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MULTIPLICITY AND THE ŁOJASIEWICZ EXPONENT

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0. Introduction

The purpose of this paper is to present some recent results concerning the Łojasiewicz exponent of a holomorphic mapping at an isolated zero. In Section 1 we recall well-known properties of multiplicity which we need later. Some basic facts about the Łojasiewicz exponent obtained by M. Lejeune-Jalabert and B. Teissier in their seminar at École Polytechnique in 1974 are presented in Section 2. Our approach is different from the original one: no use of the technique of normalized blowing-up will be made.

In Section 3 which is principal for this paper we compare two invariants of a holomorphic mapping f: its multiplicity $m_0(f)$ and Łojasiewicz exponent $l_0(f)$. Roughly speaking we are interested in the following question: what can be said about $l_0(f)$ when $m_0(f)$ is given?

As corollaries of results presented in this part of the paper we obtain some properties of Łojasiewicz exponents. For illustration let us quote the following: a rational number is equal to the Łojasiewicz exponent of a holomorphic mapping of C^2 if and only if it appears in the sequence

$$1,\ 2,\ 3,\ 3\tfrac{1}{2},\ 4,\ 4\tfrac{1}{3},\ 4\tfrac{1}{2},\ 4\tfrac{2}{3},\ 5,\ \ldots$$

Note that the fractional parts of this Łojasiewicz exponents and the number 1 form Farey's sequences

$$F_2 = \{0, \tfrac{1}{2}, 1\}, \quad F_3 = \{0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, 1\}, \ \ldots \quad \text{(cf. [Co])}.$$

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1. The multiplicity of a holomorphic mapping

If h is a nonzero holomorphic function defined in an open neighbourhood of the origin $0 \in C^n$, we denote by $\operatorname{ord} h$ its order, by $\operatorname{in} h$ the initial form of h, i.e., if $h = \sum_{i \geq m} h_i$, $h_m \neq 0$, is the expansion of h in a series of homogeneous polynomials then $\operatorname{ord} h = m$, in $h = h_m$. By definition, we put $\operatorname{ord} 0 = +\infty$, in $0 = 0$. For any holomorphic mapping $f = (f_1, \ldots, f_m)$: $(C^n, 0) \to (C^m, 0)$ (this notation means that f is defined near 0 and $f(0) = 0$) we define $\operatorname{ord} f = \min_{i=1}^{m} (\operatorname{ord} f_i)$ and $\inf = (\inf_1, \ldots, \inf_m)$. It is easy to check the following characterisation of the order.

PROPERTY 1.1. *Let $f = (f_1, \ldots, f_m)$: $(C^n, 0) \to (C^m, 0)$ be a nonzero holomorphic mapping. Then $\operatorname{ord} f$ is the largest number $q \in R$ such that $|f(z)| \leq C|z|^q$ near 0 for some constant $C > 0$.*

Note, that we shall use $|z|$ to denote the maximum norm $\max_{i=1}^{n} |z_i|$. Let $f = (f_1, \ldots, f_n)$: $(C^n, 0) \to (C^n, 0)$ be a holomorphic mapping. We say that f is finite if 0 is an isolated point of $f^{-1}(0)$. If f is finite then there exist arbitrary small neighbourhoods U and V of the origin such that $U \ni z \to f(z) \in V$ is a proper mapping from U to V which is an unramified covering over an open, dense connected subset of V. We define the multiplicity $m_0(f)$ of f to be the number of sheets of this covering. This notion of multiplicity extends easily to the case of mappings between analytic sets (cf. [M]). Let us recall two useful estimates of multiplicity.

PROPOSITION 1.2 (cf. [Č], [P$_2$]). *Let $f = (f_1, \ldots, f_n)$: $(C^n, 0) \to (C^n, 0)$ be a finite mapping. Then $m_0(f) \geq \prod_{i=1}^{n} \operatorname{ord} f_i$ with equality if and only if $(\inf)^{-1}(0) = \{0\}$.*

PROPOSITION 1.3 (cf. [P$_3$]). *Suppose that $g = (g_1, \ldots, g_n)$ is a polynomial mapping, finite at $0 \in C^n$. Then*

$$m_0(g) \leq \prod_{i=1}^{n} \deg g_i.$$

One can compute the multiplicity $m_0(f)$ by taking restriction of f to a certain analytic curve. By a local (analytic) curve we mean an analytic 1-dimensional subset of an open neighbourhood of the origin. If a local curve $S \subset C^n$ is irreducible at 0 then there exists a holomorphic injective mapping p: $(C, 0) \to (C^n, 0)$ such that S near 0 is the image under p of an open neighbourhood of $0 \in C$.

