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Fixed point index theory for a class of nonacyclic multivalued maps

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CONTENTS

0. Introduction

In 1946, S. Eilenberg and D. Montgomery [10] made the important observation that, by using an old theorem of L. Vietoris [41], [1] as a tool, several results of fixed point theory for singlevalued mappings can be carried over to the case of multivalued acyclic mappings (i.e. maps for which the image of every point is an acyclic compact set (e.g. a compact convex set)). The Lefschetz Fixed Point Theorem for compact ANR's was extended by these authors to arbitrary acyclic mappings. As in the singlevalued case, fixed point theorems for multivalued mappings prove useful in many branches of mathematics. These theorems are applicable, for instance, in game theory (see [3], [12]) and more recently also in ordinary differential equations [27] and optimal control theory [28].

Later, similar generalizations of the Lefschetz Fixed Point Theorem and of other topological theorems were given by several authors (see [4], [17], [18] for further references). A class of admissible mappings (see [18]) containing all compositions of acyclic mappings proved to be quite useful.

Constructing a fixed point index theory for multivalued mappings is a more general problem than obtaining some fixed point results. First results concerning the fixed point index for multivalued mappings were given in [7] and [25] where a fixed point index for admissible maps was obtained. Unfortunately, it is still not known whether this index satisfies the commutativity property. Therefore it cannot be carried over to arbitrary ANR's. Recently, Siegberg and Skordev [38] have proposed an index theory for acyclic mappings of polyhedra which does satisfy the commutativity property. Later, by using the results of [38] the index theory was extended to compact acyclic mappings of complete metric ANR's (see [39]). Note that the Siegberg–Skordev method is completely different from those presented by Calvert and Kucharski. In fact, the method used in [7] and [25] generalizes Dold's approach for singlevalued mappings (see [8]), and so it is a consequence of the Eilenberg–Steenrod axioms for homology. On the other hand, in [38] the older Lefschetz–O'Neill simplicial methods are used (cf. [30], see also [14]). It is quite interesting that the simplicial methods are more convenient for multivalued mappings.

In 1957 B. O'Neill introduced a class of continuous mappings for which the images of points consist of one or m acyclic components (with m fixed) and he proved the Lefschetz fixed point theorem for such mappings from a compact polyhedron into itself. The class of mappings considered by O'Neill contains in particular the m-valued transformations considered in [32]. In 1978, L. Górniewicz posed the problem whether O'Neill's theorem can be generalized to ANR's. He also asked whether an index theory can be constructed for this class of mappings. In the present work we positively answer these two questions. We introduce a class $\mathscr{P}$ of multivalued mappings which contains finite compositions of acyclic or O'Neill's mappings. The aim of this paper is to present a fixed point index teory for $\mathscr{P}$ and a Nielsen theory for some maps in $\mathscr{P}$.

The main results of this work are the following:

1. The fixed point index theory for maps $\Phi: X \to X$ from the class $\mathscr{P}$ where X is an arbitrary metric ANR-space and Φ is a map of compact attraction (see Ch. II for the definitions).

2. The Lefschetz fixed point theorem for a compact attraction map in $\mathscr{P}$ of an arbitrary metric ANR into itself (Theorem 8.11). This gives an answer to Problem 9 in [17] and generalizes the result of [15].

3. Using our fixed point index theory we deduce:
– the existence of common fixed points for multivalued semiflows (a generalization of the result of [19]),
– the existence of nonrepulsive and nonejective fixed points (an extension of the results of [13] and [20]),
– the Lefschetz theorem for pairs of spaces (an extension of the results obtained in [5] and [26]).

All the above results are presented in Chapter VIII.

4. In the last chapter of this paper we describe, a Nielsen theory for some class contained in $\mathscr{P}$. This is a generalization of the results of [23], [34] and [35].

It remains an open problem whether a Nielsen theory can be constructed for all O'Neill's mappings. Another open problem is whether the index described here is equal to the one given in [7] and [25] in the case of admissible mappings.

Let us remark that some other applications of the fixed point index are possible, but this will be the subject of another paper. For a more complete bibliography we recommend [4], [17], [18].

The author is very indebted to Professor L. Górniewicz for great help and many valuable discussions.

I. Homology

In this chapter we consider the Čech homology functor with coefficients in a field F. The general reference for this material is [11]. We make only a few remarks concerning this subject which will be necessary in the material which follows. All spaces considered are assumed to be Hausdorff topological.

1. Coverings and Čech homology. Let X be a compact space. By Cov X we denote the family of all open finite coverings of X. Let $A \subset X$. The *star* of A with respect to a covering α is defined by

$$\mathrm{St}(A, \alpha) := \bigcup \{U \in \alpha\colon\ U \cap A \neq \emptyset\}.$$

The *k-th star* is defined inductively:

$$\mathrm{St}^k(A, \alpha) := \mathrm{St}(\mathrm{St}^{k-1}(A, \alpha), \alpha).$$

One associates with a given covering α an abstract simplicial complex $N(\alpha)$ called the *nerve* of α. The vertices of $N(\alpha)$ are the sets $A \in \alpha$. The sets $A_0, A_1, \ldots, A_n$ form a simplex in $N(\alpha)$ provided $A_0 \cap A_1 \cap \ldots \cap A_n \neq \emptyset$.

If $\sigma = \{A_0, \ldots, A_n\}$ is an n-simplex in $N(\alpha)$ then the *support* of σ is the set $\mathrm{supp}\, \sigma := \bigcup_{i=0}^{n} A_i$. Let $C_*(N(\alpha))$ be the complex of oriented chains. Then we define the *support* of a chain $c \in C_*(N(\alpha))$ by $\mathrm{supp}\, c := \bigcup_i \mathrm{supp}\, \sigma_i$, where $c = \sum k_i \sigma_i$ is a nondegenerate representation of c (i.e. $k_i \neq 0$ for each i).

Let $\alpha, \beta \in \mathrm{Cov}\, X$ and assume that β is a refinement of α . Then there is a simplicial map $\pi_\alpha^\beta\colon N(\beta) \to N(\alpha)$ defined on vertices as follows: because β refines α, for each vertex w_0 of $N(\beta)$ we can find a vertex v_0 of $N(\alpha)$ such that $\mathrm{supp}\, w_0 \subset \mathrm{supp}\, v_0$; we fix for any vertex w of $N(\beta)$ such a vertex v of $N(\alpha)$ and put $\pi_\alpha^\beta(w) := v$. Of course, π_α^β is not unique, but all such maps are contiguous (see [11]) and therefore they induce the same homomorphism of homology groups.

The set Cov X is directed with the quasi-order relation:

$$\alpha \geqslant \beta \quad \text{iff} \quad \alpha \text{ refines } \beta.$$

The Čech homology groups of X are defined as the inverse limit

$$\check{H}_q(X) := \varprojlim_{\mathrm{Cov}\, X} H_q(N(\alpha)).$$

If A is a closed subset of X then every covering $\tilde{\alpha} \in \mathrm{Cov}\, A$ can be obtained from a covering $\alpha \in \mathrm{Cov}\, X$ satisfying $\tilde{U}_i \in \tilde{\alpha}$ iff $\tilde{U}_i = A \cap U_i$, where $U_i \in \alpha$. It is known that

$$\check{H}_q(A) = \varprojlim_{\alpha} H_q(N(\tilde{\alpha})).$$

Now, we would like to describe $\check{H}_q(A)$ using coverings in Cov X only. If B is a subset of X then by $N(\alpha)|_B$ we denote the subcomplex of $N(\alpha)$ which consists of all simplexes σ with $\mathrm{supp}\, \sigma \subset B$. We shall prove the following

1.1. PROPOSITION. $\check{H}_q(A) = \varprojlim_{\operatorname{Cov} X} H_q(N(\alpha)|_{\operatorname{St}(A,\alpha)})$.

Before the proof of 1.1 we need

1.2. LEMMA. *Let* $\alpha = \{U_1, \ldots, U_k\}$ *be an open covering of* X *and let* A *be a closed subset of* X. *Then there exists a finite refinement* $\beta = \{V_j\}_{j=1}^n$ *of* α *which satisfies the following property for each* $p = 1, 2, \ldots, n$:

1.2.1. *If* $V_1, \ldots, V_p \in \beta$ *are such that* $V_i \cap A \neq \emptyset$ *for each* i *and* $\bigcap_{i=1}^{p} V_i \neq \emptyset$, *then* $\bigcap_{i=1}^{p} V_i \cap A \neq \emptyset$.

Proof. We will correct the given covering α in a number of steps.

First step: $p = 2$. Let $U_1 \cap A \neq \emptyset$, $U_2 \cap A \neq \emptyset$, $U_1 \cap U_2 \neq \emptyset$ and $U_1 \cap U_2 \cap A = \emptyset$.

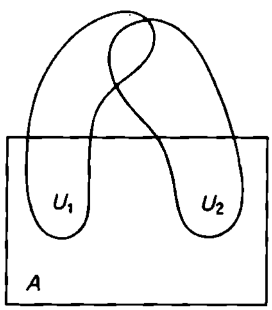


Fig. 1

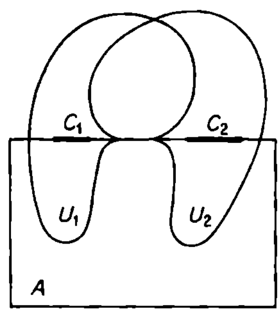


Fig. 2

If $\overline{U_1 \cap U_2} \cap \partial A = \emptyset$ (see Fig. 1), then we define

$$U_1^2 := U_1, \quad U_2^1 := U_2 - \overline{U_1 \cap U_2}$$

($\bar{U}$ denotes the closure of U and ∂U its boundary).

If $\overline{U_1 \cap U_2} \cap \partial A \neq \emptyset$ (see Fig. 2), then we consider the sets

$$C_j := \{x\colon\ x \in \partial A \cap U_j \text{ and } x \notin \bigcup_{i=3}^{k} U_i\}, \quad j = 1, 2.$$

Let W_1 be an open neighbourhood of C_1 in X such that $W_1 \subset U_1$ and $\bar{W}_1 \cap \bar{U}_2 \cap A = \emptyset$. Define then

$$U_1^2 := (U_1 \cap \operatorname{Int} A) \cup W_1.$$

Let W_2 be an open neighbourhood of C_2 such that $W_2 \subset U_2$ and $\bar{W}_2 \cap \bar{U}_1^2 = \emptyset$. We put

$$U_2^1 := (U_2 \cap \operatorname{Int} A) \cup W_2.$$

Now, define

$$V_1 := \bigcap_{i \neq 1} U_1^i, \ldots, V_k := \bigcap_{i \neq k} U_k^i,$$

$$V_{k+1} := U_1 \cap (X - A), \ldots, V_{k+k} := U_k \cap (X - A).$$

The covering $\alpha_1 = \{V_1, \ldots, V_{2k}\}$ is a refinement of α and satisfies 1.2.1 for $p = 2$.

Second step: $p = 3$. Assume that $V_1 \cap V_2 \cap V_3 \neq \emptyset$, $V_i \in \alpha_1$, $V_i \cap A \neq \emptyset$ for $i = 1, 2, 3$, and $V_1 \cap V_2 \cap V_3 \cap A = \emptyset$. We can repeat the same trick as in the first step for the sets $U_1 = V_1 \cap V_2$ and $U_2 = V_3$. If $\overline{U_1 \cap U_2} \cap \partial A = \emptyset$ then we put $V_1' := V_1$, $V_2' := V_2$ and $V_3' := V_3 - \overline{V_1 \cap V_2 \cap V_3}$. If $\overline{U_1 \cap U_2} \cap \partial A \neq \emptyset$ then $V_1' := (V_1 \cap \operatorname{Int} A) \cup W_1$, $V_2' := (V_2 \cap \operatorname{Int} A) \cup W_1$ and $V_3' := (V_3 \cap \operatorname{Int} A) \cup W_2$. The same correction is done for every triple of sets $V_i, V_j, V_l (1 \leqslant i < j < l \leqslant k)$. Taking the intersections of such V_i', one obtains a covering $\alpha_2 = \{V_1'', V_2'', \ldots, V_{k+1}'', \ldots, V_{2k}''\}$ which is a refinement of α_1 and satisfies 1.2.1 for $p \leqslant 3$. After $k-1$ such steps we obtain the desired covering β. ∎

Proof of 1.1. Let Γ denote the family of all coverings which satisfy 1.2.1. Lemma 1.2 states that Γ is a cofinal subfamily in Cov X. If $\alpha \in \Gamma$ and if we consider the induced covcering $\tilde{\alpha} \in \operatorname{Cov} A$ then 1.2.1 ensures that the simplicial complexes $N(\tilde{\alpha})$ and $N(\alpha)|_{\operatorname{St}(A,\alpha)}$ are simplicially isomorphic. Therefore $H_*(N(\tilde{\alpha})) = H_*(N(\alpha)|_{\operatorname{St}(A,\alpha)})$. Hence

$$\check{H}_*(A) = \varprojlim_{\Gamma} H_*(N(\tilde{\alpha})) = \varprojlim_{\Gamma} H_*(N(\alpha)|_{\operatorname{St}(A,\alpha)}) = \varprojlim_{\operatorname{Cov} X} H_*(N(\alpha)|_{\operatorname{St}(A,\alpha)})$$

and the proof is finished. ∎

In the above the coefficient group was inessential. From now on we assume that the coefficient group is a field F. A compact set A is *acyclic* provided

$$\check{H}_q(A) = \begin{cases} 0 & \text{for } q > 0, \\ F & \text{for } q = 0. \end{cases}$$

Denote by $\tilde{H}(X)$ the reduced homology vector space of X (see [11]).

1.3. Proposition. *Let A be a closed acyclic subset of X. Then for every covering $\alpha \in \operatorname{Cov} X$ there exists a refinement $\beta \in \operatorname{Cov} X$ of α such that the homomorphism*

$$\pi_{\alpha *}^{\beta}\colon \tilde{H}_*(N(\beta)|_{\operatorname{St}^2(A,\beta)}) \to \tilde{H}_*(N(\alpha)|_{\operatorname{St}(A,\alpha)})$$

is a trivial homomorphism of vector spaces.

Proof. We recall that the coefficients are in a field F. Hence $H_*(N(\alpha))$ are finite-dimensional graded vector spaces. Since $\tilde{H}_*(A) = 0$, by [29], II.27.13

and 1.1 we can find a covering $\gamma \in \mathrm{Cov}\, X$ such that the homomorphism

$$\pi^{\gamma}_{\alpha *}\colon \check{H}_*(N(\gamma)|_{\mathrm{St}(A,\gamma)}) \to \check{H}_*(N(\alpha)|_{\mathrm{St}(A,\alpha)})$$

is trivial. Let β be a star-refinement of γ (i.e. for each $B \in \beta$ there is $U \in \gamma$ such that $\mathrm{St}(B, \beta) \subset U$). Then

$$\mathrm{St}^2(A, \beta) \subset \mathrm{St}(A, \gamma), \qquad \pi^{\beta}_{\gamma}(N(\beta)|_{\mathrm{St}^2(A,\beta)}) \subset N(\gamma)|_{\mathrm{St}(A,\gamma)}.$$

Therefore $\pi^{\beta}_{\alpha *} = \pi^{\gamma}_{\alpha *} \circ \pi^{\beta}_{\gamma *}$ is trivial on $\check{H}_*(N(\beta)|_{\mathrm{St}^2(A,\beta)})$. ∎

2. Čech homology with compact carriers. By a *pair of spaces* (X, X_0) we understand a pair consisting of a Hausdorff topological space and its subset X_0. A pair of the form $(X, \emptyset)$ will be identified with the space X. Let (X, X_0), (Y, Y_0) be two pairs; if $X \subset Y$ and $X_0 \subset Y_0$ then we say that (X, X_0) is a *subpair of* (Y, Y_0) and write $(X, X_0) \subset (Y, Y_0)$. A pair (X, X_0) is called *compact* if X is a compact space and X_0 is a closed subset of X. By a *map* $f\colon (X, X_0) \to (Y, Y_0)$ we understand a continuous map $f\colon X \to Y$ satisfying the condition $f(X_0) \subset Y_0$. Denote by $\mathscr{C}$ the category of all pairs. We will denote by $\tilde{\mathscr{C}}$ the full subcategory of $\mathscr{C}$ consisting of all compact pairs. By $\check{H}$ we denote the Čech homology functor with coefficients in a field F from the category $\tilde{\mathscr{C}}$ to the category $\mathscr{A}$ of graded vector spaces over F and linear maps of degree zero. We recall that the functor $\check{H}$ satisfies the Eilenberg–Steenrod axioms for homology (see [11]).

Let (X, X_0) be an arbitrary pair in $\mathscr{C}$. Denote by $\mathscr{M} = \{(A_\alpha, A_{0\alpha})\}$ the directed set of all compact pairs such that $(A_\alpha, A_{0\alpha}) \subset (X, X_0)$ for each α, with the natural quasi-order relation defined by inclusion. If $(A_\alpha, A_{0\alpha}) \subset (A_\beta, A_{0\beta})$ then we shall denote by $i_{\alpha\beta}\colon (A_\alpha, A_{0\alpha}) \to (A_\beta, A_{0\beta})$ the inclusion map. The family $\{\check{H}_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}$ is a direct system in the category $\mathscr{A}$ over $\mathscr{M}$. We define the graded vector space

$$H_*(X, X_0) := \varinjlim_{\alpha} \{\check{H}_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}.$$

Let $f\colon (X, X_0) \to (Y, Y_0)$ be a map. Consider the directed sets $\mathscr{M} = \{(A_\alpha, A_{0\alpha})\}$ and $\mathscr{N} = \{(B_\gamma, B_{0\gamma})\}$ for (X, X_0) and (Y, Y_0), respectively. We define $F\colon \mathscr{M} \to \mathscr{N}$ by the formula

$$F((A_\alpha, A_{0\alpha})) = (f(A_\alpha), f(A_{0\alpha})) \qquad \text{for each } (A_\alpha, A_{0\alpha}) \in \mathscr{M}.$$

For each α, by $f_\alpha\colon (A_\alpha, A_{0\alpha}) \to (f(A_\alpha), f(A_{0\alpha}))$ we denote the map given by $f_\alpha(x) = f(x)$ for each $x \in A_\alpha$. Then the map F and the family $\{f_{\alpha *}\}$ is a map of directed systems $\{\check{H}_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}$ and $\{\check{H}_*(B_\gamma, B_{0\gamma}), i_{\gamma\sigma *}\}$. We define the induced linear map by

$$H(f) = f_* := \varinjlim_{\alpha} f_{\alpha *}.$$

Then we have $f_{*q} = \varinjlim_{\alpha} f_{\alpha * q}$ for every q. From the functoriality of $\varinjlim$ we deduce that $H: \mathscr{C} \to \mathscr{A}$ is a covariant functor. The functor H is said to be the Čech homology functor with compact carriers. We note that if (X, X_0) is a compact pair then the family consisting of the single pair (X, X_0) is a cofinal subset of $\mathscr{M}$ and hence $H_*(X, X_0) = \check{H}_*(X, X_0)$. The properties of H follow from the Eilenberg–Steenrod axioms for $\check{H}$ and the simple properties of $\varinjlim$.

An arbitrary space X is *acyclic* (F-acyclic) provided

$$H_q(X) = \begin{cases} 0 & \text{for } q > 0, \\ F & \text{for } q = 0. \end{cases}$$

II. Multivalued maps

This chapter is devoted to the definitions and basic properties of multivalued maps which will be considered in the following. The general references are [2] and [18].

1. Some generalities. Let X, Y be two spaces. We say that $\Phi\colon X \to Y$ is a *multivalued map* if for every point $x \in X$ a nonempty subset $\Phi(x)$ of Y is given. The multivalued map $\Psi\colon X \to Y$ is a *selector* of $\Phi\colon X \to Y$ provided $\Psi(x) \subset \Phi(x)$ for each $x \in X$. We will indicate it writing $\Psi \subset \Phi$.

The *image* of a subset $A \subset X$ under Φ is the set

$$\Phi(A) := \bigcup_{x \in A} \Phi(x).$$

The *small counterimage* of a subset $B \subset Y$ under Φ is

$$\Phi^{-1}(B) := \{x \in X\colon \Phi(x) \subset B\}$$

and the *big counterimage* under Φ is the set

$$\Phi^{+1}(B) := \{x \in X\colon \Phi(x) \cap B \neq \emptyset\}.$$

A point $x \in X$ is a *fixed point* of Φ provided $x \in \Phi(x)$.

A multivalued map $\Phi\colon X \to Y$ is *upper-semicontinuous* (u.s.c.) (resp. *lower-semicontinuous* — l.s.c.) provided

(i) for each $x \in X$, $\Phi(x)$ is a compact subset of Y,

(ii) for every open set $V \subset Y$ the set $\Phi^{-1}(V)$ is an open subset of X (resp. $\Phi^{+1}(V)$ is open in X).

We say that Φ is *continuous* if it is both u.s.c. and l.s.c.

Now, we list some elementary properties of u.s.c. and l.s.c. maps. The proofs are omitted since they are well known and can be found for instance in [2].

2.1. PROPOSITION. *If* Φ: $X \to Y$ *is an u.s.c. map then*

2.1.1. *The image of any compact set under* Φ *is compact;*

2.1.2. *The graph* $\Gamma_\Phi := \{(x, y) \in X \times Y : y \in \Phi(x)\}$ *is a closed subset of* $X \times Y$;

2.1.3. *The fixed point set of* Φ, Fix $\Phi := \{x \in X : x \in \Phi(x)\}$, *is a closed subset of* X *in the case* $Y = X$. ∎

2.2. PROPOSITION.

2.2.1. *The restriction of an u.s.c. (l.s.c.) map* Φ: $X \to Y$ *to a subspace* $A \subset X$ *is an u.s.c. (l.s.c.) map.*

2.2.2. *The composition* Ψ: $X \to Z$ *of u.s.c. (l.s.c.) maps* Φ_1: $X \to Y$ *and* Φ_2: $Y \to Z$ *defined by* $\Psi(x) := \Phi_2(\Phi_1(x))$ *for each* $x \in X$ *is u.s.c. (l.s.c.).*

2.2.3. *The cartesian product* Ψ: $X_1 \times X_2 \to Y_1 \times Y_2$ *defined by* $\Psi(x, y) := \Phi_1(x) \times \Phi_2(y)$ *is u.s.c. (l.s.c.).*

2.2.4. *If* Φ_1, Φ_2: $X \to Y$ *are u.s.c. (l.s.c.) then the maps* $\Phi(x) := \Phi_1(x) \cup \Phi_2(x)$ *and* $\Psi(x) := \Phi_1(x) \cap \Phi_2(x)$ *are also u.s.c. (l.s.c.).*

2.2.5. *Let* E *be a topological vector space. If* Φ_1, Φ_2: $X \to E$ *are u.s.c. (l.s.c.) then the maps* $\Phi(x) := \Phi_1(x) + \Phi_2(x)$ *and* $\Psi(x) := k\Phi_1(x)$ *are also u.s.c. (l.s.c.)* (k *is an element of the scalar field*). ∎

2. Maps of compact attraction. A multivalued map Φ: $X \to Y$ is *compact* provided $\Phi(X)$ is a relatively compact subset of Y. Evidently, if Φ: $X \to X$ is a compact u.s.c. map then by 2.1.3 the set Fix Φ is compact. We say that a map Φ: $X \to Y$ is *locally compact* if for every point $x \in X$ there exists a neighbourhood U of x such that the restriction $\Phi|_U$: $U \to Y$ is compact. Let Φ: $X \to X$. Denote by Φ^n the nth iteration of Φ defined by $\Phi^n(x) := \Phi(\Phi^{n-1}(x))$.

