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# STEENROD HOMOLOGY *

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## 1. Introduction

This paper has two purposes. The first one (§§ 2–3) is to give a brief account of the origins and development of the Steenrod homology theories. We discuss only ordinary homologies with constant coefficients and do not consider the extraordinary theories like the Steenrod $K$-theory (for this topic see [EH], [KKS] and [KS]). Moreover, we have in mind only homologies of the "first kind" and disregard the ones of the "second kind" (which belong to proper homotopy theory). This means that for complexes we always consider homologies based on finite chains and not the ones based on infinite chains.

The second and main purpose of the paper (§§ 4–10) is to outline a new approach to Steenrod homology, developed by the authors in a series of recent papers [LM3–8], [M1–3] and announced in [LM1–2]. The special feature of our homology is that it applies to arbitrary spaces and is invariant with respect to strong shape. For pairs $(X, A)$, where $X$ is paracompact and $A$ is closed, it satisfies all the Eilenberg–Steenrod axioms. Furthermore, on polyhedra it agrees with the singular theory and on metric compacta it agrees with the classical Steenrod theory [St].

We wish to express our thanks to several colleagues from whom we obtained valuable information. In particular, this applies to N. A. Berikashsvili, L. D. Mdzinarishvili, Z. R. Miminoshvili, S. V. Petkova, E. G. Sklyarenko and T. Watanabe. Our thanks also go to the Banach Center, where (during the Topology Semester 1984) many of our discussions took place.

* This paper is in final form and no version of it will be submitted for publication elsewhere.

## 2. Motivation and the various constructions

Singular homology and cohomology are well-established theories, which satisfy the seven Eilenberg–Steenrod axioms [ES]. However, they have serious defects, when applied to spaces, which are not locally nice. In particular, singular theory does not behave properly with respect to dimension, extension of maps (obstruction theory) and duality. To demostrate this, it suffices to recall the well-known examples of metric compacta $X$, which admit nontrivial singular homology groups in dimensions greater than $\dim X$ [BM].

In order to correct these defects, Čech homology and cohomology were introduced. The Čech homology group $\check{H}_p(X; G)$ (cohomology group $\check{H}^p(X; G)$) is defined as the inverse (direct) limit of singular homology groups $H_p(N; G)$ (cohomology groups $H^p(N; G)$) of the nerves $N$ of normal open coverings of the space $X$. An equivalent approach (sometimes called the functional approach), instead of the nerves, uses the Vietoris complexes associated with the coverings [Do]. For locally nice spaces, e.g., polyhedra or ANR's, Čech and singular theories coincide.

The Čech cohomology is also well-established. It satisfies the Eilenberg–Steenrod axioms and does not have the above mentioned defects. In particular, concerning duality, one has the well-known Alexander–Pontryagin theorem: For a compact subset $X \subseteq S^n$,

$$\check{H}^p(X; G) \approx H_{n-p-1}(S^n \backslash X; G), \quad 0 \leqslant p \leqslant n \tag{1}$$

(the groups $\check{H}^0$ and $H_0$ are reduced).

In distinction to cohomology, Čech homology does not satisfy the exactness axiom [ES] and therefore is not a genuine homology theory. Furthermore, using Čech homology (and singular cohomology), one cannot dualize (1). For example, for the dyadic solenoid $D \subseteq S^3$ the (reduced) Čech group $\check{H}_0(D; \mathbf{Z}) = 0$. However, one can show that $H^2(S^3 \backslash D; \mathbf{Z})$ is an uncountable group and therefore $\check{H}_0(D; \mathbf{Z}) \neq H^2(S^3 \backslash D; \mathbf{Z})$.

These were the reasons why N. E. Steenrod [St] introduced in 1940 a new kind of homology group for metric compacta, here denoted by $H_p^S(X; G)$ and called the Steenrod group. Instead of using the true cycles of L. Vietoris, Steenrod used a modified version, called regular cycles. He then proved his duality theorem for compacta $X \subseteq S^n$:

$$H_p^S(X; G) \approx H^{n-p-1}(S^n \backslash X; G), \quad 0 \leqslant p \leqslant n. \tag{2}$$

In 1951 K. A. Sitnikov [Si1, 2] defined the same groups in a somewhat different way (for the equivalence of the two definitions see [Sk1]). Sitnikov proved that (2) holds for arbitrary subsets $X \subseteq S^n$, provided one uses on the right side of (2) Čech cohomology and on the left side the group

$$H_p^{Sc}(X; G) = \operatorname{colim} \{H_p^S(C; G)\}, \tag{3}$$

where $C$ ranges over all compact subsets of $X$. The groups $H_p^{Sc}$ are often called the Steenrod–Sitnikov groups.

Already in 1936 A. N. Kolmogorov [Ko] proposed a definition of a Vietoris-type homology group (for locally compact spaces and compact coefficients). His definition used (finite) partitions instead of coverings of the space. Since partitions are disjoint coverings, they have the advantage that projections from a finer to a coarser partition are unique. This fact has been used later by many authors. In 1940 a Čech-type version of the Kolmogorov group was defined by G. S. Čogoshvili [C1, 2]. In 1972 L. D. Mdzinarishvili [Md2, 3] showed that the groups of Steenrod, Kolmogorov and Čogoshvili (with arbitrary coefficients) coincide.

