POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

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On n-ary equivalence relations and their application to geometry

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Representation theorem for the theory of E-sequences

WARSZAWA 1981
PAŃSTWOWE WYDAWNICTWO NAUKOWE

6·7133

PRINTED IN POLAND

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ISBN 83-01-02338-4 ISSN 0012-3862

WROCŁAWSKA DRUKARNIA NAUKOWA

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Foreword

The original version of the present paper by Wanda Szmielew consists of three articles [2], [3], [4] written in Polish and deposited (as a manuscript) in the library of the Institute of Mathematics of Polish Academy of Sciences in 1963. Those papers have never been published though they have proved to be very useful, especially from the methodological point of view.

After the author's death in 1976, Haragauri N. Gupta, Professor of the Regina University, Canada, suggested to translate and publish those papers. He himself translated the first article and completed the proof of T.2.2.2 which was missing in the manuscript.

The other two articles were translated by Jerzy Bednarczuk, Assistant Professor of Warsaw University. He also checked very carefully all the proofs and rearranged the work as a whole. So, the first chapter of the present paper corresponds to [2] and a part of [4], the second one consists of [3] and another part of [4], and finally the third chapter contains the remaining part of [4].

In 1977 J. Bednarczuk proved (as a part of his doctoral dissertation) a representation theorem for Wanda Szmielew's theory of n-ary equivalences. His results are presented in Appendix.

Maria Moszyńska

Warszawa, October 27, 1979.

Introduction

By an equivalence relation on a set S one usually means a binary relation R which is reflexive, symmetric and transitive.

This can be formulated by the following two conditions:

(i) $$a = b \Rightarrow R(ab),$$

(ii) $$R(ax) \wedge R(ay) \Rightarrow R(xy)$$

for arbitrary elements a, b, x, y of the set S. The part played by equivalence relations in various branches of mathematics is well known. Especially in geometry, a good number of concepts are defined as equivalence classes with respect to a certain equivalence relation R, (e.g. free vectors, directions, orientations of spaces).

Bachmann in [1] p. 64 introduced the concept of a ternary equivalence relation R (on a set S) defined by the conditions:

(i') $$a = b \vee b = c \vee c = a \Rightarrow R(abc),$$

(ii') $$a \neq b \wedge R(abx) \wedge R(aby) \wedge R(abz) \Rightarrow R(xyz)$$

for arbitrary elements a, b, c, x, y, z of the set S (1). Using this concept, he defined some notions of 2-dimensional geometry in terms of equivalence classes.

In the present paper we generalize the notions of binary and ternary equivalence relations to the notion of an n-ary equivalence relation, more precisely to the concept of an equivalence sequence which will be called an E-sequence (E for equivalence relation). This concept turns out to be very convenient for the development of geometry, especially for the theory of dimension. We hope that it will turn out to be a convenient tool in other areas of mathematics as well.

We use the following notation: T for theorem, C for corollary, T.1.2 for Theorem 2 in § 1 of this chapter, T.1.3.2 for Theorem 2 in § 3 of Chapter 1. The symbol "$\bigwedge_{i=1}^{n} \varphi_i$" denotes the conjunction of the formulae $\varphi_1, \ldots, \varphi_n$.

(1) In fact, the conditions given by Bachmann are slightly different but equivalent to (i'), (ii').

Chapter I

n-ary equivalence relations

In §1 we define the E-sequence and enumerate its basic properties.

In §2 we give alternative conditions for an E-sequence.

In §3 we discuss some special relational sequences and investigate the conditions under which such sequences may be E-sequences.

§1

DEFINITION. Let S be a non-empty set and (R_n) a finite or infinite sequence of relations $R_2, \ldots, R_n, \ldots$ where $R_n \subset S^n$, i.e. for each n, R_n is an n-ary relation on S.

The sequence (R_n) is an *E-sequence* if and only if R_2 is the identity relation and for $n \geqslant 3$ the following conditions hold:

I_n (*Reflexivity law*)

$$R_{n-1}(x_1 \ldots x_{i-1} x_{i+1} \ldots x_n) \Rightarrow R_n(x_1 \ldots x_n) \quad \text{for} \quad i = 1, \ldots, n.$$

II_n (*Transitivity law*)

$$\sim R_{n-1}(a_1 \ldots a_{n-1}) \wedge \bigwedge_{i=1}^{n} R_n(a_1 \ldots a_{n-1} x_i) \Rightarrow R_n(x_1 \ldots x_n).$$

In particular if $(=, R)$ is an E-sequence (finite), then the relation R is called a *ternary equivalence relation on* S (compare (i'), (ii')). Let us assume in addition that R_1 is the empty unary relation.

Properties of an E-sequence (R_n)

T.1.1. $R_k(x_1 \ldots x_k) \Rightarrow R_n(x_1 \ldots x_n)$ *for* $k = 2, \ldots, n$.

Proof by induction (downward) on k:

$$R_n(x_1 \ldots x_n) \Rightarrow R_n(x_1 \ldots x_n),$$

$$R_{k-1}(x_1 \ldots x_{k-1}) \underset{\mathrm{I}_k}{\Rightarrow} R_k(x_1 \ldots x_k) \underset{\mathrm{Ind}}{\Rightarrow} R_n(x_1 \ldots x_n). \quad \blacksquare$$

COROLLARY. *Let* $X \subset S$. *Then*

$$R_k|_X = X^k \Rightarrow R_n|_X = X^n \quad \textit{for} \quad k = 2, \ldots, n.$$

T.1.2. $x_k = x_l \Rightarrow R_n(x_1 \ldots x_n)$ *for* $k, l = 1, \ldots, n$ *and* $k \neq l$.

Proof by induction on n:

$$x_1 = x_2 \Rightarrow R_2(x_1 x_2).$$

Let $i \neq k, l$. Then

$$x_k = x_l \underset{\text{Ind}}{\Rightarrow} R_n(x_1 \dots x_{i-1} x_{i+1} \dots x_{n+1}) \underset{\Pi_{n+1}}{\Rightarrow} R_{n+1}(x_1 \dots x_{n+1}). \quad \blacksquare$$

T.1.3. $\sim R_{n-1}(a_1 \dots a_{n-1}) \wedge \bigwedge_{i=1}^{n-1} R_n(a_1 \dots a_{n-1} x_i) \Rightarrow R_n(x_1 \dots x_{n-1} a_j)$ *for* $j = 1, \dots, n-1$.

Proof.

$$\sim R_{n-1}(a_1 \dots a_{n-1}) \wedge \bigwedge_{i=1}^{n-1} R_n(a_1 \dots a_{n-1} x_i) \wedge R_n(a_1 \dots a_{n-1} a_j)$$

$$\underset{\text{(by T.1.2)}}{\Rightarrow} \underset{\Pi_n}{\Rightarrow} R_n(x_1 \dots x_{n-1} a_j). \quad \blacksquare$$

T.1.4. $\sim R_{n-1}(x_1 \dots x_{n-1}) \wedge \bigwedge_{i=1}^{n-1} R_n(a_1 \dots a_{n-1} x_i) \wedge R_n(x_1 \dots x_n)$

$$\Rightarrow R_n(a_1 \dots a_{n-1} x_n).$$

Proof. Case 1. $R_{n-1}(a_1 \dots a_{n-1}) \underset{\Pi_n}{\Rightarrow} R_n(a_1 \dots a_{n-1} x_n)$.

Case 2.

$$\sim R_{n-1}(a_1 \dots a_{n-1}) \wedge \bigwedge_{i=1}^{n-1} R_n(a_1 \dots a_{n-1} x_i) \underset{\text{T.1.3}}{\Rightarrow} \bigwedge_{j=1}^{n-1} R_n(x_1 \dots x_{n-1} a_j),$$

$$\sim R_{n-1}(x_1 \dots x_{n-1}) \wedge \bigwedge_{j=1}^{n-1} R_n(x_1 \dots x_{n-1} a_j) \wedge R_n(x_1 \dots x_{n-1} x_n)$$

$$\underset{\Pi_n}{\Rightarrow} R_n(a_1 \dots a_{n-1} x_n). \quad \blacksquare$$

Definition. Assume $\sim R_n(a_1 \dots a_n)$

$$x \in [a_1 \dots a_n] \underset{\text{def}}{\Leftrightarrow} R_{n+1}(a_1 \dots a_n x).$$

The set $[a_1 \dots a_n]$ will be called the *equivalence class of* R_{n+1} *spanned by the elements* $a_1, \dots, a_n$.

In particular,

$$x \in [a_1] \Leftrightarrow x = a_1, \quad \text{i.e.} \quad [a_1] = \{a_1\}.$$

Let E_n be the family of all equivalence classes of R_n; its members will be denoted by $A_{n-1}, B_{n-1}, \dots$, etc. Moreover, we set $E_1 = \{\emptyset\}$.

Properties of equivalence classes

In the following we assume $\sim R_{n-1}(a_1 \dots a_{n-1})$.

T.1.5. $a_1, \dots, a_{n-1} \in [a_1 \dots a_{n-1}]$.

Proof. By T.1.2, $R_n(a_1 \dots a_{n-1} a_i)$ for $i = 1, \dots, n-1$. $\blacksquare$

T.1.6. $x_1, \dots, x_n \in [a_1 \dots a_{n-1}] \Rightarrow R_n(x_1 \dots x_n)$.

Proof. $\sim R_{n-1}(a_1 \dots a_{n-1}) \wedge \bigwedge_{i=1}^{n} R_n(a_1 \dots a_{n-1} x_i) \underset{\Pi_n}{\Rightarrow} R_n(x_1 \dots x_n)$. ■

T.1.7. $\sim R_{n-1}(x_1 \dots x_{n-1}) \wedge x_1 \dots x_{n-1} \in [a_1 \dots a_{n-1}] \wedge R_n(x_1 \dots x_n)$
$\Rightarrow x_n \in [a_1 \dots a_{n-1}]$.

See T.1.4. ■

T.1.8. $\sim R_{n-1}(b_1 \dots b_{n-1}) \wedge b_1 \dots b_{n-1} \in [a_1 \dots a_{n-1}] \Rightarrow [a_1 \dots a_{n-1}]$
$\subset [b_1 \dots b_{n-1}]$.

Proof. Let $x \in [a_1 \dots a_{n-1}]$.

$$\sim R_{n-1}(a_1 \dots a_{n-1}) \wedge \bigwedge_{i=1}^{n-1} R_n(a_1 \dots a_{n-1} b_i) \wedge R_n(a_1 \dots a_{n-1} x)$$
$$\underset{\Pi_n}{\Rightarrow} R_n(b_1 \dots b_{n-1} x) \Rightarrow x \in [b_1 \dots b_{n-1}]. \quad ■$$

T.1.9. $[a_1 \dots a_{n-1}] \subset [b_1 \dots b_{n-1}] \Rightarrow [a_1 \dots a_{n-1}] = [b_1 \dots b_{n-1}]$.

Proof. $a_1, \dots, a_{n-1} \underset{\text{T.1.5}}{\in} [a_1 \dots a_{n-1}] \subset [b_1 \dots b_{n-1}]$. Therefore

$$\sim R_{n-1}(a_1 \dots a_{n-1}) \wedge a_1, \dots, a_{n-1} \in [b_1 \dots b_{n-1}]$$
$$\underset{\text{T.1.8}}{\Rightarrow} [b_1 \dots b_{n-1}] \subset [a_1 \dots a_{n-1}]. \quad ■$$

T.1.10. $\sim R_{n-1}(b_1 \dots b_{n-1}) \wedge b_1, \dots, b_{n-1} \in [a_1 \dots a_{n-1}]$
$\Rightarrow [a_1 \dots a_{n-1}] = [b_1 \dots b_{n-1}]$.

See T.1.8 and T.1.9. ■

T.1.11. $[a_1 \dots a_{n-2}] \subset [a_1 \dots a_{n-1}]$.

Proof. Let $x \in [a_1 \dots a_{n-2}]$.

$$R_{n-1}(a_1 \dots a_{n-2} x) \underset{\Pi_n}{\Rightarrow} R_n(a_1 \dots a_{n-2} a_{n-1} x) \underset{\text{def}}{\Rightarrow} x \in [a_1 \dots a_{n-1}]. \quad ■$$

T.1.12. $\sim R_{k-1}(b_1 \dots b_{k-1}) \wedge b_1, \dots, b_{k-1} \in [a_1 \dots a_{n-1}]$
$\Rightarrow [b_1 \dots b_{k-1}] \subset [a_1 \dots a_{n-1}]$ *for* $k = 3, \dots, n$.

Proof by induction (downward) on k. In case $k = n$, we apply T.1.8 and T.1.9. If $R_{k-1}(b_1 \dots b_{k-2} a_i)$ for all i, $1 \leqslant i \leqslant k-1$, then

$$\sim R_{k-2}(b_1 \dots b_{k-2}) \wedge \bigwedge_{i=1}^{k-1} R_{k-1}(b_1 \dots b_{k-2} a_i) \underset{\Pi_{k-1}}{\Rightarrow} R_{k-1}(a_1 \dots a_{k-1})$$
$$\underset{\text{T.1.1}}{\Rightarrow} R_{n-1}(a_1 \dots a_{n-1}),$$

a contradiction.

So, we may assume $\sim R_{k-1}(b_1 \dots b_{k-2} a_l)$ for some l, $1 \leqslant l \leqslant k-1$. Then

$$[b_1 \dots b_{k-2}] \underset{\text{T.1.11}}{\subset} [b_1 \dots b_{k-2} a_l].$$

But $b_1, \ldots, b_{k-2}, a_l \in [a_1 \ldots a_{n-1}]$ by assumption and T.1.5. Hence

$$\begin{aligned} \sim R_{k-1}(b_1 \ldots b_{k-2} a_l) \wedge b_1, \ldots, b_{k-2}, a_l \in [a_1 \ldots a_{n-1}] \\ \underset{\text{Ind}}{\Rightarrow} [b_1 \ldots b_{k-2} a_l] \subset [a_1 \ldots a_{n-1}] \\ \underset{\text{T.1.11}}{\Rightarrow} [b_1 \ldots b_{k-2}] \subset [b_1 \ldots b_{k-2} a_l] \subset [a_1 \ldots a_{n-1}]. \quad \blacksquare \end{aligned}$$

DEFINITION. Let $X \subset S$.

$$\varrho(X) = k \Leftrightarrow \begin{cases} k = 0 \text{ and } X = \emptyset, \\ k = 1 \text{ and } X = \{x\}, \\ k > 1 \text{ and } R_k|_X \neq X^k \text{ and } R_{k+1}|_X = X^{k+1}. \end{cases}$$

The formula $\varrho(X) = k$ may be read as: the set X is of rank k. If there is no natural number k such that $\varrho(X) = k$, then we say that X is of infinite rank and we write $\varrho(X) = \infty$.

Properties

T.1.13. $X \subset Y \Rightarrow \varrho(X) \leqslant \varrho(Y)$.

T.1.14. $\varrho(A_n) = n$.

T.1.15. $\varrho(A_m \cap A_n) \leqslant \min(m, n)$.

T.1.16. $\varrho(A_m \cap A_n) = k \Rightarrow A_m \cap A_n \in E_{k+1}$.

Proof. We write A for A_m and B for A_n

$$\begin{aligned} k = 0 &\Rightarrow A \cap B = \emptyset \Rightarrow A \cap B \in E_1, \\ k = 1 &\Rightarrow A \cap B = \{c\} \Rightarrow A \cap B \in E_2, \\ k > 1 &\Rightarrow R_k|_{A \cap B} \neq (A \cap B)^k \wedge R_{k+1}|_{A \cap B} = (A \cap B)^{k+1}. \end{aligned}$$

Now

$$R_k|_{A \cap B} \neq (A \cap B)^k \Rightarrow \bigvee_{c_1, \ldots, c_k} (c_1, \ldots, c_k \in A \cap B \wedge \sim R_k(c_1 \ldots c_k)).$$

Hence

$$\sim R_k(c_1 \ldots c_k) \wedge c_1, \ldots, c_k \in A \cap B \underset{\text{T.1.12}}{\Rightarrow} [c_1 \ldots c_k] \subset A \cap B.$$

Also

$$\begin{aligned} x \in A \cap B \wedge R_{k+1}|_{A \cap B} = (A \cap B)^{k+1} \wedge c_1, \ldots, c_k \in A \cap B \\ \Rightarrow R_{k+1}(c_1 \ldots c_k x) \underset{\text{df}}{\Rightarrow} x \in [c_1 \ldots c_k]. \end{aligned}$$

Therefore $A \cap B = \underset{\text{minimum}}{[c_1 \ldots c_k]} \in E_{k+1}$. $\blacksquare$

C.1.17. $A_m \cap A_n \in \bigcup_{k=1} E_{k+1}$.

§2

Let us consider the following statements ($n = 3, \ldots$):

I_n (*Reflexivity law*)

$$R_{n-1}(x_1 \ldots x_{i-1}\, x_{i+1} \ldots x_n) \Rightarrow R_n(x_1 \ldots x_n) \quad \text{for} \quad i = 1, \ldots, n.$$

I_n^w (*Weak reflexivity law*)

$$R_{n-1}(x_1 \ldots x_{n-1}) \Rightarrow R_n(x_1 \ldots x_n).$$

II_n (*Transitivity law*)

$$\sim R_{n-1}(a_1 \ldots a_{n-1}) \wedge \bigwedge_{i=1}^{n} R_n(a_1 \ldots a_{n-1}\, x_i) \Rightarrow R_n(x_1 \ldots x_n).$$

II_n^w (*Weak transitivity law*)

$$\sim R_{n-1}(a_1 \ldots a_{n-1}) \wedge R_n(a_1 \ldots a_{n-1}\, p) \wedge R_n(a_1 \ldots a_{n-1}\, q) \Rightarrow R_n(a_2 \ldots a_{n-1}\, pq).$$

III_n (*Symmetry law*)

$$R_n(x_1 \ldots x_n) \Rightarrow R_n(x_{l_1} \ldots x_{l_n})$$

for each permutation $(l_1, \ldots, l_n)$ of the set $\{1, \ldots, n\}$.

III_n^w (*Weak symmetry law*)

$$R_n(x_1 \ldots x_n) \Rightarrow R_n(x_1 \ldots x_{i-1}\, x_{i+1}\, x_i\, x_{i+2} \ldots x_n) \quad \text{for} \quad i = 1, \ldots, n-1.$$

T.2.1. $\bigwedge_{j=3}^{n} (\mathrm{I}_j \wedge \mathrm{II}_j) \Rightarrow \mathrm{I}_n^w \wedge \mathrm{II}_n^w \wedge \mathrm{III}_n^w$.

Proof of I_n^w. $\mathrm{I}_n \Rightarrow \mathrm{I}_n^w$.

Proof of II_n^w.

$$\sim R_{n-1}(a_1 \ldots a_{n-1}) \wedge \bigwedge_{i=2}^{n-1} R_n(a_1 \ldots a_{n-1}\, a_i) \wedge R_n(a_1 \ldots a_{n-1}\, p) \wedge$$

$$\wedge R_n(a_1 \ldots a_{n-1}\, q) \underset{\mathrm{II}_n}{\underset{\mathrm{T.1.2}}{\Rightarrow}} R_n(a_2 \ldots a_{n-1}\, pq).$$

Proof of III_n^w.

$n = 2$:

$$x_1 = x_2 \Rightarrow x_2 = x_1.$$

$n \geqslant 2$:

Case 1. $i < n$,

$$R_n(x_1 \ldots x_n) \underset{\mathrm{Ind}}{\Rightarrow} R_n(x_1 \ldots x_{i+1}\, x_i \ldots x_n) \underset{\mathrm{I}_{n+1}}{\Rightarrow} R_{n+1}(x_1 \ldots x_{i+1}\, x_i \ldots x_n\, x_{n+1}),$$

$$\sim R_n(x_1 \ldots x_n) \wedge \bigwedge_{k=1}^{i-1} R_{n+1}(x_1 \underset{\mathrm{T.1.2}}{\ldots} x_n\, x_k) \wedge R_{n+1}(x_1 \ldots x_n\, x_{i+1}) \wedge$$

$$\wedge R_{n+1}(x_1 \dots x_n x_i) \underset{\text{T.1.2}}{\wedge} \bigwedge_{k=i+2}^{n} R_{n+1}(x_1 \dots x_n x_k) \underset{\text{T.1.2}}{\wedge} R_{n+1}(x_1 \dots x_{n+1}) \underset{\text{assumption}}{}$$

$$\underset{\mathrm{II}_{n+1}}{\Rightarrow} R_{n+1}(x_1 \dots x_{i+1} x_i \dots x_{n+1}).$$

Case 2. $i = n$.

$$R_n(x_1 \dots x_n) \underset{\mathrm{I}_{n+1}}{\Rightarrow} R_{n+1}(x_1 \dots x_{n-1} x_{n+1} x_n),$$

$$\sim R_n(x_1 \dots x_n) \wedge \bigwedge_{k=1}^{n-1} R_{n+1}(x_1 \dots x_n x_k) \underset{\text{T.1.2}}{\wedge} R_{n+1}(x_1 \dots x_{n+1}) \underset{\text{assumption}}{\wedge} R_{n+1}(x_1 \dots x_n x_n) \underset{\text{T.1.2}}{}$$

$$\underset{\mathrm{II}_{n+1}}{\Rightarrow} R_{n+1}(x_1 \dots x_{n-1} x_{n+1} x_n). \quad \blacksquare$$

T.2.2. $\mathrm{III}_n \Leftrightarrow \mathrm{III}_n^w$.

