^a(m+n+g_1)\}} - \chi_{\{s-g_2 < h^a(m+n)\}} = 0 \end{aligned}$$

and

$$d^\theta(m+g_1) = 0. \tag{4.4}$$

Since the $X_1, X_2, \ldots$ are assumed to be i.i.d. random variables and by the definition of $a^\theta(m, m+n)$ we get for the components $v_{n,k}^\theta(m)$, $k \in K(m)$, of the vector $v_n^\theta(m)$, defined by (3.2), for $n = 1, \ldots, g_1$

$$\begin{aligned} v_{n,k}^\theta(m) &= \sum_{k' \in \bar{K}(m+n)} c_{kk'}^\theta(m, m+n)\chi_{\{k' < h^a(m+n)\}} \\ &= \sum_{k'' \in K(m+n-1)} c_{kk''}^\theta(m, m+n-1) \sum_{k' \in \bar{K}(m+n)} c_{k''k'}^\theta(m+n-1, m+n)\chi_{\{k' < h^a(m+n)\}} \\ &= \sum_{k'' \in K(m+n-1)} c_{kk''}^\theta(m, m+n-1)\, a_{k''}^\theta(m+n-1, m+n) \\ &= a_k^\theta(m, m+n) \end{aligned} \tag{4.5}$$

with

$$a_{k''}^\theta(m+n-1, m+n) = P_\theta\big(k''+X_{m+n} < h^a(m+n)\big) = P_\theta\big(X_1 < h^a(m+n)-k''\big),$$

so that (4.2) and (4.3) hold. Then, assertion (4.1) follows by Theorem 3.1, (4.4) and (4.5).

4.2. The moments of the sample number. In calculating the moments of the sample number of $T(m, k)$ we introduce the following notation:

$e_{r,k}^\theta(m) := E_\theta N^r(m, k)$ – the r^{th} moment of the sample number of $T(m, k)$, $(m, k) \in M$, $\theta \in \Theta$, $r = 0, 1, 2, \ldots$;

$e_r^\theta(m) := \{e_{r,k}^\theta(m)\}_{k \in K(m)}$;

$r_k^\theta(m, m+n)$ – the probability of acceptance of H_1 by the test $T(m, k)$ at the nth sampling stage, $n = 1, \ldots, g_1$, $k \in K(m)$;

$r^\theta(m, m+n) := \{r_k^\theta(m, m+n)\}_{k \in K(m)}$.

Let $a^\theta(m, m+n)$ be defined as in Theorem 4.1. Then the following theorem holds.

THEOREM 4.2. *Suppose that $D_\theta^2 Z_1 > 0$. Under the conditions of Theorem 3.1 we have for $m \in \Gamma_0^+$ and $r = 1, 2, \ldots$*

$$(E - C^\theta(m, m+g_1))e_r^\theta(m) = \sum_{n=1}^{g_1} n^r\big(a^\theta(m, m+n) + r^\theta(m, m+n)\big) + \sum_{j=0}^{r-1}\binom{r}{j} g_1^{r-j} C^\theta(m, m+g_1)\cdot e_j^\theta(m) \tag{4.6}$$

with

$$r^\theta(m, m+n) = C^\theta(m, m+n-1)\, r^\theta(m+n-1, m+n) \tag{4.7}$$

and

$$r_k^\theta(n+m-1, m+n) = P_\theta\big(X_1 > h^r(m+n)-k\big) \tag{4.8}$$

for $n = 1, \ldots, g_1$ *and* $k \in K(m+n-1)$.

Proof. The assumption $D_\theta^2 Z_1 > 0$ provides $E_\theta N^r(m, k) < \infty$ for $r = 1, 2, \ldots$ Now we have

$$w(n, s) = n^r \quad \text{for} \quad (n, s) \in \Gamma_0^+ \times \Gamma$$

and

$$d(n, s) = (n+g_1)^r - n^r = \sum_{j=0}^{r-1}\binom{r}{j} g_1^{r-j} n^j,$$

which is independent of s. Thus, by Theorem 2.1 and Remark 2.2, we obtain

$$d^\theta(m+g_1) = d^\theta(m) = \sum_{j=0}^{r-1}\binom{r}{j} g_1^{r-j} e_j^\theta(m). \tag{4.9}$$

Since the $X_1, X_2, \ldots$ are assumed to be i.i.d. random variables, we obtain for $v_{n,k}^\theta(m)$, $k \in K(m)$ and $n = 1, \ldots, g_1$

$$\begin{aligned} v_{n,k}^\theta(m) &= \sum_{k' \in K(m+n)} n^r c_{kk'}^\theta(m, m+n) \\ &= \sum_{k'' \in K(m+n-1)} n^r c_{kk''}^\theta(m, m+n-1)\big(a_{k''}^\theta(m+n-1, m+n) + \\ &\qquad + r_{k''}^\theta(m+n-1, m+n)\big) \\ &= n^r\big(a_k^\theta(m, m+n) + r_k^\theta(m, m+n)\big) \end{aligned} \tag{4.10}$$

with

$$r_{k''}^\theta(m+n-1, m+n) = P_\theta\big(k'' + X_{m+n} > h^r(m+n)\big) = P_\theta\big(X_1 > h^r(m+n) - k''\big),$$

so that (4.7) and (4.8) hold. Assertion (4.6) follows by Theorem 3.1, (4.9) and (4.10). ∎

To illustrate this result we consider the computation of the first and second moments of the sample number.