In the years that followed many different constructions of Steenrod-type homology groups were produced. All yield exact homology theories on various categories of spaces including metric compacta. In particular, we mention here the constructions of A. Borel and J. C. Moore (locally compact Hausdorff spaces) [BoM], J. Milnor (compact Hausdorff spaces) [Mi1], R. Deheuvels (topological spaces) [D], L. D. Mdzinarishvili (compact Hausdorff spaces) [Md1], E. G. Sklyarenko (metric spaces) [Sk1], H. N. Inasaridze (topological spaces) [I], V. I. Kuz'minov and I. A. Švedov (topological spaces) [KŠ1, 2]. For a detailed analysis of these constructions and their comparison, we refer to [Sk1, 3]. On metric compacta all these constructions yield theories, which are equivalent to the Steenrod theory. The only exception is the Borel–Moore theory, which differs from the Steenrod theory unless one assumes that $G$ is finitely generated (see [Sk3] and [K]).

A more recent approach to Steenrod homology (for locally compact Hausdorff spaces) is due to W. S. Massey [Ma]. His construction depends on an algebraic result of G. Nöbeling [N]. In a special case this result asserts that the group of all bounded integer valued functions on an arbitrary set is a free Abelian group.

The appearance of shape theory gave a new impulse to Steenrod homology. Here we mention the papers of D. A. Edwards and H. M. Hastings [EH], F. W. Bauer [B1, 2], Z. R. Miminoshvili [Mim1, 3], Yu. T. Lisitsa [L1, 2], Yu. T. Lisitsa and S. Mardešić [LM1, 6, 7, 8] and A. Koyama [Koy1, 2]. Homology groups defined in all these papers agree with the Steenrod groups on metric compacta. The only exceptions are the groups of Bauer and Koyama, which agree with the Borel–Moore groups for pointed 1-movable compacta [K]. The authors' construction is described in details in §§ 4–7.

## 3. Axiomatic characterizations

In 1960 J. Milnor [Mi1] gave the first axiomatic characterization of the Steenrod groups. His theorem asserts that on compact metric pairs (for a given group of coefficients $G$) there is a unique homology theory $H$, which satisfies the Eilenberg–Steenrod axioms and the two additional axioms:

(RH) *If* $f: (X, A) \to (Y, B)$ *is a relative homeomorphism, then* $f_*: H_p(X, A) \to H_p(Y, B)$ *is an isomorphism for all* $p$.

(W) *Let* $(X_i, *)$; $i \in N$, *be a sequence of metric compacta and let* $V(X_i, *)$ *denote the wedge (cluster) of this sequence. Then the inclusions* $(X_j, *) \to V(X_i, *)$, $j \in N$, *induce an isomorphism* $H_p(V(X_i, *)) \to \Pi H_p(X_i, *)$ *for all* $p$.

Another proof of Milnor's uniqueness theorem was given by S. V. Petkova [P].

Note that (RH) is equivalent to the requirement that the quotient map $f: (X, A) \to (X/A, *)$ induces an isomorphism of homologies. Also note that (RH) implies the usual excision axiom.

Another axiomatic characterization of the Steenrod homology on compact metric pairs is due to Sklyarenko [Sk 2]. He also has a characterization theorem for the Steenrod–Sitnikov groups $H_*^{Sc}$ on spaces whose compact subsets are metrizable [Sk 2].

In 1980 N. A. Berikašvili [Be1, 2] obtained the first uniqueness theorems for an exact homology theory $H_*$ on compact Hausdorff pairs (also see [S1]). These theorems establish uniqueness of homology under the assumption of the Eilenberg–Steenrod axioms and of some additional axioms. Especially easy to state is Berikashvili's theorem, which requires only one additional axiom. This is the existence of a functorial exact sequence

$$(4) \qquad 0 \to \mathrm{Ext}(\check{H}^{p+1}(X, A); G) \to H_p(X, A) \to \mathrm{Hom}(\check{H}^p(X, A); G) \to 0.$$

The sequence (4) relates $H_p$ to the Čech cohomology groups. The homologies of Čogoshvili [C2], Mdzinarishvili [Md1, 3], Inasaridze [I], Kuz'minov–Švedov [KŠ1, 2], Massey [Ma] and the authors [LM6–8] satisfy (4) on Hausdorff compact pairs (for the last case see § 10).

Recently, S. A. Saneblidze [S2] has extended the uniqueness theorem of Berikašvili to pairs $(X, A)$, where $X$ is paracompact and $A \subseteq X$ is closed.

Inasaridze and Mdzinarishvili [IM], [Md4] have proved another uniqueness theorem for compact Hausdorff pairs. Their only additional axiom is the axiom of partial continuity. It requires the existence of a functorial exact sequence

$$(5) \qquad 0 \to \lim{}^1 \{H_{p+1}((K_\lambda, L_\lambda); G)\} \to H_p(X, A) \to \check{H}_p(X, A; G) \to 0,$$

where $\{(K_\lambda, L_\lambda)\}$ is any inverse system of finite polyhedral pairs with limit $(X, A)$. The sequence (5) relates $H_*$ to the Čech homology. For many of the constructions mentioned above, (5) is known to hold.

## 4. Outline of the authors' construction of $H_*^S$

In this section we describe the main steps of our construction of the Steenrod homology groups $H_p^S$ for arbitrary spaces. Some of the needed tools are discussed in more details in § 5–8.

