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Decomposition of topologies on lattices and hyperspaces

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Abstract

The notion of decomposable topology is introduced in a partially ordered set and, in particular, in the lattice $\mathcal{C}(X)$ of all closed subsets (ordered by reverse inclusion) of a topological space X, which is also called the hyperspace of X. This notion is closely related to the concepts, defined in the same framework, of lower, upper and strong upper topology.

We investigate decomposability and unique decomposability of the main hyperspace topologies, and of topologies which are defined on some quite natural lattices or semilattices.

Introduction

Let X be a topological space. The Vietoris topology V (and many others, such as the Hausdorff metric topology if X is metric) can be defined, on the collection $\mathcal{C}(X)$ of all closed subsets of X (the hyperspace), in terms of two distinct sets of subcollections of $\mathcal{C}(X)$. By considering them separately one constructs two new topologies V^+ and V^-: namely, V^+ (V^-) is the topology generated by all subcollections of $\mathcal{C}(X)$ of the form $G^+ = \{C \in \mathcal{C}(X) \mid C \subset G\}$ ($G^- = \{C \in \mathcal{C}(X) \mid C \cap G \neq \emptyset\}$), where G runs over the open subsets of X. All V^+-open (V^--open) subcollections of $\mathcal{C}(X)$ turn out to be stable under subsets (supersets); moreover, in the lattice of all topologies on $\mathcal{C}(X)$, we have $\mathsf{V}^+ \wedge \mathsf{V}^- = \mathbf{0}$ (the trivial topology) and $\mathsf{V}^+ \vee \mathsf{V}^- = \mathsf{V}$. Therefore we say that the Vietoris topology is decomposable.

This notion, in its general formulation, gives rise to several natural questions, that we have partially tackled in this paper. In particular, a number of problems, related to possible properties of unicity for the decomposition of the most important hyperspace topologies, stood as an open field of investigation for a few years, among specialists in hypertopologies, and are solved in the last section of this paper.

There are several reasons for studying decomposability of hypertopologies. For instance, in the very definition of the Hausdorff metric topology (as well as the Wijsman topology, the Kuratowski convergence, and others) some duality features show up, which makes it natural to regard most hyperspace topologies as suprema of an upper and a lower part (see e.g. [9]). It turns out that several hypertopologies have the lower Vietoris topology as their lower part, a fact which is often useful in investigating their properties. Moreover, the technique of decomposition proved to play a crucial role for results concerning comparisons and infima of hyperspace topologies (see [4, 5, 6]). Finally, it is worth noticing that one can build up new fruitful hypertopologies from old ones, by simply mixing together the upper and the lower part of different topologies (see e.g. [2, §2] and [3, Thms. 6.1 and 6.2]).

The concept of decomposability in hyperspaces stems from the natural partial-order structure of $\mathcal{C}(X)$, the ordering being reverse inclusion. Thus we have tried to investigate this property in the abstract setting, in order to find general results to be applied both to the hyperspace $\mathcal{C}(X)$ of a topological space X (or of a metric space (X, d)), and to lattices (or semilattices) of especially nice and well-known forms. The spirit of such an investigation consists also in treating by similar methods notions which are classically related to different branches of mathematics, thus emphasizing the role of topology in bringing them nearer.

The paper is organized in two parts, where the first one deals with topologies on lattices or partially ordered sets in general. We give the definitions and some basic facts about decomposition of topologies defined on a poset S, and we consider some necessary and sufficient conditions for a topology on S to be decomposable. We consider the special cases of semilattices, of linearly ordered sets and of lattices (in particular, those which are products of finitely many linearly ordered sets), and even of complete lattices. We introduce the concept of strong decomposition, whose importance is due to the fact that most hyperspace topologies turn out to be strongly decomposable. Finally, we address ourselves to uniqueness of decomposition in the abstract setting.

In the second part of the paper (§§9–14) we present, from the point of view of decomposability properties, some of the most important hypertopologies: the Vietoris topology, the Hausdorff metric topology, the proximal topology and the topology of the Kuratowski convergence. The last section is about uniqueness of decomposition in the realm of hyperspace topologies.

1. Decomposable topologies

Recall that a *preorder* on a set S is a reflexive and transitive relation on S. A *preordered set* is a pair $(S, \leq)$, where S is a set and $\leq$ is a preorder on S. A preorder $\leq$ on S which is also antisymmetric is called a *partial order*; in this case we say that $(S, \leq)$ is a *partially ordered set* (or *poset*, for short). We sometimes write S for $(S, \leq)$, when there is no danger of confusion.

1.1. Definition. Let $(S, \leq)$ be a preordered set and $U \subset S$. The *upper set* of U is

$$\uparrow U = \{x \in S \mid \exists u \in U\colon\ u \leq x\}.$$

A subset T of S is *upper* if there exists $U \subset S$ such that $\uparrow U = T$ (equivalently if $\uparrow T = T$). The *lower sets* and the operator $\downarrow$ are defined analogously.

One easily proves that a subset T of a preordered set S is upper iff its complement $S \setminus T$ is lower, and that for every family $\{T_i\}_{i \in I}$ of subsets of S one has $\uparrow \bigcup_{i \in I} T_i = \bigcup_{i \in I} \uparrow T_i$.

Therefore two topologies γ and λ are naturally defined on S. The open sets of γ are the upper sets and the closed sets are the lower ones, and conversely for λ. Moreover, the closure operators for γ and λ are $\downarrow$ and $\uparrow$ respectively.

We call γ the *upper set topology* and λ the *lower set topology*; a topology on S is *upper* (resp. *lower*) if it is coarser than γ (resp. λ).

Given a topology τ on a set S, letting $x \leq_\tau y$ if and only if $x \in \tau\text{-cl}(\{y\})$ defines a preorder on S, which is called the *specialization* of τ. It is easily seen that τ is always an upper topology on the preordered set $(S, \leq_\tau)$; moreover, τ coincides with the upper set topology if and only if the union of any collection of τ-closed sets is τ-closed.