2.3. DEFINITION. Let Φ: $X \to X$ be u.s.c., $x \in X$, $K \subset X$, $A \subset X$.

2.3.1. A *absorbs* K if there is an integer $n_0 \in N$ such that $\Phi^n(K) \subset A$ whenever $n \geqslant n_0$.

2.3.2. A *attracts* x if for each neighbourhood U of A there is $n \in N$ such that $\Phi^n(x) \subset U$.

2.3.3. A is an *attractor* for Φ if it attracts all points in X.

2.3.4. A is a *stable compact attractor* for Φ if it is a compact attractor for Φ and if A has arbitrarily small neighbourhoods U such that $\Phi(U) \subset U$ and the restriction $\Phi|_U$: $U \to U$ is compact.

2.3.5. Φ is *of compact attraction* if it has a compact attractor and is locally compact.

2.4. Remark. Observe that if Φ: $X \to X$ is an u.s.c. map of compact attraction then Fix Φ is compact since it is contained in every attractor.

The following fact is of importance (see Prop. 5.6 in [15]):

2.5. PROPOSITION. *Let Φ: $X \to X$ be an u.s.c. map of compact attraction and let A be a compact attractor for Φ. Then there is an open neighbourhood U of A in X such that $\overline{\Phi(U)}$ is a compact subset of U.* ■

2.6. COROLLARY. *Under the assumptions of 2.5 there is a stable compact attractor A_0 for Φ such that $A \subset A_0$.*

Proof. Put $A_0 := A \cup \overline{\Phi(U)}$, where U is given by 2.5. ■

3. Decompositions of multivalued maps. An u.s.c. map Φ: $X \to Y$ is said to be *acyclic* provided the set $\Phi(x)$ is acyclic for every $x \in X$. Let us denote the class of acyclic maps by $\mathscr{A}_0(X, Y)$. Let m be a positive integer.

2.7. DEFINITION. A multivalued map Φ: $X \to Y$ is in the class $\mathscr{A}_m(X, Y)$ provided

(i) Φ is continuous,

(ii) for each $x \in X$ the set $\Phi(x)$ consists of one or m acyclic components.

Note that $\mathscr{A}_1(X, Y) \subset \mathscr{A}_0(X, Y)$.

2.8. EXAMPLES.

2.8.1. Let C be the complex plane. Define a map Φ: $C \to C$ by $\Phi(x) := \{z: z^m = x\}$. It is clear that $\Phi \in \mathscr{A}_m(C, C)$.

2.8.2. Let p: $X \to B$ be a finite covering with B connected. Define the inverse map Ψ: $B \to X$ by $\Psi(b) := p^{-1}(b)$. Since p is a local homeomorphism, Ψ is a continuous map and thus $\Psi \in \mathscr{A}_n(B, X)$, where n is the number of elements in $p^{-1}(b)$.

2.8.3. Consider the map Φ: $[-1, 1] \to [-1, 1]$ defined by

$$\Phi(x) := \begin{cases} \{1, x+1\} & \text{for } x \in [-1, 0), \\ \{1, -1\} & \text{for } x = 0, \\ \{-1, x-1\} & \text{for } x \in (0, 1]. \end{cases}$$

Then Φ is u.s.c. and is 2-point-valued but not l.s.c. and hence $\Phi \notin \mathscr{A}_2([-1, 1], [-1, 1])$. Observe that Φ has no fixed points.

It is evident that the classes $\mathscr{A}_m$ are not closed under the operations mentioned in 2.2. So we shall introduce some broader but more sophisticated classes.

2.9. DEFINITION. A *decomposition* $(\Phi_1, \ldots, \Phi_n)$ of a multivalued map Φ: $X \to Y$ is a sequence of maps

$$X = X_0 \xrightarrow{\Phi_1} X_1 \xrightarrow{\Phi_2} X_2 \xrightarrow{\Phi_3} \ldots \xrightarrow{\Phi_{n-1}} X_{n-1} \xrightarrow{\Phi_n} X_n = Y,$$

where $\Phi_i \in \mathscr{A}_{m_i}(X_{i-1}, X_i)$, $\Phi = \Phi_n \circ \ldots \circ \Phi_1$. One can also say that the map Φ is *determined* by the decomposition $(\Phi_1, \ldots, \Phi_n)$. The number n is said to be the *length* of the decomposition $(\Phi_1, \ldots, \Phi_n)$. We will denote the class of decompositions by $\mathscr{D}\mathscr{A}(X, Y)$.

2.10. Remark. We will identify the two decompositions

$$X_0 \xrightarrow{\Phi_1} X_1 \xrightarrow{\Phi_2} \dots \xrightarrow{\Phi_n} X_n, \qquad X_0 \xrightarrow{\Phi_1} \Phi_1(X_0) \xrightarrow{\Phi_2} \Phi_2(\Phi_1(X_0)) \to \dots \xrightarrow{\Phi_n} X_n,$$

because they determine the same map Φ. But we do not identify a decomposition with the multivalued map it determines. The same map Φ may be determined by different decompositions.

2.11. Example. Let $f\colon X \to Y$ be a singlevalued continuous map. It admits the following decomposition:

$$\Phi_1\colon X \to X \times \{0, 1\}, \qquad \Phi_1(x) := \{(x, 0), (x, 1)\},$$

$$\Phi_2\colon X \times \{0, 1\} \to X, \qquad \Phi_2(x, t) := f(x).$$

Here $\Phi_1 \in \mathscr{A}_2(X, X \times \{0, 1\})$ and $\Phi_2 \in \mathscr{A}_1(X \times \{0, 1\}, X)$.

2.12. Remark. Elements of the classes $\mathscr{A}_i(X, Y)$ can be identified with decompositions of length one. Therefore $\mathscr{A}_i(X, Y) \subset \mathscr{D}\mathscr{A}(X, Y)$.

The *restriction* of $(\Phi_1, \dots, \Phi_n) \in \mathscr{D}\mathscr{A}(X, Y)$ to a subset $A \subset X$ and the *composition* of $(\Phi_1, \dots, \Phi_n) \in \mathscr{D}\mathscr{A}(X, Y)$ and $(\Psi_1, \dots, \Psi_n) \in \mathscr{D}\mathscr{A}(Y, Z)$ are defined in an obvious way. Now we introduce the notion of homotopy in $\mathscr{D}\mathscr{A}(X, Y)$.

2.13. Definition. Two decompositions $(\Phi_1, \dots, \Phi_n), (\Psi_1, \dots, \Psi_n) \in \mathscr{D}\mathscr{A}(X, Y)$ are *homotopic* if there exists $(\Theta_1, \dots, \Theta_n) \in \mathscr{D}\mathscr{A}(X \times [0, 1], Y)$ such that $(\Theta_1, \dots, \Theta_n)|_{X \times \{0\}} = (\Phi_1, \dots, \Phi_n)$ and $(\Theta_1, \dots, \Theta_n)|_{X \times \{1\}} = (\Psi_1, \dots, \Psi_n)$.

2.14. Proposition. *If Φ_i is homotopic to Ψ_i in $\mathscr{A}_{m_i}(X_{i-1}, X_i)$ for each $i = 1, \dots, n$, then the decompositions $(\Phi_1, \dots, \Phi_n)$ and $(\Psi_1, \dots, \Psi_n)$ are homotopic.*

Proof. Let $H_i\colon X_{i-1} \times [0, 1] \to X_i \in \mathscr{A}_{m_i}(X_{i-1} \times [0, 1], X_i)$ be a homotopy joining Φ_i and Ψ_i. Define a map $\mathscr{H}_i\colon X_{i-1} \times [0, 1] \to X_i \times [0, 1]$ by the formula $\mathscr{H}_i(x, t) := (H_i(x, t), t)$. Then $(\mathscr{H}_1, \dots, \mathscr{H}_{n-1}, H_n)$ is a homotopy joining $(\Phi_1, \dots, \Phi_n)$ and $(\Psi_1, \dots, \Psi_n)$ (comp. 2.10). ∎

4. Permissible maps. Let X, Y be two spaces.

2.15. Definition. An u.s.c. map $\Phi\colon X \to Y$ is *permissible* provided it admits a selector $\Psi\colon X \to Y$ which is determined by a decomposition $(\Psi_1, \dots, \Psi_n) \in \mathscr{D}\mathscr{A}(X, Y)$. If Φ itself is determined by a decomposition $(\Phi_1, \dots, \Phi_n)$ then it is *strongly permissible* (s-permissible).

We will denote the class of permissible maps from X into Y by $\mathscr{P}(X, Y)$ and the class of s-permissible maps by s-$\mathscr{P}(X, Y)$. An u.s.c. map $\Phi\colon X \to Y$ is *admissible* if it admits a selector $\Psi\colon X \to Y$ which is a composition of acyclic maps (see [7] and [18]). The following simple examples show that the above classes are essentially different.

2.16. Examples.

2.16.1. The map Φ: $[-1, 1] \to [-1, 1]$ defined in 2.8.3 is an u.s.c. map which is not permissible.

2.16.2. Consider the map Ψ: $[-1, 1] \to [-1, 1]$ defined by $\Psi(x) := \Phi(x) \cup \{x\}$, where Φ is the map from 2.8.3. Then Ψ is permissible because it admits the identity map as a selector. But it is not strongly permissible.

2.16.3. Let $S^1 := \{z \in C: |z| = 1\}$. The map Φ: $S^1 \to S^1$, $\Phi(z) := \{w: w^2 = z\}$, is s-permissible (it is in the class $\mathscr{A}_2(S^1, S^1)$), but not admissible.

2.17. Proposition.

2.17.1. *The composition of two permissible maps* Φ: $X \to Y$ *and* Ψ: $Y \to Z$ *is permissible.*

2.17.2. *The product of permissible maps is permissible.*

Proof. The first part is evident. We prove 2.17.2. Observe that if $\Phi \in \mathscr{A}_m(X, Y)$ then $\Phi \times \mathrm{id} \in \mathscr{A}_m(X \times I, Y \times I)$. Let the decomposition $X_0 \xrightarrow{\Phi_1} X_1 \xrightarrow{\Phi_2} \ldots \xrightarrow{\Phi_n} X_n$ determine a selector $\tilde{\Phi}$ of Φ: $X_0 \to X_n$ and the decomposition $Y_0 \xrightarrow{\Psi_1} Y_1 \xrightarrow{\Psi_2} \ldots \xrightarrow{\Psi_m} Y_m$ determine a selector $\tilde{\Psi}$ of Ψ: $Y_0 \to Y_m$. Then the decomposition

$$X_0 \times Y_0 \xrightarrow{\bar{\Phi}_1} X_1 \times Y_0 \xrightarrow{\bar{\Phi}_2} \ldots \xrightarrow{\bar{\Phi}_n} X_n \times Y_0 \xrightarrow{\bar{\Psi}_1} X_n \times Y_1 \xrightarrow{\bar{\Psi}_2} \ldots \xrightarrow{\bar{\Psi}_m} X_n \times Y_m,$$

where $\bar{\Phi}_i(x, y) := \Phi_i(x) \times \{y\}$, $\bar{\Psi}_j(x, y) := \{x\} \times \Psi_j(y)$, determines a selector of $\Phi \times \Psi$. ∎

2.18. Definition. Two permissible maps Φ, Ψ: $X \to Y$ are *homotopic* if there exists a permissible map Θ: $X \times [0, 1] \to Y$ such that $\Theta(x, 0) \subset \Phi(x)$ and $\Theta(x, 1) \subset \Psi(x)$ for $x \in X$.

2.19. Proposition. *If two permissible maps* Φ, Ψ: $X \to Y$ *are homotopic then they have selectors* $\tilde{\Psi}$, $\tilde{\Phi}$ *which are determined by homotopic decompositions.*

Proof. Let Θ: $X \times [0, 1] \to Y$ be a homotopy. Assume that there is a selector $\tilde{\Theta} \subset \Theta$ determined by a decomposition $(\Theta_1, \ldots, \Theta_n)$. Then the restrictions $(\Theta_1, \ldots, \Theta_n)|_{X \times \{0\}}$ and $(\Theta_1, \ldots, \Theta_n)|_{X \times \{1\}}$ are homotopic and determine the desired selectors $\tilde{\Psi}$ and $\tilde{\Phi}$. ∎

III. Chain approximations and index

In this chapter we recall the results obtained in [38] which we will use in the next chapters.

1. Notation. Let (K, τ) be a compact polyhedron with a fixed triangulation τ. Its nth barycentric subdivision is denoted by τ^n. A subset $U \subset K$ is *polyhedral* provided there is an integer l such that τ^l induces a triangulation of the closure $\bar{U}$ of U in K.

Let $B \subset K$ be a subset. The *k-th closed star* of B in K is defined recursively by

$$\mathrm{St}^1(B, \tau) = \mathrm{St}(B, \tau) := \bigcup \{\sigma \in \tau\colon\ \sigma \cap B \neq \emptyset\},$$
$$\mathrm{St}^k(B, \tau) := \mathrm{St}(\mathrm{St}^{k-1}(B, \tau), \tau).$$

Simplexes are always assumed to be closed.

Let l be a natural number and F a field. We denote by $C_*(K, \tau^l)$ the oriented chain complex $C_*(K, \tau^l; F)$ (see [22]). The *carrier* of $c \in C_*(K, \tau)$ (carr c) is the smallest polyhedral subset $X \subset K$ such that $c \in C_*(X, \tau)$. Note that there is a close relation between carr c and supp c defined in Ch.I. It will be explained in Ch.V. By $b\colon C_*(K, \tau) \to C_*(K, \tau^l)$ we denote the subdivision chain mapping which maps each chain into its lth barycentric subdivision (see [12]). By $\chi\colon C_*(K, \tau^l) \to C_*(K, \tau)$ we denote any chain mapping which is induced by a simplicial approximation of the map id: $(K, \tau^l) \to (K, \tau)$.

2. Approximation systems. Let $\Phi\colon (K, \tau) \to (L, \mu)$ be an u.s.c. multivalued mapping.

3.1. Definition. Let k, l be natural numbers. A chain mapping $\varphi\colon C_*(K, \tau^l) \to C_*(L, \mu^k)$ is called an *(n, k)-approximation* of Φ provided the following condition holds: for each simplex $\sigma \in \tau^l$ there exists a point $y(\sigma) \in K$ such that

$$\sigma \subset \mathrm{St}^n(y(\sigma), \tau^k), \qquad \mathrm{carr}\, \varphi\sigma \subset \mathrm{St}^n(\Phi(y(\sigma)), \mu^k).$$

3.2. Definition. A graded set $A(\Phi) = \{A(\Phi)_j\}_{j \in N}$, where $A(\Phi)_j \subset \hom(C_*(K, \tau^j), C_*(L, \mu^j))$, is called an *approximation system* (*A-system*) for Φ provided there is an integer $n = n(A)$ such that

3.2.1. If $\varphi \in A(\Phi)_j$ then $\varphi = \varphi_1 \circ b$, where φ_1 is an (n, j)-approximation of Φ;

3.2.2. For every $j \in N$ there exists $j_1 \in N$ such that for $m \geqslant l \geqslant j_1$ and for all $\varphi = \varphi_1 \circ b \in A(\Phi)_l$ and $\psi = \psi_1 \circ b \in A(\Phi)_m$ the diagram

$$\begin{array}{ccc} C_*(K, \tau^{l_1}) & \xrightarrow{\ \varphi_1\ } & C_*(L, \mu^{l}) \\ \uparrow{\scriptstyle \chi} & & \uparrow{\scriptstyle \chi} \\ C_*(K, \tau^{m_1}) & \xrightarrow[\ \psi_1\ ]{} & C_*(L, \mu^{m}) \end{array}$$

with $m_1 \geqslant l_1$ is homotopy commutative with a chain homotopy D satisfying the following *smallness* condition:

3.2.3. For any simplex $\sigma \in \tau^{m_1}$ there exists a point $z(\sigma) \in K$ such that

$$\sigma \subset \mathrm{St}^n(z(\sigma), \tau^j), \qquad \mathrm{carr}\, D\sigma \subset \mathrm{St}^n(\Phi(z(\sigma)), \mu^j).$$

3.3. DEFINITION. Let Φ_1, Φ_2: $K \to L$ be u.s.c. maps and let H: $K \times [0, 1] \to L$ be an u.s.c. homotopy joining Φ_1 and Φ_2. Let $A(\Phi_1)$ and $A(\Phi_2)$ be A-systems for Φ_1 and Φ_2, respectively. They are *H-homotopic* provided there is an integer $m \in N$ such that the following condition holds:

3.3.1. For every $j \in N$ there is $j_1 \in N$ such that for any $l \geq j_1$ there are $\varphi = \varphi_1 \circ b \in A(\Phi_1)_l$ and $\psi = \psi_2 \circ b \in A(\Phi_2)_l$ such that ψ_1, φ_1: $C_*(K, \tau^{l_1}) \to C_*(L, \mu^l)$ are chain homotopic with an *H-small* homotopy D, i.e.:

3.3.2. For $\sigma \in \tau^{l_1}$ there is a point $d(\sigma) \in K$ such that

$$\sigma \subset \mathrm{St}^m(d(\sigma), \tau^j), \qquad \mathrm{carr}\, D\sigma \subset \mathrm{St}^m(H(d(\sigma) \times I), \mu^j).$$

3.4. PROPOSITION. *Let* Φ_1: $K \to L$, Φ_2: $L \to M$ *be u.s.c. maps and let* $A(\Phi_1)$, $A(\Phi_2)$ *be A-systems for* Φ_1, Φ_2, *respectively. Then the graded set* $A = \{A_j\}$, *where*

$$A_j = A(\Phi_2)_j \circ A(\Phi_1)_j := \{\varphi = \varphi_2 \circ \varphi_1\colon\ \varphi_2 \in A(\Phi_2)_j,\ \varphi_1 \in A(\Phi_1)_j\}$$

is an A-system for the composition $\Phi_2 \circ \Phi_1$. ∎

A simple example of an A-system is the family of all chain mappings induced by simplicial approximations of a given singlevalued continuous map (cf. [38]).

3. Index of an A-system. Let $U \subset K$ be open and polyhedral and let Φ: $\bar{U} \to K$ be an u.s.c. map such that $x \notin \Phi(x)$ for $x \in \partial U$. Let $A(\Phi)$ be an A-system for Φ. Then the *index* $I_A(K, \Phi, U) \in F$ is defined as follows:

Let us denote by p_U: $C_*(K, \tau^k) \to C_*(\bar{U}, \tau^k)$ the natural linear projection. Let $\varphi \in A(\Phi)_k$. Then the "*local Lefschetz number*"

$$\lambda(p_U \circ \varphi) := \sum_{i=0}^{\dim K} (-1)^i \operatorname{tr}(p_U \circ \varphi)_i$$

is defined. It has been proved in [38] that for sufficiently large k_0 the above element of F is independent of the choice of $\varphi \in A(\Phi)_k$ ($k \geq k_0$), because all the approximations are small homotopic (see 3.2.2).

3.5. DEFINITION. $I_A(K, \Phi, U) := \lambda(p_U \circ \varphi)$ for $\varphi \in A(\Phi)_k$, $k \geq k_0$.

3.6. PROPOSITION (Additivity). *Let* U_1, U_2 *be open, disjoint and polyhedral subsets of* U *and* Φ: $\bar{U} \to K$ *an u.s.c. mapping such that* Fix $\Phi \subset U_1 \cup U_2$. *If* $A(\Phi)$ *is an A-system for* Φ *then*

$$I_A(K, \Phi, U) = I_A(K, \Phi, U_1) + I_A(K, \Phi, U_2). \ \blacksquare$$

3.7. COROLLARY (Excision). *Let* $U_1 \subset U$ *be open and polyhedral subsets of* K *and* Fix $\Phi \subset U_1$. *Then*

$$I_A(K, \Phi, U) = I_A(K, \Phi, U_1). \ \blacksquare$$

3.8. PROPOSITION (Homotopy invariance). *Let* $H\colon \bar{U}\times[0,1]\to K$ *be an u.s.c. homotopy such that* $x\notin H(x,t)$ *for* $x\in\partial U$ *and* $t\in[0,1]$. *Let* A_0, A_1 *be* H*-homotopic* A*-systems for* $H_0=H(\cdot,0)$, $H_1=H(\cdot,1)$, *respectively. Then*

$$I_{A_0}(K,H_0,U)=I_{A_1}(K,H_1,U).\ \blacksquare$$

3.9. Remark. Because of 3.7 one can define $I_A(K,\Phi,V)$, where V is open and not polyhedral, if $\Phi\colon L\to K$, $V\subset L\subset K$, and L is a subpolyhedron of K. Then one puts $I_A(K,\Phi,V):=I_A(K,\Phi,U)$ where $U\subset V$ is polyhedral and Fix $\Phi|_{\bar{V}}\subset U$.

The above properties remain true for arbitrary open sets. The proofs are given in [38] only for maps $\Phi\colon K\to K$, but one easily checks that Φ need not be defined on the whole K.