Proof – immediate.

T.2.3. $\bigwedge_{j=3}^{n} (\mathrm{I}_j^w \wedge \mathrm{II}_j^w \wedge \mathrm{III}_j^w) \Rightarrow \mathrm{I}_n \wedge \mathrm{II}_n$ *for* $n = 3, \dots$

Proof of I_n.

$$R_{n-1}(x_1 \dots x_{i-1} x_{i+1} \dots x_n) \underset{\mathrm{I}_n^w}{\Rightarrow} R_n(x_1 \dots x_{i-1} x_{i+1} \dots x_n x_i) \underset{\mathrm{III}_n^w, \text{T.2.2}}{\Rightarrow} R_n(x_1 \dots x_n).$$

Proof of II_n.

$$n = 2, \quad a_1 = x_1 \wedge a_1 = x_2 \Rightarrow x_1 = x_2.$$

Assume

$$(1) \quad \sim R_{n-1}(a_1 \dots a_{n-1}) \wedge \bigwedge_{i=1}^{n} R_n(a_1 \dots a_{n-1} x_i) \Rightarrow R_n(x_1 \dots x_n) \quad (n \geqslant 2).$$

We have to show

$$(2) \qquad \sim R_n(a_1 \dots a_n) \wedge \bigwedge_{i=1}^{n+1} R_{n+1}(a_1 \dots a_n x_i) \Rightarrow R_{n+1}(x_1 \dots x_{n+1}).$$

Case 1. $R_n(x_1 \dots x_n)$.

$$R_n(x_1 \dots x_n) \underset{\mathrm{I}_{n+1}^w}{\Rightarrow} R_{n+1}(x_1 \dots x_{n+1}).$$

Case 2. $\sim R_n(x_1 \dots x_n)$.

LEMMA. *There exists a permutation* $(l_1 \dots l_n)$ *of the set* $\{1, \dots, n\}$ *such that*

$$\sim R_n(a_k \dots a_n x_{l_1} \dots x_{l_{k-1}}) \quad \textit{for} \quad k = 1, \dots, n+1.$$

This lemma is a particular case of the following sentence:

There is a sequence $(l_1, \dots, l_m)$ *of natural numbers satisfying the conditions*:

$$(3) \qquad \neq (l_1 \dots l_m) \quad \textit{and} \quad 1 \leqslant l_1, \dots, l_m \leqslant n$$

and

(4) $$\sim R_n(a_k \dots a_n x_{l_1} \dots x_{l_{k-1}}) \quad \textit{for} \quad k = 1, \dots, m+1.$$

We shall prove this sentence by induction on m ($m = 0, 1, \dots, n$)

$$m = 0, \quad \sim R_n(a_1 \dots a_n).$$

Assume (3) and (4) for a certain $m < n$.

In particular, we get

$$\sim R_n(a_{m+1} \dots a_n x_{l_1} \dots x_{l_m}),$$

which implies (by I_n)

$$\sim R_{n-1}(a_{m+2} \dots a_n x_{l_1} \dots x_{l_m}).$$

Thus

$$\sim R_{n-1}(a_{m+2} \dots a_n x_{l_1} \dots x_{l_m}) \wedge \bigwedge_{i=1}^{n} R_n(a_{m+2} \dots a_n x_{l_1} \dots x_{l_m} x_i) \underset{\text{ind}}{\Rightarrow} R_n(x_1 \dots x_n)$$

a contradiction.

Hence $\sim \bigwedge_{i=1}^{n} R_n(a_{m+2} \dots a_n x_{l_1} \dots x_{l_m} x_i)$ and, by T.1.2,

$$R_n(a_{m+2} \dots a_n x_{l_1} \dots x_{l_m} x_i) \quad \text{for} \quad i = l_1, \dots, l_m$$

(notice that T.1.2 is based on I_n only). In conclusion, for some $l_{m+1} \neq l_1, \dots, l_m$, $1 \leqslant l_{m+1} \leqslant n$, we have

$$\sim R_n(a_{m+2} \dots a_n x_{l_1} \dots x_{l_{m+1}}).$$

Thus from (3) and (4) it follows that the sequence $(l_1 \dots l_{m+1})$ satisfies the conditions

$$\neq (l_1 \dots l_{m+1}) \quad \text{and} \quad 1 \leqslant l_1 \dots l_{m+1} \leqslant n$$

and

$$\sim R_n(a_k \dots a_n x_{l_1} \dots x_{l_{k-1}}) \quad \text{for} \quad k = 1, \dots, m+2.$$

Setting $m = n$ we obtain the lemma.

In view of III_n^w and T.2.2 we can assume that

$$(l_1, \dots, l_n) = (1, \dots, n).$$

Hence

(5) $$\sim R_n(a_k \dots a_n x_1 \dots x_{k-1}) \quad \text{for} \quad k = 1, \dots, n+1.$$

Sentence (2) is a particular case of the sentence

(6) $$\sim R_n(a_1 \dots a_n) \wedge \bigwedge_{i=1}^{k} R_{n+1}(a_1 \dots a_n x_i) \Rightarrow R_{n+1}(a_k \dots a_n x_1 \dots x_k),$$

which will be proved by induction on k ($k = 1, \ldots, n+1$)

$$k = 1, \quad \sim R_n(a_1 \ldots a_n) \wedge R_{n+1}(a_1 \ldots a_n x_1) \Rightarrow R_{n+1}(a_1 \ldots a_n x_1),$$

$$1 \leqslant k \leqslant n, \quad \sim R_n(a_1 \ldots a_n) \wedge \bigwedge_{i=1}^{k+1} R_{n+1}(a_1 \ldots a_n x_i)$$

$$\underset{\text{Ind}}{\Rightarrow} R_{n+1}(a_k \ldots a_n x_1 \ldots x_k) \wedge R_{n+1}(a_k \ldots a_n x_1 \ldots x_{k-1} x_{k+1}),$$

$$\underset{(5)}{\sim} R_n(a_k \ldots a_n x_1 \ldots x_{k-1}) \wedge R_{n+1}(a_k \ldots a_n x_1 \ldots x_k) \wedge$$

$$\wedge R_{n+1}(a_k \ldots a_n x_1 \ldots x_{k-1} x_{k+1})$$

$$\underset{\mathrm{II}^w_{n+1}}{\Rightarrow} R_{n+1}(a_{k+1} \ldots a_n x_1 \ldots x_{k+1}).$$

Setting $k = n+1$ in (6), we obtain II_n. ∎

As an immediate consequence of Theorems T.2.1 and T.2.3 we obtain

C.2.4. *Let S be a non-empty set, and let (R_n) be a sequence of relations $R_2, \ldots$ where $R_n \subset S^n$ and R_2 is the identity relation. Then,*

$$(R_n) \text{ is an E-sequence} \Leftrightarrow \mathrm{I}^w_n \wedge \mathrm{II}^w_n \wedge \mathrm{III}^w_n \text{ for } n = 3, \ldots$$

§3

DEFINITION. Let R be a ternary equivalence relation on a non-empty set S. The relational sequence (R_n): $=, R_3, R_4, \ldots$ is said to be *generated by* R if $R_3 = R$ and for $n = 4, \ldots$

$$R_n(a_1, \ldots, a_n) \underset{\text{def}}{\Leftrightarrow} \bigvee_{(l_1, \ldots, l_n)} \bigvee_x [R_{n-1}(a_{l_1} \ldots a_{l_{n-2}} x) \wedge R(a_{l_{n-1}} a_{l_n} x)],$$

where $(l_1, \ldots, l_n)$ is a permutation of the set $\{1, \ldots, n\}$.

T.3.1. *The relational sequence* (R_n) *generated by a ternary equivalence R is an E-sequence if and only if it satisfies*

II^{ww}_n (*weakest transitivity law*)

$$a_1 \neq a_2 \wedge R(a_1 a_2 p) \wedge R_n(a_1 \ldots a_n) \Rightarrow R_n(p a_2 \ldots a_n).$$

Proof. Assume II^{ww}_n for $n = 3, \ldots$ We show that the sequence (R_n) satisfies condition I^w_n.

$$R_{n-1}(a_1 \ldots a_{n-1}) \wedge R(a_{n-1} a_n a_{n-1}) \underset{\text{def}}{\overset{\text{T.1.2}}{\Rightarrow}} R_n(a_1 \ldots a_n).$$

(We take $(l_1, \ldots, l_n) = (1, \ldots, n)$ and $x = a_{n-1}$.)

The proof of III^w_n is immediate(²).

(²) Notice that an arbitrary sequence generated by a ternary equivalence R satisfies conditions I^w_n and III^w_n, and consequently also I_n and III_n.

We next show that the sequence (R_n) satisfies II_n^w. Condition II_n^w may be considered as a special case of the statements

$$\mathrm{II}_{n,k}^w \quad \sim R_{k-1}(a_1 \dots a_{k-1}) \wedge R_k(a_1 \dots a_{k-1}\, p) \wedge R_n(a_1 \dots a_{n-1}\, q) \Rightarrow R_n(a_2 \dots a_{n-1}\, pq),$$

which will be proved by double induction on n and k for $n = 3, \dots$ and $k = 3, \dots, n$.

$\mathrm{II}_{3,3}^w = \mathrm{II}_3^{ww}$. Assume $\mathrm{II}_{n,k}^w$ for some $n \geqslant 3$ and $k = 3, \dots, n$, and prove $\mathrm{II}_{n+1,k}^w$ for $k = 3, \dots, n+1$.

$\mathrm{II}_{n+1,3}^w = \mathrm{II}_{n+1}^{ww}$. Assume $\mathrm{II}_{n+1,k}^w$ for a certain $k < n+1$. To prove $\mathrm{II}_{n+1,k+1}^w$, suppose

$$\sim R_k(a_1 \dots a_k) \wedge R_{k+1}(a_1 \dots a_k\, p) \wedge R_{n+1}(a_1 \dots a_n\, q).$$

We have to show that $R_{n+1}(a_2 \dots a_n\, pq)$. Now

$$\begin{aligned} R_{k+1}(a_1 \dots a_k\, p) \Leftrightarrow & \bigvee_x \big(R(a_1\, px) \wedge R_k(a_2 \dots a_k\, x)\big) \vee && \text{(Case 1)} \\ \vee & \bigvee_x \big(R(a_2\, px) \wedge R_k(a_1\, a_3 \dots a_k\, x)\big) \vee && \text{(Case 2)} \\ \vee & \bigvee_x \big(R(a_1\, a_2\, x) \wedge R_k(a_3 \dots a_k\, px)\big) \vee && \text{(Case 3)} \\ \vee & \bigvee_x \big(R(a_2\, a_3\, x) \wedge R_k(a_1\, a_4 \dots a_k\, px)\big) \vee && \text{(Case 4)} \\ \vee & \Phi(a_1 \dots a_k\, p) \end{aligned}$$

where $\Phi(a_1 \dots a_k\, p)$ is the disjunction of all the remaining cases, which may be omitted in our proof in view of the already proved transitivity laws III_n^w for $n = 3, \dots$

Case 1. $\sim R_k(a_1 \dots a_k) \wedge R(a_1\, px) \wedge R_k(a_2 \dots a_k\, x) \wedge R_{n+1}(a_1 \dots a_n\, q)$.

If $a_1 = x$, then $R_k(a_2 \dots a_k\, x) \Rightarrow R_k(a_1 \dots a_k)$, a contradiction. Hence

$$a_1 \neq x. \tag{1}$$

Case 1.1. $R_{k-1}(a_3 \dots a_k\, x)$. Then

$$R_{k-1}(a_3 \dots a_k\, x) \wedge R(a_1\, px) \underset{\mathrm{Def}}{\Rightarrow} R_k(a_1\, a_3 \dots a_k\, p),$$

$$\sim R_k(a_1 \dots a_k) \wedge R_k(a_1\, a_3 \dots a_k\, p) \wedge R_{n+1}(a_1 \dots a_n\, q) \underset{\mathrm{Ind}}{\Rightarrow} R_{n+1}(a_2 \dots a_n\, pq).$$

Case 1.2. $\sim R_{k-1}(a_3 \dots a_k\, x)$. Then

$$\sim R_k(a_1 \dots a_k) \wedge R_k(a_2 \dots a_k\, x) \wedge R_{n+1}(a_1 \dots a_n\, q) \underset{\mathrm{Ind}}{\Rightarrow} R_{n+1}(a_1\, a_3 \dots a_n\, qx),$$

$$a_1 \underset{(1)}{\neq} x \wedge R(a_1\, px) \wedge R_{n+1}(a_1\, a_3 \dots a_n\, qx) \underset{\mathrm{II}_{n+1}^{ww}}{\Rightarrow} R_{n+1}(a_3 \dots a_n\, pqx),$$

$$\sim R_{k-1}(a_3 \dots a_k\, x) \wedge R_k(a_2 \dots a_k\, x) \wedge R_{n+1}(a_3 \dots a_n\, pqx) \underset{\mathrm{Ind}}{\Rightarrow} R_{n+1}(a_2 \dots a_n\, pq).$$

Case 2. $\sim R_k(a_1 \dots a_k) \wedge R(a_2 px) \wedge R_k(a_1 a_3 \dots a_k x) \wedge R_{n+1}(a_1 \dots a_n q)$.

If $a_2 = x$, then $R_k(a_1 a_3 \dots a_k x) \Rightarrow R_k(a_1 \dots a_k)$, a contradiction. Therefore

(2) $$a_2 \neq x.$$

$$\sim R_k(a_1 \dots a_k) \wedge R_k(a_1 a_3 \dots a_k x) \wedge R_{n+1}(a_1 \dots a_n q) \underset{\text{Ind}}{\Rightarrow} R_{n+1}(a_2 \dots a_n qx),$$

$$a_2 \underset{(2)}{\neq} x \wedge R(a_2 px) \wedge R_{n+1}(a_2 \dots a_n qx) \underset{\Pi^{ww}_{n+1}}{\Rightarrow} R_{n+1}(a_2 \dots a_n pq).$$

Case 3. $\sim R_k(a_1 \dots a_k) \wedge R(a_1 a_2 x) \wedge R_k(a_3 \dots a_k px) \wedge R_{n+1}(a_1 \dots a_n q)$.

$$R_{k-1}(a_3 \dots a_k x) \wedge R(a_1 a_2 x) \underset{\text{Def}}{\Rightarrow} R_k(a_1 \dots a_k),$$

a contradiction.

Therefore

(3) $$\sim R_{k-1}(a_3 \dots a_k x).$$

$$\sim R_k(a_1 \dots a_k) \wedge R(a_1 a_2 x) \wedge R_{n+1}(a_1 \dots a_n q) \underset{\Pi^{ww}_{n+1}}{\Rightarrow} R_{n+1}(a_2 \dots a_n qx),$$

$$\sim R_{k-1}(a_3 \underset{(3)}{\dots} a_k x) \wedge R_k(a_3 \dots a_k px) \wedge R_{n+1}(a_2 \dots a_n qx) \underset{\text{Ind}}{\Rightarrow} R_{n+1}(a_2 \dots a_n pq).$$

Case 4. $\sim R_k(a_1 \dots a_k) \wedge R(a_2 a_3 x) \wedge R_k(a_1 a_4 \dots a_k px) \wedge R_{n+1}(a_1 \dots a_n q)$.

$$R_{k-1}(a_1 a_4 \dots a_k x) \wedge R(a_2 a_3 x) \underset{\text{Def}}{\Rightarrow} R_k(a_1 \dots a_k),$$

a contradiction.

Therefore

(4) $$\sim R_{k-1}(a_1 a_4 \dots a_k x).$$

Case 4.1. $a_3 = x$.

$$\sim R_k(a_1 \dots a_k) \wedge R_k(a_1 a_3 \dots a_k p) \wedge R_{n+1}(a_1 \dots a_n q) \underset{\text{Ind}}{\Rightarrow} R_{n+1}(a_2 \dots a_n pq).$$

Case 4.2. $a_2 \neq x$.

$$\sim R_k(a_1 \dots a_k) \wedge R(a_2 a_3 x) \wedge R_{n+1}(a_1 \dots a_n q) \underset{\Pi^{ww}_{n+1}}{\Rightarrow} R_{n+1}(a_1 a_3 \dots a_n qx),$$

$$\sim R_{k-1}(a_1 a_4 \underset{(4)}{\dots} a_k x) \wedge R_k(a_1 a_4 \dots a_k px) \wedge R_{n+1}(a_1 a_3 \dots a_n qx) \underset{\text{Ind}}{\Rightarrow} R_{n+1}(a_3 \dots a_n pqx),$$

$$a_3 \neq x \wedge R(a_2 a_3 x) \wedge R_{n+1}(a_3 \dots a_n pqx) \underset{\Pi^{ww}_{n+1}}{\Rightarrow} R_{n+1}(a_2 \dots a_n pq).$$

Setting $k = n$ in $\Pi^{w}_{n,k}$, we obtain Π^{w}_{n}.

BU W

By means of C.2.4. it follows immediately from our considerations that the sequence (R_n) is an E-sequence. Consequently, II_n^{ww} for $n = 3, \ldots$ is a sufficient condition.

Assume conversely that (R_n) is an E-sequence. By virtue of C.2.4 and T.2.2 the sequence (R_n) satisfies conditions I_n^w, II_n^w and III_n for $n = 3, \ldots$ We shall now show by induction that (R_n) satisfies also condition $\mathrm{II}_{n,k}^w$ for $n = 3, \ldots$ and $k = 3, \ldots, n$. For $k = n$ we have $\mathrm{II}_{n,n}^w = \mathrm{II}_n^w$.

Assume $\mathrm{II}_{n,k}^w$ for a certain k, $3 < k \leqslant n$, and suppose

$$\sim R_{k-2}(a_1 \ldots a_{k-2}) \wedge R_{k-1}(a_1 \ldots a_{k-2}\, p) \wedge R_n(a_1 \ldots a_{n-1}\, q).$$

Case 1. $R_{k-1}(a_1 \ldots a_{k-1})$.

$$\sim R_{k-2}(a_1 \ldots a_{k-2}) \wedge R_{k-1}(a_1 \ldots a_{k-1}) \wedge R_{k-1}(a_1 \ldots a_{k-2}\, p)$$
$$\underset{\mathrm{II}_{k-1}^w}{\Rightarrow} R_{k-1}(a_2 \ldots a_{k-1}\, p) \underset{\mathrm{T.1.1,III}_n}{\Rightarrow} R_n(a_2 \ldots a_{n-1}\, pq).$$

Case 2. $\sim R_{k-1}(a_1 \ldots a_{k-1})$.

$$R_{k-1}(a_1 \ldots a_{k-2}\, p) \underset{\mathrm{I}_k^w, \mathrm{III}_k}{\Rightarrow} R_k(a_1 \ldots a_{k-1}\, p),$$

$$\sim R_{k-1}(a_1 \ldots a_{k-1}) \wedge R_k(a_1 \ldots a_{k-1}\, p) \wedge R_n(a_1 \ldots a_{n-1}\, q) \underset{\mathrm{Ind}}{\Rightarrow} R_n(a_2 \ldots a_{n-1}\, pq).$$

Setting $k = 3$ in $\mathrm{II}_{n,k}^w$, we obtain II_n^{ww}. Hence II_n^{ww} for $n = 3, \ldots$ is also a necessary condition for the sequence (R_n) generated by R to be an E-sequence. ∎

§4

T.4.1. $a_1, \ldots, a_m \in B_{m+k} \wedge \sim R_m(a_1 \ldots a_m)$

$$\Rightarrow \bigvee_{a_{m+1}, \ldots, a_{m+k}} (a_{m+1}, \ldots, a_{m+k} \in B_{m+k} \wedge \sim R_{m+k}(a_1 \ldots a_{m+k}))$$

$$m = 1, \ldots, \quad k = 0, 1, \ldots$$

Proof by induction on k. The case $k = 0$ is obvious.

Let $k \geqslant 0$ and $B_{m+k+1} = [b_1 \ldots b_{m+k+1}]$.