The first such tool is the coherent prohomotopy category CPHTop (and CPHTop$^2$), which the authors have defined and studied in [LM1–5]. The objects of CPHTop (CPHTop$^2$) are inverse systems of spaces $\underline{X}$ (of pairs of spaces $(\underline{X}, \underline{A})$). The morphisms of these categories, as well as a functor $C$ from pro-Top (pro-Top$^2$) to CPHTop (CPHTop$^2$) are described in § 5.

The second tool are homology functors $H_p^S = H_p^S(\cdot\,; G)$ from CPHTop (CPHTop$^2$) to the category of abelian groups Ab, and natural homomorphisms $\partial$: $H_{p+1}^S(\underline{X}, \underline{A}; G) \to H_p^S(\underline{A}, G)$ (see § 7).

The next ingredient needed is the notion of an ANR-resolution of a space $X$, introduced by S. Mardešić in [M1] (see § 6). It consists of an inverse system $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of ANR's $X_\lambda$ and maps $p_{\lambda\lambda'}$ and of a morphism of pro-Top $p = (p_\lambda)$: $X \to \underline{X}$ [MS], satisfying certain additional conditions. They insure that the $X_\lambda$'s give a sufficiently good approximation of $X$.

The crucial property of ANR-resolutions, fundamental to our construction of homology, is expressed by this factorization theorem, proved in [LM3]:

If $p$: $X \to \underline{X}$ is a resolution, $\underline{Y}$ is an inverse system of ANR's and $f$: $X \to \underline{Y}$ is a morphism of CPHTop, then there is a unique morphism of CPHTop $g$: $\underline{X} \to \underline{Y}$ such that $gC(p) = f$. One has analogous notions and results also for pairs of spaces [LM5].

The factorization theorem has as its immediate consequence the following fact: If $p$: $X \to \underline{X}$ and $p'$: $X \to \underline{X}'$ are two ANR-resolutions of the same space $X$, then there is a unique isomorphism of CPHTop $i$: $\underline{X} \to \underline{X}'$ such that $iC(p) = C(p')$. Clearly, $i$ induces a (canonical) isomorphism $i_*$: $H_p^S(\underline{X}; G) \to H_p^S(\underline{X}'; G)$. Since every space $X$ admits an ANR-resolution $p$: $X \to \underline{X}$ [M1], one can define the Steenrod homology group $H_p^S(X; G)$ of the space $X$ as the group $X_p^S(\underline{X}; G)$.

Moreover, every map $f$: $X \to Y$ induces a homomorphism $f_*$: $H_p^S(X; G) \to H_p^S(Y; G)$, given by the homomorphism $g_*$: $H_p^S(\underline{X}; G) \to H_p^S(\underline{Y}; G)$, where $p$: $X \to \underline{X}$, $q$: $Y \to \underline{Y}$ are ANR-resolutions and $g$: $\underline{X} \to \underline{Y}$ is the only morphism of CPHTop for which $gC(p) = C(qf)$. It is readily seen that one obtains in this way a functor $H_p^S(\cdot\,; G)$: Top $\to$ Ab. All this, repeated for pairs of spaces, yields a functor $H_p^S(\cdot\,; G)$: Top$^2 \to$ Ab [M2, LM8].

One also obtains natural homomorphisms $\partial$: $H_{p+1}^S(X, A; G) \to H_p^S(A; G)$, whenever $A \subseteq X$ is normally embedded in $X$ (i.e., when every normal covering $\mathscr{V}$ of $A$ admits a normal covering $\mathscr{U}$ of $X$ such that $\mathscr{U}|A$ refines $\mathscr{V}$). The condition of $A$ being normally embedded in $X$ is necessary and sufficient in order that the restriction $p_A$: $A \to \underline{A}$ of an ANR-resolution $p$: $(X, A) \to (\underline{X}, \underline{A})$ be itself an ANR-resolution [M2]. Note that in a paracompact space $X$ every closed subset $A$ is normally embedded.

Our definition of homology groups is not canonical, because it involves a choice of ANR-resolutions. We think that this is not a drawback, but rather an advantage, because in every case one can choose a suitable resolution. E.g., if $(X, A)$ is an ANR-pair (i.e., $X$ and $A$ are ANR's and $A$ is closed), then the rudimentary system and the identity map $1_{(X,A)}$ define a resolution of $(X, A)$. Using this resolution, it is easy to see that $H_p^S(X, A; G)$ coincides with the singular group $H_p(X, A; G)$ [LM6].

We have proved for rather general categories of pairs of spaces $(X, A)$ (in particular, for $X$ paracompact and $A$ closed) that the functors $H_p^S$ and the boundary homomorphisms give a homology theory, which satisfies all the Eilenberg–Steenrod axioms and has additional desirable properties mentioned in § 1. This is discussed in §§ 8–10.