3.10. PROPOSITION (Commutativity). *Let* $W\subset K$ *be open and let* $\Phi_1\colon K\to L$, $\Phi_2\colon L\to K$ *be u.s.c. maps. Assume that* $x\notin\Phi_2\circ\Phi_1(x)$ *for* $x\in\partial W$ *and* $y\notin\Phi_1\circ\Phi_2(y)$ *for* $y\in\partial(\Phi_2^{-1}(W))$. *Assume further that if* $y\in\text{Fix}\,\Phi_1\circ\Phi_2-\overline{\Phi_2^{-1}(W)}$ *then* $\Phi_2(y)\cap\text{Fix}\,\Phi_2\circ\Phi_1|_{\bar{W}}=\emptyset$. *Then for any* A*-systems* $A_1=A(\Phi_1)$, $A_2=A(\Phi_2)$

$$I_{A_1\circ A_2}\big(L,\Phi_1\circ\Phi_2,\Phi_2^{-1}(W)\big)=I_{A_2\circ A_1}(K,\Phi_2\circ\Phi_1,W).\ \blacksquare$$

3.11. PROPOSITION (Mod-p property). *Let* $F=\mathbf{Z}_p$, p *prime. Let* $W\subset K$ *be open and* $\Phi\colon K\to K$ *an u.s.c. map such that* $x\notin\Phi^p(x)$ *for* $x\in\partial W$. *Assume that if* $y\in Fix\,\Phi^p-W$ *then* $\Phi^k(y)\cap\text{Fix}\,\Phi^p|_{\bar{W}}=\emptyset$ *for* $k<p$. *Then*

$$I_A(K,\Phi,W)=I_{A^p}(K,\Phi^p,W).\ \blacksquare$$

The detailed proofs of the above properties are given in [38].

IV. Chain approximations of decompositions of maps

In this chapter we use nerves of coverings for constructing chain approximations of decompositions of multivalued maps, defined in Ch.II. The approximations will be a basic tool for the definition of an index. The ideas of proofs are taken from [1], [38], [39]. All spaces considered are assumed to be compact.

1. Chain approximations of maps in $\mathscr{A}_i(X,Y)$. We will use here the notation introduced in Ch.I. The symbol π^α_β will stand for the simplicial map $N(\alpha)\to N(\beta)$ defined there as well as for the induced map of chain complexes. Let us prove the following technical lemma:

4.1. LEMMA. *Let* $\Phi\in\mathscr{A}_m(X,Y)$, $\alpha_0\in\text{Cov}\,X$, $\beta_0\in\text{Cov}\,Y$, $n\in N$. *There exist sequences of coverings* $\alpha_i\in\text{Cov}\,X$, $\beta_i\in\text{Cov}\,Y$,

$$\alpha_{n+1}\geqslant\alpha_n\geqslant\ldots\geqslant\alpha_0,\qquad \beta_{n+1}\geqslant\beta_n\geqslant\ldots\geqslant\beta_0,$$

such that for each simplex $s \in N(\alpha_i)$ *there are a point* $a(s) \in X$ *and a covering* $\beta_{i-1}(s) \in \mathrm{Cov}\, Y$ $(\beta_i \geqslant \beta_{i-1}(s) \geqslant \beta_{i-1})$ *with the following properties*:

4.1.1. $\mathrm{supp}\, s \subset \mathrm{St}(a(s), \alpha_{i-1})$;

4.1.2. $\Phi(\mathrm{St}(\mathrm{supp}\, s, \alpha_i)) \subset \mathrm{St}(\Phi(a(s)), \beta_{i-1}(s))$;

4.1.3. *If* $C_j(a(s))$ *are the components of* $\Phi(a(s))$ *then the sets* $\mathrm{St}^2(C_j(a(s)), \beta_{i-1}(s))$ *are pairwise disjoint*;

4.1.4. *For* $y \in \mathrm{St}(\mathrm{supp}\, s, \alpha_i)$ *and* $j = 1, \ldots, m$

$$\Phi(y) \cap \mathrm{St}(C_j(a(s)), \beta_{i-1}(s)) \neq \emptyset;$$

4.1.5. $$\pi^{\beta_{i-1}(s)}_{\beta_{i-1}*}\colon \tilde{H}_*(N(\beta_{i-1}(s))|_{\mathrm{St}^2(C_j(a(s)), \beta_{i-1}(s))}) \to \tilde{H}_*(N(\beta_{i-1})|_{\mathrm{St}(\Phi(a(s)), \beta_{i-1})})$$

is a zero homomorphism.

Proof. Let $n = 0$. Let $x \in X$. Since every component C_j of $\Phi(x)$ is acyclic, by 1.3 there is $\beta = \beta_0(x) \in \mathrm{Cov}\, Y$ such that the sets $\mathrm{St}^2(C_j, \beta)$ are pairwise disjoint and the homomorphisms

$$\pi^{\beta}_{\beta_0}\colon \tilde{H}_*(N(\beta)|_{\mathrm{St}^2(C_j, \beta)}) \to \tilde{H}_*(N(\beta_0)|_{\mathrm{St}(\Phi(x), \beta_0)})$$

are trivial. By the continuity of Φ there is a neighbourhood $\mathcal{O}_x$ of x such that:

(i) $\Phi(\mathcal{O}_x) \subset \mathrm{St}(\Phi(x), \beta)$,

(ii) for each $y \in \mathcal{O}_x$, $\Phi(y) \cap \mathrm{St}(C_j, \beta) \neq \emptyset$,

(iii) the covering $\{\mathcal{O}_x\}_{x \in X}$ is a refinement of α_0.

We choose a finite subcovering $\{\mathcal{O}_{x_i}\}_{i=1}^k$. Let α_1 be a star-refinement of $\{\mathcal{O}_{x_i}\}$. For a simplex $s \in N(\alpha_1)$ we define $a(s) := x_i$ if $\mathrm{supp}\, s \subset \mathcal{O}_{x_i}$ and $\beta_0(s) := \beta_0(x_i)$. Let β_1 be a common refinement of all $\beta_0(x_i)$. The above procedure can be continued inductively. ∎

Such sequences (α_i, β_i) of coverings as in Lemma 4.1 will be called *squeezing sequences* for $(\Phi, \alpha_0, \beta_0)$.

Recall that the *Kronecker index* of a 0-chain $c = \sum c_i \sigma_i$ is the sum $\sum c_i$ (see [22]). Let us denote by $N^{(n)}(\alpha)$ the n-skeleton of the simplicial complex $N(\alpha)$.

4.2. Definition. Let $\alpha, \bar{\alpha} \in \mathrm{Cov}\, X$, $\beta, \bar{\beta} \in \mathrm{Cov}\, Y$ and let $\Phi \in \mathscr{A}_m(X, Y)$. A chain map $\varphi\colon C_*(N^{(n)}(\bar{\alpha})) \to C_*(N^{(n)}(\bar{\beta}))$ is called an (α, β)-*approximation* of Φ provided

4.2.1. φ multiplies the Kronecker index by m;

4.2.2. For each simplex $s \in N^{(n)}(\bar{\alpha})$ there is a point $p(s) \in X$ such that

$$\mathrm{supp}\, s \subset \mathrm{St}(p(s), \alpha), \quad \mathrm{supp}\, \varphi s \subset \mathrm{St}(\Phi(p(s)), \beta);$$

4.2.3. If $\dim s = 0$ then for every component $C_j = C_j(p(s))$ of $\Phi(p(s))$, $\mathrm{supp}\, \varphi s \subset \mathrm{St}(C_j, \beta) \neq \emptyset$.

4.3. Theorem. *Let* $\Phi \in \mathscr{A}_m(X, Y)$, $\alpha \in \mathrm{Cov}\, X$, $\beta \in \mathrm{Cov}\, Y$. *For every* $n \in N$

there exist a refinement $\bar{\alpha}$ *of* α *and an* (α, β)*-approximation* $\varphi\colon C_*(N^{(n)}(\bar{\alpha})) \to C_*(N^{(n)}(\beta))$ of Φ.

Proof. From 4.1 we obtain a squeezing sequence (α_i, β_i) with $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Define $\bar{\alpha} := \alpha_{n+1}$. We will construct the desired chain map φ inductively.

$k = 0$: Let s_0 be a vertex of $N(\bar{\alpha})$. By 4.1 we obtain a point $a(s_0) \in X$. If the set $\Phi(a(s_0))$ is connected then we define $\varphi_0 s_0 := m\bar{a}$, where $\bar{a}$ is an arbitrary vertex of $N(\beta_{n+1})$ with $\operatorname{supp} \bar{a} \subset \operatorname{St}(\Phi(a(s_0)), \beta_n(s_0))$. If $\Phi(a(s_0))$ consists of m components then $\varphi_0 s_0 := a_1 + \ldots + a_m$, where a_i are vertices of $N(\beta_{n+1})$ such that $\operatorname{supp} a_i \subset \operatorname{St}(C_i(a(s_0)), \beta_n(s_0))$. So we have defined $\varphi_0\colon C_0(N(\alpha_{n+1})) \to C_0(N(\beta_{n+1}))$. We would like to extend it to 1-chains.

Let s be a 1-simplex in $N(\bar{\alpha})$. Then $\partial s = s_1 - s_0$. Since $a(s_0)$ and $a(s_1)$ belong to $\operatorname{St}(\operatorname{supp} s, \alpha_n)$,

$$\Phi(a(s_0)) \cup \Phi(a(s_1)) \subset \operatorname{St}(\Phi(a(s)), \beta_{n-1}(s)).$$

Let $\varphi_0 \partial s = \sum a_i - \sum b_i$ with $a_i, b_i \in C_0(N(\beta_{n+1}))$. If $\Phi(a(s))$ is connected then by 4.1.5

$$\pi^{\beta_{n+1}}_{\beta_{n-1}}\left(\sum (a_i - b_i)\right) = \sum \partial c_i, \quad \text{where } c_i \in C_1(N(\beta_{n-1})).$$

If $\Phi(a(s)) = \bigcup_{i=1}^{m} C_i(a(s))$ then for each pair a_i, b_i

$$\operatorname{supp}(a_i - b_i) \subset \operatorname{St}(C_i(a(s)), \beta_{n-1}(s)).$$

Thus by 4.1.5 we obtain

$$\pi^{\beta_{n+1}}_{\beta_{n-1}}(a_i - b_i) = \partial c_i$$

for some $c_i \in C_1(N(\beta_{n-1}))$ such that $\operatorname{supp} c_i \subset \operatorname{St}(C_i(a(s)), \beta_{n-1})$. Let us define $\varphi_1 s := \sum c_i$. Hence we have obtained the following commutative diagram:

$$\begin{array}{ccccc}
C_0(N(\alpha_{n+1})) & \xrightarrow{\varphi_0} & C_0(N(\beta_{n+1})) & \xrightarrow{\pi^{\beta_{n+1}}_{\beta_{n-1}}} & C_0(N(\beta_{n-1})) \\
\uparrow \partial & & & & \uparrow \partial \\
C_1(N(\alpha_{n+1})) & & \xrightarrow[\varphi_1]{} & & C_1(N(\beta_{n-1}))
\end{array}$$

Therefore a chain map $\varphi_1\colon C_*(N^{(1)}(\alpha_{n+1})) \to C_*(N^{(1)}(\beta_{n-1}))$ is defined (on 0-chains $\varphi_1 c := \pi\varphi_0 c$).

Now, assume that $\varphi_{k-1}\colon C_*(N^{(k-1)}(\alpha_{n+1})) \to C_*(N^{(k-1)}(\beta_{n-k+1}))$ is defined and satisfies conditions 4.2 with $\alpha = \alpha_{n+1}$ and $\beta = \beta_{n-k+1}$. As in the step $0 \Rightarrow 1$, we will define $\varphi_k\colon C_k(N(\alpha_{n+1})) \to C_k(N(\beta_{n-k}))$. Let $s \in N(\alpha_{n+1})$ be a k-simplex and $\bar{s} := \pi_{\alpha_{n-k+1}}^{\alpha_{n+1}}(s)$. From 4.1 we obtain a point $a(\bar{s})$ and a covering $\beta_{n-k}(\bar{s})$. Let $\partial s = \sum s_i$; $\dim s_i = k-1$. By our inductive assumption we have points $p(s_i)$ such that

$$\operatorname{supp} s_i \subset \operatorname{St}\big(p(s_i), \alpha_{n-k+1}\big), \quad \operatorname{supp} \varphi_{k-1} s_i \subset \operatorname{St}\big(\Phi(p(s_i)), \beta_{n-k+1}\big).$$

But we also have the following inclusion:

$$\Phi\big(p(s_i)\big) \subset \operatorname{St}\big(\Phi(a(s)), \beta_{n-k}(\bar{s})\big).$$

Therefore

$$\operatorname{supp} \varphi_{k-1} \partial s \subset \operatorname{St}^2\big(\Phi(a(s)), \beta_{n-k}(\bar{s})\big).$$

Because of our assumption, $\varphi_{k-1} \partial s \in C_{k-1}\big(N(\beta_{n-k+1})\big)$ is a cycle. Hence by 4.1.5 there is a chain $c \in C_k\big(N(\beta_{n-k})|_{\operatorname{St}(\Phi(a(\bar{s})), \beta_{n-k})}\big)$ such that

$$\pi_{\beta_{n-k}}^{\beta_{n-k+1}} \varphi_{k-1} \partial s = \partial c.$$

Let us define $\varphi_k s := c$ and $p(s) := a(\bar{s})$. For chains of lower dimension we put $\varphi_k := \pi \circ \varphi_{k-1}$. Then φ_n is the desired approximation φ. ■

4.4. Definition. An (α, β)-approximation φ: $C_*\big(N^{(n)}(\tilde{\alpha})\big) \to C_*\big(N^{(n)}(\beta)\big)$ of the map Φ is *n-close* provided there is a squeezing sequence $(\alpha_i, \beta_i)_{i=0}^{n+1}$ such that $\alpha_0 = \alpha$, $\beta_0 = \beta$ and $\varphi = \pi_{\beta_0}^{\beta_{n+1}} \circ \bar{\varphi}$, where $\bar{\varphi}$: $C_*\big(N^{(n)}(\tilde{\alpha})\big) \to C_*\big(N^{(n)}(\beta_{n+1})\big)$ is an $(\alpha_{n+1}, \beta_{n+1})$-approximation of Φ.

4.5. Theorem. *Let $\Phi \in \mathscr{A}_m(X, Y)$. For all $\alpha \in \operatorname{Cov} X$, $\beta \in \operatorname{Cov} Y$ and $n \in N$ there is an n-close $(\bar{\alpha}, \beta)$-approximation φ: $C_*\big(N^{(n)}(\tilde{\alpha})\big) \to C_*\big(N^{(n)}(\beta)\big)$ of Φ for some $\tilde{\alpha} \geqslant \bar{\alpha} \geqslant \alpha$.*

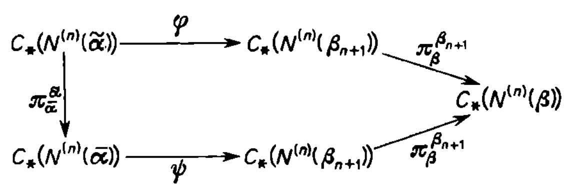
Proof. From 4.1 we obtain a squeezing sequence (α_i, β_i) with $\alpha_0 = \alpha$, $\beta_0 = \beta$. Let $\bar{\alpha} = \alpha_{n+1}$. By 4.3 there are $\tilde{\alpha} \geqslant \bar{\alpha}$ and an $(\bar{\alpha}, \beta_{n+1})$-approximation of Φ, $\bar{\varphi}$: $C_*\big(N^{(n)}(\tilde{\alpha})\big) \to C_*\big(N^{(n)}(\beta_{n+1})\big)$. Putting $\varphi = \pi_\beta^{\beta_{n+1}} \circ \bar{\varphi}$ we obtain the desired n-close $(\bar{\alpha}, \beta)$-approximation of Φ. ■

4.6. Definition. A chain homotopy D: $C_*\big((N(\bar{\alpha})\big) \to C_*\big(N(\bar{\beta})\big)$ is (Φ, α, β)-*small* provided for each simplex $s \in N(\bar{\alpha})$ there exists a point $c(s) \in X$ such that

$$\operatorname{supp} s \subset \operatorname{St}\big(c(s), \alpha\big), \quad \operatorname{supp} Ds \subset \operatorname{St}\big(\Phi(c(d)), \beta\big).$$

Let $\Phi \in \mathscr{A}_m(X, Y)$. Let us prove the following

4.7. Proposition. *For every two n-close (α, β)-approximations $\pi_\beta^{\beta_{n+1}} \circ \varphi$, $\pi_\beta^{\beta_{n+1}} \circ \psi$ of Φ with the same squeezing sequences $(\alpha_i, \beta_i)_{i=0}^{n+1}$ the diagram is*



homotopy commutative with a (Φ, α, β)-small homotopy D.

Proof. Let s_0 be a vertex in $N(\tilde{\alpha})$ and let $\bar{s}_0 := \pi^{\tilde{\alpha}}_{\bar{\alpha}}(s_0)$, $\tilde{s}_0 := \pi^{\tilde{\alpha}}_{\alpha_{n+1}}(s_0)$. Since φ and ψ are $(\alpha_{n+1}, \beta_{n+1})$-approximations of Φ, we have points $p(s_0)$, $p(\bar{s}_0)$ from 4.2. From 4.1 we obtain a point $a(\tilde{s}_0)$. Since $\operatorname{supp} s_0 \subset \operatorname{supp} \bar{s}_0 \subset \operatorname{supp} \tilde{s}_0$, it follows that $p(s_0), p(\bar{s}_0) \in \operatorname{St}(\operatorname{supp} \tilde{s}_0, \alpha_{n+1})$. Therefore

$$\Phi\big(p(s_0) \cup p(\bar{s}_0)\big) \subset \operatorname{St}\big(\Phi(a(\tilde{s}_0)), \beta_n(\tilde{s}_0)\big).$$

Since φ and ψ are approximations of Φ,

$$\operatorname{supp} \varphi s_0 \subset \operatorname{St}\big(\Phi(p(s_0)), \beta_{n+1}\big), \quad \operatorname{supp} \psi s_0 \subset \operatorname{St}\big(\Phi(p(\bar{s}_0)), \beta_{n+1}\big).$$

Therefore

$$\operatorname{supp} \varphi s_0 \cup \operatorname{supp} \psi s_0 \subset \operatorname{St}^2\big(\Phi(a(\tilde{s}_0)), \beta_n(\tilde{s}_0)\big).$$

Moreover, conditions 4.1.3 and 4.1.4 ensure that if $\varphi s_0 = \sum_{i=1}^{m} a_i$ and $\psi \bar{s}_0 = \sum_{i=1}^{m} b_i$ then there are m cycles of the form $a_i - b_i$ with

$$\operatorname{supp}(a_i - b_i) \subset \operatorname{St}^2\big(C_i(a(\tilde{s}_0)), \beta_n(\tilde{s}_0)\big).$$

Because of 4.1.5 there is a chain $c \in C_1\big(N(\beta_n)\big)$ such that

$$\operatorname{supp} c \subset \operatorname{St}\big(\Phi(a(\tilde{s}_0)), \beta_n\big), \quad \partial c = \sum_{i=1}^{m} (a_i - b_i) = \varphi s_0 - \psi \bar{s}_0.$$

We define $D_0 s_0 := c$ and $c(s_0) := a(\tilde{s}_0)$. The construction of the chain homotopy follows inductively: Assume that $D_i\colon C_i\big(N^{(n)}(\tilde{\alpha})\big) \to C_{i+1}\big(N^{(n)}(\beta_{n-k+1})\big)$ are defined for $i < k$ and satisfy the conditions:

4.7.1. $\pi^{\beta_{n+1}}_{\beta_{n-k+1}} \varphi c - \pi^{\beta_{n+1}}_{\beta_{n-k+1}} \psi \pi^{\tilde{\alpha}}_{\bar{\alpha}} c = \partial D_i c + D_{i-1} \partial c$;

4.7.2. For each simplex $s \in N(\tilde{\alpha})$, $\dim s = i$, there is a point $c(s) \in X$ such that

$$\operatorname{supp} s \subset \operatorname{St}\big(c(s), \alpha_{n-i}\big), \quad \operatorname{supp} D_i s \subset \operatorname{St}\big(\Phi(c(s)), \beta_{n-i}\big).$$

Let $s \in N(\tilde{\alpha})$ be a k-simplex. We consider the simplexes $\bar{s} := \pi^{\tilde{\alpha}}_{\bar{\alpha}} s$ and $\tilde{s} := \pi^{\tilde{\alpha}}_{\alpha_{n-k+1}} s$. From 4.2 we obtain points $p(s)$ and $p(\bar{s})$. From 4.1 we have $a(\tilde{s})$. They satisfy

$$p(s), p(\bar{s}) \in \operatorname{St}(\operatorname{supp} \tilde{s}, \alpha_{n-k+1}), \quad \Phi\big(p(s) \cup p(\bar{s})\big) \subset \operatorname{St}\big(\Phi(a(\tilde{s})), \beta_{n-k}(\tilde{s})\big).$$

Therefore

$$\operatorname{supp} \pi^{\beta_{n+1}}_{\beta_{n-k+1}} \varphi s \subset \operatorname{St}^2\big(\Phi(a(\tilde{s})), \beta_{n-k}(\tilde{s})\big),$$
$$\operatorname{supp} \pi^{\beta_{n+1}}_{\beta_{n-k+1}} \psi \bar{s} \subset \operatorname{St}^2\big(\Phi(a(\tilde{s})), \beta_{n-k}(\tilde{s})\big).$$

Let $\partial s = \sum s_j$, $\dim s_j = k-1$. By the inductive assumption we obtain

$\operatorname{supp} s_j \subset \operatorname{St}(c(s_j), \alpha_{n-k+1})$. Hence $c(s_j) \in \operatorname{St}(\operatorname{supp} \tilde{s}, \alpha_{n-k+1})$ and thus $\Phi(c(s_j)) \subset \operatorname{St}(\Phi(a(\tilde{s})), \beta_{n-k}(\tilde{s}))$. Therefore

$$\operatorname{supp} D_{k-1} s_j \subset \operatorname{St}(\Phi(c(s_j)), \beta_{n-k+1}) \subset \operatorname{St}^2(\Phi(a(\tilde{s})), \beta_{n-k}(\tilde{s})).$$

Let us consider the chain

$$c := \pi^{\beta_{n+1}}_{\beta_{n-k+1}} \varphi s - \pi^{\beta_{n+1}}_{\beta_{n-k+1}} \psi \tilde{s} - D_{k-1} \partial s.$$

We deduce from the above that

$$\operatorname{supp} c \subset \operatorname{St}^2(\Phi(a(\tilde{s})), \beta_{n-k}(\tilde{s})).$$

On the other hand, we have

$$\begin{aligned}\partial c &= \partial \pi \varphi s - \partial \pi \psi \pi s - \partial D_{k-1} \partial s = \pi \partial \varphi s - \pi \partial \psi \pi s - \partial D_{k-1} \partial s \\ &= \pi \varphi \partial s - \pi \psi \pi \partial s - \partial D_{k-1} \partial s = \partial D_{k-1} \partial s + D_{k-2} \partial \partial s - \partial D_{k-1} \partial s = 0.\end{aligned}$$

Therefore by 4.1.5 there exists a chain $\bar{c} \in C_{k+1}(N(\beta_{n-k}))$ such that $\partial \bar{c} = c$ and $\operatorname{supp} \bar{c} \subset \operatorname{St}(\Phi(a(\tilde{s})), \beta_{n-k})$. We define $D_k s := \bar{c}$ and $c(s) := a(\tilde{s})$. For $i < k$ we compose D_i with $\pi^{\beta_{n-k+1}}_{\beta_{n-k}}$. Since π can only enlarge supports, the D_i satisfy 4.7.1 and 4.7.2. In the nth step one obtains a desired (Φ, α, β)-small homotopy D. ∎

4.8. PROPOSITION. *For arbitrary two n-close (α, β)-approximations $\varphi\colon C_*(N^{(n)}(\bar{\alpha})) \to C_*(N^{(n)}(\beta))$ and $\psi\colon C_*(N^{(n)}(\tilde{\alpha})) \to C_*(N^{(n)}(\beta))$ of Φ there exists $\alpha' \in \operatorname{Cov} X$ such that $\varphi \circ \pi^{\alpha'}_{\bar{\alpha}}$ and $\psi \circ \pi^{\alpha'}_{\tilde{\alpha}}$ are (Φ, α, β)-small homotopic.*

Proof. Let $\varphi = \pi^{\beta_{n+1}}_{\beta} \circ \varphi_1$ and $\psi = \pi^{\bar{\beta}_{n+1}}_{\beta} \circ \psi_1$. There exist common refinements: $\bar{\bar{\alpha}}$ of $\bar{\alpha}, \tilde{\alpha}, \alpha_{n+1}, \bar{\alpha}_{n+1}$, and $\bar{\bar{\beta}}$ of $\beta_{n+1}, \bar{\beta}_{n+1}$. We construct an $(\bar{\bar{\alpha}}, \bar{\bar{\beta}})$-approximation $\varrho\colon C_*(N^{(n)}(\alpha')) \to C_*(N^{(n)}(\bar{\bar{\beta}}))$ of Φ using 4.3. From 4.7 we deduce that $\varphi \circ \pi^{\alpha'}_{\bar{\alpha}}$ is small homotopic to $\pi^{\bar{\bar{\beta}}}_{\beta} \circ \varrho$ with a homotopy D_1, $\psi \circ \pi^{\alpha'}_{\tilde{\alpha}}$ is small homotopic to $\pi^{\bar{\bar{\beta}}}_{\beta} \circ \varrho$ with a homotopy D_2. Therefore $D_1 + D_2$ is a small homotopy joining $\varphi \circ \pi^{\alpha'}_{\bar{\alpha}}$ and $\psi \circ \pi^{\alpha'}_{\tilde{\alpha}}$. ∎

4.9. Remark. Note that the only place where the continuity of the map Φ is important is the proof of 4.1.3 and 4.1.4. If Φ is an acyclic map then we can assume it to be u.s.c. only (cf. [38]).