Case 1. $\bigwedge_{i=1}^{m+1} R_{m+1}(a_1 \ldots a_m\, b_i)$.

$$\bigwedge_{i=1}^{m+1} R_{m+1}(a_1 \ldots a_m\, b_i) \wedge \sim R(a_1 \ldots a_m)$$
$$\underset{\mathrm{II}_{m+1}}{\Rightarrow} R_{m+1}(a_1 \ldots b_{m+1}) \Rightarrow R_{m+k+1}(b_1 \ldots b_{m+k+1}),$$

a contradiction.

Case 2. $\sim R_{m+1}(a_1 \dots a_m b_i)$ for a certain i, $1 \leqslant i \leqslant m+1$.
Let $a_{m+1} = b_i$. Then

$$a_1, \dots, a_{m+1} \in B_{(m+1)+k} \wedge \sim R_{m+1}(a_1 \dots a_{m+1})$$
$$\underset{\text{Ind}}{\Rightarrow} \bigvee_{a_{(m+1)+1}, \dots, a_{(m+1)+k}} (a_{(m+1)+1}, \dots, a_{(m+1)+k} \in B_{(m+1)+k} \wedge$$
$$\wedge \sim R_{m+k+1}(a_1 \dots a_{(m+1)+k})). \quad \blacksquare$$

C.1.10 and T.4.1 imply

T.4.2. $a_1 \dots a_m \in B_{m+k} \wedge \sim R_m(a_1 \dots a_m)$
$$\Rightarrow \bigvee_{a_{m+1} \dots a_{m+k}} B_{m+k} = [a_1 \dots a_{m+k}], \quad m, k = 1, \dots$$

T.4.3. $\sim R_{m+1}(xa_1 \dots a_m) \wedge R_{m+k+1}(xa_1 \dots a_m b_1 \dots b_k) \wedge R_{m+2}(xa_1 \dots a_m c)$
$$\Rightarrow R_{m+k+1}(a_1 \dots a_m b_1 \dots b_k c), \quad m = 0, \dots, \quad k = 1, \dots$$

Proof by induction on k.
$k = 1$.

$$\sim R_{m+1}(xa_1 \dots a_m) \wedge R_{m+2}(xa_1 \dots a_m b_1) \wedge R_{m+2}(xa_1 \dots a_m c)$$
$$\underset{\mathrm{II}^w_{m+2}}{\Rightarrow} R_{m+1+1}(a_1 \dots a_m b_1 c).$$

$k \geqslant 1$.
Case 1. $R_{m+k+1}(xa_1 \dots a_m b_1 \dots b_k)$.

$$\sim R_{m+1}(xa_1 \dots a_m) \wedge R_{m+k+1}(xa_1 \dots a_m b_1 \dots b_k) \wedge R_{m+2}(xa_1 \dots a_m c)$$
$$\underset{\text{Ind}}{\Rightarrow} R_{m+k+1}(a_1 \dots a_m b_1 \dots b_k c) \underset{\mathrm{I}_{m+k+2}}{\Rightarrow} R_{m+k+2}(a_1 \dots a_m b_1 \dots b_{k+1} c).$$

Case 2. $\sim R_{m+k+1}(xa_1 \dots a_m b_1 \dots b_k)$.

$$\sim R_{m+k+1}(xa_1 \dots a_m b_1 \dots b_k) \wedge R_{m+k+2}(xa_1 \dots a_m b_1 \dots b_{k+1}) \wedge$$
$$\wedge R_{m+k+2}(xa_1 \dots a_m b_1 \dots b_k c) \underset{\mathrm{II}^w_{m+k+2}}{\Rightarrow} R_{m+k+2}(a_1 \dots a_m b_1 \dots b_{k+1} c). \quad \blacksquare$$

T.4.4. $\sim R_{m+1}(xa_1 \dots a_m) \wedge R_{m+k+1}(xa_1 \dots a_m b_1 \dots b_k) \wedge$
$$\wedge R_{m+l+1}(xa_1 \dots a_m c_1 \dots c_l) \Rightarrow R_{m+k+l}(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l),$$
$$k, l = 1, 2, \dots, \quad m = 0, \dots$$

Proof by induction on l. For $l = 1$ see T.4.3.
Let $l \geqslant 1$.
Case 1. $R_{m+l+1}(xa_1 \dots a_m c_1 \dots c_l)$.

$$\sim R_{m+1}(xa_1 \dots a_m) \wedge R_{m+k+1}(xa_1 \dots a_m b_1 \dots b_k) \wedge R_{m+l+1}(xa_1 \dots a_m c_1 \dots c_l)$$
$$\underset{\text{Ind}}{\Rightarrow} R_{m+k+l}(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l) \underset{\mathrm{I}_{m+k+l+1}}{\Rightarrow} R_{m+k+l+1}(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_{l+1}).$$

Case 2. $\sim R_{m+l+1}(xa_1 \dots a_m c_1 \dots c_l)$.

$$\sim R_{m+l+1}(xa_1 \dots a_m c_1 \dots c_l) \wedge R_{m+l+k+1}(xa_1 \dots a_m c_1 \dots c_l b_1 \dots b_k) \wedge$$
$$\wedge R_{m+l+2}(xa_1 \dots a_m c_1 \dots c_{l+1}) \underset{\text{T.4.3}}{\Rightarrow} R_{m+k+l+1}(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_{l+1}). \blacksquare$$

DEFINITIONS.

$$W_R(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l)$$
$$\overset{\text{Def}}{\Leftrightarrow} [(\sim R_{m+k}(a_1 \dots a_m b_1 \dots b_k) \wedge \sim R_{m+l}(a_1 \dots a_m c_1 \dots c_l) \wedge$$
$$\wedge R_{m+k+l}(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l))$$
$$\Rightarrow \bigvee_x (\sim R_{m+1}(a_1 \dots a_m x) \wedge R_{m+k+1}(a_1 \dots a_m b_1 \dots b_k x) \wedge$$
$$\wedge R_{m+l+1}(a_1 \dots a_m c_1 \dots c_l x))]$$
$$m, k, l = 1, \dots,$$

$$W_R(m, k, l) \overset{\text{Def}}{\Leftrightarrow} \bigwedge_{a_1 \dots a_m} \bigwedge_{b_1 \dots b_k} \bigwedge_{c_1 \dots c_l} W_R(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l),$$
$$m, k, l = 1, \dots,$$

$$Z_R(m, r, s, p) \overset{\text{Def}}{\Leftrightarrow} \bigwedge_{A_m B_r C_s D_p} (A_m = B_r \cap C_s \wedge B_r \cup C_s \subset D_p \Rightarrow m \geqslant r+s-p)$$
$$m, r, s, p = 1, \dots$$

Evidently, the antecedent of this implication can hold only in the case $m \leqslant r, s \leqslant p$.

T.4.5. $m = r \Rightarrow Z_R(m, r, s, p)$, $m, r, s, p = 1, \dots$

Proof. Let $m = r$. We have $0 \geqslant s-p$. Hence $m \geqslant r+s-p$. $\blacksquare$

T.4.6. $m = s \Rightarrow Z_R(m, r, s, p)$, $m, r, s, p = 1, \dots$

Proof. Let $m = s$. We have $0 \geqslant r-p$. Hence $m \geqslant r+s-p$. $\blacksquare$

T.4.7. *For* $k, l, m = 1, \dots$

$$W_R(m, k, l) \Rightarrow Z_R(m, m+k, m+l, p) \quad \textit{for} \quad p = 1, \dots$$

Proof. Assume $W_R(m, k, l)$. Let $A_m, B_{m+k}, C_{m+l}, D_p$ be arbitrary equivalence classes satisfying the condition

(1) $$A_m = B_{m+k} \cap C_{m+l} \wedge B_{m+k} \cup C_{m+l} \subset D_p.$$

Let $A_m = [a_1 \dots a_m]$. Then $a_1, \dots, a_m \in B_{m+k} \cap C_{m+l}$ and $\sim R_m(a_1 \dots a_m)$ and hence, by T.4.2, classes B_{m+k} and C_{m+l} can be represented as follows:

$$B_{m+k} = [a_1 \dots a_m b_1 \dots b_k], \quad C_{m+l} = [a_1 \dots a_m c_1 \dots c_l].$$

Moreover,

(2) $$\sim R_{m+k+l}(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l);$$

indeed,

$$\sim R_{m+k}(a_1 \dots a_m b_1 \dots b_k) \wedge \sim R_{m+l}(a_1 \dots a_m c_1 \dots c_l) \wedge$$
$$\wedge R_{m+k+l}(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l)$$
$$\underset{W_R(m,k,l)}{\Rightarrow} \bigvee_x [\sim R_{m+1}(a_1 \dots a_m x) \wedge R_{m+k+1}(a_1 \dots a_m b_1 \dots b_k x) \wedge$$
$$\wedge R_{m+l+1}(a_1 \dots a_m c_1 \dots c_l x)]$$
$$\Rightarrow \bigvee_x (x \notin A_m \wedge x \in B_{m+k} \cap C_{m+l}),$$

which contradicts (1). By (1) we have

$$a_1, \dots, a_m, b_1, \dots, b_k, c_1, \dots, c_l \in D_p.$$

By T.1.6 this condition combined with (2) implies

$$p \geqslant m+k+l,$$
$$m+p \geqslant (m+k)+(m+l),$$
$$m \geqslant (m+k)+(m+l)-p. \quad \blacksquare$$

Remark. It is easy to prove that for $k, l, m = 1, \dots$ the converse implication

$$Z_R(m, m+k, m+l, p) \Rightarrow W_R(m, k, l) \quad \text{for} \quad p = 1, \dots$$

also holds. However, we do not need it in the sequel. By T.4.5, T.4.6, and T.4.7 we obtain the following corollary:

C.4.8. *If* $W_R(m, k, l)$ *for* $m, k, l = 1, \dots,$ *then* $Z_R(m, r, s, p)$ *for* $m, r, s, p = 1, \dots$

Chapter II

Application of n-ary equivalence relations to geometry

This chapter concerns geometry, more precisely, a weak subtheory $\mathscr{A}$ of affine geometry. The theory $\mathscr{A}$ has the betweennes relation B as the only primitive notion. The axiom system of $\mathscr{A}$ describes some properties of the ordering of the line and plane. This system does not contain either the continuity axiom or Euclid's axiom; it even does not contain the dimension axioms.

We define the collinearity relation L in terms of B as usual. We show that the sequence (L_n) generated by L is already an E-sequence. This theorem enables us to define hyperplanes as equivalence classes and so to apply here

theorems on equivalence classes established in Chapter I. Consequently, basic statements on hyperplanes follow directly from the definition.

Section 1 contains the "linear" axioms of $\mathscr{A}$; in Section 2 we define the collinearity relation L and the sequence (L_n) generated by L; in Section 3 we give the "plane" axioms of $\mathscr{A}$. In Section 4 we define the interior of a simplex and establish its basic properties. These properties yield, i.a., interesting multidimensional forms of the Pasch axiom. In Section 5 the linear dependence of points is expressed by means of the interior of a simplex. On the basis of Sections 4 and 5 we prove in Section 6 that (L_m) is an E-sequence. Finally, at the end of Section 6, the dimension axioms are formulated in terms of (L_n).

§1. "Linear" axioms of $\mathscr{A}$ (3)

Axioms:

A1 $B(abc) \Rightarrow B(cba)$.

A2 $B(abc) \Rightarrow \sim B(bac)$.

A3 $B(abc) \wedge B(bcd) \Rightarrow B(abd)$.

A4 $B(abd) \wedge B(bcd) \Rightarrow B(abc)$.

A5 $B(abd) \wedge B(acd) \Rightarrow b = c \vee B(abc) \vee B(acb)$.

A6 $B(abc) \wedge B(abd) \Rightarrow c = d \vee B(bcd) \vee B(bdc)$.

Properties of the betweenness relation:

T.1.1. $B(abd) \wedge B(bcd) \Rightarrow B(acd)$ (A4, A1, A3).

T.1.2. $B(abc) \wedge B(abd) \Rightarrow c = d \vee B(acd) \vee B(adc)$ (A6, T.1.1).

T.1.3. $B(abc) \wedge B(bcd) \Rightarrow B(acd)$ (A3, T.1.1).

§2. Collinearity

DEFINITION.

$$L(a_1 a_2 a_3) \overset{\mathrm{df}}{\Leftrightarrow} a_1 = a_2 \vee a_2 = a_3 \vee a_3 = a_1 \vee B(a_1 a_2 a_3) \vee \vee B(a_2 a_3 a_1) \vee B(a_3 a_1 a_2).$$

Properties of the collinearity relation:

T.2.1. *The relation L is a ternary equivalence relation.*

Proof (see C.1.2.4).

(3) In contradistinction to the later papers by the author, in this article the betweenness relations do not contain degenerate triples (Ed.).

I_3^w $\quad a_1 = a_2 \vee a_2 = a_3 \vee a_3 = a_1 \underset{\mathrm{Def}}{\Rightarrow} L(a_1 a_2 a_3)$.

III_3^w $\quad L(a_1 a_2 a_3) \underset{\mathrm{Def.A.1}}{\Rightarrow} L(a_{l_1} a_{l_2} a_{l_3})$ for every permutation (l_1, l_2, l_3) of the set $\{1, 2, 3\}$.

II_3^w $\quad a_1 \neq a_2 \wedge L(a_1 a_2 p) \wedge L(a_1 a_2 q) \Rightarrow L(a_2 pq)$.

Case 1. $a_1 = p$.

$$a_1 = p \wedge L(a_1 a_2 q) \Rightarrow L(pa_2 q) \underset{\mathrm{III}_3^w}{\Rightarrow} L(a_2 pq).$$

Case 2. $a_2 = p$.

$$a_2 = p \underset{\mathrm{Def}}{\Rightarrow} L(a_2 pq).$$

Case 3. $a_1 = q \vee a_2 = q$.

In this case the proof is similar.

Case 4. $\neq (a_1 a_2 p) \wedge \neq (a_1 a_2 q)$.

$$\begin{aligned} \neq (a_1 a_2 p) \wedge \neq (a_1 a_2 q) \wedge L(a_1 a_2 p) \wedge L(a_1 a_2 q) \\ \underset{\mathrm{Def}}{\Rightarrow} [B(a_1 a_2 p) \vee B(a_2 pa_1) \vee B(pa_1 a_2)] \wedge \\ \wedge [B(a_1 a_2 q) \vee B(a_2 qa_1) \vee B(qa_1 a_2)] \\ \underset{\mathrm{A3-A.6,\ T.1.1,\ T.1.2,\ A1}}{\Rightarrow} p = q \vee B(a_2 pq) \vee B(pqa_2) \vee B(qa_2 p) \\ \underset{\mathrm{Def}}{\Rightarrow} L(a_2 pq). \quad \blacksquare \end{aligned}$$

Let (L_n) be the sequence generated by L. The formula $L_n(a_1 \dots a_n)$ $(\sim L_n(a_1 \dots a_n))$ may be read as: the points $a_1, \dots, a_n$ are linearly dependent (linearly independent).

Let us recall (see T.1.3.1 – remark) that the sequence (L_n) satisfies the reflexivity and transitivity laws, i.e. for $n = 3, \dots$

I_n $\quad L_{n-1}(a_1 \dots a_{i-1} a_{i+1} \dots a_n) \Rightarrow L(a_1 \dots a_n)$ for $i = 1, \dots, n$.

III_n $\quad L_n(a_1 \dots a_n) \Rightarrow L_n(a_{l_1} \dots a_{l_n})$ for every permutation $(l_1 \dots l_n)$ of the set $\{1, \dots, n\}$.

T.2.2. $L_{k+1}(a_1 \dots a_k x) \wedge L_{n-k+1}(a_{k+1} \dots a_n x) \Rightarrow L_n(a_1 \dots a_n)$ *for* $n = 3, \dots$, $k = 1, \dots, n-1$.

Proof(4) by induction on n and for a fixed n by induction on k.

By the definition of L_n, the result already holds for $n = 3, 4, 5$. Assume then that the theorem has been established for a certain n and all k, $1 \leqslant k \leqslant n-1$. We want to show that the theorem holds for $n+1$ and all k, $1 \leqslant k \leqslant n$. For $k = 1, 2$ the implication follows directly by the definition of L_{n+1}.

---

(4) In the original text T.2.2 was stated without proof. This proof is due to H. N. Gupta (Ed.).

Having established the theorem for $n+1$ and a certain k, $k \geqslant 2$, we shall show that the theorem holds for $n+1$ and $k+1$.

So, let $L_{k+2}(a_1 \dots a_{k+1} x) \wedge L_{n-k+1}(a_{k+2} \dots a_{n+1} x)$. By the definition of L_{n+1} the result already holds for $k+1 = 2$.

So, let $k+1 > 2$. By the definition of L_{k+2} we obtain two typical cases:

Case 1. $L_{k+1}(a_1 \dots a_{k-1} xy) \wedge L(a_k a_{k+1} y)$.

$$L_{k+1}(a_1 \dots a_{k-1} xy) \wedge L_{n-k+1}(a_{k+2} \dots a_{n+1} x) \underset{\mathrm{Ind}_n}{\Rightarrow} L_n(a_1 \dots a_{k-1} a_{k+2} \dots a_{n+1} y),$$

$$L_n(a_1 \dots a_{k-1} a_{k+2} \dots a_{n+1} y) \wedge L(a_k a_{k+1} y) \underset{\mathrm{Def}}{\Rightarrow} L_{n+1}(a_1 \dots a_n).$$

Case 2. $L_{k+1}(a_1 \dots a_k y) \wedge L(a_{k+1} xy)$.

$$L_{n-k+1}(a_{k+2} \dots a_{n+1} x) \wedge L(a_{k+1} xy) \underset{\mathrm{Def}}{\Rightarrow} L_{n-k+2}(a_{k+1} \dots a_{n+1} y),$$

$$L_{k+1}(a_1 \dots a_k y) \wedge L_{n-k+2}(a_{k+1} \dots a_{n+1} y) \underset{\mathrm{Ind}_k}{\Rightarrow} L_{n+1}(a_1 \dots a_{n+1}). \quad \blacksquare$$

§3. "Plane" axioms of $\mathscr{A}$

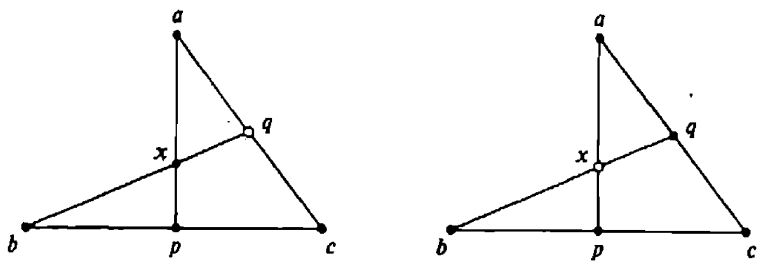
oP (*The outer form of the Pasch axiom*).
Let $\sim L(abc)$. Then

$$[B(axp) \wedge B(bpc)] \Rightarrow \bigvee_q [B(bxq) \wedge B(cqa)].$$

iP (*The inner form of the Pasch axiom*).
Let $\sim L(abc)$. Then

$$[B(cqa) \wedge B(bpc)] \Rightarrow \bigvee_x [B(bxq) \wedge B(axp)].$$

We shall use the following diagrams to express the Pasch axioms:



oP iP

The axiom oP in this form was used by Tarski in his axiom system of Euclidean geometry (see [5]). In fact, this form of the Pasch axiom is due to Schur. Consider the additional axiom

A0 $$a \neq b \Rightarrow \bigvee_c B(abc);$$

the axiom iP is dependent on the axiom system A0, A1, ..., A6. Evidently, the axiom system A1, ..., A6, oP, iP is weaker than the axiom system A0, ..., A6, oP.