## 5. Coherent prohomotopy CPHTop

We consider only inverse systems $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in Top indexed by directed cofinite sets $\Lambda$ (i.e., every element has finitely many predecessors). In order to describe morphisms $\underline{X} \to \underline{Y} = (Y_\mu, q_{\mu\mu'}, M)$, we consider for each $n \geqslant 0$ the set $M^n$ of all increasing sequences in $M$, $\underline{\mu} = (\mu_0, \ldots, \mu_n)$, $\mu_0 \leqslant \ldots \leqslant \mu_n$. A coherent map $f\colon \underline{X} \to \underline{Y}$ consists of an increasing function $\varphi\colon M \to \Lambda$ and of maps $f_{\underline{\mu}}\colon \Delta^n \times X_{\varphi(\mu_n)} \to Y_{\mu_0}$ such that

$$f_{\underline{\mu}}(\partial_j^n t, x) = \begin{cases} q_{\mu_0\mu_1} f_{\mu_1\ldots\mu_n}(t, x), & j = 0, \\ f_{\mu_0\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_n}(t, x), & 0 < j < n, \\ f_{\mu_0\ldots\mu_{n-1}}(t, p_{\varphi(\mu_{n-1})\varphi(\mu_n)}(x)), & j = n, \end{cases} \tag{6}$$

$$f_{\underline{\mu}}(\sigma_j^n t, x) = f_{\mu_0\ldots\mu_j\mu_j\ldots\mu_n}(t, x), \quad 0 \leqslant j \leqslant n; \tag{7}$$

here $\partial_j^n\colon \Delta^{n-1} \to \Delta^n$, $\sigma_j^n\colon \Delta^{n+1} \to \Delta^n$ are the usual face and degeneracy operators. Two coherent maps $f, f'$ are coherently homotopic provided there exists a coherent map $F\colon I \times \underline{X} \to \underline{Y}$, given by $\Phi$ and $F_{\underline{\mu}}$, such that $\Phi \geqslant \varphi, \varphi'$ and

$$F_{\underline{\mu}}(t, 0, x) = f_{\underline{\mu}}(t, p_{\varphi(\mu_n)\Phi(\mu_n)}(x)), \tag{8}$$

$$F_{\underline{\mu}}(t, 1, x) = f'_{\underline{\mu}}(t, p_{\varphi'(\mu_n)\Phi(\mu_n)}(x)). \tag{9}$$

Coherent homotopy is an equivalence relation.

In order to define the composition $h = gf$ of $f\colon \underline{X} \to \underline{Y}$ and $g\colon \underline{Y} \to \underline{Z} = (Z_\nu, \mu_{\nu\nu'}, N)$, one decomposes $\Delta^n$ into subpolyhedra $P_i^n \approx \Delta^i \times \Delta^{n-i}$, $0 \leqslant i \leqslant n$, where $t = (t_0, \ldots, t_n) \in P_i$ provided

$$t_0 + \ldots + t_{i-1} \leqslant \frac{1}{2} \leqslant t_0 + \ldots + t_i. \tag{10}$$

One also considers maps $\alpha_i^n\colon P_i^n \to \Delta^{n-i}$, $\beta_i^n\colon P_i^n \to \Delta^i$, given by

$$\alpha_i^n(t) = (\#, 2t_{i+1}, \ldots, 2t_n), \quad \beta_i^n(t) = (2t_0, \ldots, 2t_{i-1}, \#), \tag{11}$$

where # stands for 1 minus the sum of the remaining terms. We now put $\chi = \varphi\psi\colon N \to \Lambda$ and define $h_{\underline{\nu}}\colon \Delta^n \times X_{\chi(\nu_n)} \to Z_{\nu_0}$, $\underline{\nu} = (\nu_0, \ldots, \nu_n) \in N^n$, by

(12) $$h_{\underline{\nu}}(t, x) = g_{\nu_0 \ldots \nu_i}\big(\beta_i^n(t), f_{\psi(\nu_i) \ldots \psi(\nu_n)}(\alpha_i^n(t), x)\big), \quad t \in P_i^n.$$

The identity chain map $\underline{X} \to \underline{X}$ is given by $\varphi = 1_\Lambda$ and by $f_{\lambda_0 \ldots \lambda_n}(t, x) = p_{\lambda_0 \lambda_n}(x)$. The morphisms of CPHTop are defined as coherent homotopy classes of coherent maps and their composition is defined by composing representatives [LM3]. In an analogous way we defined CPHTop$^2$ [LM5].

A morphism of pro-Top $f\colon \underline{X} \to \underline{Y}$ is given by an increasing function $\varphi\colon M \to \Lambda$ and by maps $f_{\mu_0}\colon X_{\varphi(\mu_0)} \to Y_{\mu_0}$ such that

(13) $$f_{\mu_0} p_{\varphi(\mu_0)\varphi(\mu_1)} = q_{\mu_0 \mu_1} f_{\mu_1}, \quad \mu_0 \leqslant \mu_1$$

(see [MS]). One obtains a coherent map $\underline{X} \to \underline{Y}$ by putting

(14) $$f_{\underline{\mu}}(t, x) = f_{\mu_0} p_{\varphi(\mu_0)\varphi(\mu_n)}(x), \quad \underline{\mu} \in M^n.$$

This defines the functor $C$: pro-Top $\to$ CPHTop mentioned in § 4 (see [LM 4]).

A construction of coherent prohomotopy similar to our construction was considered by T. Porter [Po 2], who based his work on [V]. Z. R. Miminoshvili has discovered independently the category CPHTop [Mim3]. An approximation to CPHTop, involving only sequences $(\mu_0, \ldots, \mu_n)$ with $0 \leqslant n \leqslant 1$, appeared in [B1], [Mim2] and [Li1]. We also mention an entirely different construction, which however serves the same purpose. This is the construction of *Ho*(pro-Top), introduced by Edwards and Hastings [EH]. We believe that the categories *Ho*(pro-Top) and CPHTop are isomorphic.