A topology τ on $(S, \leq)$ such that $\leq_\tau = \leq$ is called "order-compatible" in [7]—see also Definition 8.2 later.

1.2. Definition. Let τ be a topology on $(S, \leq)$. We say that τ is *decomposable* if $\tau = \pi \vee \mu$, where π is upper and μ is lower. The pair (π, μ) is a *decomposition* of τ.

Recall that if τ', τ'' are topologies on any set E then $\{T' \cap T'' \mid T' \in \tau',\ T'' \in \tau''\}$ is a base for the topology $\tau' \vee \tau''$. In particular, a topology τ on $(S, \leq)$ is decomposable if and only if there exist an upper topology π and a lower topology μ such that

$$\forall x \in S\ \forall U\ \tau\text{-neighborhood of } x\ \exists V \in \pi\ \exists W \in \mu\colon \quad x \in V \cap W \subset U.$$

Of course, a given topology on a set S might be decomposable with respect to some preorder $\leq'$, and not decomposable with respect to another preorder $\leq''$. For instance, on $(S, =)$ every topology is decomposable; but when the preorder is the universal relation (i.e. the whole of $S \times S$), no topology is decomposable except $\mathbf{0} = \{\emptyset, S\}$. In general, given two preorders $\leq'$ and $\leq''$ on S, we say that $\leq''$ *refines* $\leq'$ if $\leq'' \supset \leq'$ as subsets of $S \times S$ (i.e. $x \leq' y$ implies $x \leq'' y$ for every $x, y \in S$). It is easily seen that $\leq'' \supset \leq'$ implies that every topology which is decomposable on $(S, \leq'')$ is also decomposable on $(S, \leq')$.

Later on, we shall characterize the order relations which make every topology decomposable (see Proposition 1.5).

As the trivial topology is always decomposable on $(S, \leq)$, it is natural to require also decomposabiliy of the discrete topology $\mathbf{1}$; the following proposition just gives necessary and sufficient conditions for $\mathbf{1}$ to be decomposable.

Since $\mathbf{1}$ is the upper set topology of the preordered set $(S, =)$ and $\leq \supset =$, it would be interesting to investigate the following problem: Given a preordered set $(S, \leq)$, find necessary and sufficient conditions for γ, or any other topology which is decomposable on $(S, \leq)$, to be decomposable also on $(S, \sqsubseteq)$, where $\sqsubseteq$ refines $\leq$. A particular instance of this problem is considered at the end of Section 5.

1.3. PROPOSITION. *Let $(S, \leq)$ be a preordered set. The following are equivalent*:

(1) $\mathbf{1}$ *is decomposable*;

(2) γ *is* T_0 (*or* λ *is* T_0);

(3) $\leq$ *is antisymmetric.*

PROOF. (1)⇒(3). We have $\mathbf{1} = \gamma \vee \lambda$ and then, given any $x \in S$, there exist $T = {\uparrow}T$ and $U = {\downarrow}U$ such that $T \cap U = \{x\}$. Suppose that $x \leq y$ and $y \leq x$, for a suitable $y \in S$; then $y \in T$ and $y \in U$, whence $y = x$.

(2)⇔(3). Every $s \in S$ has a smallest γ-neighborhood, that is, ${\uparrow}\{s\}$. Hence γ is T_0 if and only if, whenever x, y are distinct points of S, we have $y \notin {\uparrow}\{x\}$ (that is, $x \not\leq y$) or $x \notin {\uparrow}\{y\}$ (that is, $y \not\leq x$).

(3)⇒(1). Suppose $\leq$ is antisymmetric. Then for every $x \in S$ we have ${\uparrow}\{x\} \cap {\downarrow}\{x\} = \{x\}$, and it follows that $\gamma \vee \lambda = \mathbf{1}$. ∎

Since a topological space is T_0 if and only if, whenever $x \neq y$, we have either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$, the proof that (2)⇔(3) above can be carried out using ${\downarrow}\{s\}$ in place of ${\uparrow}\{s\}$. This slight modification makes the proof work for all (upper) topologies π such that the π-closure of a singleton $\{s\}$ is ${\downarrow}\{s\}$ (i.e. the "order-compatible" topologies of [7]), and not only for γ.

In view of the previous proposition, the symbol "$\leq$" will henceforth denote a partial order.

1.4. Remark. *Let τ be a topology on S. Then τ is decomposable if and only if*

$$(\tau \wedge \gamma) \vee (\tau \wedge \lambda) = \tau.$$

Proof. Sufficiency is obvious. So suppose $\tau = \pi \vee \mu$, where $\pi \leq \gamma$ and $\mu \leq \lambda$; we also have $\pi \leq \tau$ and $\mu \leq \tau$. Hence $\pi \leq \tau \wedge \gamma$ and $\mu \leq \tau \wedge \lambda$, so $\tau = \pi \vee \mu \leq (\tau \wedge \gamma) \vee (\tau \wedge \lambda) \leq \tau$, and equality follows. ■

The above remark allows us to characterize the posets where every topology is decomposable.

For the sake of brevity, let us call a poset $(S, \leq)$ *up-radial* (respectively, *down-radial*) if there exists $a \in S$ such that for every $x, y \in S$ with $x < y$ we have $x = a$ (resp. $y = a$).

1.5. Proposition. *For every nonempty poset $(S, \leq)$ the following are equivalent*:

(1) *every topology on S is decomposable*;

(2) *every subset of S is upper or lower*;

(3) $(S, \leq)$ *is up-radial or down-radial.*

Proof. (2)⇒(1). If τ is any topology on S, then every element of τ is in γ or in λ, and hence $\tau \leq (\tau \wedge \gamma) \vee (\tau \wedge \lambda)$.