LEMMA 1.4. *If $S = \bigcup_{i=1}^{k} S_i$ is a decomposition of a local curve S in irreducible components and if p_i is a parametrisation of S_i then for any holomorphic function h: $(C^n, 0) \to (C, 0)$ the multiplicity $m_0(h|S)$ of the* *mapping* $h|S$: $(S, 0) \to (C, 0)$ *is equal to* $\sum_{i=1}^{k} \operatorname{ord}(h \circ p_i)$. *In particular the multiplicity* $m_0(S)$ *of* S *is given by formula*

$$m_0(S) = \sum_{i=1}^{k} \operatorname{ord} p_i.$$

Now, we can state the proposition which often facilitates the computation of multiplicity.

PROPOSITION 1.5 (cf. [Č]). *Let* $f = (f_1, \ldots, f_n)$: $(C^n, 0) \to (C^n, 0)$ *be a finite holomorphic mapping such that the differentials* $df_1(z), \ldots, df_{n-1}(z)$ *are linearly independent on a dense subset of the curve* $S = \{z: f_1(z) = \ldots = f_{n-1}(z) = 0\}$. *Then*

$$m_0(f) = m_0(f_n | S).$$

2. The Łojasiewicz exponent

Let $f = (f_1, \ldots, f_n)$: $(C^n, 0) \to (C^n, 0)$ be a finite holomorphic mapping.

DEFINITION 2.1. The *Łojasiewicz exponent* $l_0(f)$ of the mapping f at $0 \in C^n$ (or, briefly, the exponent of f) is the greatest lower bound of the set of all $q > 0$ which satisfy the condition: there exists positive constants C, R such that $|f(z)| \geqslant C|z|^q$ for all $z \in C^n$ such that $|z| < R$.

From Property 1.1 and from the above definition we have $l_0(f) \geqslant \operatorname{ord} f$, hence $l_0(f) \geqslant 1$. The exponent $l_0(f)$ is an analytic invariant: if φ and ψ are local biholomorphisms then $l_0(\psi \circ f \circ \varphi) = l_0(f)$.

Moreover, one can check that $l_0(f)$ like $m_0(f)$ depends only on the local algebra of f. Let S be a local curve. It is useful to define $l_0(f|S)$ by replacing in Definition 2.1 the condition "for all $z \in C^n$" by "for all $z \in S$". Obviously $l_0(f) \geqslant l_0(f|S)$. One checks easily

LEMMA 2.2. *If* $S = \bigcup_{i=1}^{k} S_i$ *is the decomposition of* S *into irreducible components and if* p_i *is a parametrisation of* S_i *then*

$$l_0(f|S) = \min_{i=1}^{k} \left(\frac{\operatorname{ord}(f \circ p_i)}{\operatorname{ord} p_i} \right).$$

Combining Lemmas 1.4 and 2.2, we get

$$l_0(f|S) = \min_{i=1}^{k} \left(\frac{m_0(f|S_i)}{m_0(S_i)} \right)$$

where

$$m_0(f|S_i) = \min_{j=1}^{n} (m_0(f_j|S_i)).$$

If $f=(f_1, \ldots, f_n)\colon (\mathbf{C}^n, 0)\to(\mathbf{C}^n, 0)$ is a finite mapping then for any direction $l=(l_1:l_2:\ldots:l_n)\in \mathbf{P}^{n-1}$ the set $f^{-1}(\mathbf{C}l)$ is a local curve described near 0 by equations

$$l_i f_j(z)-l_j f_i(z)=0 \quad \text{for } i, j=1, \ldots, n.$$

The expression "for almost every $l\in \mathbf{P}^{n-1}$" will mean "there exist a Zariski open subset $\Omega\subset \mathbf{P}^{n-1}$ such that for every $l\in\Omega$". The following theorem is due to M. Lejeune-Jalabert and B. Teissier.