## 6. Homology $H_*^S$ on CPHTop

Following [LM6–8], we now describe the homology functors $H_*^S(\cdot, G)$ on CPHTop and CPHTop$^2$. We first associate with every inverse system $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of topological spaces a chain complex $C_\#(\underline{X}; G)$. By definition, $C_p(\underline{X}; G) = 0$ for $p < 0$. For $p \geqslant 0$ a strong $p$-chain $x$ is a function which assigns to every $\underline{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \Lambda^n$, $\lambda_0 \leqslant \ldots \leqslant \lambda_n$, $n \geqslant 0$, a singular $(p+n)$-chain $x_{\underline{\lambda}}$ of $X_{\lambda_0}$ with coefficients in $G$. By definition, $(x + x')_{\underline{\lambda}} = x_{\underline{\lambda}} + x'_{\underline{\lambda}}$. The boundary operator $d\colon C_{p+1}(\underline{X}; G) \to C_p(\underline{X}; G)$, $p \geqslant 0$, is defined by

(15) $$(-1)^n (dx)_{\underline{\lambda}} = \partial(x_{\underline{\lambda}}) - (\delta x)_{\underline{\lambda}},$$

where $\delta x \in C_p(\underline{X}; G)$ is given by

(16) $$(\delta x)_{\lambda_0 \ldots \lambda_n} = p_{\lambda_0 \lambda_1 \#} x_{\lambda_1 \ldots \lambda_n} + \sum_{j=1}^{n-1} (-1)^j x_{\lambda_0 \ldots \lambda_{j-1} \lambda_{j+1} \ldots \lambda_n} + (-1)^n x_{\lambda_0 \ldots \lambda_{n-1}}$$

(if $n = 0$, then (15) has only the first term). By definition, $H_p^S(\underline{X}; G)$ is the homology (in the algebraic sense) of the chain complex $C_\#(\underline{X}; G)$.

In the case of pairs $(\underline{X}, \underline{A})$, $C_{\#}(\underline{A}; G)$ is a subcomplex of $C_{\#}(\underline{X}; G)$ and we define $C_{\#}(\underline{X}, \underline{A}; G)$ as the quotient $C_{\#}(\underline{X}; G)/C_{\#}(\underline{A}; G)$. By definition, $H_p^S(\underline{X}, \underline{A}; G)$ is the homology of the chain complex $C_{\#}(\underline{X}, \underline{A}; G)$. Clearly, there is a long exact sequence

$$(17)\quad \ldots \to H_{p+1}^S(\underline{A}; G) \to H_{p+1}^S(\underline{X}; G) \to H_{p+1}^S(\underline{X}, \underline{A}; G) \xrightarrow{\partial} H_p^S(\underline{A}; G) \to \ldots$$

$\delta$ in formula (16) is the coboundary operator, which was used by J. E. Roos in [R] to give explicit formulas for the higher derived limits $\lim^i$ (also see [D]). In [Po 1], T. Porter has associated with every inverse system $\underline{C}$ of chain complexes a chain complex holim $\underline{C}$. If one takes for $\underline{C}$ the inverse system $(C(X_\lambda; G), p_{\lambda\lambda'}, \Lambda)$ of singular chain complexes of $X_\lambda$, then holim $\underline{C}$ is isomorphic with our chain complex $C_{\#}(\underline{X}; G)$. The only apparent difference consists in the different choice of the signs. In order to obtain the same signs, one must replace our boundary operator $d$ by a new boundary operator $d'$, given by $d'\colon C_{p+1}(\underline{H}; G) \to C_p(\underline{H}; G)$

$$(15)'\qquad (d'x)_{\boldsymbol{\lambda}} = \partial(x_{\boldsymbol{\lambda}}) + (-1)^{p+1}(\delta x)_{\boldsymbol{\lambda}}.$$

However, the new chain complex $C'_{\#}$, which one then obtains, is isomorphic to our complex $C_{\#} = C_{\#}(\underline{X}; G)$. Indeed, an isomorphism $h\colon C_{\#} \to C'_{\#}$ is obtained by putting

$$(18)\qquad (hx)_{\boldsymbol{\lambda}} = (-1)^{pn} x_{\boldsymbol{\lambda}}, \qquad x \in C_p(\underline{X}; G),\ \boldsymbol{\lambda} \in \Lambda^n.$$

This is so because $d'h = dh$. Miminoshvili in his recent work [Mim 3] also uses the complex $C'_{\#}$.