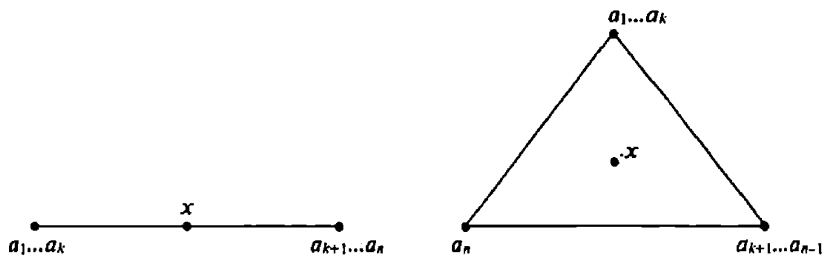
§4. Interior of a simplex

DEFINITION. Let $\sim L_n(a_1 \dots a_n)$. The *interior of an ordered* ($(n-1)$-*dimensional*) *simplex with vertices* $a_1 \dots a_n$ is understood as the set $I(a_1 \dots a_n)$ defined by the conditions:

$$\text{for } n = 2, \quad x \in I(a_1 a_2) \underset{\text{Def}}{\Leftrightarrow} B(a_1 x a_2),$$

$$\text{for } n \geqslant 3, \quad x \in I(a_1 \dots a_n) \underset{\text{Def}}{\Leftrightarrow} \bigvee_y [y \in I(a_1 \dots a_{n-1}) \wedge B(a_n x y)].$$

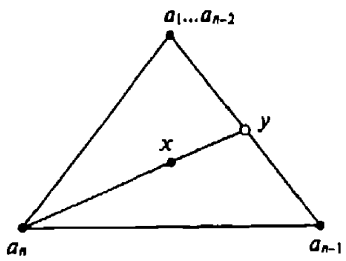
To express the formula $x \in I(a_1 \dots a_n)$ we shall use one of the following diagrams:



In fact, we need these diagrams only for $k = 1$ and $k = n-2$. When writing $I(a_1 \dots a_n)$ we always assume $\sim L_n(a_1 \dots a_n)$. We shall express the implication

$$x \in I(a_1 \dots a_n) \Rightarrow \bigvee_y [y \in I(a_1 \dots a_{n-1}) \wedge B(a_n x y)] \quad \text{for} \quad n \geqslant 3$$

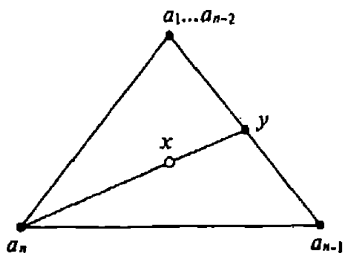
by the diagram



and the implication

$$\sim L_n(a_1 \dots a_n) \wedge y \in I(a_1 \dots a_{n-1}) \wedge B(a_n x y) \Rightarrow x \in I(a_1 \dots a_n) \quad \text{for} \quad n \geqslant 3$$

by the diagram



The last implication follows directly from the definition of the interior of a simplex.

T.4.1. $x \in I(a_1 \dots a_n) \Rightarrow L_{n+1}(a_1 \dots a_n x)$.

Proof by induction on n.

$n = 2$:

$$B(a_1 x a_2) \underset{\text{Def}}{\Rightarrow} L(a_1 a_2 x),$$

$n \geqslant 3$:

$$y \in I(a_1 \dots a_{n-1}) \wedge B(a_n x y) \underset{\text{Ind,Def}}{\Rightarrow} L_n(a_1 \dots a_{n-1} y) \wedge L(a_n x y)$$

$$\underset{\text{Def}}{\Rightarrow} L_{n+1}(a_1 \dots a_n x). \quad \blacksquare$$

T.4.2. $\sim L_n(a_1 \dots a_n) \wedge x \in I(a_1 \dots a_{n-2}) \Rightarrow \sim L(a_{n-1} a_n x)$, $n = 4, \dots$

Proof.

$$x \in I(a_1 \dots a_{n-2}) \underset{\text{T.4.1}}{\Rightarrow} L_{n-1}(a_1 \dots a_{n-2} x),$$

$$L_{n-1}(a_1 \dots a_{n-2} x) \wedge L(a_{n-1} a_n x) \underset{\text{Def}}{\Rightarrow} L_n(a_1 \dots a_n),$$

a contradiction. $\blacksquare$

T.4.3. $I(a_1 \dots a_n) \subset I(a_1 \dots a_{k-1} a_n a_{k+1} \dots a_{n-1} a_k)$, *for* $k = 1, \dots, n$.

Proof by induction on n. Let $k < n$

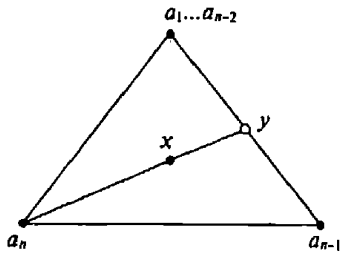
$n = 2$:

$$x \in I(a_1 a_2) \underset{\text{Def,A1}}{\Rightarrow} x \in I(a_2 a_1),$$

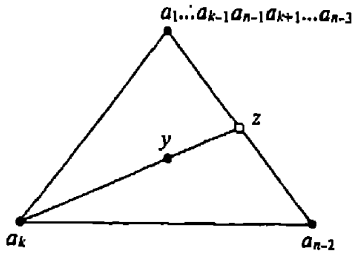
$n = 3$:

$$x \in I(a_1 a_2 a_3) \underset{\text{Def,A1}}{\Rightarrow} x \in I(a_2 a_1 a_3) \underset{\text{Def,oP}}{\Rightarrow} x \in I(a_1 a_3 a_2) \underset{\text{Def,oP}}{\Rightarrow} x \in I(a_3 a_2 a_1).$$

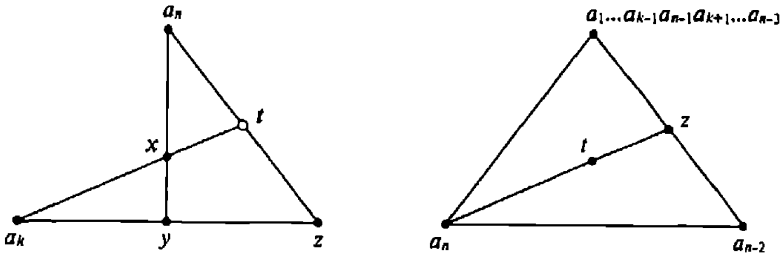
$n \geqslant 4$: Let $x \in I(a_1 \dots a_n)$



$$y \in I(a_1 \dots a_{n-1}) \underset{\text{Ind}}{\Rightarrow} y \in I(a_1 \dots a_{k-1} a_{n-1} a_{k+1} \dots a_{n-2} a_k)$$

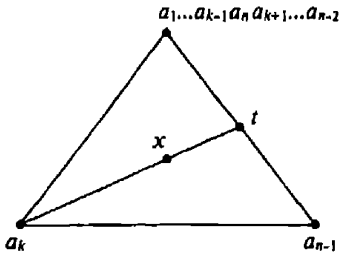


$$\sim L_n(a_1 \ldots a_n) \wedge z \in I(a_1 \ldots a_{k-1}\, a_{n-1}\, a_{k+1} \ldots a_{n-2}) \underset{\mathrm{III}_n, \mathrm{T}.4.2}{\Rightarrow} \sim L(a_n\, a_k\, z)$$



oP

$$t \in I(a_1 \ldots a_{k-1}\, a_{n-1}\, a_{k+1} \ldots a_{n-2}\, a_n) \underset{\mathrm{Ind}}{\Rightarrow} t \in I(a_1 \ldots a_{k-1}\, a_n\, a_{k+1} \ldots a_{n-1})$$



Hence $x \in I(a_1 \ldots a_{k-1}\, a_n\, a_{k+1} \ldots a_{n-1}\, a_k)$. ■

T.4.4. $I(a_1 \ldots a_n) \subset I(a_{l_1} \ldots a_{l_n})$ *for every permutation* $(l_1 \ldots l_n)$ *of the set* $\{1, \ldots, n\}$.

It is an immediate consequence of T.4.3.

T.4.5. $I(a_1 \ldots a_n) = I(a_{l_1} \ldots a_{l_n})$ *for every permutation* $(l_1 \ldots l_n)$ *of the set* $\{1, \ldots, n\}$.

It is an immediate consequence of T.4.4.

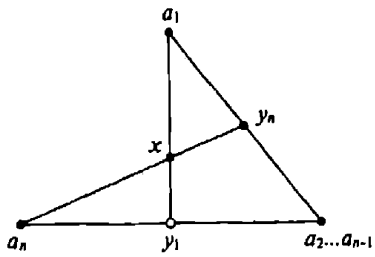
By virtue of T.4.5 the interior of a simplex is independent of the ordering of the vertices; therefore the set $I(a_1 \ldots a_n)$ will be called the interior of a simplex with vertices $a_1, \ldots, a_n$.

Setting $k = 1$ in T.4.3, we immediately obtain the following theorem, which is a generalization of the outer Pasch axiom:

oP_1^n (*The first outer form of the Pasch theorem for an* $(n-1)$-*simplex*).

Let $\sim L_n(a_1 \dots a_n)$. *Then*

$$[y_n \in I(a_1 \dots a_{n-1}) \wedge B(a_n x y_n)] \Rightarrow \bigvee_{y_1} [y_1 \in I(a_2 \dots a_n) \wedge B(a_1 x y_1)], \qquad n = 3, \dots$$



T.4.6. (The alternative definition of the interior of a simplex). *Let* $\sim L_n(a_1 \dots a_n)$. *Then*

$$x \in I(a_1 \dots a_n) \Leftrightarrow \bigvee_y [x \in I(a_1 \dots a_{n-2}\, y) \wedge B(a_{n-1}\, y a_n)], \qquad n = 3, \dots$$

Proof. $\Rightarrow$

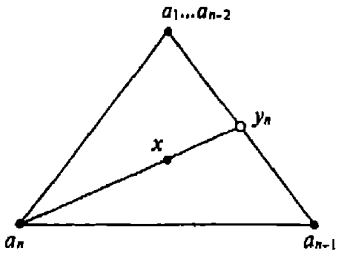
$n = 3$:

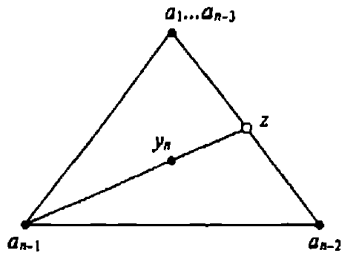
$$x \in I(a_1\, a_2\, a_3) \underset{\text{T.4.5}}{\Leftrightarrow} x \in I(a_2\, a_3\, a_1)$$

$$\Leftrightarrow \bigvee_y [y \in I(a_2\, a_3) \wedge B(a_1\, x y)]$$

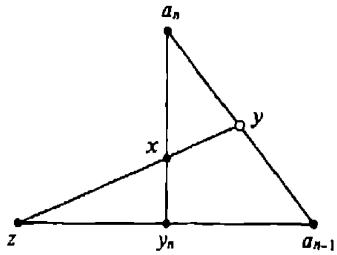
$$\Leftrightarrow \bigvee_y [x \in I(a_1\, y) \wedge B(a_2\, y a_3)]$$

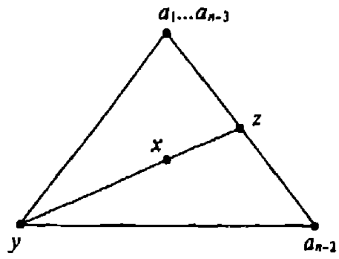
$n \geqslant 4$: Let $x \in I(a_1 \dots a_n)$. Then





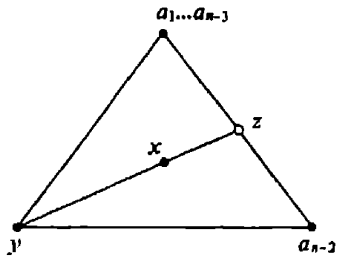
$$\sim L_n(a_1 \dots a_n) \wedge z \in I(a_1 \dots a_{n-2}) \underset{\text{T.4.2}}{\Rightarrow} \sim L(a_{n-1}\, a_n\, z).$$



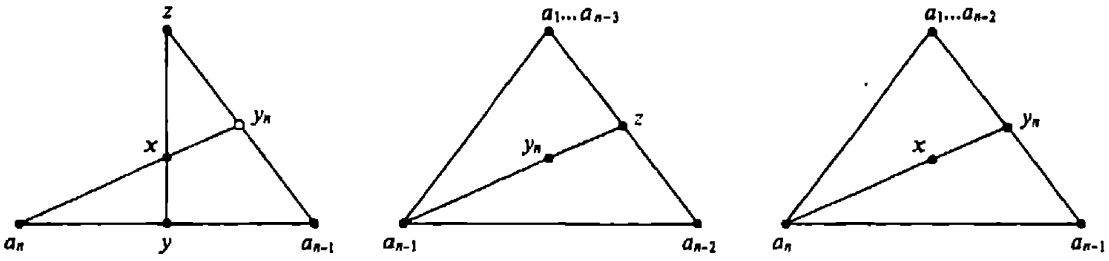


Hence $x \in I(a_1 \dots a_{n-2}\, y) \wedge B(a_{n-1}\, y a_n)$.

$\Leftarrow$ Conversely, let $x \in I(a_1 \dots a_{n-2}\, y) \wedge B(a_{n-1}\, y a_n)$ for a certain y. Then



$$\sim L_n(a_1 \dots a_n) \wedge z \in I(a_1 \dots a_{n-2}) \underset{\text{T.4.2}}{\Rightarrow} \sim L(a_{n-1}\, a_n\, z).$$



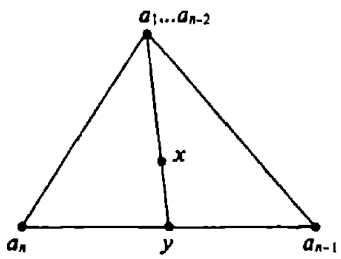
Hence $x \in I(a_1 \dots a_n)$. ∎

The implication

$$\sim L_n(a_1 \dots a_n) \wedge x \in I(a_1 \dots a_{n-2}\, y) \wedge B(a_{n-1}\, y a_n) \Rightarrow x \in I(a_1 \dots a_n), \qquad (n = 3, \dots)$$

is an immediate consequence of T.4.6.

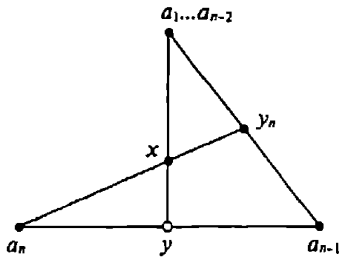
To express this implication we shall use the following diagram:



Theorem T.4.6 can be written as the conjunction of the following two implications oP_2^n and oP_3^n.

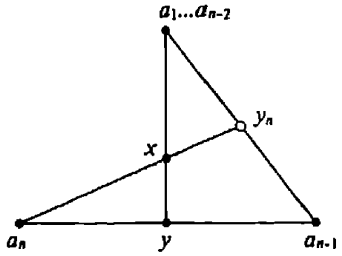
oP_2^n *(The second outer form of the Pasch theorem for an $(n-1)$-simplex). Let* $\sim L_n(a_1 \dots a_n)$. *Then*

$$[B(a_n\, x y_n) \wedge y_n \in I(a_1 \dots a_{n-1})] \Rightarrow \bigvee_y [x \in I(a_1 \dots a_{n-2}\, y) \wedge B(a_{n-1}\, y a_n)], \qquad n = 3, \dots$$



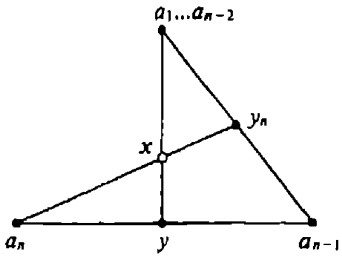
oP_3^n *(The third outer form of the Pasch theorem for an* $(n-1)$*-simplex). Let* $\sim L_n(a_1 \ldots a_n)$. *Then*

$$[x \in I(a_1 \ldots a_{n-2}\, y) \wedge B(a_{n-1}\, y a_n)] \Rightarrow \bigvee_{y_n} [B(a_n x y_n) \wedge y_n \in I(a_1 \ldots a_{n-1})], \qquad n = 3, \ldots$$

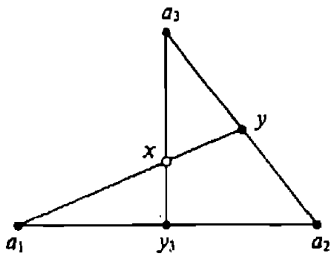


iP^n *(The inner form of the Pasch theorem for an* $(n-1)$*-simplex). Let* $\sim L_n(a_1 \ldots a_n)$. *Then*

$$[B(a_{n-1}\, y a_n) \wedge y_n \in I(a_1 \ldots a_{n-1})] \Rightarrow \bigvee_{x} [x \in I(a_1 \ldots a_{n-2}\, y) \wedge B(a_n x y_n)], \qquad n = 3, \ldots$$

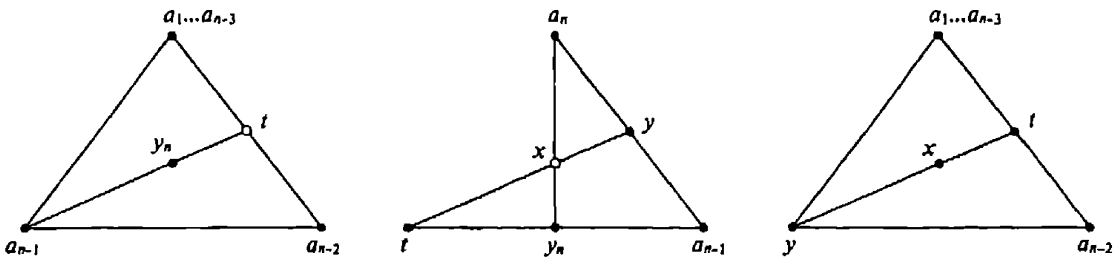


Proof.
$n = 3$:



iP

$n \geqslant 4$:



iP ■

Let $n \geqslant 3$ and $\sim L_n(a_1 \ldots a_n)$. By T.4.6 the interior of the simplex $I(a_1 \ldots a_n)$ may be defined in two ways.

DEFINITION 1. $x \in I(a_1 \ldots a_n) \Leftrightarrow \bigvee_y [y \in I(a_1 \ldots a_{n-1}) \wedge x \in I(a_n y)]$.

DEFINITION 2. $x \in I(a_1 \ldots a_n) \Leftrightarrow \bigvee_y [y \in I(a_1 a_2) \wedge x \in I(a_3 \ldots a_n y)]$.

Each of these definitions is a particular case of the general definition of the interior of a simplex([5]).

T.4.7. $x \in I(a_1 \ldots a_n) \Leftrightarrow \bigvee_y [y \in I(a_1 \ldots a_k) \wedge x \in I(a_{k+1} \ldots a_n y)]$,

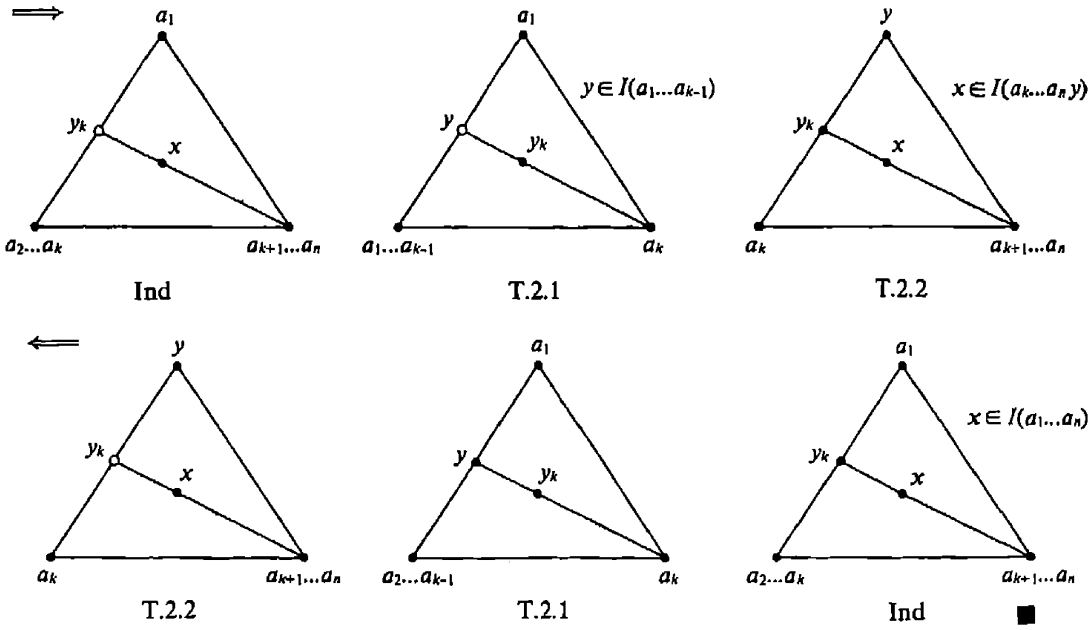
$k = 2, \ldots, n-1$.

Proof by downward induction on k:

for $k = 2$, see Definition 2;

for $k = n-1$, see Definition 1.

Let $2 < k \leqslant n-1$.

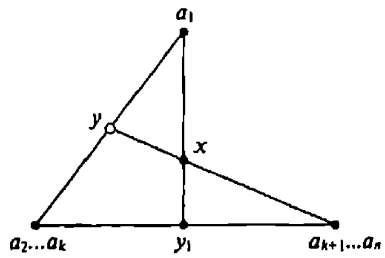


■

([5]) We use the expression "the definition of the interior of a simplex" instead of "a necessary and sufficient condition for a point to belong to the interior of a simplex".