(1)⇒(2). By contradiction, suppose there exists $A \subset S$ which is neither upper nor lower. Then the topology $\tau = \{\emptyset, A, S\}$ on S is such that $\tau \wedge \gamma = \mathbf{0}$ and $\tau \wedge \lambda = \mathbf{0}$, whence $(\tau \wedge \gamma) \vee (\tau \wedge \lambda) = \mathbf{0} < \tau$.

(3)⇒(2). If $a \in S$ is such that $\forall x, y \in S : \ (x < y \Rightarrow x = a)$, then every subset of S containing a is lower, and every subset not containing a is upper. A dual argument holds if $(S, \leq)$ is down-radial.

(2)⇒(3). By contradiction, suppose $(S, \leq)$ is neither up-radial nor down-radial. Since S is nonempty, there must be $a, b \in S$ with $a < b$. Let $u, v \in S$ with $u < v$ and $u \neq a$. If $b \neq v$ then the set $\{b, u\}$ is neither upper nor lower while, if $b = v$, we may find $x, y \in S$ with $x < y$ and $y \neq v$. As $u \neq a$, we have $x \neq u$ or $x \neq a$, and hence the set $\{x, v\}$ is neither upper nor lower. ■

The above proposition also leads to an alternative proof that the lattice of all topologies on a set S having at least three distinct elements a, b, c is not distributive. Indeed, taking into account Remark 1.4, it suffices to endow S with any order relation $\leq$ for which $a \leq b \leq c$, so that $(S, \leq)$ is neither up-radial nor down-radial.

We say that a poset $(S, \leq)$ is *order-connected* if, given any two elments $a, b \in S$, there exists a finite sequence $a = s_0, s_1, \ldots, s_n = b$ in S where s_i is comparable with s_{i-1} for every $i = 1, \ldots, n$ (this is equivalent to saying that the topology $\gamma \wedge \lambda$ is connected, as one can easily deduce from the following proposition).

On any poset $(S, \leq)$, the topology $\mathbf{0}$ clearly has a unique decomposition, and one may expect that the same holds for γ and λ; but we are going to see that this is not the case, unless the poset is order-connected.

1.6. Proposition. *The following are equivalent for a nontrivial poset $(S, \leq)$*:

(1) γ *is uniquely decomposable* (*or λ is uniquely decomposable*);

(2) $\gamma \wedge \lambda = \mathbf{0}$;
(3) $(S, \leq)$ *is order-connected.*

Proof. (1)⇔(2). Every decomposition of γ must be of the form (π, μ), where $\mathbf{0} \leq \mu \leq \gamma \wedge \lambda$; moreover, $\pi \leq \gamma$, and $\pi = \gamma$ when $\mu = \mathbf{0}$. Hence if $\gamma \wedge \lambda = \mathbf{0}$ then γ is uniquely decomposable; conversely, if $\mathbf{0} \neq \gamma \wedge \lambda$, then $(\gamma, \mathbf{0})$ and $(\gamma, \gamma \wedge \lambda)$ are two different decompositions of γ.

(2)⇔(3). Let $\tau = \gamma \wedge \lambda$. Observe that, given any $x \in S$, the set of all elements of S which are comparable with x is $\downarrow\{x\} \cup \uparrow\{x\}$. Thus it will suffice to prove that, for every $A \subset S$, the τ-closure $\overline{A}$ of A is $\bigcup_{k=0}^{\infty} A_k$, where $A_0 = A$ and $A_{k+1} = \downarrow A_k \cup \uparrow A_k$ for $k \in \mathbb{N}$. Indeed if $(S, \leq)$ is order-connected then clearly the unique τ-closed nonempty set is the whole of S; conversely, if $\tau = \mathbf{0}$ then for every $a, b \in S$ we have $b \in \overline{\{a\}}$ $(= S)$, i.e. $b \in \{a\}_k$ for some k, so that $(S, \leq)$ is order-connected.

Since $\overline{A}$ is γ-closed, for every $B \subset A$ we have $\downarrow B \subset \overline{A}$; similarly, as $\overline{A}$ is λ-closed, we have $\uparrow B \subset \overline{A}$. Hence $\downarrow B \cup \uparrow B \subset \overline{A}$, so that if $A_k \subset \overline{A}$ then $A_{k+1} \subset \overline{A}$ as well; but clearly $A_0 = A \subset \overline{A}$ and therefore $\bigcup_{k=0}^{\infty} A_k \subset \overline{A}$.

It remains to prove that $\bigcup_{k=0}^{\infty} A_k$ is τ-closed, i.e. it is both an upper set and a lower set. Let $z \in \bigcup_{k=0}^{\infty} A_k$. Then $z \in A_h$ for some $h \in \mathbb{N}$ and therefore $\uparrow\{z\} \subset \uparrow A_h \subset A_{h+1} \subset \bigcup_{k=0}^{\infty} A_k$, so that $\bigcup_{k=0}^{\infty} A_k$ is upper. To prove that $\bigcup_{k=0}^{\infty} A_k$ is lower we argue in a similar way. ∎

In the sequel, when we consider a poset, we will always assume that it is order-connected. Examples of order-connected posets are directed sets and semilattices; the latter ones will be considered in Section 3.

1.7. Definition. We say that a topology τ on a poset S is *upper modular* if it is upper and $(\tau \vee \lambda) \wedge \gamma \leq \tau$ (or equivalently, if $(\tau \vee \lambda) \wedge \gamma = \tau$). Dually, τ is *lower modular* if it is lower and $(\tau \vee \gamma) \wedge \lambda \leq \tau$.