If $f\colon \underline{X} \to \underline{Y}$ is a coherent map, given by $\varphi\colon M \to \Lambda$ and $f_{\boldsymbol{\mu}}\colon \Delta^n \times X_{\varphi(\mu_n)} \to Y_{\mu_0}$, then we associate with $f$ a chain mapping $f_{\#}\colon C_{\#}(\underline{X}; G) \to C_{\#}(\underline{Y}; G)$, given by

$$(19)\qquad (f_{\#} x)_{\boldsymbol{\mu}} = \sum_{i=0}^{n} f_{\mu_0\ldots\mu_i\#}(\Delta^i \times X_{\varphi(\mu_i)\ldots\varphi(\mu_n)}).$$

It was proved in [LM 7] that coherently homotopic coherent maps induce chain homotopic chain maps. Therefore, morphisms of CPHTop $f\colon \underline{X} \to \underline{Y}$ induce homomorphisms $f_*\colon H_p^S(\underline{X}; G) \to H_p^S(\underline{Y}; G)$. Similarly, $f\colon (\underline{X}, \underline{A}) \to (\underline{Y}, \underline{B})$ induces a chain mapping $f_{\#}\colon C_{\#}(\underline{X}, \underline{A}; G) \to C_{\#}(\underline{Y}, \underline{B}; G)$ and one obtains homomorphisms $f_*\colon H_p^S(\underline{X}, \underline{A}; G) \to H_p^S(\underline{Y}, \underline{B}; G)$.

## 7. Resolutions of spaces

A morphism $p\colon X \to \underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of pro-Top consists of maps $p_\lambda\colon X \to X_\lambda$ such that $p_{\lambda\lambda'} p_{\lambda'} = p_\lambda$, $\lambda \leqslant \lambda'$. Such a morphism $p$ is a resolution provided the following two conditions hold for any ANR $P$ and any open covering $\mathscr{V}$ of $P$ [M1], [MS]:

(R1) *Every map* $f: X \to P$ *admits a* $\lambda \in \Lambda$ *and a map* $g: X_\lambda \to P$ *such that the maps* $f$ *and* $gp_\lambda$ *are* $\mathscr{V}$*-near.*

(R2) *There exists an open covering* $\mathscr{V}'$ *of* $P$ *such that whenever* $\lambda \in \Lambda$ *and maps* $g, g': X_\lambda \to P$ *have the property that* $gp_\lambda$ *and* $g'p_\lambda$ *are* $\mathscr{V}'$*-near maps, then there exists a* $\lambda' \geqslant \lambda$ *such that* $gp_{\lambda\lambda'}$ *and* $g'p_{\lambda\lambda'}$ *are* $\mathscr{V}$*-near maps.*

These conditions modify the well-known conditions of K. Morita (see [MS]). If $X$ and all $X_\lambda$ are compact, then $p$ is a resolution if and only if $p$ is the inverse limit of $\underline{X}$. If $X$ is paracompact and $p: X \to \underline{X}$ is an ANR-resolution, then $p$ is an inverse limit [MS]. Therefore, one can view resolutions as special types of inverse limits. The notion of resolution makes sense and its basic properties remain valid also in the case of pairs of spaces [MS], [M2]. For further information on resolutions also see [M3], [W1] and [Mo].

## 8. Strong shape invariance

The strong shape category SSh was defined and studied by the authors in [LM1–5]. The objects of SSh are all topological spaces. A morphism $X \to Y$ is a class of equivalent triples $(p, q, g)$, where $p: X \to \underline{X}$ and $q: Y \to \underline{Y}$ are ANR-resolutions and $g: \underline{X} \to \underline{Y}$ is a morphism of CPHTop. Another such triple $(p', q', g')$ is considered equivalent to $(p, q, g)$ provided $g'i = jg$, where $i: \underline{X} \to \underline{X}'$, $j: \underline{Y} \to \underline{Y}'$, are the canonical isomorphisms described in § 4.

We now define the strong shape functor $S_1$: HTop → SSh from the homotopy category of spaces to the strong shape category. If $f: X \to Y$ is a map and $p: X \to \underline{X}$, $q: Y \to \underline{Y}$ are ANR-resolutions, then the factorization theorem (§ 4) yields a unique morphism of CPHTop $g: \underline{X} \to \underline{Y}$ such that $gC(p) = C(qf)$. We then associate with the homotopy class of $f$ the strong shape morphism, given by the triple $(p, q, g)$ (see [LM3]).

The construction can be repeated for pairs and one obtains a category $\mathrm{SSh}^2$ and a functor $S_1$: $\mathrm{HTop}^2 \to \mathrm{SSh}^2$ [LM5]. For other work on strong shape see [MS] and [LM3]. We mention here only the approach of F. W. Cathey and J. Segal [CS], who have defined a strong shape category using resolutions and *Ho*(pro-Top).

We now generalize the definition of the homology functor $H_p^S$: Top → Ab, given in § 4, to obtain a functor $H_p^S$: SSh → Ab. If a strong shape morphism $X \to Y$ is given by a triple $(p, q, g)$, where $g: \underline{X} \to \underline{Y}$, then the induced homomorphism: $H_p^S(X; G) \to H_p^S(Y; G)$ is defined as the homomorphism $g_*: H_p^S(\underline{X}; G) \to H_p^S(\underline{Y}; G)$.

If one composes the homotopy functor $H$: Top → HTop with $S_1$: HTop → SSh and then with $H_p^S$: SSh → Ab, one obtains the functor $H_p^S$: Top → Ab of § 4. This shows that our homology is an invariant of strong shape and a fortiori satisfies the homotopy axiom.

All this also holds for the strong shape category of pairs $SSh^2$. In particular, if a pair $(X, A)$ has the homotopy type of an ANR-pair (e.g., if $(X, A)$ is a pair of CW-complexes), then $H_p^S(X, A; G)$ coincides with the singular group.