For $r = 1$ we obtain

$$(E - C^\theta(m, m+g_1)) \cdot e_1^\theta(m) = \sum_{n=1}^{g_1} n\big(a^\theta(m, m+n) + r^\theta(m, m+n)\big) + g_1 \cdot C^\theta(m, m+g_1) \cdot 1$$

with $1 = \{1\}_{k \in K(m)}$. For $r = 2$ we obtain

$$(E - C^\theta(m, m+g_1)) \cdot e_2^\theta(m) = \sum_{n=1}^{g_1} n^2\big(a^\theta(m, m+n) + r^\theta(m, m+n)\big) + g_2^2 C^\theta(m, m+g_1) \cdot 1 + 2g_1 C^\theta(m, m+g_1) e_1^\theta(m).$$

This means that we may compute successively, beginning with the average sample numbers $e_{1,k}(m)$ for $k \in K(m)$, the moments $e_{2,k}(m)$, $e_{3,k}(m)$, ... for $k \in K(m)$. In doing this, we must solve step by step systems of linear equations.

References

[1] L. A. Aroian, *Sequential analysis: Direct method*, Technometrics **10** (1968), 125–132.

[2] W. Bartky, *Multiple sampling with constant probability*, Ann. Math. Statist. **14** (1953), 363–377.

[3] P. Bernstein, *Gruppierte Sequenzteste zur Prüfung des Mittelwertes einer 0–1-verteilten Zufallsvariablen*, Abh. Dtsch. Akad. Wiss. Bln. Kl. Math. Phys. Tech. **4** (1964), 31–33.

[4] J. P. Burman, *Sequential sampling formulae for a binomial population*, J. Roy. Statist. Soc. Suppl. **8** (1946), 98–103.

[5] K.-H. Eger, *A direct method of the computation of the OC and of the moments of the sample number for SPRTs in the case of discrete random variables*, Math. Operationsforsch. Statist. Ser. Statist. **11** (1980), 499–514.

[6] –, *Eine direkte Methode zur Berechnung der Charakteristiken des Sobel–Wald Tests zur Entscheidung zwischen k Hypothesen*, Wiss. Z. d. Tech. Hochsch. Karl-Marx-Stadt, **23** (1981), 361–367.

[7] K.-H. Eger and W. Fleischer, *Sequentielle Prüfpläne für Poisson verteilte Fehlermerkmale, in Qualitätsanalyse 1981*, Verlag Kammer der Technik Frankfurt/Oder, Frankfurt/Oder 1981.

[8] B. K. Ghosh, *Sequential Test of Statistical Hypotheses*, Reading, MA, 1970.

[9] M. A. Girshick, *Contributions to the theory of sequential analysis, II* and *III*. Ann. Math. Statist. **17** (1946) 282–298.

[10] Z. Govindarajulu, *Sequential Statistical Procedures*, Academic Press, New York 1975.

[11] –, *The Sequential Statistical Analysis of Hypothesis-Testing Point and Interval Estimation, and Decision Theory*, American Science Press, Columbus Ohio, 1981.

[12] A. Hald and U. Møller, *Multiple sampling planes of given strength for the Poisson and binomial distributions*, Preprint No. 11, Inst. Math. Statist., Univ. Copenhagen 1976.

[13] J. E. Jackson, *Bibliography on sequential analysis*, J. Amer. Statist. Assoc. **55** (1960), 561–580.

[14] N. J. Johnson, *Sequential analysis: a survey*, J. Roy. Statist. Soc. Ser. A **124** (1961), 372–411.

[15] H. L. Jones, *Formulas for the group sequential sampling of attributes*, Ann. Math. Statist. **23** (1952) 72–87.

[16] M. G. Pólya, *Exact formulas in the sequential analysis of attributes*, Univ. California Publ. in Mathem., New Ser. **1** (1948), 229–239.
[17] A. Wald, *Cumulative sums of random variables*, Ann. Math. Statist. **15** (1944), 283–296.
[18] —, *Sequential Analysis*, New York 1947.
[19] A. M. Walker, *Note on sequential sampling formulae for a binomial population*, J. Roy. Statist. Soc. Ser. B **12** (1950), 201–207.

Presented to the semester
Sequential Methods in Statistics
September 7–December 11, 1981