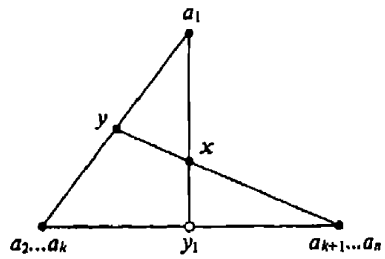
By T.2.1, the equivalence T.4.7 can be expressed as the conjunction of the following two implications:

T.4.8. $[B(a_1 xy_1) \wedge y_1 \in I(a_2 \dots a_n)] \Rightarrow \bigvee_y [y \in I(a_1 \dots a_k) \wedge$
$\wedge\, x \in I(a_{k+1} \dots a_n y)], \quad k = 2, \dots, n-1$



$\mathrm{oP}^{k,n}$

T.4.9. $[y \in I(a_1 \dots a_k) \wedge x \in I(a_{k+1} \dots a_n y)]$
$\Rightarrow \bigvee_{y_1} [B(a_1 xy_1) \wedge y_1 \in I(a_2 \dots a_n)], \quad k = 2, \dots, n-1.$



Notice that setting $k = n-1$ or $k = 2$ in T.4.8 we obtain oP_1^n or oP_2^n, and setting $k = n-1$ or $k = 2$ in T.4.9 we obtain oP_1^n or oP_3^n, respectively.

In the sequel we shall need only the implication T.4.8, which will be denoted by $\mathrm{oP}^{k,n}$ and called "the general outer form of the Pasch theorem".

§5. Connections between linear dependence and the interior of a simplex

T.5.1. *Let* $\sim L_{n-1}(a_1 \dots a_{i-1} a_{i+1} \dots a_n)$ *for* $i = 1, \dots, n$. *Then for* $n = 4, \dots$

$$L_n(a_1 \dots a_n) \Leftrightarrow \bigvee_{i=1}^{n} a_i \in I(a_1 \dots a_{i-1} a_{i+1} \dots a_n) \vee$$
$$\vee \bigvee_{(l_1 \dots l_n)} \bigvee_{k=2}^{n-2} \bigvee_x [x \in I(a_{l_1} \dots a_{l_k}) \wedge x \in I(a_{l_{k+1}} \dots a_{l_n})].$$

Proof. $\Leftarrow$

$$a_i \in I(a_1 \dots a_{i-1}\, a_{i+1} \dots a_n) \underset{\text{T.4.1, III}_n}{\Rightarrow} L_n(a_1 \dots a_n),$$

$$x \in I(a_{l_1} \dots a_{l_k}) \wedge x \in I(a_{l_{k+1}} \dots a_{l_n})$$
$$\underset{\text{T.4.1}}{\Rightarrow} L_{k+1}(a_{l_1} \dots a_{l_k}\, x) \wedge L_{n-k+1}(a_{l_{k+1}} \dots a_{l_n}\, x)$$
$$\underset{\text{T.2.2}}{\Rightarrow} L_n(a_{l_1} \dots a_{l_n}) \underset{\text{III}_n}{\Rightarrow} L_n(a_1 \dots a_n).$$

$\Rightarrow$ Conversely, assume $L_n(a_1 \dots a_n)$.
For $n = 4$,

$$L_4(a_1\, a_2\, a_3\, a_4) \underset{\text{def}}{\Rightarrow} \bigvee_y [L(a_1\, a_2\, y) \wedge L(a_3\, a_4\, y)] \vee \Phi(a_1\, a_2\, a_3\, a_4)$$

where $\Phi(a_1\, a_2\, a_3\, a_4)$ is the disjunction of all the remaining cases, which may be omitted in our proof in view of III_n. In the sequel such inessential members of a disjunction will be replaced by dots.

$$y = a_1 \wedge L(a_3\, a_4\, y) \Rightarrow L(a_3\, a_4\, a_1), \quad \text{a contradiction.}$$

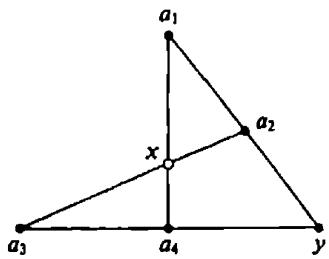
Hence $y \neq a_1$. Analogously $y \neq a_2, a_3, a_4$. Thus $\neq (a_1\, a_2\, y)$ and $\neq (a_3\, a_4\, y)$ and therefore

$$\overset{1}{B}(a_1\, a_2\, y) \vee \overset{2}{B}(a_2\, y a_1) \vee \overset{3}{B}(y a_1\, a_2)$$

and

$$\overset{1}{B}(a_3\, a_4\, y) \vee \overset{2}{B}(a_4\, y a_3) \vee \overset{3}{B}(y a_3\, a_4).$$

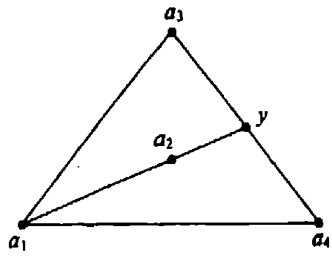
Case$^1_1$.



iP

$$x \in I(a_3\, a_2) \wedge x \in I(a_1\, a_4)$$

In the cases $\frac{1}{3}$, $\frac{3}{1}$, $\frac{3}{3}$ the proof is analogous.
Case$^1_2$.



$$a_2 \in I(a_3\, a_4\, a_1)$$

In the cases $\frac{2}{1}$, $\frac{2}{3}$, $\frac{3}{2}$ the proof is analogous.
Case $\frac{2}{2}$. Setting $x = y$ we have

$$x \in I(a_2\, a_1) \wedge x \in I(a_4\, a_3).$$

For $n \geqslant 5$,

$$L_n(a_1 \dots a_n) \underset{\text{def}}{\Rightarrow} \bigvee_y [L(a_1\, a_2\, y) \wedge L_{n-1}(a_3 \dots a_n\, y)] \vee \dots L_{n-1}(a_3 \dots a_n\, y)$$

$$\underset{\text{Ind}}{\Rightarrow} a_3 \in I(a_4 \dots a_n\, y) \vee \dots \qquad \text{Case 1}$$

$$\vee\, y \in I(a_3 \dots a_n) \vee \qquad \text{Case 2}$$

$$\vee \bigvee_z [z \in I(a_3 \dots a_k) \wedge z \in I(a_{k+1} \dots a_n\, y)] \vee \dots \qquad \text{Case 3}$$

$$k = 4, \dots, n-1,$$

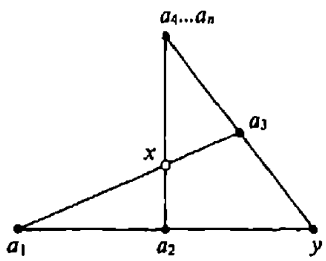
$$y = a_1 \wedge L_{n-1}(a_3 \dots a_n\, y) \underset{\text{III}_n}{\Rightarrow} L_{n-1}(a_1\, a_3 \dots a_n),$$

a contradiction.

Thus $\neq (a_1\, a_2\, y)$ and hence

$$\overset{1}{B}(a_1\, a_2\, y) \vee \overset{2}{B}(a_2\, y a_1) \vee \dots$$

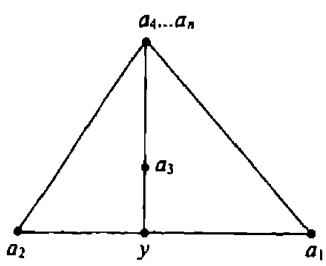
Case 1.1.



$$x \in I(a_4 \dots a_n\, a_2) \wedge x \in I(a_1\, a_3)$$

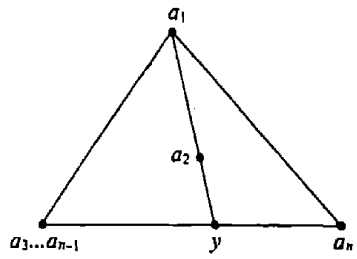
iP^{n-1}

Case 1.2.



$$a_3 \in I(a_4 \dots a_n\, a_1\, a_2)$$

Case 2.1.

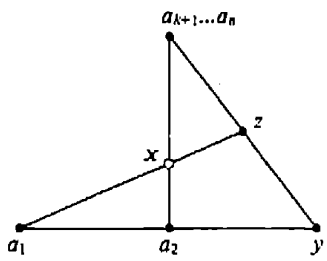


$$a_2 \in I(a_1\, a_3 \ldots a_n)$$

Case 2.2. Setting $x = y$ we get

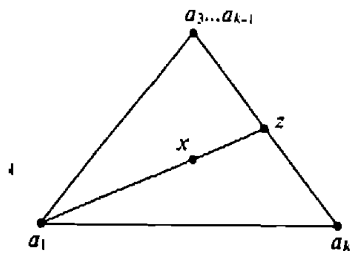
$$x \in I(a_3 \ldots a_n) \wedge x \in I(a_2\, a_1).$$

Case 3.1.



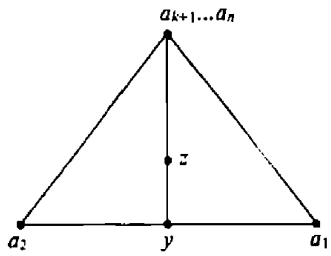
$$x \in I(a_{k+1} \ldots a_n\, a_2)$$

iP^{n-k+2}



$$x \in I(a_3 \ldots a_k\, a_1)$$

Case 3.2.



Setting $x = z$ we get

$$x \in I(a_3 \ldots a_k) \wedge x \in I(a_1\, a_2\, a_{k+1} \ldots a_n). \quad \blacksquare$$

§6. Properties of the sequence generated by a collinearity

We shall prove that the sequence (L_n) satisfies the weakest transitivity law:

II_n^{ww} $\quad a_1 \neq a_2 \wedge L(a_1 a_2 p) \wedge L_n(a_1 \dots a_n) \Rightarrow L_n(pa_2 \dots a_n)$.

Proof by induction on n.

For $n = 3$,

$$a_1 \neq a_2 \wedge L(a_1 a_2 p) \wedge L(a_1 a_2 a_3) \underset{\mathrm{T.2.1}}{\Rightarrow} L(pa_2 a_3).$$

Take $n > 3$ and assume II_m^{ss} for $3 \leqslant m < n$; let

$$a_1 \neq a_2 \wedge L(a_1 a_2 p) \wedge L_n(a_1 \dots a_n).$$

Case 1. $a_1 = p$.

$$a_1 = p \wedge L_n(a_1 \dots a_n) \Rightarrow L_n(pa_2 \dots a_n).$$

Case 2. $a_2 = p$.

$$a_2 = p \underset{\mathrm{I}_i \text{ for } i=3,\dots,n}{\Rightarrow} L_n(pa_2 \dots a_n).$$

Case 3. $L_{n-1}(a_1 \dots a_{n-1})$.

$$a_1 \neq a_2 \wedge L(a_1 a_2 p) \wedge L_{n-1}(a_1 \dots a_{n-1}) \underset{\mathrm{Ind}}{\Rightarrow} L_{n-1}(pa_2 \dots a_{n-1}) \underset{\mathrm{I}_n}{\Rightarrow} L_n(pa_2 \dots a_n).$$

The cases $L_{n-1}(a_1 \dots a_{i-1} a_{i+1} \dots a_n)$ for $i = 3, \dots, n-1$ do not differ essentially from Case 3, because II^{ww} is symmetrical with respect to $(a_3 \dots a_n)$.

Case 4. $L_{n-1}(a_2 \dots a_n)$.

$$L_{n-1}(a_2 \dots a_n) \underset{\mathrm{I}_n}{\Rightarrow} L_n(pa_2 \dots a_n).$$

Case 5. $L_{n-1}(a_1 a_3 \dots a_n)$.

$$L_{n-1}(a_1 a_3 \dots a_n) \wedge L(a_1 a_2 p) \underset{\mathrm{def,III}_{n-1}}{\Rightarrow} L_n(pa_2 \dots a_n).$$

Case 6. $\neq (a_1 a_2 p) \wedge \sim L_{n-1}(a_1 \dots a_{i-1} a_{i+1} \dots a_n)$ for $i = 1, \dots, n$.

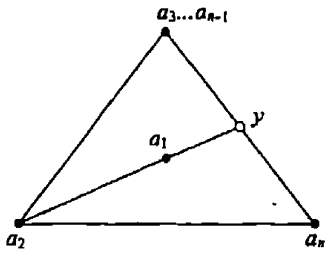
$$\begin{aligned}
L_n(a_1 \dots a_n) \underset{\mathrm{T.5.1}}{\Rightarrow} \; & a_1 \in I(a_2 \dots a_n) \vee && \text{Case 6.1} \\
& \vee\, a_2 \in I(a_1 a_3 \dots a_n) \vee && \text{Case 6.2} \\
& \vee\, a_n \in I(a_1 \dots a_{n-1}) \vee && \text{Case 6.3} \\
& \vee \bigvee_x [x \in I(a_1 \dots a_k) \wedge x \in I(a_{k+1} \dots a_n)] \vee && \text{Case 6.4} \\
& && (2 \leqslant k \leqslant n-2) \\
& \vee \bigvee_x [x \in I(a_1 a_3 \dots a_k) \wedge x \in I(a_2 a_{k+1} \dots a_n)] \vee && \text{Case 6.5} \\
& && (3 \leqslant k \leqslant n-1) \\
& \vee\, \Phi(a_1 \dots a_n),
\end{aligned}$$

where $\Phi(a_1 \dots a_n)$ is the disjunction of all the remaining cases.

This disjunction may be omitted in view of the symmetry of II_n^{ww} (compare Case 3).

Case 6.1. $a_1 \neq a_2 \wedge L(a_1\, a_2\, p) \wedge a_1 \in I(a_2 \ldots a_n)$.

$$a_1 \in I(a_2 \ldots a_n) \underset{\mathrm{T.4.4}}{\Rightarrow} a_1 \in I(a_3 \ldots a_n\, a_2)$$



$$B(a_2\, a_1\, y) \underset{\mathrm{def}}{\Rightarrow} L(a_1\, a_2\, y),$$

$$a_1 \neq a_2 \wedge L(a_1\, a_2\, p) \wedge L(a_1\, a_2\, y) \underset{\mathrm{T.2.1}}{\Rightarrow} L(a_2\, p y),$$

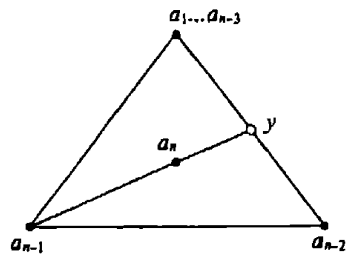
$$y \in I(a_3 \ldots a_n) \underset{\mathrm{T.4.1}}{\Rightarrow} L_{n-1}(a_3 \ldots a_n\, y),$$

$$L_{n-1}(a_3 \ldots a_n\, y) \wedge L(a_2\, p y) \underset{\mathrm{def, III}_n}{\Rightarrow} L_n(p a_2 \ldots a_n).$$

Case 6.2. $a_1 \neq a_2 \wedge L(a_1\, a_2\, p) \wedge a_2 \in I(a_1\, a_3 \ldots a_n)$.

Proof as for Case 6.1.

Case 6.3. $a_1 \neq a_2 \wedge L(a_1\, a_2\, p) \wedge a_n \in I(a_1 \ldots a_{n-1})$.



$$a_1 \neq a_2 \wedge L(a_1\, a_2\, p) \wedge L_{n-1}(a_1 \ldots a_{n-2}\, y) \underset{\mathrm{Ind}}{\Rightarrow} L_{n-1}(p a_2 \ldots a_{n-2}\, y),$$

$$L_{n-1}(p a_2 \ldots a_{n-2}\, y) \wedge L(a_{n-1}\, a_n\, y) \underset{\mathrm{def}}{\Rightarrow} L_n(p a_2 \ldots a_n).$$

Case 6.4. $a_1 \neq a_2 \wedge L(a_1\, a_2\, p) \wedge x \in I(a_1 \ldots a_k) \wedge x \in I(a_{k+1} \ldots a_n)$, $2 \leqslant k \leqslant n-2$.

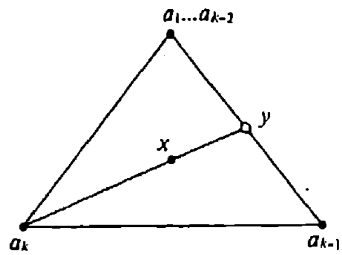
$k = 2$:

$$x \in I(a_1\, a_2) \underset{\mathrm{T.4.1}}{\Rightarrow} L(a_1\, a_2\, x),$$

$$a_1 \neq a_2 \wedge L(a_1\, a_2\, p) \wedge L(a_1\, a_2\, x) \underset{\mathrm{T.2.1}}{\Rightarrow} L(p a_2\, x),$$

$$L_{n-1}(a_3 \ldots a_n\, x) \wedge L(p a_2\, x) \underset{\mathrm{def}}{\Rightarrow} L_n(p a_2 \ldots a_n).$$

$k \geqslant 3$:



$$a_1 \neq a_2 \wedge L(a_1\, a_2\, p) \wedge L_k(a_1 \ldots a_{k-1}\, y) \underset{\text{Ind}}{\Rightarrow} L_k(pa_2 \ldots a_{k-1}\, y),$$

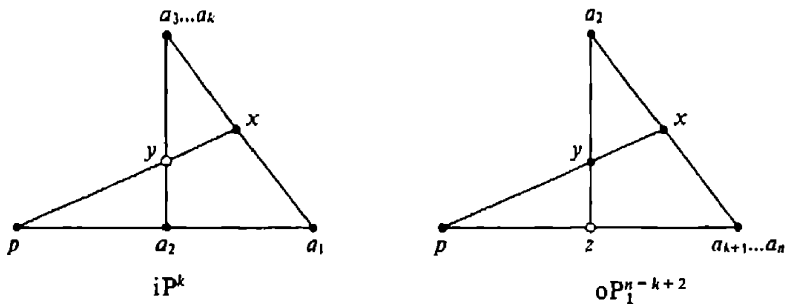
$$L_k(pa_2 \ldots a_{k-1}\, y) \wedge L(a_k\, xy) \underset{\text{def}}{\Rightarrow} L_{k+1}(pa_2 \ldots a_k\, x),$$

$$L_{k+1}(pa_2 \ldots a_k\, x) \wedge L_{n-k+1}(a_{k+1} \ldots a_n\, x) \underset{\text{T.2.2}}{\Rightarrow} L_n(pa_2 \ldots a_n).$$

Case 6.5. $\neq (a_1\, a_2\, p) \wedge L(a_1\, a_2\, p) \wedge x \in I(a_1\, a_3 \ldots a_k) \wedge x \in I(a_2\, a_{k+1} \ldots a_n)$,
$3 \leqslant k \leqslant n-1$.

$$\neq (a_1\, a_2\, p) \wedge L(a_1\, a_2\, p) \Rightarrow \overset{1}{B(a_1\, a_2\, p)} \vee \overset{2}{B(a_2\, pa_1)} \vee \overset{3}{B(pa_1\, a_2)}.$$

Case 6.5.1.

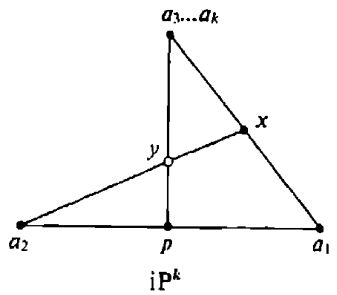


$$B(a_2\, yz) \underset{\text{A2}}{\Rightarrow} y \neq a_2 \wedge L(ya_2\, z),$$

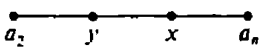
$$y \neq a_2 \wedge L(ya_2\, z) \wedge L_k(ya_2 \ldots a_k) \underset{\text{Ind}}{\Rightarrow} L_k(za_2 \ldots a_k),$$

$$L_k(a_2 \ldots a_k\, z) \wedge L_{n-k+2}(pa_{k+1} \ldots a_n\, z) \underset{\text{T.2.2}}{\Rightarrow} L_n(pa_2 \ldots a_n).$$

Case 6.5.2.