1.8. Proposition. *Let π be an upper modular topology on S. If $\widetilde{\pi}$ is an upper topology for which there exist lower topologies μ', μ'' such that $\pi \vee \mu' = \widetilde{\pi} \vee \mu''$, then $\widetilde{\pi} \leq \pi$.*

Proof. Denote by τ the topology $\pi \vee \mu' = \widetilde{\pi} \vee \mu''$; since $\mu' \leq \lambda$ and $\widetilde{\pi} \leq \tau$, we have

$$\widetilde{\pi} = \widetilde{\pi} \wedge \gamma \leq \tau \wedge \gamma = (\pi \vee \mu') \wedge \gamma \leq (\pi \vee \lambda) \wedge \gamma \leq \pi$$

as claimed. ∎

1.9. Corollary. *If every upper topology is upper modular and every lower topology is lower modular, then every decomposable topology is uniquely decomposable.*

Proof. This follows immediately from the previous proposition. ∎

The above corollary applies in particular when the lattice of all topologies on S is modular. Since in general this is not the case (as is well known), one expects that not every upper topology is upper modular and not every lower topology is lower modular.

An example of a lower topology (defined on a lattice) which is not lower modular will be given in 10.3 later. By reversing the order one also obtains an upper topology which is not upper modular.

The set of all decomposable topologies is easily seen to be closed under suprema, but not under infima in general (not even finite infima; see 12.4). For upper modular topologies the converse holds.

1.10. Proposition. *Let $\mathcal{A}(S)$ be the set of all upper modular topologies on S. Then*:

(1) $\mathbf{0} \in \mathcal{A}(S)$ *and* $\gamma \in \mathcal{A}(S)$;

(2) *if S is a lattice and $U \subset S$ is an upper set, then the topology τ_U generated by $\{U\}$ belongs to $\mathcal{A}(S)$*;

(3) *$\mathcal{A}(S)$ is closed under infima, but in general it is not closed under suprema.*

Proof. (1) This is trivial.

(2) We have to show that every upper set T which is open in $\tau_U \vee \lambda$ belongs to τ_U; we may assume $T \neq \emptyset$ and $T \neq S$. The subset T is of the form $(L \cap U) \cup M$, with $L, M \in \lambda$. Let $x \notin T$. Since T is upper, for $t \leq x$ we have $t \notin T$. Suppose $M \neq \emptyset$ and let $m \in M$. If $t = m \wedge x$ we have $t \in M \subset T$, a contradiction. Hence $M = \emptyset$, and thus $T = L \cap U$. Now, if $L \neq S$, let $y \notin L$ and $z \in T$. If $s = y \vee z$, then $s \notin L$ but $s \in T$ because T is upper. This is impossible and therefore $L = S$. So we conclude that $T = U$.

(3) Let $\mathcal{P} = \{\pi_j\}_{j \in J}$ be a family of upper modular topologies on S and let $\pi = \inf \mathcal{P}$. Clearly, π is an upper topology. Moreover, $\pi \leq \pi_j$, and hence $(\pi \vee \lambda) \wedge \gamma \leq (\pi_j \vee \lambda) \wedge \gamma = \pi_j$, for every $j \in J$. It follows that $(\pi \vee \lambda) \wedge \gamma \leq \pi$.

To see that $\mathcal{A}(S)$ is not closed under suprema, consider a lattice S and any upper topology π on S such that $\pi \notin \mathcal{A}(S)$ (this topology surely exists: see the remarks following 1.9); then $\pi = \sup_{U \in \pi} \tau_U$ where, for each $U \in \pi$, we have $\tau_U \in \mathcal{A}(S)$ by (2) above. ∎

1.11. Definition. A topology τ on a poset S is *upper complete* if it is an upper topology such that, for every $U \in \tau \vee \lambda$, we have $\uparrow U \in \tau$. A *lower complete* topology is defined dually.

1.12. Remark. (1) *If π is an upper complete topology, then it is also upper modular.*

(2) *The collection of all upper complete topologies is closed under infima.*

Proof. (1) If $U \in \pi \vee \lambda$ and $\uparrow U = U$ then $U \in \pi$, and therefore $(\pi \vee \lambda) \wedge \gamma \leq \pi$.

(2) Let $\tau = \inf_{j \in J} \tau_j$, where τ_j is upper complete for each $j \in J$. Clearly, τ is upper. Now, if $U \in \tau \vee \lambda$ then, for every $j \in J$, we have $U \in \tau_j \vee \lambda$ and hence $\uparrow U \in \tau_j$; thus $\uparrow U \in \tau$. ∎

For a general bibliographical reference about topologies on lattices and semilattices the reader may consult [10], from which we have also taken Definition 1.1. Propositions 1.3 and 1.6 are taken from [17].

2. Locally convex topologies

2.1. Definition. A subset C of S is *order-convex* or, more briefly, *convex* if, given any $a, b \in C$ and any $c \in S$ with $a \leq c \leq b$, we have $c \in C$. For every $A \subset S$, the *convex hull* of A (i.e. the smallest convex set containing A) is $\mathrm{co}(A) = \uparrow A \cap \downarrow A$. We say that a topology τ on S is *locally convex* if the convex τ-open sets form a base for τ.

It is easily seen that the supremum of any collection of locally convex topologies is locally convex. On the other hand, let $S = [0,1]\times[0,1]$ with the product order, and define two topologies on S as follows:

τ_1: the topology generated by $\big\{\{(\frac{2}{3},0),(0,\frac{2}{3})\},\{(\frac{1}{3},1),(1,\frac{1}{3})\}\big\}$,

τ_2: the topology generated by $\big\{\{(\frac{2}{3},0),(\frac{1}{3},1)\},\{(1,\frac{1}{3}),(0,\frac{2}{3})\}\big\}$.

Then τ_1 and τ_2 are locally convex, while

$$\tau_1\wedge\tau_2 = \big\{\emptyset,\{(\tfrac{2}{3},0),(0,\tfrac{2}{3}),(\tfrac{1}{3},1),(1,\tfrac{1}{3})\},S\big\}$$

is not.

A less trivial example of two locally convex topologies whose infimum is not locally convex is the following.