THEOREM 2.3 (cf. [L-J-T]). *Let* $f=(f_1, \ldots, f_n)\colon (\mathbf{C}^n, 0)\to(\mathbf{C}^n, 0)$ *be a finite holomorphic mapping. Then*:

(i) *The exponent* $l_0(f)$ *is a rational number. Moreover, the least upper bound in the definition of Łojasiewicz exponent is attained.*

(ii) *For almost every* $l\in \mathbf{P}^{n-1}$ *the exponent* $l_0(f)$ *is attained on the curve* $f^{-1}(\mathbf{C}l)$: $l_0(f)=l_0(f\,|\,f^{-1}(\mathbf{C}l))$.

Our proof of Theorem 2.3 is based on the following observation.

LEMMA 2.4 (cf. [P$_1$]). *Let* $P(T)=T^m+a_1T^{m-1}+\ldots+a_m$ *be a distinguished polynomial at* $0\in\mathbf{C}^n$ (i.e., $a_1, \ldots, a_m$ *are holomorphic near* 0 *and* $a_1(0)=\ldots=a_m(0)=0$). *Then* $\min_{i=1}^{m}\left(\frac{1}{i}\operatorname{ord} a_i\right)$ *is the largest number* $q\in\mathbf{R}$ *such that there exist a constant* $C>0$ *and a neighbourhood* V *of the origin such that*

$$\{(w, t)\in V\times\mathbf{C}\colon P(w, t)=0\}\subset\{(w, t)\in V\times\mathbf{C}\colon |t|\leqslant C|w|^q\}.$$

The proof of Lemma 2.4 is given in [P$_1$]. Let h be a holomorphic function defined near $0\in\mathbf{C}^n$. To prove Theorem 2.3 we define: $O(f, h)=$ the least upper bound of the set of all $q>0$ which satisfy the condition: there exist positive constants C, R such that $|h(z)|\leqslant C|f(z)|^q$ for all $z\in\mathbf{C}^n$ such that $|z|<R$. Analogously we define $O(f\,|\,S, h)$ for any local curve $S\subset\mathbf{C}^n$.

One sees easily that the following equality holds

$$l_0(f)=\frac{1}{\min_{i=1}^{n}(O(f, z_i))} \tag{1}$$

where $z_i\colon \mathbf{C}^n\to\mathbf{C}$ are coordinate functions. Thus, the statements (i) and (ii) of Theorem 2.3 will follow from the properties:

(2) The number $O(f, h)$ is rational and the least upper bound in the definition of $O(f, h)$ is attained.

(3) $O(f, h)=O(f\,|\,f^{-1}(\mathbf{C}l), h)$ for almost every $l\in\mathbf{P}^{n-1}$.

In order to check (2) and (3) let us consider the characteristic polynomial $P_h(T)=T^m+a_{1,h}T^{m-1}+\ldots+a_{m,h}$ of h relatively to f. The distinguished polynomial $P_h(T)$ with holomorphic coefficients has the properties: a) $P_h(T)$ is of degree $m = m_0(f)$; b) there exist arbitrary small neighbourhoods U_0, V_0 of the origin $0 \in C^n$ such that the set $\{(w, t) \in V_0 \times C\colon P_h(w, t) = 0\}$ is the image of U_0 under the mapping $z \to (f(z), h(z))$. Therefore the inequality $|h(z)| \leqslant C|f(z)|^q$, $z \in U_0$, is equivalent to the estimate

$$\{(w, t) \in V_0 \times C\colon P_h(w, t) = 0\} \subset \{(w, t) \in V_0 \times C\colon |t| \leqslant C|w|^q\}.$$

Hence, by Lemma 2.4 the least upper bound in the definition of Łojasiewicz exponent is attained. Moreover, we get the equality

$$O(f, h) = \min_{i=1}^{m} \left(\frac{1}{i} \operatorname{ord} a_{i,h}\right) \tag{4}$$

which implies the rationality of $O(f, h)$. Hence (2) is established. The proof of (3) is similar.

We observe that the image of $f^{-1}(Cl)$ under the mapping $z \to (f(z), h(z))$ is given by equations $P_h(w, t) = 0$, $l_i w_j - l_j w_i = 0$; hence by Lemma 2.4 we get

$$O(f \mid f^{-1}(Cl), h) = \min_{i=1}^{m} \left(\frac{1}{i} \operatorname{ord}(a_{i,h} \mid Cl)\right). \tag{5}$$

Property (3) follows now from (4) and (5) since the set

$$\Omega = \{l \in P^{n-1}\colon \operatorname{ord} a_{i,h} = \operatorname{ord}(a_{i,h} \mid Cl) \text{ for } i = 1, \ldots, n\}$$

is open in Zariski topology.

