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PRODUCTS WITH NON-LINEAR FINITE-SET-APOSYNDETIC CONTINUA

BY

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1. Introduction. Let $\mathscr{C}$ denote the collection to which a compac- Hausdorff continuum M belongs only if, for each compact Hausdorft continuum N, the Cartesian product $M\times N$ is finite-set-mutually apof syndetic. The simple closed curve and certain other n-mutually aposyndetic continua are known to be members of the collection $\mathscr{C}$ ([9], p. 250). Also, since the product of any three regular Hausdorff continua is finite-set--mutually aposyndetic ([9], p. 245), it follows that the product of any two compact Hausdorff continua is a member of $\mathscr{C}$. In this paper we show that each finite-set-aposyndetic compact metric continuum except an arc is an element of $\mathscr{C}$, thus obtaining the above-mentioned results as corollaries (in the compact metric case). This is a natural extension of the known result for $n=2$ (mutual aposyndesis), that if M is any aposyndetic compact metric continuum except an arc, then $M\times N$ is mutually aposyndetic for each continuum N ([9], p. 249).

The concepts of n-aposyndesis [11], mutual aposyndesis [3] and n-mutual aposyndesis [9] have originated in an attempt to fill the gap between aposyndesis and local connectedness in the spectrum of continua ([6], p. 144). Finite-set-mutual aposyndesis (n-mutual aposyndesis for each $n\geqslant 2$) is the strongest of these, yet still weaker than local connectedness, since the product of any three regular Hausdorff continua (e.g., three copies of the pseudo-arc (see [1], p. 43) is finite-set-mutually aposyndetic ([9], p. 245), but, clearly, not locally connected.

The study of the various forms of aposyndesis in products of continua began when Jones ([5], p. 406) proved that the product of any two regular Hausdorff continua is aposyndetic. Two decades later the study of n-aposyndesis in products ([11], [2], and [9], p. 246) led to the result that the product of any two regular Hausdorff continua is finite-set-aposyndetic. The investigation of mutual aposyndesis in products was begun by Hagopian ([3], p. 616) and extended to n-mutual aposyndesis by the author [9].

2. Definitions. A *continuum* is a non-degenerate closed connected set. The closure, interior and boundary of a set A will be denoted by $\mathrm{Cl}A$, A° and $\mathrm{Bd}A$, respectively. Given points x and y in the continuum M, M is *aposyndetic at x with respect to y (a set A)* if there exists a subcontinuum H of M such that $x \in H^\circ$ and $y \notin H$ (respectively, $H \cap A = \emptyset$). If, for each pair of distinct points $x, y \in M$, M is aposyndetic at x with respect to y, then we say that M is *aposyndetic*. For $n \geqslant 1$, M is *n-aposyndetic* if M is aposyndetic at each point x with respect to each n-point set which does not contain x. If M is n-aposyndetic for each $n \geqslant 1$, then we say that M is *finite-set-aposyndetic*. Given $n \geqslant 2$, if for each n-point set A there exist n disjoint subcontinua each containing a point of A in its interior, then M is *n-mutually aposyndetic*. If M is n-mutually aposyndetic for each $n \geqslant 2$, then M is *finite-set-mutually aposyndetic*.

The set D is said to *separate* a point x from a set Y if $M - D = A \cup B$ separated, (i. e., A and B being separated sets), with $x \in A$ and $Y \subset B$. The set D is said to *cut* x from Y if every subcontinuum of M intersecting both $\{x\}$ and Y intersects D also.

3. Results. The main tool in the proof of the theorem is an interesting method for constructing subcontinua in products:

LEMMA 1. *Let $n \geqslant 1$ and suppose that*

(1) *$D_1, \ldots, D_n$ are disjoint closed subsets of a compact Hausdorff continuum N;*

(2) *$A_1, \ldots, A_{n+1}$ are separated sets in N such that*

$$N - \bigcup_{1}^{n} D_i = \bigcup_{1}^{n+1} A_i;$$

(3) *$\mathrm{Bd}A_1 \subset D_1$, $\mathrm{Bd}A_{n+1} \subset D_n$ and, for $2 \leqslant i \leqslant n$, $\mathrm{Bd}A_i \subset D_{i-1} \cup D_i$;*