2.2. Example. *Let $(S,\leq)$ denote $\mathbb{R}^2$ endowed with the product order (i.e. $(a,b)\leq(c,d)$ $\Leftrightarrow a\leq c$ and $b\leq d$), and let*

$$\beta_1 = \{\{(a,b)\}\mid a\neq 0 \text{ or } b\neq 0\}\cup\{\mathrm{co}\{(0,0),(r,r)\}\mid r>0\}, \tag{2.1}$$

$$\beta_2 = \{\{(a,b)\}\mid a\neq 0 \text{ or } b\neq 0\}\cup\{\mathrm{co}\{(0,0),(s,s)\}\mid s<0\}. \tag{2.2}$$

Then β_1 and β_1 are bases for two locally convex metrizable topologies on S, denoted by τ_1 and τ_2 respectively, such that $\tau=\tau_1\wedge\tau_2$ is not locally convex.

Proof. First, it is clear that τ_1 and τ_2 are locally convex metrizable strongly zero-dimensional topologies on S. Now consider the τ-open set

$$A = \mathrm{co}\{(0,0),\,(1,1)\}\cup\mathrm{co}\{(0,0),(-1,-1)\}$$

containing $(0,0)$. We will prove that no convex τ-neighborhood of $(0,0)$ can be contained in A. Indeed, let U be such a neighborhood. As U is both a τ_1-neighborhood and a τ_2-neighborhood of $(0,0)$, there exist $r>0$ and $s<0$ such that $\mathrm{co}\{(0,0),(r,r)\}\subset U$ and $\mathrm{co}\{(0,0),(s,s)\}\subset U$. Hence, if U were contained in A, we would have, in particular, $(r,r)\in A$ and $(s,s)\in A$; by convexity this implies that $(r,s)\in A$, which is impossible. ∎

2.3. Remark. *Every decomposable topology τ on S is locally convex.*

Proof. Indeed, if $\tau=\pi\vee\mu$, then $\{T\cap U\mid T\in\pi,\ U\in\mu\}$ is a base for τ consisting of convex sets. ∎

It is easy to define a locally convex topology which is not decomposable: for example, $\{\emptyset,\{b\},S\}$, where S is any poset containing three points a,b,c with $a<b<c$. Since this topology has no separation properties, one may ask if local convexity implies decomposability for T_2-topologies. The answer is negative, as we are going to see (but compare with 4.8 and 5.7).

2.4. Example. *Let $S=[0,1]\times[0,1]$ with the product order. Let $P'=(1/4,1/4)$, $V'(\eta)=\{(1,y)\mid 0<y<\eta\}$, $P''=(1/2,1/2)$, $V''(\eta)=\{(3/4,y)\mid 0<y<\eta\}$; consider the topology τ generated by all sets of the form $W'(\eta)=\{P'\}\cup V'(\eta)$ and $W''(\eta)=\{P''\}\cup V''(\eta)$, where $\eta<1/4$, together with the singletons of all points of S which are different from P' and P'' (see Fig. 1). Then τ is a locally convex metrizable topology which is not decomposable.*

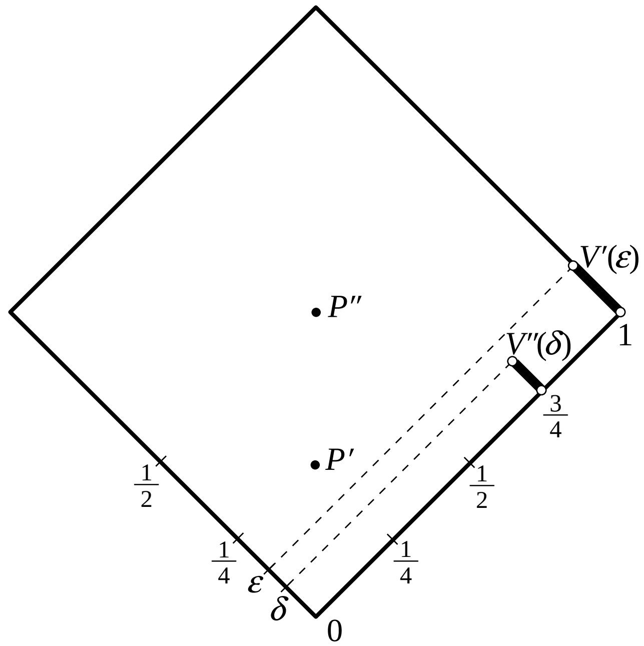

Fig. 1. A locally convex non-decomposable metrizable topology on a lattice

Proof. It is clear that τ is a locally convex metrizable topology. Suppose that τ is decomposable. Then, given any open neighborhood N of P'', there should exist an open upper set U and an open lower set L such that $P'' \in U \cap L \subset N$, and we may clearly assume that N has the form $W''(\eta)$ for some positive $\eta < 1/4$.

Since L is open and lower, it must contain P' together with $V'(\varepsilon)$ for some positive $\varepsilon < 1/4$; on the other hand, U will contain $V''(\delta)$ for a positive $\delta < \varepsilon$, hence it will contain also $V'(\delta)$. It follows that $V'(\delta) \subset U \cap L \subset N$, but this is impossible. ■

2.5. Proposition. *For a topology τ on a poset S the following are equivalent*:

(1) *for every $V \in \tau$, the upper set of V is τ-open*;

(2) *for every upper set $U \subset S$ the τ-interior of U is an upper set.*

Proof. (1)⇒(2). Let $U \subset S$ be an upper set, and denote by V the τ-interior of U. Since the upper set of V is τ-open and contained in $\uparrow U = U$, we have $\uparrow V \subset \tau\text{-int}(U) = V$.

(2)⇒(1). Let V be a τ-open subset of S and let $U = \uparrow V$. Since the τ-interior of U is an upper set and contains V, we have $\tau\text{-int}(U) \supset \uparrow V = U$ and therefore U is τ-open. ■

2.6. Definition. We say that a topology τ on a poset S is *upper-saturated* if it satisfies any of the two equivalent conditions above. A *lower-saturated* topology is defined dually. We call a topology *saturated* if it is both upper-saturated and lower-saturated.

A poset endowed with a saturated topology is called an "I-space" in [16].