2. Chain approximations of decompositions of maps.

4.10. LEMMA. *Let $\Phi_1\colon X_1 \to X_2$, $\Phi_2\colon X_2 \to X_3$ be u.s.c. maps, X_1, X_2, X_3 compact spaces and $\alpha \in \operatorname{Cov} X_1$, $\gamma \in \operatorname{Cov} X_3$. There exists $\beta \in \operatorname{Cov} X_2$ such that for each $y \in X_1$ there is a point $u(y) \in X_1$ such that*

$$y \in \operatorname{St}(u(y), \alpha), \qquad \Phi_2(\operatorname{St}^2(\Phi_1(y), \beta)) \subset \operatorname{St}(\Phi_2 \circ \Phi_1(u(y)), \gamma).$$

Proof. The set $\operatorname{St}(\Phi_2 \circ \Phi_1(y), \gamma)$ is a neighbourhood of the compact set $\Phi_2 \circ \Phi_1(y)$ in X_3. Thus for every $x \in \Phi_1(y)$ there exists a covering $\bar{\beta}(x) \in \operatorname{Cov} X_2$ such that $\Phi_2(\operatorname{St}(x, \bar{\beta}(x))) \subset \operatorname{St}(\Phi_2 \circ \Phi_1(y), \gamma)$. So the set

$U := \bigcup \{\mathrm{St}(x, \bar{\beta}(x)) : x \in \Phi_1(y)\}$ is a neighbourhood of $\Phi_1(y)$ such that $\Phi_2(U) \subset \mathrm{St}(\Phi_2 \circ \Phi_1(y), \gamma)$. Since $\Phi_1(y)$ is a compact set, there is $\beta(y) \in \mathrm{Cov}\, X_2$ such that $\mathrm{St}^3(\Phi_1(y), \beta(y)) \subset U$. Hence

$$\Phi_2(\mathrm{St}^3(\Phi_1(y), \beta(y))) \subset \mathrm{St}(\Phi_2 \circ \Phi_1(y), \gamma).$$

Now, it is sufficient to show that there is $\beta \in \mathrm{Cov}\, X_2$ such that for every $y \in X_1$ there is $u(y) \in X_1$ such that

$$y \in \mathrm{St}(u(y), \alpha), \qquad \mathrm{St}^2(\Phi_1(y), \beta) \subset \mathrm{St}^3(\Phi_1(u(y)), \beta(u(y))).$$

Suppose on the contrary that for every $\beta \in \mathrm{Cov}\, X_2$ there is $y_\beta \in X_1$ such that for each $z \in \mathrm{St}(y_\beta, \alpha)$

4.10.1. $\mathrm{St}^2(\Phi_1(y_\beta), \beta) \not\subset \mathrm{St}^3(\Phi_1(z), \beta(z))$.

We can assume that the net $\{y_\beta\}$ converges to y_0, because X_1 is compact. The set $\mathrm{St}(\Phi_1(y_0), \beta(y_0))$ is a neighbourhood of $\Phi_1(y_0)$ in X_2. Since Φ_1 is u.s.c., there is a covering $\alpha_0 \geqslant \alpha$ such that $\Phi_1(\mathrm{St}(y_0, \alpha_0)) \subset \mathrm{St}(\Phi_1(y_0), \beta(y_0))$. There exists $\beta_1 \in \mathrm{Cov}\, X_2$ such that, for every $\bar{\beta} \geqslant \beta_1$, $y_{\bar{\beta}} \in \mathrm{St}(y_0, \alpha_0)$, and consequently

$$\Phi_1(y_{\bar{\beta}}) \subset \mathrm{St}(\Phi_1(y_0), \beta(y_0)).$$

Hence

$$\mathrm{St}^2(\Phi_1(y_{\bar{\beta}}), \bar{\beta}) \subset \mathrm{St}^3(\Phi_1(y_0), \beta(y_0))$$

and $y_0 \in \mathrm{St}(y_{\bar{\beta}}, \alpha_0) \subset \mathrm{St}(y_{\bar{\beta}}, \alpha)$. We have arrived at a contradiction with 4.10.1. ∎

4.11. Definition. Let $(\Phi_1, \ldots, \Phi_k) \in \mathcal{D}\mathcal{A}(X_0, X_k)$, $\Phi_i \in \mathcal{A}_{m_i}(X_{i-1}, X_i)$. Let $\alpha, \bar{\alpha} \in \mathrm{Cov}\, X_0$, $\beta, \bar{\beta} \in \mathrm{Cov}\, X_k$. A chain map $\varphi\colon C_*(N^{(n)}(\bar{\alpha})) \to C_*(N^{(n)}(\bar{\beta}))$ is an *(α, β)-approximation* of $(\Phi_1, \ldots, \Phi_k)$ provided

4.11.1. φ multiplies the Kronecker index by $m = m_1 \ldots m_k$;

4.11.2. For each simplex $s \in N(\bar{\alpha})$ there is a point $p(s) \in X_0$ and $r \in N$ such that

$$\mathrm{supp}\, s \subset \mathrm{St}^r(p(s), \alpha), \qquad \mathrm{supp}\, \varphi s \subset \mathrm{St}^r(\Phi(p(s)), \beta),$$

where Φ is the map determined by $(\Phi_1, \ldots, \Phi_k)$.

4.12. Proposition. *Let $\Phi_1 \in \mathcal{D}\mathcal{A}(X_1, X_2)$, $\Phi_2 \in \mathcal{D}\mathcal{A}(X_2, X_3)$. Let the coverings $\alpha \in \mathrm{Cov}\, X_1$ and $\gamma \in \mathrm{Cov}\, X_3$ be given. There is a covering $\beta \in \mathrm{Cov}\, X_2$ such that if $\varphi\colon C_*(N^{(n)}(\bar{\alpha})) \to C_*(N^{(n)}(\bar{\beta}))$ is an (α, β)-approximation of Φ_1 and $\psi\colon C_*(N^{(n)}(\bar{\beta})) \to C_*(N^{(n)}(\gamma))$ is a (β, γ)-approximation of Φ_2 then $\psi \circ \varphi$ is an (α, γ)-approximation of $\Phi_2 \circ \Phi_1 \in \mathcal{D}\mathcal{A}(X_1, X_3)$.*

Proof. 4.11.1 is evident. We prove 4.11.2. For simplicity we assume that $r_1 = r_2 = 1$. Let $\beta \in \mathrm{Cov}\, X_2$ be obtained from 4.10. If φ is an (α, β)-

approximation of Φ_1, then for each simplex $s \in N(\bar{\alpha})$ there is a point $p(s) \in X_1$ such that

$$\operatorname{supp} s \subset \operatorname{St}(p(s), \alpha), \quad \operatorname{supp} \varphi s \subset \operatorname{St}(\Phi_1(p(s)), \beta).$$

From 4.10 we obtain a point $u = u(p(s)) \in X_1$ such that

$$p(s) \in \operatorname{St}(u, \alpha), \quad \Phi_2(\operatorname{St}^2(\Phi_1(p(s)), \beta)) \subset \operatorname{St}(\Phi_2 \circ \Phi_1(u), \gamma).$$

Therefore $\operatorname{supp} s \subset \operatorname{St}^2(u, \alpha)$.

Let $\varphi s = \sum a_i s_i$. For every s_i there is $p(s_i) \in X_2$ such that

$$\operatorname{supp} s_i \subset \operatorname{St}(p(s_i), \beta), \quad \operatorname{supp} \psi s_i \subset \operatorname{St}(\Phi_2(p(s_i)), \gamma).$$

Hence

$$p(s_i) \in \operatorname{St}^2(\Phi_1(p(s)), \beta),$$

$$\operatorname{supp} \psi s_i \subset \operatorname{St}(\Phi_2(\operatorname{St}^2(p(s), \beta)), \gamma) \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma).$$

Since $\psi\varphi s = \sum a_i \psi s_i$, we obtain

$$\operatorname{supp} s \subset \operatorname{St}^2(u, \alpha), \quad \operatorname{supp} \psi\varphi s \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma).$$

The proof is finished. ∎

4.13. Corollary. *Let* $(\Phi_1, \ldots, \Phi_k) \in \mathscr{D}\mathscr{A}(X_0, X_k)$. *Let* $\alpha \in \operatorname{Cov} X_0$, $\beta \in \operatorname{Cov} X_k$, $n \in N$. *Then there exists an* (α, β)*-aproximation* $\varphi\colon C_*(N^{(n)}(\bar{\alpha})) \to C_*(N^{(n)}(\beta))$ *of* $(\Phi_1, \ldots, \Phi_k)$ *which is a composition of chain approximations of the* Φ_i.

Proof. The proof proceeds by induction on the length k. For $k = 1$ the assertion follows from 4.3, and 4.12 is used in the inductive step. ∎

4.14. Proposition. *Compositions of small homotopic and sufficiently fine chain approximations are also small homotopic.*

Proof. Let $\Phi_1 \in \mathscr{D}\mathscr{A}(X_1, X_2)$, $\Phi_2 \in \mathscr{D}\mathscr{A}(X_2, X_3)$ and let $\alpha, \bar{\alpha} \in \operatorname{Cov} X_1$, $\beta, \bar{\beta} \in \operatorname{Cov} X_2$, $\gamma \in \operatorname{Cov} X_3$. Assume that the following diagram is given:

$$\begin{array}{ccccc}
C_*(N^{(n)}(\bar{\alpha})) & \xrightarrow{\varphi_1} & C_*(N^{(n)}(\bar{\beta})) & \xrightarrow{\varphi_2} & C_*(N^{(n)}(\gamma)) \\
\downarrow \mathrm{id} & \mathrm{I} & \downarrow \mathrm{id} & \mathrm{II} & \downarrow \mathrm{id} \\
C_*(N^{(n)}(\bar{\alpha})) & \xrightarrow[\psi_1]{} & C_*(N^{(n)}(\bar{\beta})) & \xrightarrow[\psi_2]{} & C_*(N^{(n)}(\gamma))
\end{array}$$

where I is homotopy commutative with a (Φ_1, α, β)-small homotopy D_1 and II is homotopy commutative with a (Φ_2, β, γ)-small homotopy D_2 (cf. 4.6). Let the covering β be fine enough to satisfy 4.10. We prove that $D := \psi_2 D_1 + D_2 \varphi_1$ is a chain homotopy joining $\varphi_2 \varphi_1$ and $\psi_2 \psi_1$ and that it is (Φ, α, γ)-small, where $\Phi = \Phi_2 \circ \Phi_1$.

Let $s \in N^{(n)}(\bar{\alpha})$ be a simplex. Then

$$\begin{aligned}\psi_2 \psi_1 s - \varphi_2 \varphi_1 s &= \psi_2 \psi_1 s - \psi_2 \varphi_1 s + \psi_2 \varphi_1 s - \varphi_2 \varphi_1 s \\ &= \psi_2(\partial D_1 s + D_1 \partial s) + \partial D_2 \varphi_1 s + D_2 \partial \varphi_1 s \\ &= \partial(\psi_2 D_1 s + D_2 \varphi_1 s) + (\psi_2 D_1 + D_2 \varphi_1) \partial s.\end{aligned}$$

Therefore D is a chain homotopy. Now, we check the *smallness condition.* From 4.6 we obtain a point $c(s) \in X_1$ such that

$$\operatorname{supp} s \subset \operatorname{St}(c(s), \alpha), \quad \operatorname{supp} D_1 s \subset \operatorname{St}(\Phi_1(c(s)), \beta).$$

By 4.10 we find a point $u = u(c(s)) \in X_1$ such that

$$c(s) \in \operatorname{St}(u, \alpha), \quad \Phi_2(\operatorname{St}^2(\Phi_1(c(s)), \beta)) \subset \operatorname{St}(\Phi_2 \circ \Phi_1(u), \gamma).$$

Thus $\operatorname{supp} s \subset \operatorname{St}^2(u, \alpha)$. We show that $\operatorname{supp} Ds \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma)$. But $\operatorname{supp} Ds = \operatorname{supp} \psi_2 D_1 s \cup \operatorname{supp} D_2 \varphi_1 s$. Let $D_1 s = \sum a_i s_i$. For each s_i there is $p(s_i) \in X_2$ such that

$$\operatorname{supp} s_i \subset \operatorname{St}(p(s_i), \beta), \quad \operatorname{supp} \psi_2 s_i \subset \operatorname{St}(\Phi_2(p(s_i)), \gamma).$$

Hence $p(s_i) \in \operatorname{St}^2(\Phi_1(c(s)), \beta)$ and

$$\operatorname{supp} \psi_2 s_i \subset \operatorname{St}(\Phi_2 \operatorname{St}^2(\Phi_1(c(s)), \beta), \gamma) \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma).$$

Therefore

$$\operatorname{supp} \psi_2 D_1 s = \bigcup_i \operatorname{supp} \psi_2 s_i \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma).$$

Note that $\operatorname{supp} \varphi_1 s \subset \operatorname{supp} D_1 s$. Let $\varphi_1 s = \sum b_j s_j$. By 4.6 for each $s_j \in N^{(n)}(\bar{\beta})$ there is a point $c(s_j) \in X_2$ such that

$$\operatorname{supp} s_j \subset \operatorname{St}(c(s_j), \beta), \quad \operatorname{sup} D_2 s_j \subset \operatorname{St}(\Phi_2(c(s_j)), \gamma).$$

But we note that $c(s_j) \in \operatorname{St}^2(\Phi_1(c(s)), \beta)$, and consequently

$$\operatorname{supp} D_2 s_j \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma).$$

Therefore

$$\operatorname{supp} D_2 \varphi_1 s = \bigcup_j \operatorname{supp} D_2 s_j \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma)$$

and our proof is finished. ∎

V. Index of decompositions for compact polyhedra

In this chapter we use the previous results to construct a fixed point index theory on compact polyhedra.

1. The induced homology homomorphism. Let X and Y be two compact spaces. Let $(\Phi_1, \ldots, \Phi_k) \in \mathcal{DA}(X, Y)$. Then each chain approximation $\varphi\colon C_*(N^{(n)}(\alpha)) \to C_*(N^{(n)}(\beta))$ of $(\Phi_1, \ldots, \Phi_k)$ induces a homomorphism

$$\varphi_*\colon H_*(N^{(n)}(\alpha)) \to H_*(N^{(n)}(\beta)).$$

Assume that the chain approximations considered are compositions of n-close approximations of the Φ_i. From 4.8 and 4.14 we deduce that for sufficiently fine coverings α, β the diagram

$$\begin{array}{ccc} H_*(N^{(n)}(\bar{\alpha})) & \xrightarrow{\varphi_*} & H_*(N^{(n)}(\bar{\beta})) \\ {\scriptstyle \pi^{\tilde{\alpha}}_{\bar{\alpha}*}} \uparrow & & \uparrow {\scriptstyle \pi^{\tilde{\beta}}_{\bar{\beta}*}} \\ H_*(N^{(n)}(\tilde{\alpha})) & \xrightarrow[\psi_*]{} & H_*(N^{(n)}(\tilde{\beta})) \end{array}$$

commutes, where $\tilde{\alpha} \geqslant \bar{\alpha} \geqslant \alpha$, $\tilde{\beta} \geqslant \bar{\beta} \geqslant \beta$ and φ, ψ are (α, β)-approximations of $(\Phi_1, \ldots, \Phi_k)$. Therefore one can define the *induced homomorphism*

$$(\Phi_1, \ldots, \Phi_k)_*\colon \check{H}_*(X) \to \check{H}_*(Y)$$

by the formula

5.1. $(\Phi_1, \ldots, \Phi_k)_q := \varprojlim \{\varphi_{*q}\colon H_q(N^{(n)}(\alpha)) \to H_q(N^{(n)}(\beta))\}$

for $q < n$. This homomorphism is nontrivial in the sense of O'Neill (see [31]), i.e. it is a nonzero homomorphism of the 0-th homology vector spaces.

Let (X, A) and (Y, B) be compact pairs and let Φ be a map determined by $(\Phi_1, \ldots, \Phi_k) \in \mathcal{DA}(X, Y)$ such that $\Phi(A) \subset B$. Let $\alpha \in \operatorname{Cov} X$ and $\beta \in \operatorname{Cov} Y$ be such that $\Phi(\operatorname{St}^2(A, \alpha)) \subset \operatorname{St}(B, \beta)$. Let $\bar{\beta} \in \operatorname{Cov} Y$ be a star-refinement of β. Consider a chain map $\varphi\colon C_*(N^{(n)}(\bar{\alpha})) \to C_*(N^{(n)}(\bar{\beta}))$ which is an $(\alpha, \bar{\beta})$-approximation of $(\Phi_1, \ldots, \Phi_k)$.

If a chain $c \in C_*(N^{(n)}(\bar{\alpha}))$ is such that $\operatorname{supp} c \subset \operatorname{St}(A, \bar{\alpha})$ then $\operatorname{supp} \varphi c \subset \operatorname{St}^2(B, \bar{\beta}) \subset \operatorname{St}(B, \beta)$. This follows from 4.11.2 (without loss of generality we can assume that φ satisfies 4.11.2 with $r = 1$). Therefore $\bar{\varphi} = \pi^{\bar{\beta}}_{\beta} \circ \varphi$ is a chain map of the relative chain complexes:

$$\bar{\varphi}\colon C_*(N^{(n)}(\bar{\alpha}), N^{(n)}(\bar{\alpha})|_{\operatorname{St}(A,\bar{\alpha})}) \to C_*(N^{(n)}(\beta), N^{(n)}(\beta)|_{\operatorname{St}(B,\beta)}).$$

Now, a formula similar to 5.1 gives the definition of the relative induced homomorphism (cf. [11]):

5.2. $(\Phi_1, \ldots, \Phi_k)_*\colon \check{H}_*(X, A) \to \check{H}_*(Y, B)$.

Let $(X, A), (Y, B)$ be arbitrary pairs. If Φ is determined by $(\Phi_1, \ldots, \Phi_k) \in \mathcal{DA}(X, Y)$ then by 2.1.1 the image $\Phi(C)$ of any compact set $C \subset X$ is a compact subset of Y. Assume that $\Phi(A) \subset B$. Then the procedure given in Section 2 of Chapter I can be applied. Therefore we obtain the induced homology homomorphism $(\Phi_1, \ldots, \Phi_k)_*\colon H_*(X, A) \to H_*(Y, B)$, where H is the Čech homology functor with compact carriers and coefficients in the field F.

2. Induced A-systems. Let (K, τ) be a polyhedron with a fixed triangulation τ. The *covering associated with the triangulation* τ is $\alpha(\tau) := \{\mathrm{st}(v_i, \tau) := \mathrm{Int}\,\mathrm{St}(v_i, \tau)\}$, where v_i are vertices of τ. There are simplicial maps $\theta\colon (K, \tau) \to N(\alpha(\tau))$ and $\lambda\colon N(\alpha(\tau)) \to (K, \tau)$ defined on vertices by $\theta(v) := \mathrm{st}(v, \tau)$ and $\lambda(\mathrm{st}(v, \tau)) := v$. These maps define the canonical simplicial isomorphism between the complexes (K, τ) and $N(\alpha(\tau))$. Moreover, carr $s \subset \mathrm{supp}\,\theta s$ and $\mathrm{supp}\,\sigma \subset \mathrm{St}(\mathrm{carr}\,\lambda\sigma, \alpha(\tau))$.