Note. Recently, J. Chądzyński and T. Krasiński (cf. [Ch–K]) showed, using the method of "horn neighbourhoods" due to Kuo (cf. [K–L]), that the exponent $l_0(f)$ of the finite mapping $f = (f_1, f_2)\colon (C^2, 0) \to (C^2, 0)$ is attained on one of the curves $f_1 = 0$ or $f_2 = 0$. This result does not extend to the case of three or more variables.

Indeed, if $f\colon C^3 \to C^3$ is given by $f(x, y, z) = (x^2, y^3, z^3 - xy)$, then $l_0(f) = 18/5$, $l_0(f \mid \{f_i = f_j = 0\}) \leqslant 3$; hence $l_0(f)$ is not attained on the curves $f_i = f_j = 0$, $i \neq j$.

Combining Lemma 2.2 and Theorem 2.3(ii), we get

COROLLARY 2.5. *Let Π be the set of all analytic paths $p\colon (C, 0) \to (C^n, 0)$.* Then

$$l_0(f) = \sup_{p \in \Pi} \left(\frac{\operatorname{ord}(f \circ p)}{\operatorname{ord} p}\right).$$

For almost all $l \in P^{n-1}$ the sup *is attained on the parametrisation of an irreducible component of the curve $f^{-1}(Cl)$.*

The proposition given below completes Corollary 2.5 in the case where f is the gradient of a holomorphic function.

PROPOSITION 2.6 (cf. [T]). *Let h be a holomorphic function near $0 \in C^n$ having at 0 an isolated singularity. Then for almost every $l \in P^{n-1}$: if p is a parametrisation of an irreducible component of the curve $P_l = (\operatorname{grad} h)^{-1}(Cl)$ then*

$$\operatorname{ord}(h \circ p) = \operatorname{ord}((\operatorname{grad} h) \circ p) + \operatorname{ord} p.$$

Proof. Put $H_l = \{z \in C^n\colon\ l_1 z_1 + \ldots + l_n z_n = 0\}$. It is a standard property of polar curves (cf. [T]) that the tangent cone $C_0(P_l)$ and the hyperplane H_l intersect only at the origin $0 \in C^n$ for almost all $l \in P^{n-1}$. Let $l \in P^{n-1}$ be such that $C_0(P_l) \cap H_l = \{0\}$ and let p be the parametrisation of a component of the curve P_l. Differentiating and taking orders give

$$\operatorname{ord}(h \circ p) = \operatorname{ord}((\operatorname{grad} h) \circ p) + \operatorname{ord}(l_1 p_1 + \ldots + l_n p_n).$$

On the other hand, the condition $C_0(P_l) \cap H_l = \{0\}$ implies that $\operatorname{ord}(l_1 p_1 + \ldots + l_n p_n) = \operatorname{ord} p$. Therefore we have

$$\operatorname{ord}(h \circ p) = \operatorname{ord}((\operatorname{grad} h) \circ p) + \operatorname{ord} p$$

and Proposition 2.6 is established. □

Using Proposition 2.6, the exponent $l_0(\operatorname{grad} h)$ can be computed in terms of analytic invariants of the singularity (cf. [T]). In the case $n = 2$ an interesting formula for $l_0(\operatorname{grad} h)$ was given by Kuo and Lu (cf. [K–L], [T]).

3. Multiplicity and the Łojasiewicz exponent

Let $f = (f_1, \ldots, f_n)\colon (C^n, 0) \to (C^n, 0)$ be a finite holomorphic mapping. From formulae (1) and (4) in the proof of Theorem 2.3 we obtain

PROPOSITION 3.1 (cf. [P$_1$]). *If $l_0(f) = p/q$ where $p, q > 0$ are relative prime integers then $1 \leqslant q \leqslant p \leqslant m_0(f)$.*

The above property shows that for a given $m \geqslant 1$, the set of all numbers $l \in R$ such that there is a holomorphic mapping f satisfying the conditions $l_0(f) = l$ and $m_0(f) = m$ is finite. We shall determine these sets later for small values of $m_0(f)$.

In the proposition below, $[x]$ denotes the integral part of the number x.

PROPOSITION 3.2 (cf. [A], [P$_3$]). *For any finite holomorphic mapping $f\colon (C^n, 0) \to (C^n, 0)$ we have*

$$m_0(f) \leqslant ([l_0(f)])^n.$$

The proof of Proposition 3.2 is based on a lemma that is of independent interest.