(4) *$T_1, \ldots, T_n$ are subcontinua of a compact Hausdorff continuum M;*

(5) *$x_1, \ldots, x_{n+1}$ are points of M (not necessarily distinct) such that $\{x_i, x_{i+1}\} \subset T_i$ for each i;*

(6) *L denotes the set*

$$\Big[\bigcup_{1}^{n} (T_i \times D_i)\Big] \cup \Big[\bigcup_{1}^{n+1} (\{x_i\} \times A_i)\Big].$$

Then

(i) *L is a continuum, and*

(ii) *for $k \leqslant n$, if $z \in A_k$ and U is an open set such that $x_k \in U \subset T_k$, then $L \cup (\mathrm{Cl}\, U \times A_k)$ is a continuum which contains the point (x_k, z) in its interior.*

Proof. It is clear that L is closed. To show that L is connected, let $f : L \to N$ denote the projection map π_2 (from $M \times N$ to N) restricted to L. Since $f^{-1}(y)$ is connected for each y, the map f is monotone. Thus $L = f^{-1}(N)$ is a continuum.

Part (ii) of the conclusion easily follows, since the set in question is, simply, a union (over $\mathrm{Cl}\, U$) of continua similar to L.

LEMMA 2. *Suppose that $n \geqslant 1$. Let M be an n-aposyndetic compact Hausdorff continuum and let D be an n-point subset of M. Then D cuts a point x from a set Y if and only if D separates x from Y.*

Proof. The sufficiency is trivial. To prove the necessity, suppose that D cuts x from Y, where $x \notin Y$ and $D \cap (\{x\} \cup Y) = \emptyset$. Let A denote the set of all points $z \in M$ such that there exists a subcontinuum T of M for which $\{x, z\} \subset T$ and $T \cap D = \emptyset$. Let $B = M - (A \cup D)$. These are non-empty sets, since $x \in A$ and $Y \subset B$. We shall show that A and B are separated sets. Suppose that $z \in (\mathrm{Cl} A) \cap B$. Since M is n-aposyndetic, there exists a subcontinuum H such that $z \in H^\circ$ and $H \cap D = \emptyset$. Since z is a limit point of A, there is a point $a \in A \cap H^\circ$. By the definition of A, there is a subcontinuum T which contains $\{x, a\}$ and is disjoint from D. Then $T \cup H$ is a subcontinuum which contains $\{x, z\}$ and is disjoint from D. Thus z must be an element of A. But this contradicts the fact that $z \in B$, and so $(\mathrm{Cl} A) \cap B = \emptyset$.

Now suppose that $z \in A \cap \mathrm{Cl} B$. There is a subcontinuum H such that $z \in H^\circ$ and $H \cap D = \emptyset$. Since z is a limit point of B, there exists a point $b \in H^\circ \cap B$. Since $z \in A$, there is a subcontinuum T containing $\{x, z\}$ and disjoint from D. But then, since the continuum $H \cup T$ contains $\{x, b\}$ and is disjoint from D, we infer that $b \in A$, contrary to the fact that $b \in B$. Thus $A \cap \mathrm{Cl} B = \emptyset$.

Consequently, $M - D$ is the union of the two separated sets A and B, so D separates x from Y.

LEMMA 3. *Suppose that $n \geqslant 1$, M is an n-aposyndetic compact metric continuum, D is a subcontinuum of M, and $x_1, \ldots, x_n \in M - D$ such that, for each $j \leqslant n$, $\{x_i \mid i > j\}$ does not cut x_j from D. Let k be a positive integer less than n, and let U be an open set containing $\{x_i \mid i > k\}$. Then there exists a point $z \in U$ such that $\{x_i \mid i > k\}$ does not cut z from D and, for each $j \leqslant k$, $\{z\} \cup \{x_i \mid i > j\}$ does not cut x_j from D.*