Let $(\Phi_1, \ldots, \Phi_k) \in \mathscr{D}\mathscr{A}(K, L)$. Let τ be a triangulation of K and μ a triangulation of L. We define $A_j(\Phi_1, \ldots, \Phi_k)$ to be the set of chain maps $\varphi\colon C_*(K, \tau^j) \to C_*(L, \mu^j)$ which are of the form $\varphi = \lambda \circ \varphi_k \circ \ldots \circ \varphi_1 \circ \theta \circ b$, where b is the standard subdivision map (cf. Ch. III), λ and θ are induced by the above-named isomorphism for μ^j, τ^j respectively, and φ_i are n-close chain approximations of Φ_i; $n = \dim K$.

5.3. Proposition. *The graded set $\{A_j(\Phi_1, \ldots, \Phi_k)\}_j$ is an A-system for the map Φ determined by $(\Phi_1, \ldots, \Phi_k)$.*

Proof. For 4.5 and 4.13 we deduce that the sets $A_j(\Phi_1, \ldots, \Phi_k)$ are nonempty for arbitrarily large j. Condition 3.2.2 follows from 4.8 and 4.14. ∎

5.4. Definition. The above A-system is said to be an *induced A-system* of the decomposition $(\Phi_1, \ldots, \Phi_k)$. We denote it by $A_*(\Phi_1, \ldots, \Phi_k)$.

Since every element φ of $A_j(\Phi_1, \ldots, \Phi_k)$ induces a homology homomorphism, using 3.2.2 we obtain the induced homomorphism, $(A_*(\Phi_1, \ldots, \Phi_k))_*\colon H_*(K) \to H_*(L)$. A straightforward consequence of definitions is given by

5.5. Proposition. *If X and Y are compact polyhedra and $(\Phi_1, \ldots, \Phi_k) \in \mathscr{D}\mathscr{A}(X, Y)$ then*

$$(\Phi_1, \ldots, \Phi_k)_* = (A_*(\Phi_1, \ldots, \Phi_k))_*. \quad ∎$$

3. A fixed point index. Let X be a compact polyhedron and let Φ be a map determined by a decomposition $(\Phi_1, \ldots, \Phi_k) \in \mathscr{D}\mathscr{A}(X, X)$. Let U be an open subset of X such that $x \notin \Phi(x)$ for $x \in \partial U$.

5.6. Definition. We define the *fixed point index* $i(X, (\Phi_1, \ldots, \Phi_k), U)$ of $(\Phi_1, \ldots, \Phi_k)$ with respect to U by

$$i(X, (\Phi_1, \ldots, \Phi_k), U) := I_{A_*(\Phi_1, \ldots, \Phi_k)}(X, \Phi, U).$$

5.7. Proposition. *Definition* 5.6 *does not depend on the triangulation τ of X.*

Proof. Let $A_*(\Phi_1, \ldots, \Phi_k)$ and $\bar{A}_*(\Phi_1, \ldots, \Phi_k)$ be constructed for triangulations τ and τ_0, respectively. Given $j \in N$ there is an integer j_1 such that $\alpha(\tau_0^{j_1}) \geqslant \alpha(\tau^j)$. Let

$$\psi\colon C_*\left(N^{(n)}(\alpha(\tau_0^{j_1}))\right) \to C_*\left(N^{(n)}(\alpha(\tau_0^{j_1}))\right)$$

be an approximation of Φ such that

$$\lambda \circ \psi \circ \theta \circ b \in \bar{A}_{j_1}(\Phi_1, \ldots, \Phi_k).$$

Let l be an integer such that $\alpha(\tau^l) \geqslant \alpha(\tau_0^{l_1})$. We define a chain map

$$\varphi\colon\ C_*\big(N^{(n)}(\alpha(\tau^l))\big) \to C_*\big(N^{(n)}(\alpha(\tau^j))\big)$$

by the formula

$$\varphi := \pi_{\alpha(\tau^j)}^{\alpha(\tau_0^{j_1})} \circ \psi \circ \pi_{\alpha(\tau_0^{l_1})}^{\alpha(\tau^l)}.$$

It is also an approximation of Φ and

$$\lambda \circ \varphi \circ \theta \circ b \in A_j(\Phi_1, \ldots, \Phi_k).$$

Note that the only difference between φ and ψ is in the natural maps π. Therefore one easily verifies that A_* and $\bar{A}_*$ have the same index (cf. [38]). ∎

5.8. Lemma. *Let $H\colon X \times I \to Y \in \mathscr{A}_m(X \times I, Y)$ be a homotopy* ($I = [0, 1]$). *Then for every pair $\alpha \in \operatorname{Cov} X$ and $\beta \in \operatorname{Cov} Y$ there are (α, β)-approximations φ_0 of H_0 and φ_1 of H_1 which are chain homotopic with an (H, α, β)-small homotopy D* (see 3.3.2).

Proof. For every $x \in X$ there is a neighbourhood $\mathcal{O}_x \subset X$ of x such that $H(\mathcal{O}_x \times I) \subset \operatorname{St}(H(\{x\} \times I), \beta)$. We can assume that the covering $\{\mathcal{O}_x\}_{x \in X}$ is a refinement of α. Let $\{\mathcal{O}_{x_i}\}_{i=1}^k$ be a finite subcovering of $\{\mathcal{O}_x\}$. Let $\bar{\alpha}$ be a star-refinement of $\{\mathcal{O}_{x_i}\}$. Now, we consider a covering γ of $X \times I$, $\gamma := \{U \times I,\ U \in \bar{\alpha}\}$. There exists a (γ, β)-approximation of H, $\psi\colon C_*\big(N^{(n+1)}(\tilde{\gamma})\big) \to C_*\big(N^{(n+1)}(\beta)\big)$. We can assume that $\tilde{\gamma}$ consists of sets of the form $U \times V$ where $U \subset X$ and $V \subset I$. Then $N(\tilde{\gamma}) = N(\tilde{\alpha}) \times N(\eta)$, where $\tilde{\alpha} = \{U\}$, $\eta = \{V\}$ and $\tilde{\gamma} = \{U \times V\}$. We can also assume that the complex $N(\eta)$ is 1-dimensional. Let us define, for $i = 0, 1$, $\varphi_i := \psi|_{N(\tilde{\alpha}) \times \{V_i\}}$, where V_i are such that $i \in V_i$. Obviously $N(\tilde{\alpha}) \times \{V_i\} \approx N(\tilde{\alpha})$ and one easily verifies that the φ_i are (α, β)-approximations of H_i, $i = 0, 1$.

Let $s \in N(\tilde{\alpha})$ be a simplex and consider the chain $s \times I = \sum s_j$ where s_j are simplexes of $N(\tilde{\gamma})$. We define $Ds := \psi(s \times I)$. Since ψ is an approximation of H, for each s_j there is a point $p(s_j) \in X \times I$ such that

$$\operatorname{supp} s_j \subset \operatorname{St}(p(s_j), \gamma), \qquad \operatorname{supp} \psi s_j \subset \operatorname{St}\big(H(p(s_j)), \beta\big).$$

Since $\tilde{\alpha} \geqslant \bar{\alpha}$, there is a set $\mathcal{O}_{x_i}$ such that $\operatorname{St}(\operatorname{supp} s, \bar{\alpha}) \subset \mathcal{O}_{x_i}$. Therefore

$$H\big(\operatorname{St}(\operatorname{supp} s, \bar{\alpha}) \times I\big) \subset \operatorname{St}\big(H(\{x_i\} \times I), \beta\big).$$

Let $p(s_j) = (p_j, t_j)$. Then $p_j \in \operatorname{St}(\operatorname{supp} s, \bar{\alpha})$ and hence

$$\operatorname{supp} \psi s_j \subset \operatorname{St}\big(H(\{x_i\} \times I), \beta\big).$$

We put $d(s) := x_i$ (see 3.3.2) and the proof is complete. ∎

5.9. PROPOSITION. *Let X and Y be compact polyhedra and let $(\Phi_1, \ldots, \Phi_k), (\Psi_1, \ldots, \Psi_k) \in \mathscr{D}\mathscr{A}(X, Y)$ be homotopic. If $H: X \times I \to Y$ is the map determined by the joining homotopy then the induced A-systems $A_*(\Phi_1, \ldots, \Phi_k)$ and $A_*(\Psi_1, \ldots, \Psi_k)$ are H-homotopic* (see 3.3).

Proof. Without loss of generality we can assume that $\Phi_2 = \Psi_2, \ldots, \Phi_k = \Psi_k$ and Φ_1 is homotopic to Ψ_1 in $\mathscr{A}_{m_1}(X, X_1)$. By 5.8 there are arbitrarily fine approximations of Φ_1 and Ψ_1 which are h_1-small homotopic (where H is determined by $(h_1, \ldots, h_k)$). We obtain the desired H-small homotopic approximations of $(\Phi_1, \ldots, \Phi_k)$ and $(\Psi_1, \ldots, \Psi_k)$ by composing those from 5.8 with the approximations of the maps $h_2, \ldots, h_k$. ∎

5.10. THEOREM. *The fixed point index satisfies the following properties:*

5.10.1. Additivity. *Let Φ be determined by the decomposition $(\Phi_1, \ldots, \Phi_k) \in \mathscr{D}\mathscr{A}(X, X)$, where X is a compact polyhedron. Assume that U_1 and U_2 are open disjoint subsets of an open set $U \subset X$ and* $\operatorname{Fix} \Phi|_{\bar{U}} \subset U_1 \cup U_2$. *Then*

$$i(X, (\Phi_1, \ldots, \Phi_k), U) = i(X, (\Phi_1, \ldots, \Phi_k), U_1) + i(X, (\Phi_1, \ldots, \Phi_k), U_2).$$

5.10.2. Homotopy invariance. *Let $(\Phi_1, \ldots, \Phi_k)$ and $(\Psi_1, \ldots, \Psi_k)$ be homotopic and let the map $H: X \times [0, 1] \to Y$ determined by the homotopy satisfy $x \notin H(x, t)$ for $x \in \partial U$ and $t \in [0, 1]$. Then*

$$i(X, (\Phi_1, \ldots, \Phi_k), U) = i(X, (\Psi_1, \ldots, \Psi_k), U).$$

5.10.3. Normalization. *If $(\Phi_1, \ldots, \Phi_k) \in \mathscr{D}\mathscr{A}(X, X)$ and*

$$\lambda(\Phi_1, \ldots, \Phi_k)_* = \sum (-1)^j \operatorname{tr}((\Phi_1, \ldots, \Phi_k)_*)_j$$

then

$$i(X, (\Phi_1, \ldots, \Phi_k), X) = \lambda(\Phi_1, \ldots, \Phi_k)_*.$$

5.10.4. Commutativity. *Let X, Y be compact polyhedra and $\Phi \in \mathscr{D}\mathscr{A}(X, Y)$, $\Psi \in \mathscr{D}\mathscr{A}(Y, X)$. Denote by the same letters the maps determined by Φ and Ψ. Let W be an open subset of X such that $x \notin \Psi \circ \Phi(x)$ for $x \in \partial W$, $y \notin \Phi \circ \Psi(y)$ for $y \in \partial(\Psi^{-1}(W))$ and*

$$\Psi(\operatorname{Fix} \Phi \circ \Psi - \Psi^{-1}(W)) \cap \operatorname{Fix} \Psi \circ \Phi|_{\bar{W}} = \emptyset.$$

Then

$$i(X, \Psi \circ \Phi, W) = i(Y, \Phi \circ \Psi, \Psi^{-1}(W)).$$

5.10.5. Mod-p property. *Let $F = Z_p$, p prime. Let $\Phi \in \mathscr{D}\mathscr{A}(X, X)$ and denote by the same letter the map determined by Φ. Let $W \subset X$ be open and assume that $x \notin \Phi^p(x)$ for $x \in \partial W$ and*

$$\Phi^k(\operatorname{Fix} \Phi^p - W) \cap \operatorname{Fix} \Phi^p|_{\bar{W}} = \emptyset \quad \text{for } k < p.$$

Then

$$i(X, \Phi, W) = i(X, \Phi^p, W).$$

Proof. 5.10.1 is an immediate consequence of 3.6 (see 3.9). The homotopy invariance follows from 3.8 and 5.9. The normalization property is a consequence of 5.5 and the Hopf Trace Theorem (see [6] or [9]). The last two properties follow from 3.10 and 3.11, respectively. ∎

VI. Index of decompositions for compact ANR's

In this chapter we extend the index theory to the case of compact ANR's. We try to do it in a way similar, as far as possible, to the one in [6].

1. Preliminaries. Let X and Y be two spaces. We say that the space X *ε-dominates* Y provided there are continuous mappings $r\colon X \to Y$ and $s\colon Y \to X$ such that $r \circ s\colon Y \to Y$ is ε-homotopic to the identity map, i.e. there is a map $H\colon Y \times I \to Y$ such that $H(x, 0) = r \circ s(x)$, $H(x, 1) = x$ and $\operatorname{diam}(H(\{x\} \times I)) < \varepsilon$ for each $x \in X$. The following well-known fact will be a crucial tool for this chapter.

6.1. Theorem ([6], p.41). *Given X, a compact metric ANR, and $\varepsilon > 0$, there exists a compact polyhedron P which ε-dominates X.* ∎

Let $d\colon X \times X \to \mathbf{R}$ be a metric in X and let $A, B \subset X$. We will denote the distance between A and B by $d(A, B)$.

6.2. Lemma. *Let A be a closed subset of a compact metric space X. Let $\Phi\colon X \to X$ be an u.s.c. mapping without fixed points in A. There exists $\delta = \delta(\Phi, A) > 0$ such that*

$$\text{if} \quad d(x, A) < \delta \quad \text{then} \quad d(x, \Phi(x)) > \delta.$$

Proof. Because of 2.2.2 and 2.2.3 the mapping $d \circ (\Phi, \mathrm{id}_X)\colon X \to \mathbf{R}$ is u.s.c. Since A is compact, the set $d \circ (\Phi, \mathrm{id}_X)(A) \subset (0, \infty)$ is compact, hence contained in some interval $[a, b]$, where $a, b > 0$. Let

$$V := [d \circ (\Phi, \mathrm{id}_X)]^{-1}((a/2, 2b)).$$

Then $A \subset V$ and V is open in X. We define $\delta := \min\{a/2, d(A, X - V)\}$. ∎

Let us introduce the following useful notion. If A is a subset of a metric space X then the *ε-neighbourhood* $O_\varepsilon(A)$ of A is the set $O_\varepsilon(A) := \{x \in X\colon d(x, A) < \varepsilon\}$.

6.3. Proposition. *Let U be an open subset of a compact ANR X and let $\Phi\colon X \to X$ be an u.s.c. map such that* $\operatorname{Fix} \Phi \cap \partial U = \emptyset$. If $\varepsilon < \delta(\Phi, \partial U)$ *and the polyhedron P ε-dominates X with maps $r\colon P \to X$ and $s\colon X \to P$ then the map $\Psi = s \circ \Phi \circ r\colon P \to P$ has no fixed points in $\partial(f^{-1}(U))$.*

Proof. Suppose $y \in \partial(r^{-1}(U))$ is such that $y \in \Psi(y)$. Then $r(y) \in r \circ \Psi(y)$. By the definition of ε-domination we obtain $r \circ \Psi(y) \subset O_\varepsilon(\Phi \circ r(y))$, and hence $d(r(y), \Phi \circ r(y)) < \varepsilon$. But $y \in \partial(r^{-1}(U))$ implies that $r(y) \in \partial U$, so

$$d(r(y), \Phi \circ r(y)) > \delta(\Phi, \partial U) > \varepsilon$$

and we have arrived at a contradiction. ∎

The following is a metric version of 4.10.

6.4. Lemma. *Let X_1, X_2, X_3 be compact metric spaces, Φ_1: $X_1 \to X_2$ and Φ_2: $X_2 \to X_3$ u.s.c. maps, and $\varepsilon_0 > 0$. There exists $\varepsilon > 0$ such that for every point $y \in X_1$ there is a point $u(y) \in X_1$ such that*

$$y \in O_{\varepsilon_0}(u(y)), \quad \Phi_2(O_{2\varepsilon}(\Phi_1(y))) \subset O_{\varepsilon_0}(\Phi_2 \circ \Phi_1(u(y))). \ ∎$$

6.5. Lemma. *Let X_1, X_2 be two compact metric spaces and let Φ_1: $X_1 \to X_2$ and Φ_2: $X_2 \to X_1$ be u.s.c. maps. Then for each $\varepsilon > 0$ there is $\varepsilon_0 > 0$ such that for $\eta \in (0, \varepsilon_0)$ and u.s.c. maps $\Phi_{1\eta}$: $X_1 \to X_2$, $\Phi_{2\eta}$: $X_2 \to X_1$ with $\Phi_{i\eta}(x) \subset O_\eta(\Phi_i(x))$ we have*

$$\mathrm{Fix}\ \Phi_{2\eta} \circ \Phi_{1\eta} \subset O_\varepsilon(\mathrm{Fix}\ \Phi_2 \circ \Phi_1). \ ∎$$

We omit the proof of 6.5 since it is an easy exercise. The following lemma is also elementary.

6.6. Lemma. *Let U be an open subset of a metric space Z and let $C \subset Z$ be such that $C \cap \bar{U} = \emptyset$. For $\delta < d(C, U)$ and a continuous singlevalued map f: $Z \to Z$ such that $d(x, f(x)) < \delta$ we have $f^{-1}(U) \cap C = \emptyset$.* ∎

2. Index in $\mathcal{DA}(X, X)$. In this section we will denote with the same letter a decomposition $\Phi \in \mathcal{DA}(X, X)$ and the map Φ: $X \to X$ it determines. Usually the assumptions are for the map, but the index is defined for decompositions.

Let X be a compact metric ANR and $\Phi \in \mathcal{DA}(X, X)$. Let U be an open subset of X such that $\mathrm{Fix}\ \Phi \cap \partial U = \emptyset$.

6.7. Definition. Let $\varepsilon < \frac{1}{4}\delta(\Phi, \partial U)$. Let P be a finite polyhedron ε-dominating X with maps r: $P \to X$ and s: $X \to P$. We define the *fixed point index* of the decomposition Φ with respect to U by

$$i(X, \Phi, U) := i(P, \Psi, r^{-1}(U)),$$

where $\Psi = s \circ \Phi \circ r$.

Lemma 6.3 ensures that the right side is defined.

6.8. Proposition. *Definition 6.7 does not depend on the choice of an ε-domination.*

Proof. Let $\varepsilon_1, \varepsilon_2 < \frac{1}{4}\delta(\Phi, \partial U)$ and let r_i: $P_i \to X$, s_i: $X \to P_i$ be such that $r_i \circ s_i$ is ε_i-homotopic to id_X, $i = 1, 2$. We shall prove that

$$i(P_1, s_1 \circ \Phi \circ r_1, r_1^{-1}(U)) = i(P_2, s_2 \circ \Phi \circ r_2, r_2^{-1}(U))$$

in four steps.

6.8.1. $i(P_2, s_2 \circ \Phi \circ r_2, r_2^{-1}(U)) = i(P_2, T, r_2^{-1}(U))$,

where $T := s_2 \circ r_1 \circ s_1 \circ \Phi \circ r_1 \circ s_1 \circ r_2$.

Let $H\colon X \times I \to X$ be an ε_1-homotopy with $H(x, 0) = x$ and $H(x, 1) = r_1 \circ s_1(x)$. Consider the composition

$$P_2 \times I \xrightarrow{r'_2} X \times I \xrightarrow{H'} X \times I \to \overset{\Phi'}{\ldots} \to X \times I \xrightarrow{H} X \xrightarrow{s_2} P_2,$$

where $r'_2 = (r_2, \mathrm{id})$, $H' = (H, \mathrm{id})$, $\Phi' = (\Phi, \mathrm{id})$. In order to apply the homotopy property 5.10.2 we have to show that for each $t \in [0, 1]$ the map

$$\Psi := s_2 \circ H_t \circ \Phi \circ H_t \circ r_2\colon P_2 \to P_2$$

has no fixed points in the set $\partial(r_2^{-1}(U))$. Note that $\partial(r_2^{-1}(U)) \subset r_2^{-1}(\partial U)$. Actually, we will prove that $y \notin \Psi(y)$ for $y \in X$ such that $d(r_2(y), \partial U) < \frac{1}{4}\delta(\Phi, \partial U)$. First, note that if $x \in \partial U$ is such that $d(r_2(y), \partial U) = d(r_2(y), x)$ then

$$d(H_t \circ r_2(y), \delta U) \leqslant d(H_t \circ r_2(y), x) \leqslant d(H_t \circ r_2(y), r_2(y)) + d(r_2(y), x)$$
$$< \varepsilon_1 + \tfrac{1}{4}\delta(\Phi, \partial U) < \tfrac{1}{2}\delta(\Phi, \partial U).$$

From 6.2 we obtain $d(\Phi \circ H_t \circ r_2(y), H_t \circ r_2(y)) > \delta(\Phi, \partial U)$. Since

$$d(H_t \circ r_2(y), r_2(y)) < \varepsilon_1 < \tfrac{1}{4}\delta(\Phi, \partial U),$$

we have $d(\Phi \circ H_t \circ r_2(y), r_2(y)) > \frac{3}{4}\delta(\Phi, \partial U)$. For every $z \in \Phi \circ H_t \circ r_2(y)$ we have

$$d(H_t(z), z) < \varepsilon_1 < \tfrac{1}{4}\delta(\Phi, \partial U)$$

and therefore

$$(*) \qquad d(H_t(z), r_2(y)) > \tfrac{1}{2}\delta(\Phi, \partial U).$$

Now, suppose that for some y such that $d(r_2(y), \partial U) < \frac{1}{4}\delta(\Phi, \partial U)$ we have $y \in \Psi(y)$. Then $r_2(y) \in r_2 \circ \Psi(y)$. But for each $z \in \Phi \circ H_t \circ r_2(y)$ we have

$$(**) \qquad d(r_2 \circ s_2 \circ H_t(z), H_t(z)) < \varepsilon_2 < \tfrac{1}{4}\delta(\Phi, \partial U).$$

Now $(**)$ together with $(*)$ imply that

$$d(r_2 \circ s_2 \circ H_t(z), r_2(y)) > \tfrac{1}{4}\delta(\Phi, \partial U) > 0.$$

Therefore $d(r_2 \circ \Psi(y), r_2(y)) > 0$ which is impossible because $r_2(y) \in r_2 \circ \Psi(y)$. So the homotopy property 5.10.2 gives 6.8.1.