LEMMA 3.3 (cf. [P$_3$]). *If* g: $(C^n, 0) \to (C^n, 0)$ *is a holomorphic mapping such that* $\operatorname{ord}(g-f) > l_0(f)$ *then* g *is finite,* $l_0(g) = l_0(f)$ *and* $m_0(g) = m_0(f)$.

The proof of Lemma 3.3 is given in [P$_3$]. In order to prove Proposition 3.2 let us put g_i = the sum of all monomials of degree $\leqslant [l_0(f)]$ which appear in the Taylor series of f_i and let $g = (g_1, \ldots, g_n)$. Therefore $\operatorname{ord}(g-f) > l_0(f)$ and from Lemma 3.3 and Proposition 1.3 we get

$$m_0(f) = m_0(g) \leqslant \prod_{i=1}^{n} \deg g_i \leqslant ([l_0(f)])^n.$$

Now, let f: $(C^n, 0) \to (C^n, 0)$ be a holomorphic finite mapping and let $l_0(f) = N + b/a$ where $N = [l_0(f)]$ and a, b are relative prime integers such that $0 \leqslant b < a$. Combining Propositions 3.1 and 3.2 we get $aN+b \leqslant N^n$, whence $a < N^{n-1}$ if $b > 0$. Summarizing, we have proved the following strengthened version of Theorem 2.3(i).

THEOREM 3.4 (cf. [P$_3$]). *Let* f: $(C^n, 0) \to (C^n, 0)$ *be a holomorphic finite mapping. Then there exist integers* N, a, b *such that* $l_0(f) = N + b/a$ *with* $0 < b < a < N^{n-1}$ *or the exponent* $l_0(f)$ *is an integer.*

Let L_n be the set of all numbers $l \in \boldsymbol{R}$ which possess the following property: there exists a finite mapping f: $(C^n, 0) \to (C^n, 0)$ such that $l_0(f) = l$. Obviously $L_1 = \{1, 2, 3, \ldots\}$. From Theorem 3.4 it follows that each of the sets L_n can be arranged in an increasing sequence. The mapping $(x, y) \to (x^{a+1} + y^a, x^{N-b} y^b)$, where $0 < b < a < N$ are integers, has the exponent equal to $N + b/a$. Therefore $L_2 = \{1, 2, 3, 3\frac{1}{2}, 4, 4\frac{1}{3}, 4\frac{1}{2}, 4\frac{2}{3}, \ldots\}$.

Let us note that the evaluation of L_n $(n > 2)$ given in Theorem 3.4 is not exact. Now, we would like to present an estimate of the exponent of a holomorphic mapping in terms of the multiplicity and the orders of its components.

THEOREM 3.5 (cf. [Ch], [P$_2$]). *Let* $f = (f_1, \ldots, f_n)$: $(C^n, 0) \to (C^n, 0)$ *be a finite holomorphic mapping. Then, we have*

$$\max_{i=1}^{n}(\operatorname{ord} f_i) \leqslant l_0(f) \leqslant m_0(f) - \prod_{i=1}^{n} \operatorname{ord} f_i + \max_{i=1}^{n}(\operatorname{ord} f_i).$$

The above estimate was proved by Chądzyński in [Ch] in the case of two variables $n = 2$, the general case $n \geqslant 2$ was done in [P$_2$]. We give here a new proof which is based on Theorem 2.3.

Proof of Theorem 3.5. We may assume, without loss of generality, that $\operatorname{ord} f_i \leqslant \operatorname{ord} f_n$ for $i = 1, \ldots, n$. Let p: $(C, 0) \to (C^n, 0)$ be a parametrisation of an irreducible component of the curve $f_1 = \ldots = f_{n-1} = 0$. Then $f_n \circ p \neq 0$ near 0 and

$$\operatorname{ord}(f \circ p) = \operatorname{ord}(f_n \circ p) \geqslant (\operatorname{ord} f_n)(\operatorname{ord} p),$$

consequently we get

$$l_0(f) \geqslant \frac{\operatorname{ord}(f \circ p)}{\operatorname{ord} p} \geqslant \operatorname{ord} f_n = \max_{i=1}^{n}(\operatorname{ord} f_i).$$