Proof. Let C be the component of $M - \{x_i \mid i > k\}$ which contains $D \cup \{x_k\}$. By Lemma 2, no point of C is cut from D by $\{x_i \mid i > k\}$. Since the boundary of $M - \{x_i \mid i > k\}$ is simply $\{x_i \mid i > k\}$, there is an integer $j > k$ such that x_j is a limit point of C. Let $z_1, z_2, \ldots$ be a sequence of points in $C \cap U$ converging to x_j. Suppose that, for each positive integer m, $z_m \cup \{x_i \mid i > k\}$ cuts some point of $\{x_i \mid i \leqslant k\}$ from D. We can assume that (choosing a subsequence, if necessary) there exists an integer $v \leqslant k$ such that, for each m, $z_m \cup \{x_i \mid i > k\}$ cuts x_v from D. But since $x_j = \lim z_m$, we infer that $\{x_i \mid i > k\}$ cuts x_v from D. Since $v \leqslant k$, we have a contradiction. Thus there exists an integer q such that $z_q \cup \{x_i \mid i > k\}$ cuts no point of $\{x_i \mid i \leqslant k\}$ from D. Finally, since $z_q \in C$, z_q is not cut from D by $\{x_i \mid i > k\}$.

THEOREM. *Let M be a finite-set-aposyndetic compact metric continuum. Then $M \times N$ is finite-set-mutually aposyndetic for each compact Hausdorff continuum N if and only if M is not an arc.*

Proof. The necessity is clear, since the product of an arc with the curve $\sin(1/x)$ is not mutually aposyndetic ([8], p. 808). Thus we consider the sufficiency.

Assume M is not an arc and let N be a compact Hausdorff continuum. Suppose $n \geqslant 2$, $k \geqslant 1$, and let $\{x_1, \ldots, x_n\}$ be an n-point subset of M and $\{y_1, \ldots, y_k\}$ a k-point subset of N. We shall exhibit nk disjoint continua, each containing an (x_i, y_j) in its interior.

In case M is a simple closed curve, the conclusion follows ([9], p. 250). Thus we assume that M is not a simple closed curve. By [10], p. 455, M is either locally connected or a triod. In order to define the sets $Q, P_1, P_2, P_3, H_1, \ldots, H_n$, we consider two cases.

Case 1. M is locally connected.

By [4], p. 429, M contains a simple triod $T_0 = T^1 \cup T^2 \cup T^3$ such that each T^i is an arc and $T^1 \cap T^2 = T^1 \cap T^3 = T^2 \cap T^3 = \{q\}$. Let Q_0, S_0 and T_0^i (for $i = 1, 2, 3$) denote $\{q\}$, $\emptyset$ and T^i, respectively. For $j \geqslant 1$, assuming T_m, Q_m, T_m^i and S_m have already been defined for $m < j$, we define T_j, Q_j, T_j^i $(i = 1, 2, 3)$ and S_j as follows:

(1) let S_j be an arc irreducible from x_j to T_{j-1} (if $x_j \in T_{j-1}$, simply let $S_j = \{x_j\}$),

(2) let $T_j = T_{j-1} \cup S_j$,

(3) let Q_j be the q-component of $Q_{j-1} \cup S_j$,

(4) let T_j^i be the q-component of $T_{j-1}^i \cup S_j \cup Q_j$ for $i = 1, 2, 3$.

By the local connectedness of M, there exist subcontinua $H_1, \ldots, H_n$ such that, for each $j \leqslant n$,

$$x_j \in H_j^\circ \subset H_j \subset M - \bigcup_{i \neq j} H_i$$

and, if $x_j \notin Q_n$, then $H_j \cap Q_n = \emptyset$. Finally, let $T = T_n \cup \left(\bigcup H_i\right)$, Q be the q-component of $Q_n \cup \left(\bigcup H_i\right)$ and, for $j = 1, 2, 3$, let P_j be the q-component of $T_n^j \cup \left(\bigcup H_i\right)$. Then T is a triod, since

$$T = P_1 \cup P_2 \cup P_3 \quad \text{and} \quad P_1 \cap P_2 = P_1 \cap P_3 = P_2 \cap P_3 = P_1 \cap P_2 \cap P_3 = Q.$$

Since each $x_i \in T^\circ$, it suffices to prove that in $T \times N$ there exist nk disjoint subcontinua each containing an (x_i, y_j) in its interior. Thus we can assume that $M = T$.

Case 2. M is a triod.

There are subcontinua P_1, P_2, P_3 and Q such that

$$M = P_1 \cup P_2 \cup P_3 \quad \text{and} \quad P_1 \cap P_2 = P_1 \cap P_3 = P_2 \cap P_3 = P_1 \cap P_2 \cap P_3 = Q.$$

By the n-aposyndesis, for each $j \leqslant n$, there exists a subcontinuum H_j of M such that $x_j \in H_j^\circ$ and $H_j \subset M - \{x_i \mid i \neq j\}$. We can assume that if $x_j \in P_i - Q$, then $H_j \subset P_i$ for $j \leqslant n$ and $i \leqslant 3$.