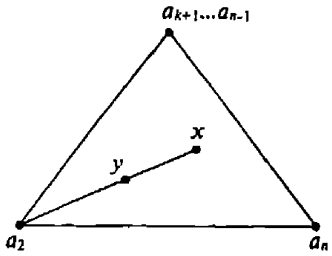


$k = n-1$:



$y \in I(a_2\, a_{k+1} \dots a_n)$

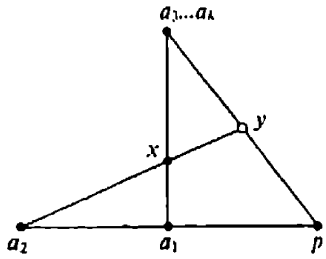
$k < n-1$:



$y \in I(a_2\, a_{k+1} \dots a_n)$

$$y \in I(pa_3 \dots a_k) \wedge y \in I(a_2\, a_{k+1} \dots a_n) \underset{\text{T.5.1}}{\Rightarrow} L_n(pa_2 \dots a_n).$$

Case 6.5.3.



oP_3^k

$k = n-1$:

$$x \in I(a_2\, a_{k+1} \dots a_n) \Rightarrow B(a_2\, xa_n),$$

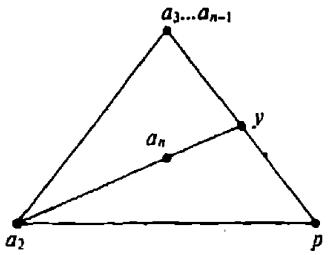
$$B(a_2\, xy) \wedge B(a_2\, xa_n) \underset{A6}{\Rightarrow} y = a_n \vee B(xya_n) \vee B(xa_n\, y),$$

$$y = a_n \Rightarrow a_n \in I(pa_3 \dots a_{n-1}) \underset{\text{T.4.1}}{\Rightarrow} L_{n-1}(pa_3 \dots a_n) \underset{\text{I}_n}{\Rightarrow} L_n(pa_2 \dots a_n),$$

$$B(xya_n) \wedge B(a_2\, xy) \underset{\text{T.1.3}}{\Rightarrow} B(a_2\, ya_n),$$

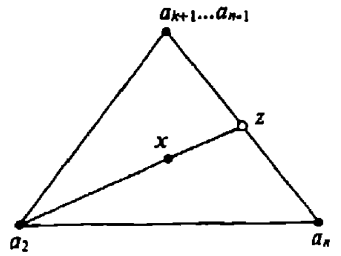
$$y \in I(a_2\, a_n) \wedge y \in I(pa_3 \dots a_{n-1}) \underset{\text{T.5.1}}{\Rightarrow} L_n(pa_2 \dots a_n),$$

$$B(xa_n\, y) \wedge B(a_2\, xy) \underset{\text{T.1.1}}{\Rightarrow} B(a_2\, a_n\, y),$$



$$a_n \in I(pa_2 \dots a_{n-1}) \underset{\text{T.4.1}}{\Rightarrow} L_n(pa_2 \dots a_n).$$

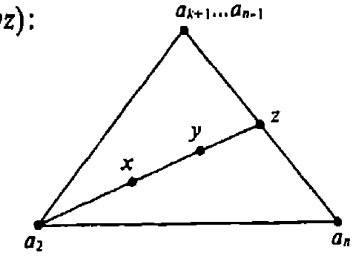
$k < n-1$:



$$B(a_2 xy) \wedge B(a_2 xz) \underset{A6}{\Rightarrow} y = z \vee B(xyz) \vee B(xzy),$$

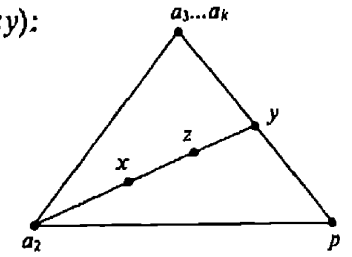
$$y = z \Rightarrow y \in I(pa_3 \dots a_k) \wedge y \in I(a_{k+1} \dots a_n) \underset{T.5.1}{\Rightarrow} L_{n-1}(pa_3 \dots a_n) \underset{I_n}{\Rightarrow} L_n(pa_2 \dots a_n),$$

$B(xyz)$:



$$y \in I(a_2 a_{k+1} \dots a_n) \wedge y \in I(pa_3 \dots a_k) \underset{T.5.1}{\Rightarrow} L_n(pa_2 \dots a_n),$$

$B(xzy)$:



$$z \in I(pa_2 \dots a_k) \wedge z \in I(a_{k+1} \dots a_n) \underset{T.5.1}{\Rightarrow} L_n(pa_2 \dots a_n). \blacksquare$$

We have proved that the sequence (L_n) satisfies condition II_n^{ww}. In view of T.1.3.1 we immediately obtain

T.6.1. Fundamental Theorem. *The sequence (L_n) is an E-sequence.*

Definition. A *k-dimensional hyperplane* is understood as an equivalence class with respect to the relation L_{k+2}. Many of the basic properties of hyperplanes turn out to be simple consequences of the Fundamental Theorem itself, and so they are of set-theoretical character.

To determine the dimension of a space, we have to add the following two dimension axioms:

$$\mathrm{D}_1^n \quad \bigvee_{a_1 \dots a_{n+1}} \sim L_{n+1}(a_1 \dots a_{n+1});$$

$$\mathrm{D}_2^n \quad \bigwedge_{a_1 \dots a_{n+2}} L_{n+2}(a_1 \dots a_{n+2}).$$

The first of these axioms restricts the dimension from below to n, while the second one restricts the dimension from above to n(6).

On the basis of the axiom system $\{A1, \ldots, A6, oP, iP, D_1^n, D_2^n\}$ we can obtain those properties of hyperplanes which depend on the dimension of the space. Those properties will be referred to as geometric properties. Such an approach to the theory of hyperplanes will be sketched in Chapter III.

Chapter III

Theory of hyperplanes in affine geometry

According to Chapter II, hyperplanes are equivalence classes with respect to the sequence (L_n); more exactly, k-dimensional hyperplanes are equivalence classes with respect to the relation L_{k+2}.

We shall use the letters $A_{k+1}, B_{k+1}, \ldots$ to denote k-dimensional hyperplanes. Hyperplanes have all the (set-theoretical) properties of equivalence classes (T.1.1.5–T.1.1.12); particularly, the non-empty intersection of two hyperplanes is always a hyperplane, and

$$(\alpha) \qquad A_m = B_r \cap C_s \Rightarrow m \leqslant \min(r, s), \quad m, r, s = 1, \ldots$$

This sentence states that the rank (dimension) of the intersection of two hyperplanes is restricted from above by the rank (dimension) of each of those hyperplanes.

Let us now consider the sentence

$$(\beta) \qquad A_m = B_r \cap C_s \wedge B_r \cup C_s \subset D_p \Rightarrow m \geqslant r+s-p; \; m, r, s, p = 1, \ldots,$$

which restricts from below the rank of the intersection A_m in the case where the classes B_r and C_s are both contained in the class D_p. This sentence is a theorem of Euclidean geometry, i.e. of the theory of the Cartesian n-space over the field of reals (see [2], p. 72). We may ask whether the sentence (β) is already a theorem of the general theory of E-sequences (and so also

(6) In particular, for $n = 2$ we obtain

$$D_1^2 \quad \bigvee_{a_1 a_2 a_3} \sim L(a_1 a_2 a_3);$$

$$D_2^2 \quad \bigwedge_{a_1 a_2 a_3 a_4} \bigvee_{x} [(L(a_1 a_2 x) \wedge L(a_3 a_4 x)) \vee (L(a_1 a_3 x) \wedge L(a_2 a_4 x)) \vee (L(a_1 a_4 x) \wedge L(a_2 a_3 x))].$$

This form of plane dimension axioms is due to M. Kordos. Tarski in his axiom system of geometry (see [4]) gives the following axiom, stronger than D_2^2:

$$\bigwedge_{a_1 a_2 a_3 a_4} \bigvee_{x} [(L(a_1 a_2 x) \wedge B(a_3 x a_4)) \vee (L(a_1 a_3 x) \wedge B(a_2 x a_4)) \vee (L(a_1 a_4 x) \wedge B(a_2 x a_3))].$$

of the set theory). The answer is negative, as the following simple example shows. Consider a pyramid abcde (in R^3) with the base abcd. Let $S = \{a, b, c, d, e\}$ and let R_n be an affine dependence restricted to the set S (for $n = 5, \ldots$ we obviously have $R_n = S^n$). The sequence (R_n) is an E-sequence on the set S. Setting

$$A_1 = [e], \quad B_3 = [eab], \quad C_3 = [ecd], \quad D_4 = [eabc],$$

we get

$$(\gamma) \qquad A_1 = B_3 \cap C_3 \wedge B_3 \cup C_3 \subset D_4 \wedge 1 < 3+3-4 = 2.$$

The question arises whether the sentence (β) is true for E-sequences (R_n) generated by some ternary equivalence relation. The answer is again negative. To show this we slightly modify the preceding model. Let us replace the set S by $S' = \{a, b, c, d, e, f\}$ where f is the point of intersection of diagonals of the base $abcd$. Let (R_n) be an E-sequence on the set S', generated by collinearity. Then the sentence (γ) remains true, whence (β) is false.

Is the sentence (β) true for an E-sequence (L_n) in every model of $\mathscr{A}$?

Here again the answer is negative. Indeed, let B be the Cartesian betweenness relation restricted to S'. Then B satisfies all the axioms A1 – A6, oP, iP, the relation L is the Cartesian collinearity restricted to S' and so the sequence (L_n) coincides with the sequence (R_n) from the previous two models. Thus again the sentence (γ) is true.

Let us now consider theory $\mathscr{A}^0$, obtained from $\mathscr{A}$ by adding the axiom

$$\text{A0} \qquad a \neq b \Rightarrow \bigvee_c B(abc).$$

(Then the axiom iP becomes dependent.) Theory $\mathscr{A}^0$ may be called absolute affine geometry without the continuity axiom.

We are going to prove that T.1.1 is a theorem of $\mathscr{A}^0$. In conclusion, the sentence (β) (equivalent to T.1.1) is also a theorem of $\mathscr{A}^0$.

The main difficulty is to proceed by induction in such a way that no argument is used twice. For this purpose we introduce some auxiliary predicates (U, V and W). Axiom A0 is used only in Lemma 9.

DEFINITIONS.

(1) $$\begin{aligned} U(a_1 \ldots a_m b_1 \ldots b_k c_1 \ldots c_l) \\ \overset{\text{def}}{\Leftrightarrow} [(\sim L_{m+k}(a_1 \ldots a_m b_1 \ldots b_k) \wedge \sim L_{m+l}(a_1 \ldots a_m c_1 \ldots c_l)) \\ \Rightarrow \bigvee_x (\sim L_{m+1}(a_1 \ldots a_m x) \wedge \\ \wedge L_{m+k+1}(a_1 \ldots a_m b_1 \ldots b_k x) \wedge L_{m+l-1}(a_1 \ldots a_m c_1 \ldots c_l x))], \\ m = 0, \ldots, k, l = 1, \ldots \end{aligned}$$

(2) $V(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l)$

$$\overset{\text{def}}{\Leftrightarrow} [(\sim L_{m+k}(a_1 \dots a_m b_1 \dots b_k) \wedge \sim L_{m+l}(a_1 \dots a_m c_1 \dots c_l) \wedge$$
$$\wedge L_{m+k+l-1}(a_1 \dots a_{m-1} b_1 \dots b_k c_1 \dots c_l))$$
$$\Rightarrow \bigvee_x (\sim L_{m+1}(a_1 \dots a_m x) \wedge L_{m+k+1}(a_1 \dots a_m b_1 \dots b_k x) \wedge$$
$$\wedge L_{m+l+1}(a_1 \dots a_m c_1 \dots c_l x))], \quad m, k, l = 1, \dots$$

(3) $W(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l)$

$$\overset{\text{def}}{\Leftrightarrow} [(\sim L_{m+k}(a_1 \dots a_m b_1 \dots b_k) \wedge \sim L_{m+l}(a_1 \dots a_m c_1 \dots c_l) \wedge$$
$$\wedge L_{m+k+l}(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l))$$
$$\Rightarrow \bigvee_x (\sim L_{m+1}(a_1 \dots a_m x) \wedge L_{m+k+1}(a_1 \dots a_m b_1 \dots b_k x) \wedge$$
$$\wedge L_{m+l+1}(a_1 \dots a_m c_1 \dots c_l x))], \quad m, k, l = 1, \dots$$

Then

(4) $V(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l)$

$$\Leftrightarrow [L_{m+k+l-1}(a_1 \dots a_{m-1} b_1 \dots b_k c_1 \dots c_l) \Rightarrow U(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l)],$$

(5) $W(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l)$

$$\Leftrightarrow [L_{m+k+l}(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l) \Rightarrow U(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l)].$$

Obviously

(6) $$W(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l) \Rightarrow V(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l).$$

Let

(7) $$V(m, k, l) \overset{\text{def}}{\Leftrightarrow} \bigwedge_{a_1 \dots a_m} \bigwedge_{b_1 \dots b_k} \bigwedge_{c_1 \dots c_l} V(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l),$$
$$m, k, l = 1, \dots,$$

(8) $$W(m, k, l) \overset{\text{def}}{\Leftrightarrow} \bigwedge_{a_1 \dots a_m} \bigwedge_{b_1 \dots b_k} \bigwedge_{c_1 \dots c_l} W(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_l),$$
$$m, k, l = 1, \dots$$

Obviously

(9) $$W(m, k, l) \Rightarrow V(m, k, l).$$

Comparing Definitions (3) and (8) with the definitions from §1.4, we obtain

$$W = W_R.$$

T.1.1. $W(m, k, l)$ *for* $m, k, l = 1, \ldots$

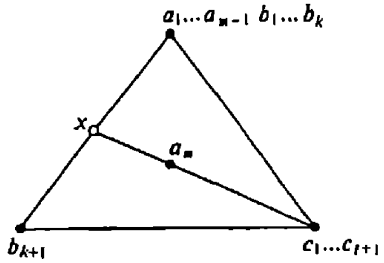
Proof by induction on m, k, l. The proof is based on the following nine lemmas:

1. $W(m, 1, l)$ for $m, l = 1, \ldots$
   $x = b_1$.
2. $W(m, k, 1)$ for $m, k = 1, \ldots$
   $x = c_1$.

In particular

2'. $W(m, k+1, 1)$ for $m, k = 1, \ldots$

3. $a_m \in I(a_1 \ldots a_{m-1} b_1 \ldots b_{k+1} c_1 \ldots c_{l+1})$
$$\Rightarrow U(a_1 \ldots a_m b_1 \ldots b_{k+1} c_1 \ldots c_{l+1}), \quad m, k, l = 1, \ldots$$

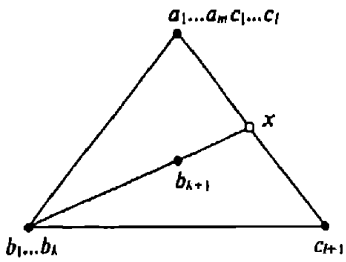


T.2.4.7

$$L_{m+k+1}(a_1 \ldots a_{m-1} b_1 \ldots b_{k+1} x) \wedge$$
$$\wedge L_{l+3}(a_m c_1 \ldots c_{l+1} x)$$

$\sim L_{m+1}(a_1 \ldots a_m x)$ because
$$a_m \neq x \wedge L_{m+1}(a_1 \ldots a_m x) \wedge L_{l+3}(c_1 \ldots c_{l+1} a_m x) \underset{\text{T.1.4.4}}{\Rightarrow} L_{m+l+1}(a_1 \ldots a_m c_1 \ldots c_{l+1})$$
a contradiction.

4. $b_{k+1} \in I(a_1 \ldots a_m b_1 \ldots b_k c_1 \ldots c_{l+1}) \Rightarrow U(a_1 \ldots a_m b_1 \ldots b_{k+1} c_1 \ldots c_{l+1})$,
$$m = 0, \ldots, \quad k, l = 1, \ldots$$



T.2.4.7

$$L_{k+2}(b_1 \ldots b_{k+1} x) \wedge$$
$$\wedge L_{m+l+2}(a_1 \ldots a_m c_1 \ldots c_{l+1} x)$$

$\sim L_{m+1}(a_1 \ldots a_m x)$ because
$$L_{m+1}(a_1 \ldots a_m x) \wedge L_{k+2}(b_1 \ldots b_{k+1} x) \underset{\text{T.2.2.2}}{\Rightarrow} L_{m+k+1}(a_1 \ldots a_m b_1 \ldots b_{k+1}),$$
a contradiction.

5. $b_{k+1} \in I(b_1 \dots b_k c_1 \dots c_{l+1}) \Rightarrow U(a_1 b_1 \dots b_{k+1} c_1 \dots c_{l+1})$
$k, l = 1, \dots$

$$\sim L_{k+2}(a_1 b_1 \dots b_{k+1}) \Rightarrow \sim L_{k+1}(b_1 \dots b_{k+1}),$$

$$\sim L_{l+2}(a_1 c_1 \dots c_{l+1}) \Rightarrow \sim L_{l+1}(c_1 \dots c_{l+1}),$$

$$b_{k+1} \in I(b_1 \dots b_k c_1 \dots c_{l+1}) \wedge \sim L_{k+1}(b_1 \dots b_{k+1}) \wedge \sim L_{l+1}(c_1 \dots c_{l+1})$$
$$\underset{4}{\Rightarrow} \bigvee_x [L_{k+2}(b_1 \dots b_{k+1} x) \wedge L_{l+2}(c_1 \dots c_{l+2} x)],$$

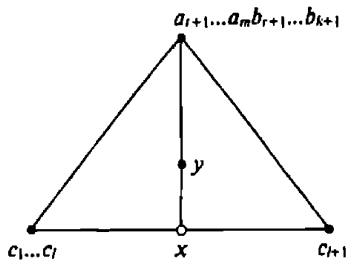
$$a_1 \neq x \quad \text{because} \quad a_1 = x \Rightarrow L_{k+2}(a_1 b_1 \dots b_{k+1}),$$

a contradiction.

6. $y \in I(b_1 \dots b_{k+1}) \wedge y \in I(c_1 \dots c_{l+1}) \Rightarrow U(b_1 \dots b_{k+1} c_1 \dots c_{l+1})$,
$x = y$. $\quad k, l = 1, \dots,$

7. $y \in I(a_1 \dots a_t b_1 \dots b_r) \wedge y \in I(a_{t+1} \dots a_m b_{r+1} \dots b_{k+1} c_1 \dots c_{l+1})$
$\Rightarrow U(a_1 \dots a_m b_1 \dots b_{k+1} c_1 \dots c_{l+1})$,
$m = 0, \dots, \quad k, l = 1, \dots,$
$t = 0, \dots, m, \quad r = 0, \dots, k+1,$
$2 \leqslant t+r \leqslant m+k.$

Remark. For the case $m = 0$, $r = k+1$ – see 6, for $m > 0$, the case $t = m$, $r = k+1$ is analogous to the case $t = 0$, $r = k+1$



$L_{l+1}(c_1 \dots c_l x)$

T.2.4.7

$$L_{t+r+1}(a_1 \dots a_t b_1 \dots b_r y) \wedge L_{m+k+3-t-r}(a_{t+1} \dots a_m b_{r+1} \dots b_{k+1} yx)$$
$$\underset{\text{T.2.2.2}}{\Rightarrow} L_{m+k+2}(a_1 \dots a_m b_1 \dots b_{k+1} x),$$

$\sim L_{m+1}(a_1 \dots a_m x)$ because
$L_{m+1}(a_1 \dots a_m x) \wedge L_{l+2}(c_1 \dots c_{l+1} x) \underset{\text{T.2.2.2}}{\Rightarrow} L_{m+l+1}(a_1 \dots a_m c_1 \dots c_{l+1})$.

a contradiction.

8. $y \in I(b_1 \dots b_r) \wedge y \in I(b_{r+1} \dots b_{k+1} c_1 \dots c_{l+1})$

$$\Rightarrow U(a_1 b_1 \dots b_{k+1} c_1 \dots c_{l+1}), \quad k, l = 1, \dots; \; r = 2, \dots, k+1,$$

$$\sim L_{k+2}(a_1 b_1 \dots b_{k+1}) \Rightarrow \sim L_{k+1}(b_1 \dots b_{k+1}),$$

$$\sim L_{l+2}(a_1 c_1 \dots c_{l+1}) \Rightarrow \sim L_{l+1}(c_1 \dots c_{l+1}),$$

$$y \in I(b_1 \dots b_r) \wedge y \in I(b_{r+1} \dots b_{k+1} c_1 \dots c_{l+1}) \wedge$$

$$\wedge \sim L_{k+1}(b_1 \dots b_{k+1}) \wedge \sim L_{l+1}(c_1 \dots c_{l+1})$$

$$\underset{6,7}{\Rightarrow} \bigvee_x [L_{k+2}(b_1 \dots b_{k+1} x) \wedge L_{l+2}(c_1 \dots c_{l+1} x)],$$

$a_1 \neq x$ because $a_1 = x \Rightarrow L_{k+2}(a_1 b_1 \dots b_{k+1})$, a contradiction.