2.7. Proposition. *If τ is a locally convex saturated topology*, *then τ is decomposable.*

Proof. Let $\mathcal{B}$ be a base for τ consisting of convex sets. Denote by π and μ the topologies generated by $\{\uparrow B \mid B \in \mathcal{B}\}$ and $\{\downarrow B \mid B \in \mathcal{B}\}$ respectively. Obviously, $\pi \leq \gamma$ and $\mu \leq \lambda$. Since every $B \in \mathcal{B}$ is convex we have $\pi \vee \mu \geq \tau$ and by saturation it follows that $\pi \leq \tau$ and $\mu \leq \tau$. Hence $\pi \vee \mu = \tau$. ∎

3. Semilattices. Strong decomposability

In this section, the poset S will be a semilattice. We will use the convention adopted also in [10] that "semilattice" always means "inf-semilattice".

The mapping $(a, b) \mapsto a \wedge b = \inf\{a, b\}$ will be referred to as the *binary operation* of the semilattice S. A topology on S which makes the binary operation jointly continuous is called a *semilattice topology*.

3.1. Proposition. *If a topology τ makes the binary operation of S separately continuous* (*in particular*, *if it is a semilattice topology*), *then it is upper-saturated.*

Proof. Let U be a τ-open set. By our assumption every set of the form $U(a) = \{x \in S \mid a \wedge x \in U\}$ is τ-open as well; now, one can easily check that $\bigcup_{a\in S} U(a) = \uparrow U$. ∎

The converse is not true, in general. Indeed, consider two incomparable elements $a, b \in S$ and let $U = \uparrow\{a, b\}$. The (upper) topology τ_U, generated by $\{U\}$, is clearly upper-saturated, but the binary operation is not separately continuous with respect to τ_U. Note that, if S is a lattice, τ_U is also an example of upper modular topology which is not upper complete (a less trivial example will be given in 6.9).

Observe further that the result in [19, Prop. 1.1.6] may be easily derived from the above proposition.

3.2. Definition. A topology on a semilattice is said to be *strong* if it admits a subbase (and hence a base) consisting of subsemilattices.

3.3. Proposition. *Let S be a semilattice. Then*:

(1) *every strong upper topology on S is a semilattice topology*;

(2) *γ and every lower topology are strong topologies*;

(3) *the set of all strong topologies is closed under suprema.*

Proof. (1) Let π be a strong upper topology and $a, b \in S$. We will show π-continuity of the binary operation $\wedge$, which we momentarily denote by $f : S \times S \to S$, at the point (a, b). Let U be a basic π-open set containing $a \wedge b$. As U is upper, both a and b belong to U, and $V = U \times U$ is a neighborhood of (a, b) in the product topology; since U is a subsemilattice of S, we have $f(V) \subset U$.

(2) The subsets $\uparrow\{x\}$, for every $x \in S$, are subsemilattices and form a base for γ; hence γ is strong. Also, since lower sets are subsemilattices, every lower topology is strong.

(3) Let $\mathcal{T} = \{\tau_j\}_{j\in J}$ be a family of strong topologies on S and, for every $j \in J$, let $\mathcal{B}_j$ be a base for τ_j consisting of subsemilattices. Then $\mathcal{B} = \bigcup_{j\in J} \mathcal{B}_j$ is a subbase for $\tau = \sup \mathcal{T}$ consisting of subsemilattices, and therefore τ is a strong topology. ∎

On the other hand, the set of strong topologies on a semilattice is not closed under infima in general, as we will see in 4.10 below.

3.4. Lemma. *Let K be a non-empty* (*convex*) *subsemilattice of S and let $T = \uparrow T$ and $U = \downarrow U$ be such that $T \cap U = K$. Then T is a subsemilattice of S if and only if $T = \uparrow K$.*

Proof. Sufficiency is clear. To prove necessity observe first that we only have to show $T \subset \uparrow K$, as the reverse inclusion always holds. So, let $x \in T$ and fix a $y \in K$. Since T is a subsemilattice, $x \wedge y \in T$ as well; on the other hand, we have $x \wedge y \in \downarrow K \subset \downarrow U = U$. It follows that $x \wedge y \in T \cap U = K$, and hence $x \in \uparrow K$. ∎

3.5. Proposition. *Every strong upper topology is upper complete*, *and hence upper modular.*

Proof. Let π be a strong upper topology, and $B \neq \emptyset$ be a basic open set of $\pi \vee \lambda$; we may assume that $B = T \cap U$, where T is a π-open subsemilattice of S and $U \in \lambda$. Hence B is a convex subsemilattice of S. Now, Lemma 3.4 implies that $\uparrow B = T$. Thus π is upper complete and, by 1.12(1), it is also upper modular. ∎

The converse is not true, as we will show in 9.6 later. On the other hand, we will see in Proposition 6.2 and Example 6.3 that there are particular lattices, such as $[0,1] \times [0,1]$, in which all the upper complete topologies are strong.

3.6. Definition. We say that a decomposition (π, μ) of a topology τ is *strong* if π and μ are strong (by 3.2(2) it suffices that π be strong). A topology admitting a strong decomposition is called *strongly decomposable*.

Observe that the strong upper topologies, the lower topologies and the discrete topology are all strongly decomposable. Moreover, 3.3(3) implies that every strongly decomposable topology is a strong topology.

Now we are going to characterize strongly decomposable topologies. Let U be a convex subset of S; denote by $\check{U}$ the set $(S \setminus \uparrow U) \cup U$. Since U is convex, $\downarrow U \setminus U \subset S \setminus \uparrow U$; so we have $\check{U} = (S \setminus \uparrow U) \cup \downarrow U$ and therefore $\check{U}$ is a lower set.

3.7. Theorem. *A topology τ on S is strongly decomposable if and only if the non-empty convex sets $U \subset S$ such that*

(1) *U is a τ-open subsemilattice of S*,
(2) *$\uparrow U$ is τ-open*, *and*
(3) *there exists a τ-open $\widetilde{U} = \downarrow \widetilde{U}$ such that $U \subset \widetilde{U} \subset \check{U}$*

form a base for τ.