6.8.2. $i(P_2, T, r_2^{-1}(U)) = i(P_2, T, S^{-1}(U))$,

where $T := s_2 \circ r_1 \circ s_1 \circ \Phi \circ r_1 \circ s_1 \circ r_2$, $S := r_1 \circ s_1 \circ r_2$.

For the proof of 6.8.2 we apply the additivity 5.10.1. First we show that the map $T\colon P_2 \to P_2$ has no fixed points in the set $A := r_2^{-1}(U) - S^{-1}(U)$. Let

$y \in A$. Then $r_2(y) \in U$ but $S(y) \notin U$. Consider the map c: $[0, 1] \to X$, $c(t) := H(r_2(y), t)$. The path $c([0, 1])$ meets U at $c(0)$ and $X - U$ at $c(1)$. So there is $t_0 \in [0, 1]$ such that $c(t_0) \in \partial U$. But

$$d(r_2(y), H(r_2(y), t_0)) < \varepsilon_1 < \tfrac{1}{4}\delta(\Phi, \partial U),$$

hence $d(r_2(y), \partial U) < \frac{1}{4}\delta(\Phi, \partial U)$. The argument from the first step implies that $y \notin T(y)$. By the additivity property 5.10.1 one obtains

$$i(P_2, T, r_2^{-1}(U)) = i(P_2, T, r_2^{-1}(U) \cap S^{-1}(U)).$$

The same argument used for the set $B := S^{-1}(U) - r_2^{-1}(U)$ implies that

$$i(P_2, T, S^{-1}(U)) = i(P_2, T, r_2^{-1}(U) \cap S^{-1}(U)).$$

The last two equalities imply 6.8.2.

6.8.3. $i(P_2, T, S^{-1}(U)) = i(P_1, R, r_1^{-1}(U))$,

where $R := s_1 \circ r_2 \circ s_2 \circ r_1 \circ s_1 \circ \Phi \circ r_1$.

For the proof of 6.8.3 we apply the commutativity 5.10.4. Let K: $X \times [0, 1] \to X$ be an ε_2-homotopy such that $K(x, 0) = x$ and $K(x, 1) = r_2 \circ s_2(x)$. We wish to show that the map

$$R_t := s_1 \circ K_t \circ H_t \circ \Phi \circ r_1 \colon P_1 \to P_1$$

has no fixed points in the set $\partial(r_1^{-1}(U))$ for $t \in [0, 1]$. Let $y \in P_1$ be such that $r_1(y) \in \partial U$. Then $d(r_1(y), \Phi \circ r_1(y)) > \delta(\Phi, \partial U)$. For each $z \in \Phi \circ r_1(y)$ we have

$$d(K_t \circ H_t(z), z) \leqslant d(K_t \circ H_t(z), H_t(z)) + d(H_t(z), z)$$
$$< \varepsilon_2 + \varepsilon_1 < \tfrac{1}{2}\delta(\Phi, \partial U).$$

Since $\delta(\Phi, \partial U) < d(r_1(y), z) \leqslant d(z, K_t \circ H_t(z)) + d(r_1(y), K_t \circ H_t(z))$, we have

$$(***) \qquad d(r_1(y), K_t \circ H_t \circ \Phi \circ r_1(y)) > \tfrac{1}{2}\delta(\Phi, \partial U).$$

Suppose that for some $y \in r_1^{-1}(\partial U)$ we have $y \in R_t(y)$. Then $r_1(y) \in r_1 \circ R_t(y)$. But for each $z \in \Phi \circ r_1(y)$ we have

$$d(r_1 \circ s_1 \circ K_t \circ H_t(z), K_t \circ H_t(z)) < \varepsilon_1 < \tfrac{1}{4}\delta(\Phi, \partial U).$$

Combining this with (***) we obtain

$$d(r_1 \circ R_t(y), r_1(y)) > \tfrac{1}{4}\delta(\Phi, \partial U) > 0,$$

which establishes the contradiction with our assumption. In particular, taking $t = 1$, we have shown that the map $R = R_1$ has no fixed points in the set $\partial(r_1^{-1}(U))$.

Let us consider the compositions $h := s_2 \circ r_1 \circ s_1 \circ \Phi \circ r_1$, $k := s_1 \circ r_2$. We have just shown that $k \circ h$ has no fixed points in the set $\partial(r_1^{-1}(U))$. Now we check the last two assumptions for the commutativity property 5.10.4.

Denote $W := r_1^{-1}(U)$. Let $y \in \partial(k^{-1}(W)) = k^{-1}(\partial W)$. If $y \in h \circ k(y) = T(y)$, then $k(y) \in s_1 \circ r_2 \circ T(y) = k \circ h \circ s_1 \circ r_2(y)$ and $s_1 \circ r_2(y) \in \partial(r_1^{-1}(U))$, which is impossible.

Let $y \in \operatorname{Fix} h \circ k - \overline{k^{-1}(W)}$. We have to prove that $k(y) \in \operatorname{Fix} k \circ h|_{\bar{W}}$. But this follows from $\operatorname{Fix} k \circ h|_{\bar{W}} \subset k^{-1}(W)$. Therefore 5.10.4 implies 6.8.3.

6.8.4. $i(P_1, R, r_1^{-1}(U)) = i(P_1, s_1 \circ \Phi \circ r_1, r_1^{-1}(U))$.

We take the homotopy

$$L\colon\ P_1 \times I \xrightarrow{r_1'} X \times I \to \overset{\Phi'}{\ldots} \to X \times I \xrightarrow{H'} X \times I \xrightarrow{K} X \xrightarrow{s_1} P_1,$$

where $r_1' = (r_1, \mathrm{id})$, $\Phi' = (\Phi, \mathrm{id})$, $H' = (H, \mathrm{id})$. We have already proved in the third step that $y \notin L(y, t)$ for $y \in \partial(r_1^{-1}(U))$ and $t \in I$. By the homotopy property 5.10.2 we obtain 6.8.4 and this completes the proof of 6.8. ∎

Now we verify the properties of the index.

6.9. PROPOSITION. *Let U be an open subset of a compact ANR X. Let $\Phi \in \mathcal{CA}(X, X)$ and let $U_1, U_2 \subset U$ be open, disjoint and such that $\operatorname{Fix} \Phi|_{\bar{U}} \subset U_1 \cup U_2$. Then*

$$i(X, \Phi, U) = i(X, \Phi, U_1) + i(X, \Phi, U_2).$$

Proof. Applying 6.2 to the map Φ and $A := \bar{U} - U_1 \cup U_2$ we find δ which is smaller than $\delta(\Phi, \partial U)$ and $\delta(\Phi, \partial U_i)$. Let $\varepsilon < \frac{1}{4}\delta$ and let P be a compact polyhedron which ε-dominates X with maps $r\colon P \to X$ and $s\colon X \to P$. Then

$$i(X, \Phi, U) = i(P, \Psi, r^{-1}(U)), \qquad i(X, \Phi, U_j) = i(P, \Psi, r^{-1}(U_j)),$$

where $\Psi = s \circ \Phi \circ r$, $j = 1, 2$. We have to show that Ψ has no fixed points in the set $r^{-1}(U) - r^{-1}(U_1 \cup U_2) = r^{-1}(U - U_1 \cup U_2)$. Suppose $y \in \Psi(y)$ for $y \in r^{-1}(U - U_1 \cup U_2)$. Then $r(y) \in r \circ \Psi(y)$, i.e. there is $z \in \Phi \circ r(y)$ such that $r(y) = r \circ s(z)$. Therefore

$$d(r(y), \Phi \circ r(y)) \leqslant d(r(y), z) = d(r \circ s(z), z) < \varepsilon < \delta/4$$

and we have arrived at a contradiction with 6.2. Hence the additivity 5.10.1 implies our assertion. ∎

6.10. PROPOSITION. *Let $H \in \mathcal{CA}(X \times I, X)$. If U is an open subset of the compact ANR X such that the maps $H_t\colon X \to X$ have no fixed points on ∂U for $t \in I$, then*

$$i(X, H_0, U) = i(X, H_1, U).$$

Proof. Let $\mathcal{H} = (H, \mathrm{id})\colon X \times I \to X \times I$. If $x \in \partial U$ then $\mathcal{H}(x, t) = (x, t)$ means that $H(x, t) = H_t(x) = x$ contrary to the assumption, and so $\mathcal{H}$ has no fixed points in the set $\partial U \times I$. Apply 6.2 to the map $\mathcal{H}$ and $A = \partial U \times I$, with the natural cartesian metric d' in $X \times I$. We find δ such that

$$d'((x, t), \partial U \times I) < \delta \quad \text{implies} \quad d'((x, t), \mathcal{H}(x, t)) > \delta.$$

But $d'((x, t), \partial U \times I) = d(x, \partial U)$ and $d'((x, t), \mathscr{H}(x, t)) = d(x, H(x, t)) = d(x, H_t(x))$. Therefore we may assume that $\delta(H_t, \partial U)$ is the same for all $t \in I$. Denote it by δ. For $\varepsilon < \delta/4$ we have a polyhedron P ε-dominating X with maps $s\colon X \to P$ and $r\colon P \to X$. By definition, $i(X, H_t, U) = i(P, s \circ H_t \circ r, r^{-1}(U))$. By 6.3 the family $s \circ H_t \circ r$ satisfies the assumption of the homotopy property 5.10.2. ∎

6.11. PROPOSITION. *Let X be a compact ANR and $\Phi \in \mathscr{D}\mathscr{A}(X, X)$. Then*

$$i(X, \Phi, X) = \lambda(\Phi_*).$$

Proof. The assertion follows from 5.10.3 and from the fact that $r_* \circ s_* = \mathrm{id}_*$ (see [6], p.3). ∎

6.12. PROPOSITION. *Let X, X' be compact ANR's and $\Phi \in \mathscr{D}\mathscr{A}(X, X')$, $\Psi \in \mathscr{D}\mathscr{A}(X', X)$. Let U be an open subset of X such that $x \notin \Psi \circ \Phi(x)$ for $x \in \partial U$, $y \notin \Phi \circ \Psi(y)$ for $y \in \partial(\Psi^{-1}(U))$ and*

$$\Psi(\operatorname{Fix} \Phi \circ \Psi - \Psi^{-1}(U)) \cap \operatorname{Fix} \Psi \circ \Phi|_{\bar{U}} = \emptyset.$$

Then

$$i(X', \Phi \circ \Psi, \Psi^{-1}(U)) = i(X, \Psi \circ \Phi, U).$$

Proof. From 6.2 we obtain $\delta(\Psi \circ \Phi, \partial U)$ and $\delta(\Phi \circ \Psi, \partial(\Psi^{-1}(U)))$. Let δ be the smaller of the two. Next, we find ε for $\varepsilon_0 = \delta/4$ from 6.4 ($\varepsilon < \varepsilon_0$) with $\Phi_1 = \Phi$ and $\Phi_2 = \Psi$. Again from 6.4 we find ε' for $\varepsilon_0 = \varepsilon$ and $\Phi_1 = \Psi$ and $\Phi_2 = \Phi$. Let P be a polyhedron ε-dominating X with maps $r_1\colon P \to X$, $s_1\colon X \to P$; let P be a polyhedron ε'-dominating X with $r_2\colon P \to X$, $s_2\colon X \to P$. Since $\varepsilon < \delta/4 < \delta(\Psi \circ \Phi, \partial U)$,

$$i(X, \Psi \circ \Phi, U) = i(P, s_1 \circ \Psi \circ \Phi \circ r_1, r_1^{-1}(U)).$$

Since $\varepsilon' < \frac{1}{4}\delta(\Phi \circ \Psi, \partial(\Psi^{-1}(U)))$,

$$i(X', \Phi \circ \Psi, \Psi^{-1}(U)) = i(P', s_2 \circ \Phi \circ \Psi \circ r_2, (\Psi \circ r_2)^{-1}(U)).$$

We have to prove that the right sides of the above equalities are equal. We will do it in four steps:

6.12.1. $i(P, s_1 \circ \Psi \circ \Phi \circ r_1, r_1^{-1}(U)) = i(P, \Psi_1 \circ \Phi_1, r_1^{-1}(U))$,

where $\Psi_1 = s_1 \circ \Psi \circ r_2$, $\Phi_1 = s_2 \circ \Phi \circ r_1$.

6.12.2. $i(P, \Psi_1 \circ \Phi_1, r_1^{-1}(U)) = i(P', \Phi_1 \circ \Psi_1, (r_1 \circ \Psi_1)^{-1}(U))$.

6.12.3. $i(P', \Phi_1 \circ \Psi_1, (r_1 \circ \Psi_1)^{-1}(U)) = i(P', \Phi_1 \circ \Psi_1, (\Psi \circ r_2)^{-1}(U))$.

6.12.4. $i(P', \Phi_1 \circ \Psi_1, (\Psi \circ r_2)^{-1}(U)) = i(P', s_2 \circ \Phi \circ \Psi \circ r_2, (\Psi \circ r_2)^{-1}(U))$.

Proof of 6.12.1. We apply the homotopy property 5.10.2. Let $H'\colon X \times I \to X$ be an ε'-homotopy such that $H'(x, 0) = x$, $H'(x, t) = r_2 \circ s_2(x)$, and $H'_t(x) := H'(x, t)$. We have to prove that $\Theta_t := s_1 \circ \Psi \circ H_t \circ \Phi \circ r_1$ has no fixed points in the set $r_1^{-1}(\partial U)$. Let $y \in r_1^{-1}(\partial U)$ be such that $y \in \Theta_t(y)$. Then $r_1(y) \in r_1 \circ \Theta_t(y)$ and therefore

$$r_1(y) \in r_1 \circ s_1 \circ \Psi\big(O_{\varepsilon'}(\Phi \circ r_1(y))\big).$$

By the definition of ε' there is a point $u = u(r_1(y))$ such that

$$r_1(y) \in r_1 \circ s_1\big(O_{\delta/4}(\Psi \circ \Phi(u))\big) \subset O_{\delta/2}(\Psi \circ \Phi(u)).$$

Therefore $u \in O_{3\delta/4}(\Psi \circ \Phi(u))$ and $d(u, \partial U) < \delta$ which contradicts the choice of δ.

Proof of 6.12.2. We verify the assumptions of the commutativity property 5.10.4. First, note that the map $\Psi_1 \circ \Phi_1$ has no fixed points in $\partial(r_1^{-1}(U))$, as has been proved above ($t = 1$). Now, denote $B_1 := \operatorname{Fix} \Psi \circ \Phi|_U$, $B_2 := \operatorname{Fix} \Phi \circ \Psi|_{\Psi^{-1}(U)}$, $B_3 := \operatorname{Fix} \Phi \circ \Psi - \Psi^{-1}(U)$. Since B_i are compact, there exists ε_1 such that

$$\begin{aligned} d(O_{\varepsilon_1}(B_1), X - U) &= \delta_1 > 0,\\ d(O_{\varepsilon_1}(B_2), X' - \Psi^{-1}(U)) &= \delta_2 > 0,\\ d(O_{\varepsilon_1}(B_3), \Psi^{-1}(U)) &= \delta_3 > 0,\\ d\big(\Psi(O_{\varepsilon_1}(B_3)), U\big) &= \delta_4 > 0,\\ d\big(\Psi(O_{\varepsilon_1}(B_2)), X - U\big) &= \delta_5 > 0. \end{aligned}$$

Let $\varepsilon_2 \leqslant \frac{1}{2}\min\{\varepsilon_1, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$. Let $\varepsilon_3 < \varepsilon_2$ satisfy 6.4 for $\varepsilon_0 = \varepsilon_2$. Without loss of generality we can assume that ε and ε' are smaller than ε_3. Suppose $y \in \Phi_1 \circ \Psi_1(y)$. Then $r_2(y) \in \operatorname{Fix} r_2 \circ \Phi_1 \circ s_1$ and therefore

$$r_2(y) \in O_{\varepsilon_2}(\operatorname{Fix} \Phi \circ \Psi) = O_{\varepsilon_2}(B_2) \cup O_{\varepsilon_2}(B_3).$$

Let $C := \overline{\Psi(O_{\varepsilon_2}(B_3))}$. We have $\varepsilon < \varepsilon_2 < \frac{1}{2}d(C, \bar{U})$ and from 6.6 we obtain $(r_1 \circ s_1)^{-1}(U) \cap C = \emptyset$. Hence $O_{\varepsilon_1}(B_3) \cap V = \emptyset$, where $V := (r_1 \circ s_1 \circ \Psi)^{-1}(U)$, and therefore $d(O_{\varepsilon_2}(B_3), V) > 0$.

Let $D := \overline{\Psi(O_{\varepsilon_1}(B_2))}$. Since $\varepsilon < \frac{1}{2}d(D, X - U)$, by 6.6 we obtain $(r_1 \circ s_1)^{-1}(X - U) \cap D = \emptyset$ and thus $D \subset (r_1 \circ s_1)^{-1}(U)$. Therefore $O_{\varepsilon_1}(B_2) \subset V$. Hence we deduce that $r_2(y) \notin \partial V$ and therefore $y \notin \partial(r_2^{-1}(V))$. It remains to prove the following implication:

$$(\Box) \quad y \in \operatorname{Fix} \Phi_1 \circ \Psi_1 - r_2^{-1}(V) \Rightarrow \Psi_1(y) \cap \operatorname{Fix} \Psi_1 \circ \Phi_1|_{r_1^{-1}(U)} = \emptyset.$$

Actually we prove a stronger one:

$$(\Box) \Rightarrow \Psi_1(y) \cap r_1^{-1}(U) = \emptyset.$$

Suppose y satisfies $(\Box)$. Then $r_2(y) \in \operatorname{Fix} r_2 \circ \Phi_1 \circ s_1 - V$. Therefore $r_2(y) \in O_{\varepsilon_2}(B_3) \subset O_{\varepsilon_1}(B_3)$. But $\Psi(O_{\varepsilon_1}(B_3)) \cap \bar{U} = \emptyset$ and by the choice of $\varepsilon' < \delta_4$ we obtain $r_1 \circ s_1 \circ \Psi(O_{\varepsilon_1}(B_3)) \cap \bar{U} = \emptyset$. Therefore $\Phi_1(y) \cap r_1^{-1}(U) = \emptyset$ and applying 5.10.4 we finish the proof of 6.12.2.

Proof of 6.12.3. Note that we have just proved above that the map $\Phi_1 \circ \Psi_1$ has no fixed points in the sets $A_1 := r_2^{-1}(V) - (\Psi \circ r_2)^{-1}(U)$ and $A_2 := (\Psi \circ r_2)^{-1}(U) - r_2^{-1}(V)$.

So applying twice the additivity property 5.10.1 one obtains 6.12.3.

Proof of 6.12.4. We apply again the homotopy property. We only have to show that the map $\Theta_t' := s_2 \circ \Phi \circ H_t \circ \Psi \circ r_2$ has no fixed points in the set $r_2^{-1}\big(\partial(\Psi^{-1}(U))\big)$. Let $y \in \Theta_t'(y)$. Then $r_2(y) \in r_2 \circ \Theta_t'(y)$ and therefore, by 6.5, $r_2(y) \in O_{\varepsilon_2}(\mathrm{Fix}\, \Phi \circ \Psi)$. The considerations in the proof of 6.12.2 imply that $r_2(y) \notin \partial\big(\Psi^{-1}(U)\big)$. This finishes the proof of 6.12. ∎

6.13. Proposition. *Let* $F = Z_p$, *p a prime number. Let* $\Phi \in \mathcal{GA}(X, X)$ *and let* U *be an open subset of* X *such that* $\mathrm{Fix}\, \Phi^p \cap \partial U = \emptyset$ *and*

$$\Phi^k(\mathrm{Fix}\, \Phi^p - U) \cap \mathrm{Fix}\, \Phi^p|_U = \emptyset \quad \text{for } k < p.$$

Then

$$i(X, \Phi^p, U) = i(X, \Phi, U).$$

Proof. As in the proof of 6.12 we choose ε so small that for the polyhedron P ε-dominating X with maps $r\colon P \to X$ and $s\colon X \to P$, the map $s \circ \Phi \circ r$ satisfies the assumptions of 5.10.5. We omit the details. ∎

VII. Index of decompositions for arbitrary ANR's

1. Index for decompositions of compact maps. Let us recall the following result of J. Girolo [16]:

7.1. Theorem. *Let* U *be open subset of a normed space* E *and let* $K \subset U$ *be a compact set. Then there exists a compact ANR* M *such that* $K \subset M \subset U$. ∎

We will use the following consequence of the above result:

7.2. Proposition. *Let* C *be a compact subset of a metric ANR* X. *Then there exists a compact ANR* N *such that* $C \subset N \subset X$.

Proof. It is well known that X may be homeomorphically embedded as a closed subset into a normed space E. Then the image $h(X)$ is a retract of some open set $U \subset E$ (see e.g. [18]). Let $r\colon U \to h(X)$ be the retraction. By 7.1 there exists a compact ANR M such that $h(C) \subset M \subset U$. Define $N' := r(M)$. We have $h(C) \subset N' \subset h(X)$ and N' is a compact ANR, being the retract of a compact ANR M. Therefore $N := h^{-1}(N')$ is the desired compact ANR such that $C \subset N \subset X$. ∎

Let X be an arbitrary metric ANR and let $\Phi \in \mathcal{GA}(X, X)$ determine a compact map $\Phi\colon X \to X$ (see Ch. II).

7.3. Definition. Let $U \subset X$ be an open subset such that $\mathrm{Fix}\, \Phi \cap \partial U = \emptyset$. We define the *index* of the decomposition Φ with respect to U by

$$i(X, \Phi, U) := i(N, \Phi|_N, U \cap N),$$

where N is a compact ANR such that $\overline{\Phi(X)} \subset N \subset X$.

The above definition does not depend on the choice of N. Let $N' \subset N$ be another one. Then by the commutativity property 6.12 we obtain $i(N, \Phi, U \cap N) = i(N, j \circ \Phi, U \cap N) = i(N', \Phi \circ j, U \cap N) = i(N, \Phi|_{N'}, U \cap N')$, where $j\colon N' \to N$ is the inclusion mapping.

Now, we recall the notion of the *generalized Lefschetz number*. Let $u\colon E \to E$ be an endomorphism of an arbitrary vector space. Put $N(u) = \{x \in E\colon u^n(x) = 0 \text{ for some } n\}$ where u^n is the nth iterate of u, and $\tilde{E} := E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\tilde{u}\colon \tilde{E} \to \tilde{E}$. We call u *admissible* provided $\dim \tilde{E} < \infty$. Let $u = \{u_q\}\colon E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call u a *Leray endomorphism* if

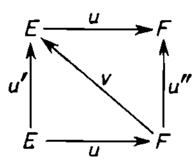
(i) all u_q are admissible,

(ii) almost all $\tilde{E}_q$ are trivial.