9. $y \in I(b_1 \dots b_r c_1 \dots c_s) \wedge y \in I(b_{r+1} \dots b_{k+1} c_{s+1} \dots c_{l+1})$

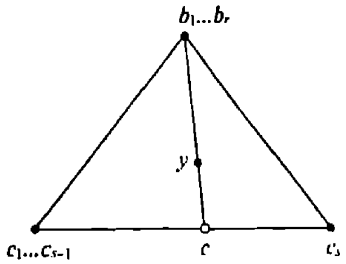
$$\Rightarrow U(a_1 b_1 \dots b_{k+1} c_1 \dots c_{l+1}),$$

$$k, l = 1, \dots; \; r = 1, \dots, k; \; s = 1, \dots, l.$$

Define the point c as follows:

$$s = 1 \Rightarrow c = c_1;$$

for $s = 2, \dots, l$



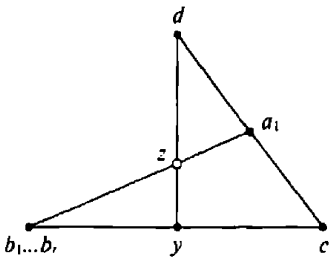
T.2.4.7

$c \neq a_1$ because

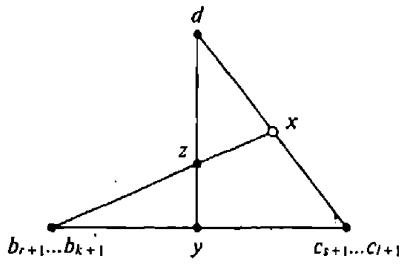
$$c = a_1 \Rightarrow L_{s+1}(a_1 c_1 \dots c_s) \Rightarrow L_{l+2}(a_1 c_1 \dots c_{l+1}),$$

a contradiction,

$$c \neq a_1 \underset{A0}{\Rightarrow} \bigvee_d B(c a_1 d)$$



iP^{r+2}



$\mathrm{oP}^{l-s+2;\,k-r+l-s+3}$

$$L_{r+2}(a_1 b_1 \dots b_r z) \wedge L_{k+3-r}(b_{r+1} \dots b_{k+1} xz) \underset{\text{T.2.2.2}}{\Rightarrow} L_{k+3}(a_1 b_1 \dots b_{k+1} x),$$

$$L(a_1 cd) \wedge L_{l+3-s}(c_{s+1} \dots c_{l+1} xd) \underset{\text{T.2.2.2}}{\Rightarrow} L_{l+4-s}(a_1 cc_{s+1} \dots c_{l+1} x),$$

$$s = 1 \Rightarrow L_{l+3}(a_1 c_1 \dots c_{l+1} x),$$

for $s = 2, \dots, l$

$$L_{s+1}(c_1 \dots c_s c) \wedge L_{l+4-s}(a_1 c_{s+1} \dots c_{l+1} xc) \underset{\text{T.2.2.2}}{\Rightarrow} L_{l+3}(a_1 c_1 \dots c_{l+1} x).$$

$a_1 \neq x$ because $a_1 = x \Rightarrow L_{k+3-r}(a_1 zb_{r+1} \dots b_{k+1})$,

$$a_1 \neq z \wedge L_{r+2}(a_1 zb_1 \dots b_r) \wedge L_{k+3-r}(a_1 zb_{r+1} \dots b_{k+1}) \underset{\text{T.1.4.4}}{\Rightarrow} L_{k+2}(a_1 b_1 \dots b_{k+1}),$$

a contradiction.

We now pass to the induction.

By 1 we have $W(m, 1, l)$ for $m, l = 1, \dots$ Assume $W(m, k, l)$ for a certain $k \geqslant 1$ and $m, l = 1, \dots$ We have to prove $W(m, k+1, l)$ for $m, l = 1, \dots$ By 2′ we have $W(m, k+1, 1)$ for $m = 1, \dots$ Assume $W(m, k+1, l)$ for a certain $l \geqslant 1$ and $m = 1, \dots$; we have to prove $W(m, k+1, l+1)$ for $m = 1, \dots$

10. $L_{m+k+l+1}(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_{l+1})$
$$\Rightarrow U(a_1 \dots a_m b_1 \dots b_{k+1} c_1 \dots c_{l+1}),\ m = 1, \dots$$

$$\sim L_{m+k}(a_1 \dots a_m b_1 \dots b_k) \wedge \sim L_{m+l+1}(a_1 \dots a_m c_1 \dots c_{l+1}) \wedge$$
$$\wedge L_{m+k+l+1}(a_1 \dots a_m b_1 \dots b_k c_1 \dots c_{l+1})$$
$$\underset{W(m,k,l+1)}{\Rightarrow} \bigvee_x [\sim L_{m+1}(a_1 \dots a_m x) \wedge L_{m+k+1}(a_1 \dots a_m b_1 \dots b_k x) \wedge$$
$$\wedge L_{m+l+2}(a_1 \dots a_m c_1 \dots c_{l+2} x)],$$

$$L_{m+k+1}(a_1 \dots a_m b_1 \dots b_k x) \Rightarrow L_{m+k+2}(a_1 \dots a_m b_1 \dots b_{k+1} x).$$

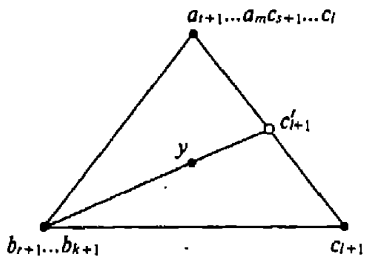
11. $V(1, k+1, l+1)$, $k, l = 1, \dots$

$$L_{k+l+1}(b_1 \dots b_k c_1 \dots c_{l+1}) \Rightarrow L_{k+l+2}(a_1 b_1 \dots b_k c_1 \dots c_{l+1})$$
$$\underset{10}{\Rightarrow} U(a_1 b_1 \dots b_{k+1} c_1 \dots c_{l+1}),$$

$$L_{k+l+2}(b_1 \dots b_{k+1} c_1 \dots c_{l+1}) \underset{4.8.9}{\Rightarrow} U(\bar{a}_1 b_1 \dots b_{k+1} c_1 \dots c_{l+1}).$$

12. $V(m, k+1, l+1) \wedge y \in I(a_1 \dots a_t b_1 \dots b_r c_1 \dots c_s) \wedge$
$$\wedge\, y \in I(a_{t+1} \dots a_m b_{r+1} \dots b_{k+1} c_{s+1} \dots c_{l+1}) \Rightarrow U(a_1 \dots a_m b_1 \dots b_{k+1} c_1 \dots c_{l+1})$$
$$m, k, l = 1, \dots;\ t = 0, \dots, m-1;\ r = 1, \dots, k;\ s = 1, \dots, l.$$

Remark. Case $t = m$ coincides with the case $t = 0$.



T.2.4.7

$$\sim L_{m+l+1}(a_1 \dots a_m c_1 \dots c_l c'_{l+1}) \text{ because}$$

$$\sim L_{m+l+1-t-s}(a_{t+1} \dots a_m c_{s+1} \dots c_l c'_{l+1}) \wedge$$
$$\wedge L_{m+l+2-t-s}(a_{t+1} \dots a_m c_{s+1} \dots c_l c'_{l+1} c_{l+1}) \wedge L_{m+l+1}(a_1 \dots a_m c_1 \dots c_l c'_{l+1})$$
$$\underset{\text{T.1.4.4}}{\Rightarrow} L_{m+l+1}(a_1 \dots a_m c_1 \dots c_{l+1}), \quad \text{a contradiction.}$$

$$L_{t+r+s+1}(a_1 \dots a_t b_1 \dots b_r c_1 \dots c_s y) \wedge L_{k+3-r}(b_{r+1} \dots b_{k+1} c'_{l+1} y)$$
$$\underset{\text{T.1.4.4}}{\Rightarrow} L_{t+k+s+2}(a_1 \dots a_t b_1 \dots b_{k+1} c_1 \dots c_s c'_{l+1})$$
$$\Rightarrow L_{m+k+l+1}(a_1 \dots a_{m-1} b_1 \dots b_{k+1} c_1 \dots c_l c'_{l+1}),$$

$$\sim L_{m+k+1}(a_1 \dots a_m b_1 \dots b_{k+1}) \wedge \sim L_{m+l+1}(a_1 \dots a_m c_1 \dots c_l c'_{l+1}) \wedge$$
$$\wedge L_{m+k+l+1}(a_1 \dots a_{m-1} b_1 \dots b_{k+1} c_1 \dots c_l c'_{l+1}) \wedge V(m, k+1, l+1)$$
$$\underset{\text{def}}{\Rightarrow} \bigvee_x [\sim L_{m+1}(a_1 \dots a_m x) \wedge L_{m+k+2}(a_1 \dots a_m b_1 \dots b_{k+1} x) \wedge$$
$$\wedge L_{m+l+2}(a_1 \dots a_m c_1 \dots c_l c'_{l+1} x)],$$

$$\sim L_{m-t+l-s+1}(a_{t+1} \dots a_m c_{s+1} \dots c_l c'_{l+1}) \wedge$$
$$\wedge L_{m-t+l-s+2}(a_{t+1} \dots a_m c_{s+1} \dots c_l c'_{l+1} c_{l+1}) \wedge L_{m+l+2}(a_1 \dots a_m c_1 \dots c_l c'_{l+1} x)$$
$$\underset{\text{T.1.4.4}}{\Rightarrow} L_{m+l+2}(a_1 \dots a_m c_1 \dots c_{l+1} x).$$

13. $W(1, k+1, l+1)$, $k, l = 1, \dots$

See 11, 10, 2, 3, 5, 12 and (5) and T.2.5.1. Assume $W(m, k+1, l+1)$ for a certain $m \geqslant 1$. We have to prove $W(m+1, k+1, l+1)$.

14. $V(m+1, k+1, l+1)$.

$$\sim L_{m+k+2}(a_1 \dots a_{m+1} b_1 \dots b_{k+1}) \Rightarrow \sim L_{m+k+1}(a_1 \dots a_m b_1 \dots b_{k+1}),$$
$$\sim L_{m+k+2}(a_1 \dots a_{m+1} c_1 \dots c_{l+1}) \Rightarrow \sim L_{m+k+1}(a_1 \dots a_m c_1 \dots c_{l+1}),$$

$$\sim L_{m+k+1}(a_1 \dots a_m b_1 \dots b_{k+1}) \wedge \sim L_{m+l+1}(a_1 \dots a_m c_1 \dots c_{l+1}) \wedge$$
$$\wedge L_{m+k+l+2}(a_1 \dots a_m b_1 \dots b_{k+1} c_1 \dots c_{l+1})$$
$$\underset{W(m,k+1,l+1)}{\Rightarrow} \bigvee_x [\sim L_{m+1}(a_1 \dots a_m x) \wedge L_{m+k+2}(a_1 \dots a_m b_1 \dots b_{k+1} x) \wedge$$
$$\wedge L_{m+l+2}(a_1 \dots a_m c_1 \dots c_{l+1} x)],$$

$\sim L_{m+2}(a_1 \dots a_{m+1} x)$ because

$$\sim L_{m+1}(a_1 \dots a_m x) \wedge L_{m+2}(a_1 \dots a_{m+1} x) \wedge L_{m+k+2}(a_1 \dots a_m b_1 \dots b_{k+1} x)$$
$$\underset{\text{T.1.4.4}}{\Rightarrow} L_{m+k+2}(a_1 \dots a_{m+1} b_1 \dots b_{k+1}) \quad \text{a contradiction.}$$

$$L_{m+k+2}(a_1 \dots a_m b_1 \dots b_{k+1} x) \Rightarrow L_{m+k+3}(a_1 \dots a_{m+1} b_1 \dots b_{k+1} x),$$
$$L_{m+l+2}(a_1 \dots a_m c_1 \dots c_{l+1} x) \Rightarrow L_{m+l+3}(a_1 \dots a_{m+1} c_1 \dots c_{l+1} x).$$

15. $W(m+1, k+1, l+1)$.
See 10, 3, 4, 7, 12, 14 and (5). ∎

By C.1.4.8, T.1.1 implies

T.1.2. $A_m = B_r \cap C_s \wedge B_r \cup C_s \subset D_p \Rightarrow m \geqslant r+s-p$ *for* $m, r, s, p = 1, \dots$

Appendix

by JERZY BEDNARCZUK

Representation theorem for the theory of E-sequences

We are going to prove a representation theorem for the theory of E-sequences. In fact, we shall deal with a slightly wider class of relational sequences than that of E-sequences, taking for R_2 any binary equivalence relation, not necessarily identity(1). That is why in §1 we formulate definitions of (generalized) E-sequences and their equivalence classes. For the sake of simplicity we shall refer to these generalized relational sequences as E-sequences.

We shall use some statements on E-sequences proved in Chapter I, since they remain valid for generalized E-sequences (the proofs are almost the same).

§1. E-sequences

DEFINITION. Let $E = (S, R_1, R_2, \ldots)$ where $S \neq \emptyset$ and $R_n \subset S^n$, $n = 1, \ldots$ E is an E-sequence if and only if the following conditions hold:

I $R_1 = \emptyset$,

II $R_2(aa)$ for each $a \in S$,

I_n, II_n (comp. §1 Chap. I), $n = 1, \ldots$

T.1.1. *R_2 is a binary equivalence relation.*

Proof. (a) Let $R_2(ab)$.

$$R_2(ab) \wedge R_2(aa) \underset{\mathrm{I,II}_2}{\Rightarrow} R_2(ba).$$

(b) Let $R_2(ab)$ and $R_2(bc)$,

$$R_2(ab) \underset{(a)}{\Rightarrow} R_2(ba),$$

$$R_2(ba) \wedge R_2(bc) \underset{\mathrm{I,II}_2}{\Rightarrow} R_2(ac).$$

(1) As follows from the notes of W. Szmielew, she considered the possibility of admitting such a definition of E-sequence.

DEFINITION.

$$[a_1 \dots a_{n-1}] \underset{\text{def}}{=} \{x \in S\colon R_n(a_1 \dots a_{n-1} x)\} \quad \text{for} \quad n = 2, \dots$$

DEFINITION.

$$E_1 \underset{\text{def}}{=} \{\emptyset\};$$

for $n = 2, \dots,$

$$E_n = \{[a_1 \dots a_{n-1}]\colon \sim R_{n-1}(a_1 \dots a_{n-1}) \vee R_{n-1} = S^{n-1}\}.$$

Any element $[a_1 \dots a_{n-1}]$ of E_n will be called the *equivalence class of* R_n *spanned by the elements* $a_1, \dots, a_{n-1}$.

T.1.2. $R_n = S^n \wedge m \geqslant n \Rightarrow R_m = S^m$.

Proof. It is a direct consequence of I_k $(k = 2, \dots)$.

By T.1.2 we obtain

C.1.3. $R_n = S^n \wedge m \geqslant n \Rightarrow E_m = \{S\}$.

§ 2. H-sequences

DEFINITION. Let $H = (S, H_1, H_2, \dots)$ where $S \neq \emptyset$ and $H_n \subset 2^S$, $n = 1, 2, \dots$ H is an *H-sequence*(2) if and only if the following conditions hold:

Ax1 $H_1 = \{\emptyset\}$,

$\mathrm{Ax2}_n$ $\bigwedge_{A \in H_n} \bigwedge_{a \in S} \bigvee_{B \in H_{n+1}} A \cup \{a\} \subset B$,

$\mathrm{Ax3}_n$ $\bigwedge_{A \in H_n} \{a_1, \dots, a_n\} \not\subset A \wedge B, C \in H_{n+1} \wedge \{a_1, \dots, a_n\} \subset B \cap C \Rightarrow B = C$,

$\mathrm{Ax4}_n$ $A, B \in H_n \wedge A \subset B \Rightarrow A = B$, $n = 1, 2, \dots$

Properties of H-sequences

In the following statements T.2.1 – T.2.12, $H = (S, H_1, H_2, \dots)$ is assumed to be an H-sequence.

T.2.1. $\bigwedge_{a_1 \dots a_n \in S} \bigvee_{A \in H_{n+1}} \{a_1 \dots a_n\} \subset A$.

Proof by induction on n. For $n = 1$,

$$\emptyset \in H_1 \underset{\mathrm{Ax2}_1}{\Rightarrow} \bigvee_{A \in H_2} \{a_1\} \cup \emptyset \subset A.$$

Assume $\{a_1 \dots a_n\} \subset B \in H_{n+1}$. By $\mathrm{Ax3}_{n+1}$,

$$\bigvee_{A \in H_{n+2}} B \cup \{a_{n+1}\} \subset A.$$

(2) "H" for hyperplane.

Hence $\{a_1 \dots a_{n+1}\} \subset A \in H_{n+2}$. ∎

By T.2.1 we get

C.2.2. $n \geqslant 2 \Rightarrow \bigcup H_n = S$.

T.2.3. $A, B \in H_2 \wedge A \cap B \neq \emptyset \Rightarrow A = B$.

Proof. Let $x \in A \cap B$. By Ax1 $\bigwedge_{C \in H_1} \{x\} \not\subset C$,

$$\bigwedge_{C \in H_1} \{x\} \not\subset C \wedge \{x\} \subset A \cap B \underset{\mathrm{Ax3}_1}{\Rightarrow} A = B. \quad ∎$$

T.2.4. $A \subset S \wedge \bigwedge_{B \in H_n} A \not\subset B \Rightarrow \bar{A} \geqslant n$.

Proof. It is a direct consequence of T.2.1.

T.2.5. $A \in H_n \wedge n \leqslant m \Rightarrow \bigvee_{B \in H_m} A \subset B$.

Proof. It is a direct consequence of $\mathrm{Ax2}_k$, $k = 1, \dots, m-1$.

T.2.6. $S \in H_n \wedge n \leqslant m \Rightarrow H_m = \{S\}$.

Proof.

Case 1. $n = m$; $A \in H_m \wedge S \in H_m \underset{\mathrm{Ax4}_m}{\Rightarrow} A = S$.

Case 2. $n < m$; $S \in H_n \wedge n < m \underset{\mathrm{T.2.5}}{\Rightarrow} S \in H_m$. ∎

T.2.7. $\bigwedge_{A \in H_n} \{a_1 \dots a_n\} \not\subset A \Rightarrow \bigvee_{B \in H_{n+1}} \{a_1 \dots a_n\} \subset B$.

Proof. It is a consequence of T.2.1 and $\mathrm{Ax3}_{n+1}$.

T.2.8. $A \in H_n \wedge B \in H_{n+1} \wedge B \subset A \Rightarrow A = B = S$.

Proof. (1) By T.2.5, $\bigvee_{B_1 \in H_{n+1}} A \subset B_1$. Then $B \subset B_1$

$$B \subset B_1 \wedge B, B_1 \in H_{n+1} \underset{\mathrm{Ax4}_{n+1}}{\Rightarrow} B = B_1,$$

$$B \subset A \subset B_1 \wedge B = B_1 \Rightarrow A = B.$$

(2) Let $a \in S$. By $\mathrm{Ax2}_n$,

$$\bigvee_{B_1 \in H_{n+1}} A \cup \{a\} \subset B_1.$$

Then $B \subset A \subset B_1$. Hence $\bigwedge_{a \in S} a \in B$, and thus $S = B$. ∎

By T.2.5 and T.2.8 we obtain

C.2.9. $A \in H_n \wedge B \in H_m \wedge n < m \wedge B \subset A \Rightarrow A = B = S$.

T.2.10. $A \subset S \wedge \bigwedge_{B \in H_n} A \not\subset B \Rightarrow \bigvee_{a_1 \dots a_n \in A} \bigwedge_{C \in H_n} \{a_1 \dots a_n\} \not\subset C$.