Proof. Suppose $\tau = \pi \vee \mu$, where π is a strong upper topology and $\mu \leq \lambda$; denote by $\mathcal{B}$ the collection of all π-open subsemilattices of S (which is a base for π), and let

$$\mathcal{U} = \{P \cap M \mid P \in \mathcal{B},\ M \in \mu\}.$$

Then $\mathcal{U}$ is a base for τ which consists of convex sets. Given any $U = P \cap M \in \mathcal{U}$, where $P \in \mathcal{B}$ and $M \in \mu$, we immediately see that U satisfies (1). Also, Lemma 3.4 implies that $\uparrow U = P$, and thus we get (2). Finally, let $\widetilde{U} = M$. Then $\widetilde{U} = \downarrow \widetilde{U}$ and $U \subset \widetilde{U}$. Now we have

$$\widetilde{U} \setminus U = M \setminus (P \cap M) = M \setminus P \subset S \setminus P = S \setminus {\uparrow} U;$$

therefore $\widetilde{U} \subset (S \setminus {\uparrow} U) \cup U = \check{U}$, and U satisfies (3).

Conversely, suppose that there exists a base $\mathcal{U}$ for τ such that every $U \in \mathcal{U}$ is a convex set and satisfies (1)–(3). Denote by π and μ the topologies generated by $\{{\uparrow} U \mid U \in \mathcal{U}\}$ and $\{\widetilde{U} \mid U \in \mathcal{U}\}$ respectively. Then π is a strong upper topology and μ is a lower topology. Moreover, $\pi \vee \mu \leq \tau$, and it remains to prove the reverse inequality; to do this, we take a τ-open set U and show that $U \in \pi \vee \mu$. We may assume that $U \in \mathcal{U}$. Then, since U is convex,

$$U = {\uparrow} U \cap {\downarrow} U \subset {\uparrow} U \cap {\downarrow} \widetilde{U} = {\uparrow} U \cap \widetilde{U} \subset {\uparrow} U \cap \check{U} = {\uparrow} U \cap ((S \setminus {\uparrow} U) \cup U) = U;$$

therefore $U = {\uparrow} U \cap \widetilde{U} \in \pi \vee \mu$. ■

3.8. Corollary. *A strong decomposable topology on S is strongly decomposable if and only if it is upper-saturated.*

Proof. Let τ be a strongly decomposable topology. By the above theorem, there exists a base $\mathcal{U}$ for τ such that ${\uparrow} U \in \tau$ for every $U \in \mathcal{U}$. It follows that τ is upper-saturated, because the operator $\uparrow$ commutes with set union.

Conversely, let τ be a strong decomposable upper-saturated topology. Since τ is decomposable, $((\tau \wedge \gamma), (\tau \wedge \lambda))$ is a decomposition of τ and, by upper saturation, a base $\mathcal{B}$ for $\tau \wedge \gamma$ is obtained by taking all sets of the form ${\uparrow} U$, where U runs over some base $\mathcal{U}$ of τ. As τ is strong, we may choose $\mathcal{U}$ to be the collection of all τ-open subsemilattices; hence $\mathcal{B}$ consists of semilattices as well, and therefore $\tau \wedge \gamma$ is strong. ■

One can anyway ask: Could every strong decomposable topology be strongly decomposable? Even after the previous corollary we cannot exclude this so far. But the answer is negative, as we will see in 4.9 later on.

We conclude this section giving a uniqueness property for strong decompositions.

3.9. Proposition. *If (π', μ') and (π'', μ'') are strong decompositions of the same topology τ, then $\pi' = \pi'' = \tau \wedge \gamma$.*

Proof. This follows from 3.5 and 1.8. ■

4. Convex topologies

We have seen that saturation is a sufficient condition for a locally convex topology to be decomposable. This condition can be weakened in some sense, as we are going to see in this section.

4.1. Proposition. *For a topology τ on a poset S the following are equivalent*:

(1) *for every $U \in \tau$, the convex hull of U is τ-open*;

(2) *for every convex $C \subset S$ the τ-interior of C is convex.*

Proof. (1)⇒(2). Let C be a convex subset of S and denote by U the τ-interior of C. Since the convex hull of U is τ-open and contained in C, we have $\mathrm{co}(U) \subset \tau\text{-int}(C) = U$ and therefore U is convex.

(2)⇒(1). Let U be a τ-open subset of S and denote by C the convex hull of U. Since the τ-interior of C is convex and contains U, we have $\tau\text{-int}(C) \supset \mathrm{co}(U) = C$ and therefore C is τ-open. ∎

4.2. Definition. Let τ be a topology on a poset S. We say that τ is *convex* if it satisfies any of the two equivalent conditions above.

Observe that a convex topology need not be locally convex (see Corollary 12.5).

We say that a topology τ on a poset S is *weakly locally convex* if each point of S has a τ-neighborhood base consisting of convex (not necessarily open) sets. Clearly, a convex topology is locally convex if and only if it is weakly locally convex.

An example of a weakly locally convex topology which is not locally convex may be found in [19, Ex. 1.1.11(e)]; we also give a metrizable example on the lattice $\mathbb{R}^2$, in 6.11 below.

Of course, a saturated topology is convex, so we can put "weakly locally convex" instead of "locally convex" in the assumptions of Proposition 2.7.

On the other hand, a convex topology need not be saturated, in general.

4.3. Example. *Let* $S = [0,1] \times [0,1]$ *with the product order. Let* τ *be the topology on* S *generated by all sets of the form* $[0,1/n[\times [0,1/n[$ *and* $]1-1/n,1] \times]1-1/n,1]$, *where* n *is a positive integer*, *together with the singletons of all points different from* $(0,0)$ *and* $(1,1)$. *Then* τ *is a convex topology which is neither upper-saturated nor lower-saturated.*

Proof. Let A be any open subset of S and let $C = \mathrm{co}(A)$. We show that C is open. Indeed, given $z \in C$, if z is different from $(0,0)$ and $(1,1)$ then $\{z\}$ is a τ-neighborhood of z contained in C; if z is $(0,0)$ or $(1,1)$ then necessarily z was already in A, hence A is such a neighborhood.