Thus in either of these two cases M is a triod. We shall show that the x_i's can be relabelled so that, for each $j \leqslant n$, $\{x_i \mid i > j\}$ does not cut x_j from Q, and so that if $x_i \in Q$ and $x_j \notin Q$, then $i < j$. The latter is obvious. To prove the former, let j be the smallest positive integer m for which $\{x_i \mid i > m\}$ cuts x_m from Q, and suppose the x_i's have been relabelled so that j is as large as possible, yet less than n. By Lemma 2, $\{x_i \mid i > j\}$ separates x_j from Q, and hence $M - \{x_i \mid i > j\} = A \cup B$, separated, with $x_j \in A$ and $Q \subset B$. Since $\{x_i \mid i > j\} \cup B$ has at most $n-j$ components, there exists a $j' > j$ such that $x_{j'}$ is a limit point of the component of B containing Q. Then, by interchanging x_j and $x_{j'}$, we have a relabelling of the x_i's in which, for each $m \leqslant j$, $\{x_i \mid i > m\}$ does not cut x_m from Q. This contradicts the original choice of j, and hence establishes the claim. We can also assume that if $x_j \in Q$, then $\{x_i \mid i > j$ and $x_i \in Q\}$ does not cut x_j from $P_1 - Q$ in the continuum P_1 (this can be seen using an argument similar to the preceding one).

From Lemma 3 it follows that there exist points z_{ij}, subcontinua T_{ij} of P_i (for $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant nk$) and subcontinua H'_m of P_1 (for $x_m \in Q$) such that

(1) if $x_m \in P_i - Q$, then $x_m \in \{z_{ij} \mid j \leqslant nk\}$;

(2) if $x_m = z_{ij}$ and $v > j$, then $z_{iv} \notin H_m$;

(3) T_{ij} is irreducible about $Q \cup \{z_{ij}\}$ and $T_{ij} \cap \{z_{im} \mid m > j\} = \emptyset$;

(4) for $x_m \in Q$, H'_m is irreducible about $H_m \cup \{z_{ij}\}$ for some $j \leqslant nk$, and $H'_m \cap (\{z_{1v} \mid v > j\} \cup \{x_v \mid v > m$ and $x_v \in Q\}) = \emptyset$.

Let $U_1, \ldots, U_n$ be open sets in M such that, for $j \leqslant n$,

(i) $x_j \in U_j \subset H_j$;

(ii) $H_i \cap \mathrm{Cl}\, U_j = \emptyset$ for $i \neq j$;

(iii) if $x_j \notin Q$, then $Q \cap \mathrm{Cl}\, U_j = \emptyset$;

(iv) $\mathrm{Cl}\, U_j \cap \{z_{iv} \mid v \leqslant n, x_j \neq z_{iv}\} = \emptyset$;

(v) if $x_j = z_{im}$, then $\mathrm{Cl}\, U_j \cap T_{iv} = \emptyset$ for each $v < m$;

(vi) if $x_j \in Q$, then $\mathrm{Cl}\, U_j \cap H'_m = \emptyset$ for each $m < j$ such that $x_m \in Q$.

For $x_m = z_{ij}$, $H'_m = H \ \cup T_{ij}$. Thus, for $x_m = z_{ij}$, we have

$$H_m \cup Q \subset H'_m, \quad H'_m \subset P_i \quad \text{and} \quad H'_m \cap \{z_{iv} \mid v > j\} = \emptyset.$$

For $j<k$ and $\theta \leqslant n(k+1)-n_1$, let $A_{j\theta}$, $B_{j\theta}$, $C_{j\theta}$ and $E_{j\theta}$ be subsets of N such that

(1) $y_j \in B_{j\theta}$,

(2) $D_{j\theta}$ and $E_{j\theta}$ are closed sets,

(3) $N-D_{j\theta} = A_{j\theta} \cup (B_{j\theta} \cup E_{j\theta} \cup C_{j\theta})$, separated,

(4) $N-E_{j\theta} = C_{j\theta} \cup (A_{j\theta} \cup D_{j\theta} \cup B_{j\theta})$, separated,

(5) $A_{j,\theta+1} \subset A_{j\theta}$ and $C_{j,\theta+1} \subset C_{j\theta}$,

(6) $D_{j\theta} \cup B_{j\theta} \cup E_{j\theta} \subset B_{j,\theta+1}$,

(7) $(D_{j\theta} \cup B_{j\theta} \cup E_{j\theta}) \cap (D_{i\theta} \cup B_{i\theta} \cup E_{i\theta}) = \emptyset$ for $i \neq j$.