In order to prove the second estimate let us consider the curves $S_k = f^{-1}(Ck + C)$ where $k = (k_1, \ldots, k_{n-1}) \in C^{n-1}$. Then S_k is an analytic curve given near 0 by equations

$$f_1 - k_1 f_n = \ldots = f_{n-1} - k_{n-1} f_n = 0.$$

Applying Sard's theorem to the mapping

$$U \setminus f_n^{-1}(0) \ni z \to \left(\frac{f_1(z)}{f_n(z)}, \ldots, \frac{f_{n-1}(z)}{f_n(z)}\right) \in C^{n-1},$$

and Theorem 2.3, we find a point $k = (k_1, \ldots, k_{n-1}) \in C^{n-1}$ such that the following conditions hold:

(1) the differentials $df_1(z) - k_1 df_n(z), \ldots, df_{n-1}(z) - k_{n-1} df_n(z)$ are linearly independent for $z \in S_k \setminus \{0\}$;

(2) $$\operatorname{ord}(f_i - k_i f_n) = \operatorname{ord} f_i \text{ for } i = 1, \ldots, n-1;$$

(3) $$l_0(f) = l_0(f \mid S_k).$$

Form (1) and Proposition 1.5 it follows that for any holomorphic function h: $(C^n, 0) \to (C, 0)$ we have

(4) $$m_0(f_1 - k_1 f_n, \ldots, f_{n-1} - k_{n-1} f_n, h) = m_0(h \mid S_k).$$

Therefore we get

(5) $$m_0(S_k) \geqslant \operatorname{ord} f_1 \ldots \operatorname{ord} f_{n-1}.$$

Indeed, putting in (4) $h =$ a linear form with sufficiently general coefficients we get with the help of Proposition 1.2:

$$\begin{aligned} m_0(S_k) = m_0(h \mid S_k) &= m_0(f_1 - k_1 f_n, \ldots, f_{n-1} - k_{n-1} f_n, h) \\ &\geqslant \operatorname{ord}(f_1 - k_1 f_n) \ldots \operatorname{ord}(f_{n-1} - k_{n-1} f_n) \operatorname{ord} h \\ &= \operatorname{ord} f_1 \ldots \operatorname{ord} f_{n-1}. \end{aligned}$$

On the other hand let us note that setting $h = f_n$ in (4) gives

(6) $$m_0(f_n \mid S_k) = m_0(f).$$

Let $S_k = \bigcup_j S_k^{(j)}$ be a decomposition of S_k into irreducible components. We may assume that $l_0(f\,|\,S_k) = l_0(f\,|\,S_k^{(1)})$. Now, we have

$$m_0(f) - \prod_{i=1}^{n} \operatorname{ord} f_i + \max_{i=1}^{n}(\operatorname{ord} f_i) = m_0(f) - \Big(\prod_{i=1}^{n-1} \operatorname{ord} f_i\Big)\cdot \operatorname{ord} f_n + \operatorname{ord} f_n$$

$$\geqslant m_0(f_n|S_k) - m_0(S_k)\operatorname{ord} f_n + \operatorname{ord} f_n = \sum_j \big(m_0(f_n\,|\,S_k^{(j)}) - m_0(S_k^{(j)})\operatorname{ord} f_n\big) + \operatorname{ord} f_n$$

$$\geqslant m_0(f_n\,|\,S_k^{(1)}) - m_0(S_k^{(1)})\operatorname{ord} f_n + \operatorname{ord} f_n$$

$$\geqslant \frac{m_0(f_n\,|\,S_k^{(1)})}{m_0(S_k^{(1)})} = l_0(f_n\,|\,S_k^{(1)}) = l_0(f_n\,|\,S_k) = l_0(f).$$

The estimate in Theorem 3.5 is the best possible. □

EXAMPLE 3.6. Let m_1, m_2, $m \geqslant 1$ be integers such that $m \geqslant m_1 m_2$ and $m_2 \leqslant m_1$. Let $f(z_1, z_2) = (z_1^{m_1} + z_2^{m_1+m-m_1m_2}, z_1 z_2^{m_2-1})$. Then $\operatorname{ord} f_1 = m_1$, $\operatorname{ord} f_2 = m_2$, $m_0(f) = m$ and $l_0(f) = m - m_1 m_2 + m_1$. Similarly we construct examples for $n > 2$ (cf. [P₂]).