Proof. By T.2.4 we have $\bar{A} \geqslant n$. Assume

$$A \subset S \wedge \bigwedge_{B \in H_n} A \not\subset B \wedge \bigwedge_{a_1 \dots a_n \in A} \bigvee_{C \in H_n} \{a_1 \dots a_n\} \subset C.$$

We shall prove by induction (downward) on k that

$$1 \leqslant k \leqslant n \Rightarrow \bigwedge_{a_1 \ldots a_k \in A} \bigvee_{B \in H_k} \{a_1 \ldots a_k\} \subset B.$$

By the assumption, the implication holds for $n = k$. Assume then that for a certain k $(1 < k \leqslant n)$

$$\bigwedge_{a_1 \ldots a_k \in A} \bigvee_{B \in H_k} \{a_1 \ldots a_k\} \subset B.$$

We want to show that

$$\bigwedge_{a_1 \ldots a_{k-1} \in A} \bigvee_{C \in H_{k-1}} \{a_1, \ldots, a_{k-1}\} \subset C.$$

Let $a_1, \ldots, a_{k-1} \in A$

$$\bar{A} \geqslant n > k-1 \Rightarrow A \backslash \{a_1, \ldots, a_{k-1}\} \neq \emptyset.$$

Let $b_1 \in A \backslash \{a_1, \ldots, a_{k-1}\}$. By induction we have

$$\bigvee_{B_1 \in H_k} \{a_1, \ldots, a_{k-1}, b_1\} \subset B_1,$$

$$\bigwedge_{B \in H_n} A \not\subset B \underset{\text{T.2.5}}{\Rightarrow} A \backslash B_1 \neq \emptyset.$$

Let $b_2 \in A \backslash B_1$. By induction we get

$$\bigvee_{B_2 \in H_k} \{a_1, \ldots, a_{k-1}, b_2\} \subset B_2$$

$$b_2 \in B_2 \backslash B_1 \Rightarrow B_1 \neq B_2,$$

$$\{a_1, \ldots, a_{k-1}\} \subset B_1 \cap B_2 \wedge B_1, B_2 \in H_k \wedge B_1 \neq B_2$$

$$\underset{\text{Ax}3_{k-1}}{\Rightarrow} \bigvee_{C \in H_{k-1}} \{a_1, \ldots, a_{k-1}\} \subset C.$$

In particular $\bigwedge_{a_1 \in A} \bigvee_{D \in H_1} a_1 \in D$,

$$\bigwedge_{a \in A} \bigvee_{D \in H_1} a_1 \in D \underset{\text{Ax1}}{\Rightarrow} A = \emptyset \Rightarrow \bigwedge_{B \in H_n} A \subset B,$$

a contradiction. ∎

T.2.11. $H_n \neq \{S\} \wedge A \in H_{n+1} \Rightarrow \bigvee_{a_1, \ldots, a_n \in A} \bigwedge_{C \in H_n} \{a_1 \ldots, a_n\} \not\subset C.$

Proof. $A \in H_{n+1} \wedge H_n \neq \{S\} \underset{\text{T.2.8}}{\Rightarrow} \bigwedge_{B \in Hn} A \not\subset B$

$$\underset{\text{T.2.10}}{\Rightarrow} \bigvee_{a_1, \ldots, a_n \in A} \bigwedge_{C \in H_n} \{a_1, \ldots, a_n\} \not\subset C. \quad ∎$$

The statement T.2.11 it is a theorem on the existence of a "base".

T.2.12. $H_n = \{S\} \Leftrightarrow \bigwedge_{a_1, \ldots, a_n \in S} \bigvee_{A \in H_n} \{a_1, \ldots, a_n\} \subset A.$

Proof. $\Rightarrow$ is obvious.

$\Leftarrow$

$$\bigwedge_{a_1,\dots,a_n\in S} \bigvee_{A\in H_n} \{a_1,\dots,a_n\} \subset A \underset{\text{T.2.10}}{\Rightarrow} S \not\subset S \vee \bigvee_{B\in H_n} S \subset B$$

$$\Rightarrow \bigvee_{B\in H_n} S \subset B \Rightarrow S\in H_n \underset{\text{T.2.6}}{\Rightarrow} H_n = \{S\}. \quad \blacksquare$$

§ 3. Functions F_n and G_n

Given a natural n, we define two functions, F_n and G_n. The first one to every set of subsets of the universe S assigns some n-ary relation R on S. The second one, conversely, to every n-ary relation on S assigns some set of subsets of S.

Definition. For any natural n consider the function $F_n\colon 2^{2^S} \to 2^{S^n}$ defined as follows:

for any $H \subset 2^S$ let $F_n(H) = R$, where

$$R(a_1,\dots,a_n) \underset{\text{def}}{\Leftrightarrow} \bigvee_{A\in H} \{a_1,\dots,a_n\} \subset A.$$

T.3.1. *If $H = (S, H_1, H_2, \dots)$ is an H-sequence, then $(S, F_1(H_1), F_2(H_2), \dots)$ is an E-sequence.*

Proof.

I $H_1 = \{\emptyset\} \Rightarrow F(H_1) = \emptyset$.

II By C.2.2, $\bigcup H_2 = S$,

$$\bigcup H_2 = S \Rightarrow \bigwedge_{a\in S} F_2(H_2)(a,a).$$

I_n It is a simple consequence of $\mathrm{Ax2}_n$.

II_n Let $\sim F_n(H_n)(a_1,\dots,a_n)$ and $F_{n+1}(H_{n+1})(a_1,\dots,a_n,x_i)$ for $i = 1,\dots,n+1$.

$$F_{n+1}(H_{n+1})(a_1,\dots,a_n,x_i) \Rightarrow \bigvee_{B_i\in H_{n+1}} \{a_1,\dots,a_n,x_i\} \subset B_i, \qquad i = 1,\dots,n+1,$$

$$\sim F_n(H_n)(a_1,\dots,a_n) \Rightarrow \bigwedge_{A\in H_n} \{a_1,\dots,a_n\} \not\subset A,$$

$$\bigwedge_{A\in H_n} \{a_1,\dots,a_n\} \not\subset A \underset{\text{T.2.7}}{\Rightarrow} \dot{\bigvee}_{B\in H_{n+1}} \{a_1,\dots,a_n\} \subset B,$$

$$\dot{\bigvee}_{B\in H_{n+1}} \{a_1,\dots,a_n\} \subset B \bigwedge_{i=1}^{n+1} \{a_1,\dots,a_n,x_i\} \subset B_i \in H_{n+1} \Rightarrow B_1 = \dots = B_{n+1} = B.$$

Hence $\{x_1,\dots,x_{n+1}\} \subset B \in H_{n+1}$ and thus $F_{n+1}(H_{n+1})(x_1,\dots,x_{n+1})$. $\blacksquare$

We shall refer to the sequence $(S, F_1(H_1), F_2(H_2), \dots)$ as the E-sequence induced by $(S, H_1, H_2, \dots)$.

DEFINITION. For any natural n consider the function $G_n\colon 2^{S^n} \to 2^{2^S}$ defined as follows:

for any $R \subset S^n$ let $G_n(R) = H$, where

$$A \in H \underset{\text{def}}{\Leftrightarrow} \Big[(R \neq S^n \Rightarrow A \neq S) \wedge \bigvee_{a_1,\ldots,a_{n-1} \in S} A = \{x \in S\colon R(a_1,\ldots,a_{n-1})\}\Big].$$

T.3.2. *If* $H = (S, H_1, H_2, \ldots)$ *is an* H-*sequence, then* $G_n F_n(H_n) = H_n$, $n = 1, 2, \ldots$

Proof.

$n = 1$:

$$H_1 = \{\emptyset\} = G_1 F_1(H_1).$$

$n \geqslant 2$:

Case 1. $H_{n-1} = \{S\}$.

$$H_{n-1} = \{S\} \underset{\text{T.2.6}}{\Rightarrow} H_n = \{S\} \Rightarrow F_n(H_n) = S^n \Rightarrow G_n F_n(H_n) = \{S\}.$$

Hence $G_n F_n(H_n) = \{S\} = H_n$.

Case 2. $H_{n-1} \neq \{S\}$. We shall prove that

$$A \in G_n F_n(H_n) \Leftrightarrow A \in H_n.$$

$\Rightarrow$

Let $A \in G_n F_n(H_n)$,

$$H_{n-1} \neq \{S\} \underset{\text{T.2.12}}{\Rightarrow} F_{n-1}(H_{n-1}) \neq S^{n-1},$$

$$A \in G_n F_n(H_n) \wedge F_{n-1}(H_{n-1}) \neq S^{n-1}$$

$$\underset{\text{In}}{\Rightarrow} \bigvee_{a_1,\ldots,a_{n-1}} A = \{x\colon F_n(H_n)(a_1,\ldots,a_{n-1}x) \wedge \sim F_{n-1}(H_{n-1})(a_1,\ldots,a_{n-1})\},$$

$$\sim F_{n-1}(H_{n-1})(a_1,\ldots,a_{n-1}) \underset{\text{T.2.7}}{\Rightarrow} \bigvee_{A_1 \in H_n} \{a_1,\ldots,a_{n-1}\} \subset A_1.$$

We shall prove that $A = A_1$.

Let $x \in A_1$, then $\{a_1,\ldots,a_{n-1},x\} \subset A_1$,

$$\{a_1,\ldots,a_{n-1},x\} \subset A_1 \in H_n \Rightarrow F_n(H_n)(a_1,\ldots,a_{n-1},x) \Rightarrow x \in A.$$

Let $x \in A$,

$$x \in A \Rightarrow F_n(H_n)(a_1,\ldots,a_{n-1},x) \Rightarrow \bigvee_{A_2 \in H_n} \{a_1,\ldots,a_{n-1},x\} \subset A_2.$$

Then $\{a_1,\ldots,a_{n-1}\} \subset A_1 \cap A_2$. By T.2.7 $A_1 = A_2$, therefore $x \in A_1$.

$\Leftarrow$

Let $A \in H_n$,

$$H_{n-1} \neq \{S\} \underset{\text{T.2.11}}{\Rightarrow} \bigvee_{a_1,\ldots,a_{n-1} \in A} \bigwedge_{B \in H_{n-1}} \{a_1,\ldots,a_{n-1}\} \not\subset B$$

$$\Rightarrow \sim F_{n-1}(H_{n-1})(a_1,\ldots,a_{n-1}).$$

Let $B = \{x\colon F_n(H_n)(a_1,\ldots,a_{n-1},x)\}$. Then $B \in G_n F_n(H_n)$.

We shall prove that $A = B$.

Let $x \in A$, then $\{a_1, \ldots, a_{n-1}, x\} \subset A \in H_n$,

$$\{a_1, \ldots, a_{n-1}, x\} \subset A \in H_n \Rightarrow F_n(H_n)(a_1, \ldots, a_{n-1}, x) \Rightarrow x \in B.$$

Let $x \in B$,

$$x \in B \Rightarrow F_n(H_n)(a_1, \ldots, a_{n-1}, x) \Rightarrow \bigvee_{A_1 \in H_n} \{a_1, \ldots, a_{n-1}, x\} \subset A_1,$$

then $\{a_1, \ldots, a_{n-1}\} \subset A \cap A_1$. By T.2.7 $A = A_1$. Therefore $x \in A$. ∎

§4. Functions G'_n and G''_n

Definition. For any natural n consider the function G''_n: $2^{S^n} \to 2^{2^S}$ defined as follows:

for any $R \subset S^n$ let $G''_n(R) = H$, where

$$A \in H \underset{\text{def}}{\Leftrightarrow} R|_A = A^n.$$

Definition. For any natural n consider the function G'_n: $2^{S^n} \to 2^{2^S}$ defined as follows:

for any $R \subset S^n$ let $G'_n(R) = H$, where

$$A \in H \underset{\text{def}}{\Leftrightarrow} \left[A \in G''_n(R) \wedge \bigwedge_B \left((B \in G''_n(H) \wedge A \subset B) \Rightarrow A = B\right)\right].$$

In the following statements T.4.1–T.4.4, $E = (S, R_1, R_2, \ldots)$ is assumed to be an E-sequence.

T.4.1. $\left(Q \subset G''_n(R_n) \wedge \bigwedge_{A, B \in Q} (A \subset B \vee B \subset A)\right) \Rightarrow \bigcup Q \in G''_n(R_n)$.

Proof. Let $\{a_1, \ldots, a_n\} \subset \bigcup Q$.

$$a_i \in A_i \bigwedge_{i=1}^{n} A_i \in Q \underset{\text{ass}}{\Rightarrow} \bigvee_{1 \leq k \leq n} \{a_1, \ldots, a_n\} \subset A_k \in G''_n(R_n) \Rightarrow R_n(a_1, \ldots, a_n). \quad ∎$$

T.4.2. $A \in G''_n(R_n) \Rightarrow \bigvee_{B \in G'_n(R_n)} A \subset B$.

Proof. By the Kuratowski–Zorn lemma, this theorem is a consequence of T.4.1.

T.4.3. $R_n(a_1, \ldots, a_n) \Leftrightarrow \bigvee_{A \in G'_n(R_n)} \{a_1, \ldots, a_n\} \subset A$.

Proof. $\Rightarrow$

$$R_n(a_1, \ldots, a_n) \underset{\text{def}}{\Rightarrow} \{a_1, \ldots, a_n\} \in G''_n(R_n) \underset{\text{T.4.2}}{\Rightarrow} \bigvee_{A \in G'_n(R_n)} \{a_1, \ldots, a_n\} \subset A.$$

$\Leftarrow$

$$\{a_1, \ldots, a_n\} \subset A \wedge A \in G'_n(R_n) \underset{\text{def}}{\Rightarrow} \{a_1, \ldots, a_n\} \subset A \wedge A \in G''_n(R_n)$$
$$\underset{\text{def}}{\Rightarrow} R_n(a_1, \ldots, a_n). \quad ∎$$

T.4.4. $(S, G_1'(R_1), G_2'(R_2), \ldots)$ *is an H-sequence.*

Proof.

Ax1 $R_1 = \emptyset \Rightarrow G_1''(R_1) = \{\emptyset\} \Rightarrow G_1'(R_1) = \{\emptyset\}$.

Ax2 Let $A \in G_n'(R_n)$, $a \in S$ and $\{a_1, \ldots, a_{n+1}\} \subset A \cup \{a\}$.

Case 1. $a_i \neq a$ for $i = 1, \ldots, n$,

$$\{a_1, \ldots, a_n\} \subset A \wedge A \in G_n'(R_n) \Rightarrow R_n(a_1, \ldots, a_n) \underset{\mathrm{I}_{n+1}}{\Rightarrow} R_{n+1}(a_1, \ldots, a_{n+1}).$$

Case 2. $a_n = a_{n+1} = a$,

$$a_n = a_{n+1} \underset{\mathrm{T.1.1.2}}{\Rightarrow} R_{n+1}(a_1, \ldots, a_{n+1}).$$

Hence

$$\{a_1, \ldots, a_{n+1}\} \subset A \cup \{a\} \Rightarrow R_{n+1}(a, \ldots, a_{n+1}).$$

Therefore $A \cup \{a\} \in G_{n+1}''(R_{n+1})$,

$$A \cup \{a\} \in G_{n+1}''(R_{n+1}) \underset{\mathrm{T.4.2}}{\Rightarrow} \bigvee_{B \in G_{n+1}'(R_{n+1})} A \cup \{a\} \subset B.$$

Ax4 Compare definition of G_n'.

Ax3 Let $B, C \in G_{n+1}'(R_{n+1})$, $\{a_1, \ldots, a_n\} \subset B \cap C$ and $\bigwedge_{A \in G_n'(R_n)} \{a_1, \ldots, a_n\} \not\subset A$.

$$\bigwedge_{A \in G_n'(R_n)} \{a_1, \ldots, a_n\} \not\subset A \underset{\mathrm{T.4.3}}{\Rightarrow} \sim R_n(a_1, \ldots, a_n).$$

Let $D = \{x \in S\colon R_{n+1}(a_1, \ldots, a_n, x)\}$. By II_n, $D \in G_{n+1}''(R_{n+1})$.

We shall prove that $D \in G_{n+1}'(R_{n+1})$.

Assume that $\bigvee_{D_1} (D_1 \in G_{n+1}''(R_{n+1}) \wedge D \subsetneqq D_1)$.

Let $x \in D_1 \backslash D$, then

$$\{a_1, \ldots, a_n, x\} \subset D_1 \wedge D_1 \in G_{n+1}''(R_{n+1}),$$

$$D_1 \in G_{n+1}''(R_{n+1}) \Rightarrow R_{n+1}(a_1, \ldots, a_n, x) \Rightarrow x \in D,$$

a contradiction.

Now we shall prove that $B = D$. Assume that $B \neq D$.

$$B \neq D \wedge B, D \in G_{n+1}'(R_{n+1}) \Rightarrow B \backslash D \neq \emptyset.$$

Let $x \in B \vee D$, then $\{a_1, \ldots, a_n, x\} \subset B \wedge B \in G_{n+1}''(R_{n+1})$.

$$\{a_1, \ldots, a_n, x\} \subset B \wedge B \in G_{n+1}''(R_{n+1}) \Rightarrow R_{n+1}(a_1, \ldots, a_n, x) \Rightarrow x \in D,$$

a contradiction.

Analogously, we can prove that $C = D$. Hence $B = C$. ∎

§5. Representation theorem for the theory of E-sequences

We are now going to prove a representation theorem for the theory of E-sequences. The theorem says that the class of models of this theory, i.e. the class of E-sequences, coincides with the class of sequences induced by some H-sequences. In fact, the theorem states something more, since it also describes the class of all H-sequences by means of E-sequences.

Let $\boldsymbol{E}$ be the class of all E-sequences, and let $\boldsymbol{H}$ be the class of all H-sequences.

Theorems T.3.1 and T.4.4 enable us to define the functions F and G':

$F\colon \boldsymbol{H} \to \boldsymbol{E}$,

$$F((S, H_1, H_2, \ldots)) \underset{\text{def}}{=} (S, F_1(H_1), F_2(H_2), \ldots),$$

$G'\colon \boldsymbol{E} \to \boldsymbol{H}$,

$$G'((S, R_1, R_2, \ldots)) \underset{\text{def}}{=} (S, G'_1(R_1), G'_2(R_2), \ldots).$$

In the following statements T.5.1–T.5.4, $E = (S, R_1, R_2, \ldots)$ and H are assumed to be an E-sequence and an H-sequence, respectively.

T.5.1, $FG'(E) = E$.

Proof. This theorem is a direct consequence of T.4.4 and T.4.3.

T.5.2. $G_n(R_n) = G'_n(R_n)$.

Proof. $G_n(R_n) \underset{\text{T.5.1}}{=} G_n F_n G'_n(R_n) \underset{\text{T.3.2}}{=} G'_n(R_n)$. ∎

Statement T.5.2 says that, for E-sequences, equivalence classes coincide with the maximal linearly dependent subsets of S.

T.5.3. $(S, G_1(R_1), G_2(R_2), \ldots)$ *is an H-sequence.*

Proof. This theorem is a direct consequence of T.4.4 and T.5.2.

Theorem T.5.3 enables us to define the function G:

$G\colon \boldsymbol{E} \to \boldsymbol{H}$,

$$G((S, R_1, R_2, \ldots)) \underset{\text{def}}{=} (S, G_1(R_1), G_2(R_2), \ldots).$$

T.5.4. $FG(E) = E$.

Proof. $FG(E) \underset{\text{T.3.T.5.2}}{=} FG'(E) \underset{\text{T.5.1}}{=} E$. ∎

T.5.5. $GF(H) = H$.

Proof. This theorem is a direct consequence of T.3.2 and T.3.1.

By T.5.4 and T.5.5 the functions $F\colon \boldsymbol{H} \to \boldsymbol{E}$ and $G\colon \boldsymbol{E} \to \boldsymbol{H}$ are mutually inverse (definitional) functions. This fact can be expressed as follows:

REPRESENTATION THEOREM. *The theory of E-sequences and the theory of H-sequences are definitionally equivalent* ([3]).

([3]) Compare [6], p. 56 or [7], Introduction.

Remark. The axiom $\mathrm{Ax3}_n$ can be replaced by the following simpler axiom $\mathrm{Ax3}'_n$:

$$\mathrm{Ax3}'_n \quad A \in H_n \wedge B \in H_m \Rightarrow A \cap B \in \bigcup_{k=1}^{\min(n,m)} H_k.$$

Proof. Assume Ax1, $\mathrm{Ax2}_n$, $\mathrm{Ax3}'_n$, $\mathrm{Ax4}_n$. We have to prove $\mathrm{Ax3}_n$. Let

$$\bigwedge_{A \in H_n} \{a_1, \ldots, a_n\} \not\subset A \wedge B, C \in H_{n+1} \wedge \{a_1, \ldots, a_n\} \subset B \cap C;$$

then by $\mathrm{Ax3}'_n$, $B \cap C \in H_k$, $k \leqslant n+1$.

Case 1. $k = n+1$.

By $\mathrm{Ax4}_{n+1}$ $B = B \cap C = C$.

Case 2. $k < n+1$.

By T.2.5, $\bigvee_{D \in H_n} B \cap C \subset D$. Then $\{a_1, \ldots, a_n\} \subset D \wedge D \in H_n$,

a contradiction.

Assume Ax1, $\mathrm{Ax2}_n$, $\mathrm{Ax3}_n$, $\mathrm{Ax4}_n$. We have to prove $\mathrm{Ax3}'_n$. Let

$$H = (S, H_1, H_2, \ldots) \wedge H \in \boldsymbol{H} \wedge A \in H_n \wedge B \in H_m.$$

By T.3.2, $A \in G_n F_n(H_n) \wedge B \in G_m F_m(H_m)$. Then by C.1.1.17

$$A \cap B \in \bigcup_{k=1}^{\min(n,m)} G_k F_k(H_k),$$

and by T.3.2

$$A \cap B \in \bigcup_{k=1}^{\min(n,m)} H_k. \quad \blacksquare$$

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