For such u we define the *generalized Lefschetz number*:

$$\Lambda(u) := \sum (-1)^n \operatorname{tr}(\tilde{u}_n).$$

7.4. Proposition (see [15], Prop. 1.3). *Suppose we are given a commutative diagram in the category of graded vector spaces:*



Then, if one of the two maps u', u'' is a Leray endomorphism, so is the other; in that case $\Lambda(u') = \Lambda(u'')$. ∎

7.5. Proposition. *The index satisfies the following properties:*

7.5.1. *If $U_1, U_2 \subset U$ are open and disjoint and if* $\operatorname{Fix} \Phi|_{\bar{U}} \subset U_1 \cup U_2$ *then*

$$i(X, \Phi, U) = i(X, \Phi, U_1) + i(X, \Phi, U_2).$$

7.5.2. *Let $H \in \mathcal{G}\mathcal{A}(X \times I, X)$ determine a compact homotopy $H\colon X \times I \to X$ such that $x \notin H(x, t)$ for all $x \in \partial U$ and $t \in I$. Then*

$$i(X, H_0, U) = i(X, H_1, U).$$

7.5.3. *Let $W \subset X$ be open and let $\Phi \in \mathcal{G}\mathcal{A}(X, Y)$, $\Psi \in \mathcal{G}\mathcal{A}(Y, X)$ determine compact maps. Assume that $x \notin \Psi \circ \Phi(x)$ for $x \in \partial W$, $y \notin \Phi \circ \Psi(y)$ for $y \in \partial(\Psi^{-1}(W))$ and*

$$\Psi(\operatorname{Fix} \Phi \circ \Psi - \Psi^{-1}(W)) \cap \operatorname{Fix} \Psi \circ \Phi|_{\bar{W}} = \emptyset.$$

Then

$$i(X, \Psi \circ \Phi, W) = i(Y, \Phi \circ \Psi, \Psi^{-1}(W)).$$

7.5.4. *Let* $F = Z_p$, *p prime. Let* $W \subset X$ *be open and such that*

$$\Phi^k(\mathrm{Fix}\, \Phi^p - W) \cap \mathrm{Fix}\, \Phi^p|_{\bar{W}} = \emptyset \quad \textit{for } k < p$$

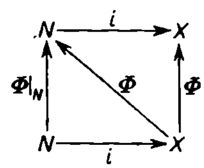
and $\mathrm{Fix}\, \Phi^p \cap \partial W = \emptyset$. *Then*

$$i(X, \Phi, W) = i(X, \Phi^p, W).$$

7.5.5. *If* $\Phi \in \mathcal{GA}(X, X)$ *determines a compact map* $\Phi \colon X \to X$ *then* Φ_* *is a Leray endomorphism and*

$$i(X, \Phi, X) = \Lambda(\Phi_*).$$

Proof. Properties 7.5.1–7.5.4 are straightforward consequences of Definition 7.3 and the respective properties of the index for compact ANR's. For the proof of 7.5.5 we consider the commutative diagram



where i is the inclusion map. Applying 7.4 to the induced homomorphisms of homology vector spaces we obtain

$$\Lambda(\Phi_*) = \lambda(\Phi|_N)_*$$

and, by 6.11, $\lambda(\Phi|_N)_* = i(N, \Phi|_N, N)$. Therefore 7.5.5 is proved. ∎

2. Index of decompositions of compact attraction maps. Let X be a metric ANR and let $\Phi \in \mathcal{GA}(X, X)$ determine a map $\Phi \colon X \to X$ which is of compact attraction (see 2.3.5). By 2.6 there exists a stable compact attractor for Φ. Let Y be an invariant neighbourhood of the stable attractor given by 2.3.4.

7.6. Definition. Let U be an open subset of X such that $x \notin \Phi(x)$ for $x \in \partial U$. Then we define

$$i(X, \Phi, U) := i(Y, \Phi|_Y, U \cap Y),$$

where Y is a neighbourhood of the stable compact attractor for Φ such that $\Phi|_Y \colon Y \to Y$ is a compact map.

We verify the independence of the choice of Y. Let Z be another set with the same properties. Consider the set $P := Z \cap Y$. Evidently $\Phi|_P \colon P \to P$ is compact and directly from the definition 7.3 of the index we obtain

$$i(Y, \Phi, U \cap Y) = i(P, \Phi, U \cap P) = i(Z, \Phi, U \cap Z).$$

7.7. Proposition (Additivity). *Let* U *be an open subset of a metric ANR* X *and let* $\Phi \in \mathcal{GA}(X, X)$ *determine a map of compact attraction. Assume that* $U_1, U_2 \subset U$ *are disjoint open sets such that* $\mathrm{Fix}\, \Phi|_{\bar{U}} \subset U_1 \cup U_2$. *Then*

$$i(X, \Phi, U) = i(X, \Phi, U_1) + i(X, \Phi, U_2).$$ ∎

7.8. PROPOSITION (Homotopy invariance). *Let* $H \in \mathcal{DA}(X \times I, X)$ *be such that the map* $\mathcal{H} = (H, \mathrm{id})\colon X \times I \to X \times I$ *is of compact attraction. If* $U \subset X$ *is open and for each* $t \in I$ *the map* $H_t\colon X \to X$ *has no fixed points in* ∂U *then*

$$i(X, H_0, U) = i(X, H_1, U). \ \blacksquare$$

Propositions 7.7 and 7.8 are easy consequences of 7.5.1 and 7.5.2, respectively. The next one follows from 7.5.4.

7.9. PROPOSITION (Mod-p property). *Let* $\Phi \in \mathcal{DA}(X, X)$ *determine a map of compact attraction and let* $W \subset X$ *be open and such that the assumptions of 7.5.4 are satisfied. Then*

$$i(X, \Phi, W) = i(X, \Phi^p, W). \ \blacksquare$$

7.10. PROPOSITION (Commutativity). *Let* X *and* Y *be metric ANR's and let* $\Phi \in \mathcal{DA}(X, Y)$, $\Psi \in \mathcal{DA}(Y, X)$ *determine locally compact maps. Let* $U \subset X$ *be open and assume that* $x \notin \Psi \circ \Phi(x)$ *for* $x \in \partial U$, $y \notin \Phi \circ \Psi(y)$ *for* $y \in \partial(\Psi^{-1}(U))$,

$$\Psi(\mathrm{Fix}\, \Phi \circ \Psi - \Psi^{-1}(U)) \cap \mathrm{Fix}\, \Psi \circ \Phi|_{\bar{U}} = \emptyset$$

and $\Psi \circ \Phi$ *has a compact attractor. Then* $\Phi \circ \Psi$ *has a compact attractor and*

$$i(X, \Psi \circ \Phi, U) = i(Y, \Phi \circ \Psi, \Psi^{-1}(U)).$$

Proof. Let A be an attractor for $\Psi \circ \Phi$. We prove that $\Phi(A)$ is an attractor for $\Phi \circ \Psi$. Let V be a neighbourhood of $\Phi(A)$. Then $\Phi^{-1}(V)$ is a neighbourhood of A. If $y \in Y$ then for every $z \in \Psi(y)$ there is $n \in N$ such that $(\Psi \circ \Phi)^n(z) \subset \Phi^{-1}(V)$. Hence there is n such that $(\Psi \circ \Phi)^n(\Psi(y)) \subset \Phi^{-1}(V)$ and therefore

$$(\Phi \circ \Psi)^{n+1}(y) \subset \Phi(\Phi^{-1}(V)) \subset V.$$

Now, let $W \subset X$ be open and such that $\Psi \circ \Phi|_W\colon W \to W$ is a compact map and W absorbs compact sets (see 2.3). For each $x \in \overline{\Psi \circ \Phi(W)}$ choose $U_x \subset W$ such that $\overline{\Phi(U_x)}$ is compact. Let

$$\overline{\Psi \circ \Phi(W)} \subset \bigcup_{j=1}^{k} U_{x_j} =: \tilde{W} \subset X.$$

We also have $\overline{\Psi \circ \Phi(\tilde{W})} \subset \Psi \circ \Phi(W) \subset \tilde{W}$ and $\tilde{W}$ absorbs compact sets in X. Let $Z := \Psi^{-1}(\tilde{W})$. Since $\Psi \circ \Phi(\tilde{W}) \subset \tilde{W}$, we have $\Psi(\overline{\Phi(\tilde{W})}) \subset \tilde{W}$ and therefore $\Phi(\tilde{W}) \subset \Psi^{-1}(\tilde{W}) = Z$. Thus

$$\overline{\Phi \circ \Psi(Z)} \subset \overline{\Phi \circ \Psi(\Psi^{-1}(\tilde{W}))} \subset \overline{\Phi(\tilde{W})} \subset Z.$$

Let K be a compact subset of Y. There is an integer n such that $(\Psi \circ \Phi)^n(\Psi(K)) \subset \tilde{W}$. Hence

$$(\Phi \circ \Psi)^{n+1}(K) = \Phi \circ (\Psi \circ \Phi)^n \circ \Psi(K) \subset \Phi(\tilde{W}) \subset W.$$

Therefore

$$i(X, \Psi\circ\Phi, U) = i(\tilde{W}, \Psi\circ\Phi|_{\tilde{W}}, \tilde{W}\cap U) = i(Z, \Phi\circ\Psi|_Z, \Psi^{-1}(U\cap\tilde{W}))$$
$$= i(Z, \Phi\circ\Psi|_Z, \Psi^{-1}(U)\cap Z) = i(Y, \Phi\circ\Psi, \Psi^{-1}(U)).$$

The proof of 7.10 is complete. ∎

7.11. PROPOSITION. *Let X be a metric ANR and let $\Phi\in\mathcal{DA}(X, X)$ determine a map of compact attraction. Then Φ_* is a Leray endomorphism and*

$$i(X, \Phi, X) = \Lambda(\Phi_*).$$

Proof. Let U be a neighbourhood of an attractor such that $\Phi|_U\colon U \to U$ is a compact map and U absorbs compact sets. Then we have three homomorphisms: $\Phi_*\colon H_*(X)\to H_*(X)$, $\Phi'_*\colon H_*(U)\to H_*(U)$ and $\Phi''_*\colon H_*(X, U)\to H_*(X, U)$. From the exactness axiom for Čech homology with compact carriers (coefficients are in a field F) and 5.1.4 we deduce that if two of the above endomorphisms are Leray endomorphisms then so is the third and

$$\Lambda(\Phi''_*) = \Lambda(\Phi_*)-\Lambda(\Phi'_*).$$

Moreover, Φ''_* is weakly nilpotent (i.e. for each $w\in H_q(X, U)$ there is n such that $(\Phi''_{*q})^n(w) = 0$). Thus by [15], 1.4, $\Lambda(\Phi''_*) = 0$. Hence we obtain $\Lambda(\Phi_*) = \Lambda(\Phi'_*)$ and, by 7.5.5, $i(U, \Phi, U) = \Lambda(\Phi'_*)$. ∎

VIII. Applications of the index to multivalued maps

1. The index theory for permissible maps. Let X be a metric ANR and let $\Phi\colon X\to X$ be a permissible map which is of compact attraction. Then every selector of Φ is a map of compact attraction. Thus, if a decomposition $(\Phi_1, \ldots, \Phi_k)\in\mathcal{DA}(X, X)$ determines a selector of Φ then $(\Phi_1, \ldots, \Phi_k)_*\colon H_*(X)\to H_*(X)$ is a Leray endomorphism (see 7.11) and the generalized Lefschetz number $\Lambda(\Phi_1, \ldots, \Phi_k)_*$ is defined.

8.1. DEFINITION. We define the *Lefschetz set* of the map $\Phi\colon X\to X$ by

$$\mathcal{L}(\Phi) := \{\Lambda(\Phi_1, \ldots, \Phi_k)_*\colon (\Phi_1, \ldots, \Phi_k)\in\mathcal{DA}(X, X) \text{ and } (\Phi_1, \ldots, \Phi_k) \text{ determines a selector of } \Phi\}.$$

8.2. DEFINITION. Let U be an open subset of X such that $\operatorname{Fix}\Phi\cap\partial U = \emptyset$. Then we define the *index set* of Φ by

$$\mathcal{I}(X, \Phi, U) := \{i(X, (\Phi_1, \ldots, \Phi_k), U)\colon \Phi_k\circ\ldots\circ\Phi_1\subset\Phi\}.$$

8.3. PROPOSITION (Selectors). *Let X be a metric ANR and $\Phi\subset\Psi\in\mathcal{P}(X, X)$. Assume that Ψ is a map of compact attraction and $U\subset X$ is open and such that* $\operatorname{Fix}\Psi\cap\partial U = \emptyset$. *Then*

$$\mathcal{I}(X, \Phi, U)\subset\mathcal{I}(X, \Psi, U). \quad ∎$$

8.4. Proposition (Additivity). *Let X be a metric ANR and $\Phi \in \mathscr{P}(X, X)$ a map of compact attraction. Let $U \subset X$ be open and let $U_1, U_2 \subset U$ be open, disjoint and such that* Fix $\Phi|_{\bar{U}} \subset U_1 \cup U_2$. *Then*

$$\mathscr{I}(X, \Phi, U) \subset \mathscr{I}(X, \Phi, U_1) + \mathscr{I}(X, \Phi, U_2) \ \blacksquare$$

8.5. Proposition (Localization). *Let $U_1 \subset U \subset X$ be open subsets of a metric ANR X and let $\Phi \in \mathscr{P}(X, X)$ be a map of compact attraction such that* Fix $\Phi|_{\bar{U}} \subset U_1$. *Then*

$$\mathscr{I}(X, \Phi, U) = \mathscr{I}(X, \Phi, U_1). \ \blacksquare$$

8.6. Corollary (Fixed points). *Let X be a metric ANR and $\Phi \in \mathscr{P}(X, X)$ a map of compact attraction. If $\mathscr{I}(X, \Phi, U) \neq \{0\}$ then Φ has a fixed point in U.* $\blacksquare$

8.7. Proposition (Homotopy). *Let $H\colon X \times [0, 1] \to X$ be a permissible map such that $\mathscr{H} = (H, \mathrm{id})\colon X \times [0, 1] \to X \times [0, 1]$ is a map of compact attraction. If $U \subset X$ is open and such that $H_t\colon X \to X$ has no fixed points in ∂U for any $t \in [0, 1]$ then*

$$\mathscr{I}(X, H_0, U) \cap \mathscr{I}(X, H_1, U) \neq \emptyset. \ \blacksquare$$

8.8. Proposition (Commutativity). *Let $U \subset X$ be open and let $\Phi \in \mathscr{P}(X, Y)$, $\Psi \in \mathscr{P}(Y, X)$ satisfy the assumption of* 7.10. *Then*

$$\mathscr{I}(Y, \Phi \circ \Psi, \Psi^{-1}(U)) \cap \mathscr{I}(X, \Psi \circ \Phi, U) \neq \emptyset. \ \blacksquare$$

8.9. Proposition (Mod-p property). *If $\Phi \in \mathscr{P}(X, X)$ satisfies the assumption of* 7.9 *then*

$$\mathscr{I}(X, \Phi, W) \subset \mathscr{I}(X, \Phi^p, W). \ \blacksquare$$

8.10. Proposition (Normalization). *If X is a metric ANR and $\Phi \in \mathscr{P}(X, X)$ is of compact attraction, then*

$$\mathscr{I}(X, \Phi, X) = \mathscr{L}(\Phi). \ \blacksquare$$

All the above properties are easy consequences of the definition and of the results of Chapter VII.

As a corollary of 8.6 and 8.10 we obtain the following generalization of the Lefschetz fixed point theorem (see [18] for admissible maps and [31] for $\mathscr{A}_m$ for a compact polyhedron).

8.11. Theorem. *Let X be a metric ANR and $\Phi\colon X \to X$ a permissible map of compact attraction. Then $\mathscr{L}(\Phi) \neq \{0\}$ implies that* Fix $\Phi \neq \emptyset$. $\blacksquare$

Since the induced homomorphisms $(\Phi_1, \ldots, \Phi_k)_*$ are nontrivial (cf. [31]), we obtain:

8.12. Corollary. *If X is an acyclic metric ANR then every permissible map $\Phi\colon X \to X$ which is of compact attraction has a fixed point.* $\blacksquare$

2. The Lefschetz theorem for pairs of spaces. Let X and A be metric ANR's, $A \subset X$. Let $\Phi \in \mathscr{P}(X, X)$ be a map of compact attraction and assume that $\Phi(A) \subset A$. If a selector $\Psi \subset \Phi$ is determined by a decomposition $(\Phi_1, \ldots, \Phi_k) \in \mathscr{D}\mathscr{A}(X, X)$, then $\Psi(A) \subset A$ and we have an endomorphism $(\Phi_1, \ldots, \Phi_k)_*\colon H_*(X, A) \to H_*(X, A)$. Since Φ and $\Phi|_A\colon A \to A$ are maps of compact attraction,

$$(\Phi_1, \ldots, \Phi_k)_{*X}\colon H_*(X) \to H_*(X) \quad \text{and} \quad (\Phi_1, \ldots, \Phi_k)_{*A}\colon H_*(A) \to H_*(A)$$

are Leray endomorphisms (cf. 7.11). Therefore by [5], 1.4, $(\Phi_1, \ldots, \Phi_k)_*$ is also a Leray endomorphism. Thus we can define the *relative Lefschetz set* of Φ by

$$\mathscr{L}_{X,A}(\Phi) := \{\Lambda(\Phi_1, \ldots, \Phi_k)_*\colon (\Phi_1, \ldots, \Phi_k)_*\colon H_*(X, A) \to H_*(X, A) \text{ and } \Phi_k \circ \ldots \circ \Phi_1 \subset \Phi\}.$$

The following theorem generalizes 4.4 in [5] (see also [26] for admissible maps).

8.13. Theorem. *Let A be an open subset of a metric ANR X and $\Phi\colon (X, A) \to (X, A)$ a permissible map of compact attraction. Then $\mathscr{L}_{X,A}(\Phi) \neq \{0\}$ implies that Φ has a fixed point in $X - A$.*

Proof. Let $(\Phi_1, \ldots, \Phi_k)$ determine a selector and $\Lambda(\Phi_1, \ldots, \Phi_k)_* \neq 0$. Consider three homomorphisms

$$(\Phi_1, \ldots, \Phi_k)_{*X}\colon H_*(X) \to H_*(X), \quad (\Phi_1, \ldots, \Phi_k)_{*A}\colon H_*(A) \to H_*(A)$$

and $(\Phi_1, \ldots, \Phi_k)_*\colon H_*(X, A) \to H_*(X, A)$. In view of [5], 1.4, their Lefschetz numbers satisfy the equality

$$\Lambda(\Phi_1, \ldots, \Phi_k)_* = \Lambda(\Phi_1, \ldots, \Phi_k)_{*X} - \Lambda(\Phi_1, \ldots, \Phi_k)_{*A}.$$

We can assume that Φ has no fixed points in $\partial(X - A)$. Then by 7.7 and 7.11 we obtain

$$\Lambda(\Phi_1, \ldots, \Phi_k)_{*X} = i\big(X, (\Phi_1, \ldots, \Phi_k), A\big) + i\big(X, (\Phi_1, \ldots, \Phi_k), X - A\big).$$

But it is clear from the definition of the index that

$$i\big(X, (\Phi_1, \ldots, \Phi_k), A\big) = \Lambda(\Phi_1, \ldots, \Phi_k)_{*A}.$$

Therefore

$$0 \neq \Lambda(\Phi_1, \ldots, \Phi_k)_* = i\big(X, (\Phi_1, \ldots, \Phi_k), \operatorname{Int}(X - A)\big)$$

and, by 8.6, Φ has a fixed point in $\operatorname{Int}(X - A)$. ∎

The next theorem generalizes 4.5 in [5]. The proof is the same as the proof of 8.13.

8.14. Theorem. *Given a permissible map $\Phi\colon (X, A) \to (X, A)$ of compact attraction, where X and A are metric ANR's and A is closed in X, the condition $\mathscr{L}_{X,A}(\Phi) \neq \{0\}$ implies that Φ has a fixed point in $\overline{X - A}$.* ∎

3. Multivalued semiflows. Let us denote by $\boldsymbol{R}_+$ the set of nonnegative real numbers.

8.15. Definition. A *multivalued semiflow* is a map $\Phi\colon \boldsymbol{R}_+ \times X \to X$ such that

8.15.1. Φ is permissible;

8.15.2. $\Phi(t_1, \Phi(t_2, x)) \subset \Phi(t_1+t_2, x)$ for $t_1, t_2 \in \boldsymbol{R}_+$, $x \in X$;

8.15.3. $\Phi(0, x) = \{x\}$ for $x \in X$.

8.16. Theorem (cf. [19], or [40] for singlevalued case). *Let X be a compact ANR with Euler characteristic $\chi(X) \neq 0$. Then for every multivalued semiflow $\Phi\colon \boldsymbol{R}_+ \times X \to X$ there is a point $x_0 \in X$ such that $x_0 \in \Phi(t, x_0)$ for each $t \in \boldsymbol{R}_+$.*

Proof. Since $\Phi_0 = \mathrm{id}_X$ and $\lambda(\mathrm{id}_X) = \chi(X) \neq 0$, $0 \notin \mathscr{L}(\Phi_0)$. Every Φ_t is homotopic to Φ_0 with the homotopy $H\colon X \times [0, 1] \to X$, $H(x, s) = \Phi((1-s)t, x)$. In view of 8.7 we have $\mathscr{L}(\Phi_t) \neq \{0\}$ and therefore, by 8.11, Fix $\Phi_t \neq \emptyset$ for each $t \in \boldsymbol{R}_+$.