For positive integers m, j, θ and sequences a, β such that $m \leqslant n$, $j \leqslant k$, and θ, a_v, $\beta_v \leqslant n(k+1)-n_1$ (for $v \leqslant k$), let $G(m, j, \theta, a, \beta)$ denote the set

$$\begin{aligned}[\mathrm{Cl}\, U_m \times B_{j\theta} \cup H'_m \times (D_{j\theta} \cup E_{j\theta})] \cup &\\ \cup \Big[\bigcup_{v=1}^{j-1} T_{2\beta_v} \times (D_{v+1,a_{v+1}} \cup E_{va_v}) \cup \{z_{2\beta_v}\} \times (A_{v+1,a_{v+1}} \cap C_{va_v})\Big] \cup &\\ \cup \Big[\bigcup_{v=j+1}^{k} T_{3\beta_v} \times (D_{va_v} \cup E_{v-1,a_{v-1}}) \cup \{z_{3\beta_v}\} \times (A_{va_v} \cap C_{v-1,a_{v-1}})\Big] \cup &\\ \cup [T_{2\beta_1} \times D_{1a_1} \cup \{z_{2\beta_1}\} \times A_{1a_1}] \cup [T_{3\beta_k} \times E_{ka_k} \cup \{z_{3\beta_k}\} \times C_{ka_k}]. &\end{aligned}$$

Then there are nk disjoint subcontinua of $M \times N$ of the following forms, each containing a point (x_m, y_j) in its interior (they are subcontinua by Lemma 1, and can be made disjoint by careful choices of the z_{ij}'s and H'_m (for $x_m \in Q$) above, and of θ, a and β depending upon m and j):

(a) for $x_m \in P_1 - Q$, $G(m, j, a_j, a, \beta)$;

(b) for $i = 2, 3$ and $x_m \in P_i - Q$,

$$G(m, j, \theta, a, \beta) \cup T_{1\gamma} \times (D_{ja_j} \cup D_{j\theta} \cup E_{j\theta} \cup E_{ja_j}) \cup \{z_{1\gamma}\} \times (B_{ja_j} - B_{j\theta}),$$

where $\theta < a_j$ and $\gamma \leqslant nk$;