## 9. Exactness and excision of $H_*^S$

If $(X, A)$ is a pair of spaces such that $A \subseteq X$ is normally embedded in $X$ (e.g., if $X$ is paracompact and $A$ is closed), then the following sequence is well-defined and exact

$$(20) \quad \ldots \to H_{p+1}^S(A; G) \to H_{p+1}^S(X; G) \to H_{p+1}^S(X, A; G) \xrightarrow{\partial} H_p^S(A; G) \to \ldots$$

This is an immediate consequence of our definitions, (17) and of the facts that $(X, A)$ admits an ANR-resolution $p\colon (X, A) \to (\underline{X}, \underline{A})$ and that the restriction $p_A\colon A \to \underline{A}$ is also an ANR-resolution (because $A$ is normally embedded) [M2].

We will now show that for arbitrary pairs $(X, A)$ one cannot have an exact sequence like (20). We first note that the inclusion $i\colon (X, A) \to (X, \bar{A})$ is always a strong shape equivalence. Indeed, if $p\colon (X, A) \to (\underline{X}, \underline{A})$ is an ANR-resolution, then it is an immediate consequence of (R1), (R2) that also $\bar{p}\colon (X, \bar{A}) \to (\underline{X}, \underline{A})$ is a resolution. Here $\bar{p}$ consists of the same maps $p_\lambda$, $\lambda \in \Lambda$, as $p$. The only difference is that we now view $p_\lambda$ as maps $(X, \bar{A}) \to (X_\lambda, A_\lambda)$ (recall that $A_\lambda$ is closed). It follows from [LM5], Lemma 3, that $C(\bar{p})C(i) = 1_{(\underline{X}, \underline{A})}C(p)$. Therefore, $S_1(i)$ is given by the triple $(p, \bar{p}, 1_{(\underline{X}, \underline{A})})$. Clearly, the triple $(\bar{p}, p, 1_{(\underline{X}, \underline{A})})$ represents a strong shape morphism $(X, \bar{A}) \to (X, A)$, which is inverse to the morphism $S_1(i)$. Since $H_p^S(\cdot; G)$ is an invariant of strong shape (§ 8), we conclude that the inclusion $i\colon (X, A) \to (X, \bar{A})$ always induces an isomorphism $H_p^S(X, A; G) \to H_p^S(X, \bar{A}; G)$.

In order to obtain a specific counter-example to (20) we put $X = B^{n+1}$, $n \geq 1$, $A = S^n \setminus \{*\}$, where $S^n = \operatorname{Bd} B^{n+1}$ and $*$ is a point of $S^n$. Then $A \approx R^n$ and $\bar{A} = S^n$. Note that

$$H_{n+1}^S(X, A) \approx H_{n+1}^S(X, \bar{A}) \approx H_{n+1}(B^{n+1}, S^n) \approx Z.$$

Since $H_{n+1}^S(X) = 0$, $H_n^S(A) = 0$, this contradicts (20).

$(X, A)$ and $(X, \bar{A})$ also have the same ordinary shape [LM3, 5] and thus pro-$H_n(X, A; G) \approx$ pro-$H_n(X, \bar{A}; G)$ (see [MS]). Therefore, in the above example pro-$H_{n+1}(X, A) \approx Z$, pro-$H_{n+1}(X) = 0$, pro-$H_n(A) = 0$. This shows that for arbitrary pairs $(X, A)$ one cannot have an exact sequence of homology pro-groups.

Our homology $H_*^S$ satisfies the excision axiom in the following form.

Let $(X, A)$ be a pair consisting of a normal space $X$ and of a closed subset $A \subseteq X$. Let $U \subseteq X$ be an open set such that $\bar{U} \subseteq A$ and let $X \setminus U$ be normally embedded in $X$. Then the inclusion $i$: $(X\backslash U, A\backslash U) \to (X, A)$ induces an isomorphism of groups

$$(21) \qquad i_*: H_p^S(X\backslash U, A\backslash U; G) \to H_p^S(X, A; G).$$

In particular, (21) is an isomorphism if $X$ is paracompact, $A \subseteq X$ is closed, $U \subseteq X$ is open and $U \subseteq A$.

In order to prove this excision theorem, we put $A' = X\backslash U$ and we choose an ANR-resolution of triads $p$: $(X, A, A') \to (\underline{X}, \underline{A}, \underline{A}')$, such that $X_\lambda = \operatorname{Int} A_\lambda \cup \operatorname{Int} A'_\lambda$, $\lambda \in \Lambda$ [M3]. The inclusions $i_\lambda$: $(A'_\lambda, A_\lambda \cap A'_\lambda) \to (X_\lambda, A_\lambda)$ define a morphism of CPHTop$^2$ $j$: $(\underline{X}, \underline{A}) \to (\underline{A}', \underline{A} \cap \underline{A}')$, where $(\underline{A}', \underline{A} \cap \underline{A}') = ((A'_\lambda, A_\lambda \cap A'_\lambda), p_{\lambda\lambda'}, \Lambda)$. The morphisms $p_{X,A}$: $(X, A) \to (\underline{X}, \underline{A})$, $p_{A',A\cap A'}$: $(A', A \cap A') \to (\underline{A}', \underline{A} \cap \underline{A}')$ of pro-Top$^2$ satisfy $C(p_{X,A}\, i) = jC(p_{A',A\cap A'})$, because $p_\lambda i = p_\lambda i_\lambda$. We now conclude (by [M3], Corollaries 1, 2) that $p_{X,A}$ and $p_{A',A\cap A'}$ are ANR-resolutions. Therefore, $i_*$: $H_p^S(A', A \cap A'; G) \to H_p^S(X, A; G)$ is given by $j_*$: $H_p^S(\underline{A}', \underline{A} \cap A'; G) \to H_p^S(\underline{X}, \underline{A}; G)$. However, by [LM8], $j_*$ is an isomorphism.