Denote $B_n := \mathrm{Fix}\, \Phi_{2^{-n}}$. We prove that $B_{n+1} \subset B_n$. Let $x \in \Phi(2^{-n-1}, x)$. Then $x \in \Phi(2^{-n} - 2^{-n-1}, \Phi(2^{-n-1}, x))$ and from 8.15.2 we obtain

$$x \in \Phi(2^{-n} - 2^{-n-1} + 2^{-n-1}, x) = \Phi(2^{-n}, x).$$

Therefore the set $B := \bigcap_{n=1}^{\infty} B_n$ is nonempty. From 8.15.2 we deduce that, for every $x \in B$, $x \in \mathrm{Fix}\, \Phi_t$, where $t = m \cdot 2^{-n}$, $m, n \in \boldsymbol{N}$. Let $t > 0$ be a real number and $x \in B$. Since Φ is an u.s.c. map, for every neighbourhood U of $\Phi(t, x)$ there is a neighbourhood V of t such that $\Phi(t', x) \subset U$ for $t' \in V$. Therefore we obtain $x \in \Phi(m \cdot 2^{-n}, x) \subset U$ and thus $x \in \Phi(t, x)$. The proof is complete. ■

The *Euler characteristic* of a pair (X, A) is defined to be the Lefschetz number of the identity on (X, A):

$$\chi(X, A) := \lambda(\mathrm{id}_{(X, A)}).$$

Using 8.14 instead of 8.11 we obtain the relative version of the preceding theorem, which generalizes 5.3 in [5].

8.17. Theorem. *Let (X, A) be a pair of compact metric ANR's with $\chi(X, A) \neq 0$. If $\Phi\colon (\boldsymbol{R}_+ \times X, \boldsymbol{R}_+ \times A) \to (X, A)$ is a multivalued semiflow, then there exists in $\overline{X - A}$ a common fixed point for all Φ_t.* ■

4. Repulsive and ejective points. Let X be a topological space, $\Phi\colon X \to X$ an u.s.c. map, $x_0 \in \mathrm{Fix}\, \Phi$ and let U be a neighbourhood of x_0. Then x_0 is a *repulsive fixed point with respect to* U if for any neighbourhood V of x_0 there is $n_0 \in \boldsymbol{N}$ such that $\Phi^n(X - V) \subset X - U$ whenever $n \geqslant n_0$. The point x_0 is an *ejective fixed point with respect to* U if for any $x \in U - \{x_0\}$ there is $n \in \boldsymbol{N}$ such that $\Phi^n(x) \subset X - U$. The point x_0 is a *repulsive* (*ejective*) fixed point of Φ provided it is repulsive (ejective) with respect to some neighbourhood U

(cf. [13]). We denote by $\text{Fix}_r\, \Phi$ and $\text{Fix}_e\, \Phi$ the sets of all repulsive and ejective fixed points of Φ, respectively.

8.18. PROPOSITION ([13], p. 484). *Let X be a metric space and $\Phi\colon X \to X$ an u.s.c. map of compact attraction.*

8.18.1. *If $x_0 \in X$ is a repulsive fixed point of Φ with respect to U, and V is a neighbourhood of x_0 such that $V \subset U$, then $\Phi|_{X-\{x_0\}}\colon X-\{x_0\} \to X-\{x_0\}$ has a stable compact attractor $A \subset X-\bar{V}$.*

8.18.2. *Let $\Phi(X-\{x_0\}) \subset X-\{x_0\}$. Then x_0 is an ejective fixed point of Φ if and only if the map $\Phi|_{X-\{x_0\}}\colon X-\{x_0\} \to X-\{x_0\}$ is of compact attraction.*

8.19. PROPOSITION (cf. [13], p. 480). *Let X be a metric ANR and $\Phi\colon X \to X$ a strongly permissible map of compact attraction. Let $(\Phi_1, \ldots, \Phi_k) \in \mathcal{S}\mathcal{A}(X, X)$ determine Φ. Let $F \subset \text{Fix}\, \Phi$ be open and closed in $\text{Fix}\, \Phi$. Assume that $\Phi(X-F) \subset X-F$ and the map $\Phi|_{X-F}\colon X-F \to X-F$ is of compact attraction. Let W be a neighbourhood of F such that $\bar{W} \cap (\text{Fix}\, \Phi - F) = \emptyset$. Then*

$$i(X, (\Phi_1, \ldots, \Phi_k), W) = \Lambda(\Phi_1, \ldots, \Phi_k)_{*X} - \Lambda(\Phi_1, \ldots, \Phi_k)_{*X-F}.$$

Proof. From 7.7 and 7.11 we obtain

$$\Lambda(\Phi_1, \ldots, \Phi_k)_{*X} = i(X, (\Phi_1, \ldots, \Phi_k), W) + i(X, (\Phi_1, \ldots, \Phi_k), X-\bar{W}).$$

Since $\Phi(X-W) \subset X-F$ and $X-W \subset X-F$,

$$i(X, (\Phi_1, \ldots, \Phi_k), X-\bar{W}) = i(X-F, (\Phi_1, \ldots, \Phi_k), X-\bar{W}).$$

Now, since there are no fixed points in $(X-F)-(X-\bar{W})$,

$$i(X-F, (\Phi_1, \ldots, \Phi_k), X-\bar{W}) = \Lambda(\Phi_1, \ldots, \Phi_k)_{*X-F}.$$

Hence we obtain the desired equality. ∎

8.20. THEOREM (cf. [13], p.481). *Let the assumptions of 8.19 be fulfilled and let A be a stable compact attractor for $\Phi|_{X-F}$. Assume that one of the following conditions holds:*

8.20.1. $H_*(X, X-F) = 0$;

8.20.2. *There is a neighbourhood U of F such that $\bar{U} \cap A = \emptyset$ and $H_*(X, X-U) = 0$;*

8.20.3. *There is a neighbourhood V of A such that $\bar{V} \cap F = \emptyset$ and $H_*(X, V) = 0$.*

Then $i(X, (\Phi_1, \ldots, \Phi_k), W) = 0$. If in addition $\Lambda(\Phi_1, \ldots, \Phi_k)_{*X} \neq 0$ then Φ has a fixed point outside F.

Proof. The assumption 8.20.1 implies that the right side of the equality in 8.19 is zero.

Assume 8.20.2. Choose a neighbourhood Y of A such that $\bar{Y} \cap \bar{U} = \emptyset$, $\overline{\Phi(Y)}$ is a compact subset of Y and Y absorbs compact sets in $X-F$. Then we obtain

$$\Lambda(\Phi_1, \ldots, \Phi_k)_{*X} = i(X, (\Phi_1, \ldots, \Phi_k), U) + i(X, (\Phi_1, \ldots, \Phi_k), Y)$$
$$= i(X, (\Phi_1, \ldots, \Phi_k), W) + \Lambda(\Phi_1, \ldots, \Phi_k)_{*Y}.$$

On the other hand, since Φ: $(X, Y) \to (X, Y)$, by [5], 1.4 we have

$$\Lambda(\Phi_1, \ldots, \Phi_k)_{*X} = \Lambda(\Phi_1, \ldots, \Phi_k)_{*(X,Y)} + \Lambda(\Phi_1, \ldots, \Phi_k)_{*Y}.$$

But the assumption 8.20.2 ensures that the endomorphism $(\Phi_1, \ldots, \Phi_k)_{*(X,Y)}$ is weakly nilpotent, since Y absorbs compact sets in $X - F$ and the inclusion j: $(X - U, Y) \to (X, Y)$ induces isomorphisms in H_* (see [13] for details). Therefore, by [15], 1.4, $\Lambda(\Phi_1, \ldots, \Phi_k)_{*(X,Y)} = 0$.

A similar argument works if we assume 8.20.3. The second assertion follows from 8.11. ∎

Now, we note that a repulsive or an ejective fixed point of Φ must be isolated (because it is a unique fixed point in U). Since we are interested in the existence of nonejective and nonrepulsive points, we can assume without loss of generality that Φ has a finite number of such points. Then $\mathrm{Fix}_e \Phi$ and $\mathrm{Fix}_r \Phi$ are closed and open in $\mathrm{Fix}\,\Phi$. Summing up we obtain

8.21. PROPOSITION. *Let X be a metric ANR and Φ: $X \to X$ a map of compact attraction determined by $(\Phi_1, \ldots, \Phi_k) \in \mathcal{SA}(X, X)$. If for every ejective fixed point of Φ we have*

$$\Phi(X - \{x_0\}) \subset X - \{x_0\} \quad \text{and} \quad H_*(X, X - \{x_0\}) = 0,$$

then $\Lambda(\Phi_1, \ldots, \Phi_k)_{*X} \neq 0$ implies that Φ has a nonejective fixed point. ∎

See [13] and [20] for some special situations where the assumptions of 8.21 are satisfied.

8.22. PROPOSITION. *Let X be a metric ANR and Φ: $X \to X$ a map of compact attraction determined by $(\Phi_1, \ldots, \Phi_k) \in \mathcal{SA}(X, X)$. Assume that for every fixed point of Φ which is repulsive with respect to U there is a neighbourhood V such that $V \subset U$ and $H_*(X, X - V) = 0$. Then $\Lambda(\Phi_1, \ldots, \Phi_k)_{*X} \neq 0$ implies that Φ has a nonrepulsive fixed point.*

We recommend [13] and [20] for further literature concerning this interesting subject.

IX. The Nielsen theory

It is difficult to construct the Nielsen relation for multivalued maps in general. We cannot do this for the whole class $\mathcal{A}$ and even for $\mathcal{A}_n$. But if one considers a special class of maps in $\mathcal{A}_n$ called *n-acyclic*, i.e. such that the images of points consist of exactly n acyclic components, then this can be done. The results of this chapter are closely related to those of [23], [35]. In the proofs we also follow [34]. The general references concerning the material of this chapter are also [6] and [24].

1. The Nielsen relation in $\mathrm{Fix}\,\Phi$. Let us start with the following lemma:

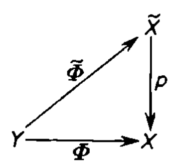
9.1. LEMMA (see [35], [36]). *Let Φ: $X \to Y$ be an n-acyclic continuous map. If X is path connected and simply connected then Φ splits into n disjoint u.s.c. acyclic maps (i.e. such that $\Phi_i(x) \cap \Phi_j(x) = \emptyset$ for each $x \in X$ whenever $i \neq j$).*

Proof. By [31], Lemma 4, $\mathrm{graph}\,\Gamma_\Phi \subset X \times Y$ has exactly n components C_i. Denote by p_X, p_Y the projections of $X \times Y$ onto X and Y, respectively. The continuity of Φ implies that the restriction p_i of p_X to each C_i is closed and surjective. Thus $\Phi = \Phi_1 \cup \ldots \cup \Phi_n$ with $\Phi_i(x) = p_Y(p_i^{-1}(x))$ is the desired splitting. ∎

9.2. DEFINITION. An n-acyclic continuous map Φ: $X \to Y$ is called a *Nielsen n-map* provided for each $x \in X$ every component $A_i(x)$ of $\Phi(x)$ has a neighbourhood U_x^i such that every loop from U_x^i is contractible in Y.

For example, if Y is 1-connected then every n-acyclic map is a Nielsen n-map. If $n = 1$ then we call them simply *Nielsen maps*; the u.s.c. assumption can be dropped in this case (see [23]).

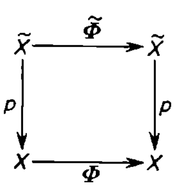
If X is a connected ANR, then it admits a universal covering p: $\tilde{X} \to X$ (see [40]). Let Φ: $Y \to X$ be a Nielsen map. A Nielsen map $\tilde{\Phi}$: $Y \to \tilde{X}$ such that the diagram



commutes will be called a *lift* of the map Φ. We will use the following

9.3. PROPOSITION ([23]). *If Y is path connected and 1-connected then for any Nielsen map Φ: $Y \to X$ and points $y_0 \in Y$, $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) \in \Phi(y_0)$ there is a unique lift $\tilde{\Phi}$: $Y \to \tilde{X}$ satisfying $\tilde{x}_0 \in \tilde{\Phi}(y_0)$.* ∎

9.4. COROLLARY ([23]). *Let X be a connected ANR and Φ: $X \to X$ a Nielsen map. Let $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$ be such that $p(\tilde{x}_2) \in \Phi(p(\tilde{x}_1))$. Then there is a unique Nielsen map $\tilde{\Phi}$: $\tilde{X} \to \tilde{X}$ for which $\tilde{x}_2 \in \tilde{\Phi}(\tilde{x}_1)$ and the diagram*



commutes.

Proof. Apply 9.3 to the map $\Phi \circ p$: $\tilde{X} \to X$. ∎

If Φ: $X \to X$ is a Nielsen map then we have the *Nielsen relation* in $\mathrm{Fix}\,\Phi$ (introduced in [23]):

9.5. $x \sim_N y$ iff there is a lift $\tilde{\Phi}$: $\tilde{X} \to \tilde{X}$ such that $x, y \in p(\mathrm{Fix}\,\tilde{\Phi})$.

Note that a Nielsen n-map Φ: $X \to X$ need not be globally splitted. But if X is an ANR then Φ can be locally splitted into n disjoint Nielsen maps. As an example, consider Φ: $S^1 \to S^1$ defined by $\Phi(z) := \{w \in S^1 : w^n = z\}$.

9.6. LEMMA. *Let X be an ANR and Φ: $X \to X$ a Nielsen n-map. For each $x \in \mathrm{Fix}\,\Phi$ there is a neighbourhood V_x of x in X such that if $\Phi = \Phi_1 \cup \ldots \cup \Phi_n$ is the splitting of $\Phi|_{V_x}$ then $x \in \Phi_i(x)$ implies that $V_x \cap \mathrm{Fix}\,\Phi = V_x \cap \mathrm{Fix}\,\Phi_i$.*

Proof. Let $\Phi(x) = \Phi_1(x) \cup \Phi_2(x) \cup \ldots \cup \Phi_n(x)$ and $x \in \Phi_1(x)$. Since the sets $\Phi_i(x)$ are compact and disjoint, there are neighbourhoods V_i of $\Phi_i(x)$ such that $V_i \cap V_j = \emptyset$ for $i \neq j$. Let V be a neighbourhood of x such that $\Phi|_V = \Phi_1 \cup \ldots \cup \Phi_n$. Then we define $V_x := \bigcap_{i=1}^{n} \Phi_i^{-1}(V_i)$. ∎

Now, let us consider the map $\Phi \circ p$: $\tilde{X} \to X$. By 9.1 we have the splitting $\Phi \circ p = \Phi_1 \cup \ldots \cup \Phi_n$, where Φ_i are distinct Nielsen maps.

9.7. DEFINITION. Let $x, y \in \mathrm{Fix}\,\Phi$. We define the *Nielsen relation*: $x \sim_N y$ iff there is a lift $\tilde{\Phi}_i$: $\tilde{X} \to \tilde{X}$ of one the maps Φ_i: $\tilde{X} \to X$ such that $x, y \in p(\mathrm{Fix}\,\tilde{\Phi}_i)$.

9.8. LEMMA. *If $x \in \mathrm{Fix}\,\Phi$, then there is a neighbourhood V_x of x such that $x \sim_N y$ for $y \in V_x \cap \mathrm{Fix}\,\Phi$.*

Proof. Let $U_x = U_x^1 \cup \ldots \cup U_x^n$ be a neighbourhood of $\Phi(x)$ given by 9.2. We can assume that $U_x^i \cap U_x^j = \emptyset$ for $i \neq j$ and $x \in U_x^1$. Let V_x' be given by 9.6 and consider the local splitting $\Phi|_{V_x'} = \Phi_1' \cup \ldots \cup \Phi_n'$. Assume that $\Phi_1'(V_x) \subset U_x^1$. We take V_x to be any path connected neighbourhood of x contained in $U_x^1 \cap V_x$. If $y \in V_x \cap \mathrm{Fix}\,\Phi$ then $y \in V_x \cap \mathrm{Fix}\,\Phi_1'$. Note that if Φ_1 is chosen from the splitting of $\Phi \circ p$ then $\Phi_1' \circ p|_{p^{-1}(V_x)} = \Phi_1|_{p^{-1}(V_x)}$. Fix a point $\tilde{x} \in p^{-1}(x)$. By 9.4 we have a lift $\tilde{\Phi}_1$: $\tilde{X} \to \tilde{X}$ such that $\tilde{x} \in \tilde{\Phi}_1(\tilde{x})$. We now consider the restriction $\tilde{\Phi}_1$: $p^{-1}(V_x) \to p^{-1}(U_x^1)$. The last two sets are disjoint sums of connected components, each of them homeomorphic by p with V_x and U_x^1, respectively. Denote by $\tilde{V}_x$ and $\tilde{U}_x^1$ the components containing $\tilde{x}$. We obtain the commutative diagram

$$\begin{array}{ccc} \tilde{V}_x & \xrightarrow{\tilde{\Phi}_1} & \tilde{U}_x^1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ V_x & \xrightarrow[\Phi_1']{} & U_x^1 \end{array}$$

Obviously, if $y \in V_x \cap \mathrm{Fix}\,\Phi_1'$ and $\tilde{y} \in \tilde{V}_x$ with $p(\tilde{y}) = y$, then $\tilde{y} \in \tilde{\Phi}_1(\tilde{y})$. Thus $x, y \in p(\mathrm{Fix}\,\tilde{\Phi}_1)$ and the proof is complete. ∎

9.9. COROLLARY. *If $\mathrm{Fix}\,\Phi$ is compact then the number of Nielsen classes is finite.* ∎

9.10. LEMMA. *Let* $H: X\times[0,1]\to X$ *be a Nielsen n-map and* $\mathscr{H} = (H, \mathrm{id}): X\times[0,1]\to X\times[0,1]$. *Let* A *be a Nielsen class of* $\mathscr{H}$. *For* $t\in[0,1]$ *the set* $A_t = \{x: (x,t)\in A\}$ *is empty or is a Nielsen class of* $H_t: X \to X$, $H_t(x) = H(x,t)$.

Proof. This follows easily from the fact that $(X\times[0,1])^{\sim} = \tilde{X}\times\times[0,1]$. ∎

9.11. COROLLARY. *If* H *is a homotopy and* F *is a Nielsen fixed point class of* H_t *for some* $t\in[0,1]$, *then there is a unique Nielsen fixed point class* $\mathscr{F}$ *of* $\mathscr{H}$ *such that* $F\times\{t\} = \mathscr{F}\cap X\times\{t\}$. ∎

2. The Nielsen number. Let X be a metric ANR and let $\Phi: X\to X$ be a Nielsen n-map of compact attraction. Then the set $\mathrm{Fix}\,\Phi$ is compact and by 9.9 the number of Nielsen classes is finite. Let F be a Nielsen class of $\mathrm{Fix}\,\Phi$. We can choose an open set U such that $F\subset U$ and $\mathrm{Fix}\,\Phi\cap\bar{U} = F$. Thus the index $i(X,\Phi,U)$ is defined (Φ regarded as a decomposition of length 1) and is independent of the choice of U by 7.7. So we can simply write it $i(F)$. We call the class F *essential* provided $i(F)\neq 0$; otherwise it is *inessential.*

9.12. DEFINITION. The *Nielsen number* $N(\Phi)$ of Φ is the number of essential fixed point classes.

In the case of a singlevalued map this is the usual Nielsen number (see [23] and [24]). The following is an immediate consequence of 7.7.

9.13. THEOREM. If $\Phi: X\to X$ *is a Nielsen n-map of compact attraction then it has at least* $N(\Phi)$ *fixed points.* ∎

9.14. LEMMA. *Let* $H: X\times[0,1]\to X$ *be a Nielsen n-map such that* $\mathscr{H} = (H,\mathrm{id}): X\times[0,1]\to X\times[0,1]$ *is a map of compact attraction. Let* $\mathscr{F}$ *be a Nielsen class of* $\mathscr{H}$. *Then* $i(F_0) = i(F_1)$, *where* $F_t = \mathscr{F}\cap X\times\{t\}$.

Proof. Since $\mathscr{H}$ is of the form (H, id), every invariant neighbourhood of an attractor must be of the form $U\times[0,1]$, where U is an open subset of X invariant for H_t for each $t\in[0,1]$. Thus without loss of generality we can assume that H is a compact map, because $\mathrm{Fix}\,\mathscr{H}\subset U\times[0,1]$. Choose an open set $V\subset X\times[0,1]$ such that $F\subset V$ and $\bar{V}\cap\mathrm{Fix}\,\mathscr{H} = \mathscr{F}$. If $t\in[0,1]$ then $F_t\subset V_t$ and also $\bar{V}_t\cap\mathrm{Fix}\,H_t = F_t$. Thus $i(F_t) = i(X, H_t, V_t)$. Let $p_X: X\times\times[0,1]\to X$ be the projection. Consider the compact set $K = p_X(\mathscr{F}) - V_t$. There is $\delta>0$ such that $d(H_t(x), x)>\delta$ for each $x\in K$. Since K is compact, there is $\varepsilon>0$ such that if $|t-s|<\varepsilon$ then $H(x,s)\subset O_\delta(H(x,t))$ for $x\in K$. Therefore H_s has no fixed points on K for $|t-s|<\varepsilon$. Since $F_s\subset p_X(\mathscr{F})$, we must have $F_s\subset V_t$ and $\bar{V}_t\cap\mathrm{Fix}\,H_s = F_s$. Thus $i(F_s) = i(X, H_s, V_t)$. By the homotopy property 7.8 we obtain

$$i(F_s) = i(X, H_s, V_t) = i(X, H_t, V_t) = i(F_t).$$

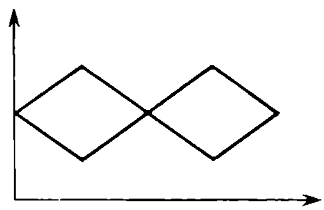
Now, the connectedness of $[0,1]$ implies the desired equality. ∎

9.15. THEOREM. *Let* $H: X\times[0,1]\to X$ *be a Nielsen n-map such that*

$\mathscr{H} = (H, \mathrm{id})\colon X \times [0, 1] \to X \times [0, 1]$ *is a map of compact attraction. Then* $N(H_0) = N(H_1)$.

Proof. For a Nielsen class F_0 of H_0 we find a unique Nielsen class $\mathscr{F}$ of $\mathscr{H}$ using 9.11. In view of 9.14 we have $i(F_0) = i(F_1)$, where $F_1 = \mathscr{F} \cap X \times \{1\}$. Therefore $N(H_0) \leqslant N(H_1)$. Analogously one obtains $N(H_1) \leqslant N(H_0)$. ■

We have just noted that the relation $\sim_N$ for singlevalued maps is the same as the one in [6] (see also [34]). If Φ is an n-valued continuous map then it coincides with the one defined in [35]. It remains an open question whether a Nielsen relation can be introduced for the class $\mathscr{A}_n(X, X)$. But it seems that the technique of splitting is not applicable in this case. Consider the map with the following graph:



Obviously it cannot be splitted into two disjoint continuous maps. If one wants to split it into not disjoint maps, then the splitting is nonunique.

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