Remark 3.7. In the notations of the proof of Theorem 3.5 we have

$$m_0(f) = m_0(f_n|S_k) = \sum_j m_0(f_n\,|\,S_k^{(j)}) = \sum_j l_0(f_n\,|\,S_k^{(j)})\,m_0(S_k^{(j)})$$

$$\leqslant \sum_j l_0(f)\,m_0(S_k^{(j)}) = l_0(f)\,m_0(S_k) \quad \text{for almost all } k \in C^{n-1}.$$

If $n = 2$ then $m_0(S_k) = \operatorname{ord} f$ (for almost all k) and we get the estimate

$$l_0(f) \geqslant \frac{m_0(f)}{\operatorname{ord} f}.$$

Suppose that $f = (f_1, \ldots, f_n)$: $(C^n, 0) \to (C^n, 0)$ is a holomorphic mapping such that $(\operatorname{in} f)^{-1}(0) = \{0\}$. Then

$$m_0(f) = \prod_{i=1}^{n} \operatorname{ord} f_i,$$

by Proposition 1.2 and from Theorem 3.5 we get

COROLLARY 3.8. *If* $f = (f_1, \ldots, f_n)$: $(C^n, 0) \to (C^n, 0)$ *is a holomorphic mapping such that* $(\operatorname{in} f)^{-1}(0) = \{0\}$ *then*

$$l_0(f) = \max_{i=1}^{n}(\operatorname{ord} f_i).$$

Using Proposition 3.2 and Theorem 3.5 we obtain similarly

COROLLARY 3.9. *If* $f = (f_1, \ldots, f_n)$: $(C^n, 0) \to (C^n, 0)$ *is finite,* $\operatorname{ord} f_1 = \ldots = \operatorname{ord} f_n = k$ *and* $m_0(f) = k^n + 1$, *then* $l_0(f) = k + 1$.

We conclude this paper by computing the exponent $l_0(f)$ for small values of $m_0(f)$.

PROPOSITION 3.10. *The list below gives the exact evaluations of the exponent $l_0(f)$ for $m_0(f) \leqslant 9$.*

$m_0(f)$	1	2	3	4	5	6	7	8	9
$l_0(f)$	1	2	3	2, 4	3, 5	3, 4, 6	$3\frac{1}{2}$, 4, 5, 7	2, 4, 5, 6, 8	3, $4\frac{1}{2}$, 5, 6, 7, 9

We need three lemmas. We omit the standard proof of the following

LEMMA 3.11. *If $f = (f_1, \ldots, f_n)$: $(C^n, 0) \to (C^n, 0)$ is a finite mapping such that $r = \operatorname{rank}(df(0)) < n$ then there exists a holomorphic mapping*

$$\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_{n-r}) \colon (C^{n-r}, 0) \to (C^{n-r}, 0)$$

such that $m_0(f) = m_0(\tilde{f})$, $l_0(f) = l_0(\tilde{f})$ and $\operatorname{ord} \tilde{f} \geqslant 2$.

LEMMA 3.12. *If $f = (f_1, f_2)$: $(C^2, 0) \to (C^2, 0)$ is finite of multiplicity $m_0(f) = m$ and if $\operatorname{ord} f = 2$ then*

(*) $\qquad l_0(f) \in \{\frac{1}{2}m\} \cup \{\text{all integers } l \text{ such that } \frac{1}{2}m \leqslant l \leqslant m-2\}$.

This evaluation is exact if $m \neq 5$.

Proof. In virtue of Theorem 3.5 and Remark 3.7 we have

$$\frac{m_0(f)}{\operatorname{ord} f} \leqslant l_0(f) \leqslant m_0(f) - \operatorname{ord} f_1 \operatorname{ord} f_2 + \max(\operatorname{ord} f_1, \operatorname{ord} f_2).$$

If $\operatorname{ord} f = 2$ and $m_0(f) = m$ then we get

$$\tfrac{1}{2}m \leqslant l_0(f) \leqslant m-2.$$

Hence if $l_0(f)$ is an integer then (*) holds. If $l_0(f)$ is not an integer then we write $l_0(f) = b/a$, $2 \leqslant a \leqslant b$ with a, b relative prime.