The following example shows that excision does not hold if one only requires that $U$ is a subset of a closed set $A$. Indeed, let $X$ be the wedge $S^1 \vee S^1$ and let $U = A$ be the first copy of $S^1$. Then $(X\backslash U, A\backslash U) \approx (R, \emptyset)$ so that $H_1^S(X\backslash U, A\backslash U) = 0$. However, $H_1^S(X, A) \approx Z$.

## 10. Additional properties of $H_*^S$

On compact metric spaces our homology $H_*^S$ coincides with the Steenrod homology of [St]. In order to prove this, it suffices to show that the additional Milnor axioms (RH) and (W) are satisfied.

We first give a proof of (RH) for compact metric pairs $(X, A)$. Let $cA = A \times I/A \times 1$ denote the cone over $A$. The quotient map $f$: $(X, A) \to (X/A, *)$ is the composition of the inclusion $i$: $(X, A) \to (X \cup cA, cA)$ and of the quotient map $g$: $(X \cup cA, cA) \to (X/A, *)$, which collapses $cA$ to a point. It therefore suffices to prove that the induced homomorphisms of homology groups $i_*$ and $g_*$ are isomorphisms. The assertion for $i_*$ follows by excising $U = cA\backslash(A \times 0)$. In order to see that also $g_*$ is an isomorphism, it suffices to show that $g$: $X \cup cA \to X/A$ is a strong shape equivalence. Indeed, this assertion implies that $g_*$: $H_p^S(X \cup cA; G) \to H_p^S(X/A, *; G)$ is an isomorphism. The analogous statement for the relative groups follows then by exactness and naturality, using the five lemma. Now, it is well known that the collapsing of a compact subset of trivial shape induces an isomorphism of ordinary shape (see, e.g., [MS, III, 1.3]). Therefore, $g$: $X \cup cA \to X/A$ is a hereditary shape equivalence. However, hereditary shape equivalences are strong shape equivalences (see, e.g. [MS, III. 10, Theorem 4]). Note that for compact metric spaces strong shape in our sense and in the sense of [MS, III, 10] coincide as it was shown in [LM4].

The wedge axiom for our homology has been verified directly by T. Watanabe [W2]. He actually showed a more general result. In particular, the wedge axiom holds for any collection of compact Hausdorff spaces.

Alternate proofs of (RH) and (W) can be obtained by first verifying (5) for compact metric pairs and inverse sequences of polyhedra. This is not difficult because one can use a simplified version of homology. It is a homology of order $m = 1$ in the sense that it involves only single indexes $\lambda_0$ and pairs of indexes $(\lambda_0, \lambda_1)$ [LM6]. Now one can use the work of L. D. Mdzinarishvili [Md5] to derive (RH) and (W) from (5).

Z. R. Miminoshvili has informed the authors that for compact Hausdorff pairs $H_*^S$ satisfies (4) and therefore, by [Be1, S1], is the unique Steenrod homology theory on the category of compact Hausdorff pairs. The main step in his proof consists in showing that also for compact Hausdorff spaces the homology $H_*^S$ agrees with a simpler homology $H_*^1$ of order 1. More generally, for every $m \geqslant 0$, one can define a homology $H_*^m$ of order $m$ (see [L3], [Mi3]). It is based on $p$-chains, which consist of a collection of singular $(p+n)$-chains $x_{\underline{\lambda}} \in C_{p+n}(X_{\lambda_0}; G)$, $\underline{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \Lambda^n$, where one now has $0 \leqslant n \leqslant m$. Then, for compact Hausdorff spaces,

$$H_p^1(X; G) \approx H_p^2(X; G) \approx \ldots \approx H_p^S(X; G).$$

Beyond the compact case our homology does not coincide with the unique theory to which Saneblidze refers in [S2]. This is so because already on infinite polyhedra (4) can fail. Indeed, if $X$ is the wedge of a sequence of projective planes endowed with the CW-topology, then $H^3(X) = 0$, $H^2(X) \approx \Pi \mathbf{Z}_2$ and $H_2(X, \mathbf{Z}_2) \approx \oplus \mathbf{Z}_2$. Therefore, (4) would imply $\mathrm{Hom}(\Pi \mathbf{Z}_2, \mathbf{Z}_2) \approx \oplus \mathbf{Z}_2$. However, if one views $\Pi \mathbf{Z}_2$ as a vector space over the field $\mathbf{Z}_2$, then this is an infinite-dimensional vector space. Therefore, the set of homomorphisms $\mathrm{Hom}(\Pi \mathbf{Z}_2, \mathbf{Z}_2)$ is uncountable. On the other hand, $\oplus \mathbf{Z}_2$ is a countable set. More work remains to be done to fully understand the behaviour of the groups $H_p^S(X; G)$ on noncompact spaces $X$.

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