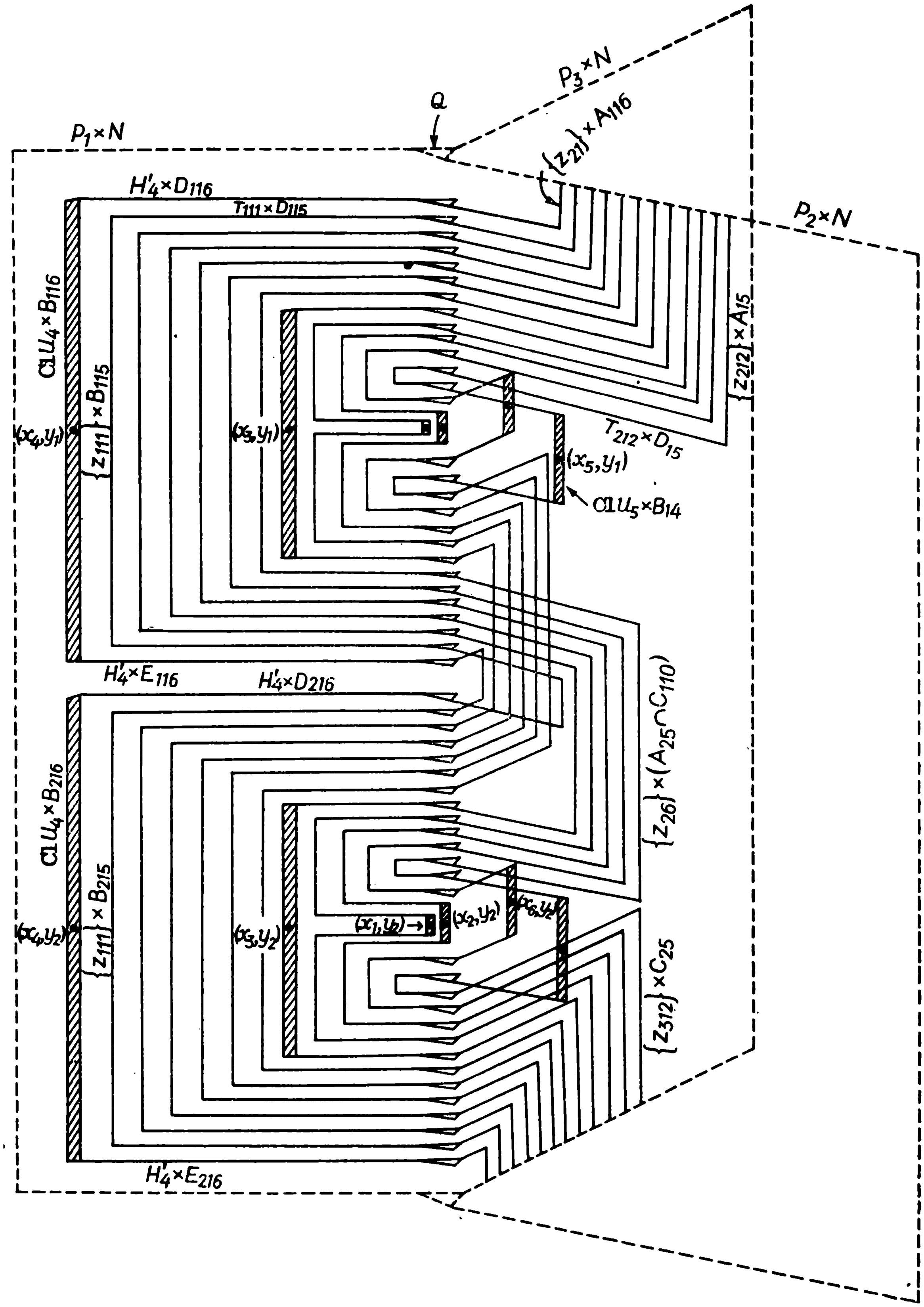
(c) for $x_m \in Q$,

$$G(m, j, \theta, a, \beta) \cup T_{1\gamma} \times (D_{ja_j} \cup E_{ja_j}) \cup \{z_{1\gamma}\} \times (B_{ja_j} - B_{j\theta}),$$

where $\theta < a_j$ and $\gamma \leqslant nk$.

The existence of these nk disjoint continua completes the proof. Fig. 1 illustrates the case where $k = 2$, $n = 6$, with $2, 2, 1$ and 1 x_i's in $Q, P_1 - Q, P_2 - Q$ and $P_3 - Q$, respectively.

COROLLARY. *If L, M, N are compact metric continua, then $L \times M \times N$ is finite-set-mutually aposyndetic.*



$P_3 \times N$
Q
$P_1 \times N$
$\{z_{21}\} \times A_{116}$
$H'_4 \times D_{116}$
$T_{111} \times D_{115}$
$P_2 \times N$
$\mathrm{Cl}\,U_4 \times B_{116}$
$\{z_{111}\} \times B_{115}$
(x_4, y_1)
(x_3, y_1)
$\{z_{212}\} \times A_{15}$
$T_{212} \times D_{15}$
(x_5, y_1)
$\mathrm{Cl}\,U_5 \times B_{14}$
$H'_4 \times E_{116}$
$H'_4 \times D_{216}$
$\{z_{26}\} \times (A_{25} \cap C_{110})$
$\mathrm{Cl}\,U_4 \times B_{216}$
$\{z_{111}\} \times B_{215}$
(x_4, y_2)
(x_3, y_2)
$(x_1, y_2) \rightarrow$
(x_2, y_2)
(x_6, y_2)
$\{z_{312}\} \times C_{25}$
$H'_4 \times E_{216}$

Fig. 1

Proof. The 2-product $L \times M$ is finite-set-aposyndetic [2] yet, clearly, not an arc. Hence the conclusion follows directly from the theorem.

QUESTION. In the hypothesis of the theorem, can "finite-set-aposyndetic" be replaced with "aposyndetic"? (**P 926**)

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