Then we have $b \leqslant m$ by Proposition 3.1 and $b/a \geqslant m/2$. Therefore we get $a = 2$ and $b = m$. This proves the evaluation (*). Now, let $m \geqslant 4$ be an integer (we take $m \geqslant 4$ because $m_0(f) \geqslant (\operatorname{ord} f)^2 = 2^2 = 4$) and let l be an integer such that $\frac{1}{2}m \leqslant l \leqslant m-2$. Then $m-l \geqslant 2$ and $2(m-l) \leqslant m$, so by Example 3.6 there is a holomorphic mapping $f = (f_1, f_2)$ such that $\operatorname{ord} f_1 = 2$, $\operatorname{ord} f_2 = m-l$, $m_0(f) = m$ and $l_0(f) = m - 2(m-l) + \max(2, m-l) = l$. If $m \geqslant 7$ is an odd integer then for the mapping $f(x, y) = (y^2 - x^3, xy^{N-1})$, where $N = [\frac{1}{2}m]$, we have $m_0(f) = m$, $\operatorname{ord} f = 2$ and $l_0(f) = \frac{1}{2}m$. This shows that the evaluation (*) is exact. □

LEMMA 3.13. *Suppose that $f = (f_1, \ldots, f_n)$: $(C^n, 0) \to (C^n, 0)$ is finite. Let $r = \operatorname{rank}(df(0))$. Then $m_0(f) \geqslant 2^{n-r}$.* If $m_0(f) = 2^{n-r}$, *then $l_0(f) = 2$, if $m_0(f) = 2^{n-r} + 1$, then $l_0(f) = 3$.*

Proof. By Lemma 3.11 there is a holomorphic mapping $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_{n-r})$ such that $l_0(\tilde{f}) = l_0(f)$, $m_0(\tilde{f}) = m_0(f)$ and $\operatorname{ord}\tilde{f} \geqslant 2$. Then

$$m_0(f) = m_0(\tilde{f}) \geqslant \operatorname{ord}\tilde{f}_1 \ldots \operatorname{ord}\tilde{f}_{n-r} \geqslant 2^{n-r}$$

according to Proposition 1.2.

If $m_0(f) = 2^{n-r}$, then $(\operatorname{in} f)^{-1}(0) = \{0\}$ by Proposition 1.2 and $l_0(f) = l_0(\tilde{f}) = 2$ by Corollary 3.8. If $m_0(f) = m_0(\tilde{f}) = 2^{n-r}+1$, then $\operatorname{ord} f_1 = \ldots = \operatorname{ord} f_{n-r} = 2$ by Proposition 1.2 and we get $l_0(f) = l_0(\tilde{f}) = 3$ by Corollary 3.9. □

Proof of Proposition 3.10. Let f: $(C^n, 0) \to (C^n, 0)$ be a finite holomorphic mapping. Put $r_0(f) = \operatorname{rank}\,(df(0))$ and assume $m_0(f) \leqslant 9$. Then by Lemma 3.13 we have $r_0(f) \geqslant n-3$. Let us distinguish three cases.

Case 1. $r_0(f) \geqslant n-1$. According to Lemma 3.11 we may assume $n = 1$, hence $l_0(f) = m_0(f)$.

Case 2. $r_0(f) = n-2$. By Lemma 3.13 we get $m_0(f) \geqslant 4$. Moreover $l_0(f) = 2$ if $m_0(f) = 4$ and $l_0(f) = 3$ if $m_0(f) = 5$. Assume that $m_0(f) \geqslant 6$. According to Lemma 3.11 we may assume $n = 2$, hence $r_0(f) = 0$, i.e. $\operatorname{ord} f \geqslant 2$. From the inequalities $m_0(f) \geqslant (\operatorname{ord} f)^2$, $\operatorname{ord} f \geqslant 2$ we get $\operatorname{ord} f = 2$ or $\operatorname{ord} f = 3$, since $m_0(f) \leqslant 9$. We have $\operatorname{ord} f = 3$ only if $m_0(f) = 9$ and $\operatorname{ord} f_1 = \operatorname{ord} f_2 = 3$, hence $l_0(f) = 3$ by Corollary 3.8. Then we may assume $\operatorname{ord} f = 2$. From Lemma 3.12 we get $l_0(f) = \frac{1}{2}m_0(f)$ or $l_0(f)$ is an integer from the interval $[\frac{1}{2}m_0(f), m_0(f)-2]$.

Case 3. $r_0(f) = n-3$. Then by Lemma 3.13 we have $m_0(f) \geqslant 8$ with $l_0(f) = 2$ (if $m_0(f) = 8$) or $l_0(f) = 3$ (if $m_0(f) = 9$).

Summing up the results of the above reasoning we get Proposition 3.10. □

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