To complete the proof it suffices to observe that $\{(0,1)\}$ is τ-open while $\uparrow\{(0,1)\} = [0,1] \times \{1\}$ and $\downarrow\{(0,1)\} = \{0\} \times [0,1]$ are not. ∎

Using the formulation 4.1(1) it is easily seen that the collection of all convex topologies on a poset S is closed under infima. On the other hand, even the supremum of two convex topologies need not be convex. Indeed let $a,b,c,d \in S$ and suppose that $a < b < c$, while either $a \not\leq d$ or $d \not\leq c$. The topologies $\tau' = \{\emptyset, \{a\}, S\}$ and $\tau'' = \{\emptyset, \{c\}, S\}$ are convex, but $\tau' \vee \tau'' = \{\emptyset, \{a\}, \{c\}, \{a,c\}, S\}$ is not. For a less trivial example, see 6.8.

We are going to give a decomposability criterion which improves 2.7. In order to show a somewhat general statement, we require some new terminology.

Recall that a subset T of a poset S is *cofinal* if for every $s \in S$ there exists $t \in T$ with $s \leq t$; equivalently, $\downarrow T = S$.

4.4. Definition. A topology τ on a poset S is called *upper-regular* if there exists a filter base $\mathcal{M}$, consisting of cofinal τ-open upper sets, whose cluster points have a local base of τ-open upper sets. Such a filter base will be called *τ-fundamental*.

The next two propositions show that under suitable assumptions on S there are simple sufficient conditions for a topology on S to be upper-regular.

4.5. Proposition. *Let T be a cofinal subset of the poset S and let τ be a topology on S. Suppose that τ satisfies the following conditions*:

(1) *every point of T has a local base of τ-open upper sets*;
(2) *for each $c \notin T$ there exist $U, V \in \tau$ such that $T \subset U$, $c \in V$ and $U \cap V = \emptyset$.*

Then τ is upper-regular. Moreover, *if each $t \in T$ is maximal in S and τ is* T_1, *the converse also holds.*

Proof. Suppose that (1) and (2) hold. Let $\mathcal{M}$ be the filter base of all τ-open upper sets containing T. Every $M \in \mathcal{M}$ is cofinal and (2) implies that the set of τ-cluster points of $\mathcal{M}$ is T. Hence $\mathcal{M}$ is τ-fundamental by (1).

Conversely, suppose that τ is upper-regular and let $\mathcal{M}$ be a τ-fundamental filter base. We claim that $T \subset M$ for each $M \in \mathcal{M}$ (so that (1) holds). Indeed, let $t \in T$. As M is cofinal, there exists $m \in M$ such that $t \leq m$; since t is maximal, we have $t = m$.

Now let $c \notin T$. Since τ is T_1, c cannot have a local base of τ-open upper sets and therefore it is not a cluster point of $\mathcal{M}$. Hence, for a suitable $U \in \mathcal{M}$, there exists a τ-open neighborhood V of c which is disjoint from U, and (2) follows. ∎

4.6. Proposition. *If the poset S has a maximum*, *then every locally convex* T_2-*topology τ on S is upper-regular.*

Proof. Let 1 denote the greatest element of S and take $\mathcal{M}$ to be the filter base of all convex τ-open neighborhoods of 1. Every $M \in \mathcal{M}$ is cofinal and upper and $\mathcal{M}$ is a τ-base at 1, by local convexity; moreover, as τ is T_2, the only τ-cluster point of $\mathcal{M}$ is 1. ∎

4.7. Theorem. *Let S be a poset with minimum. A convex upper-regular* T_2-*topology τ on S is decomposable if and only if it is* (*weakly*) *locally convex.*

Proof. Necessity is obvious, so we prove sufficiency. Given any $x \in S$ and any basic τ-open neighborhood A of x (which we may assume to be convex) we have to show that

$$\exists U, L \in \tau\colon \quad U = \uparrow U, \quad L = \downarrow L, \quad x \in U \cap L \subset A. \tag{4.1}$$

Denote by 0 the least element of S and by $\mathcal{M}$ a τ-fundamental filter base. Since A is convex, if $x = 0$ we have $\downarrow A = A$ so we may take $U = S$ and $L = A$. If x is a τ-cluster point for $\mathcal{M}$, we may take a τ-open upper set $U \subset A$ and let $L = S$.

Now suppose that $x \neq 0$ and x is not a τ-cluster point for $\mathcal{M}$. Since S is not a singleton, it also follows that 0 is not a τ-cluster point for $\mathcal{M}$, otherwise τ cannot be T_2 (not even T_1). Hence there are $M', M'' \in \mathcal{M}$ such that for suitable τ-open sets N', N'', where $0 \in N'$ and $x \in N''$, we have $M' \cap N' = \emptyset$ and $M'' \cap N'' = \emptyset$. As τ is a T_2-topology, we may as well assume that $N' \cap N'' = \emptyset$.

Let V be a convex τ-open neighborhood of 0 contained in N', let $G = N'' \cap A$, and let W be an element of $\mathcal{M}$ contained in $M' \cap M''$. The τ-open sets V, G, W are pairwise disjoint, and moreover, V is lower and W is upper.

Take $U = \mathrm{co}(G \cup W)$ and $L = \mathrm{co}(V \cup G)$. As τ is convex we have $U, L \in \tau$. Moreover, $G \cup W$ is cofinal (because W is); this means that $\downarrow(G \cup W) = S$ and therefore $U =$

$\uparrow(G\cup W) = \uparrow G \cup W$. Similarly, $\uparrow(V \cup G) = S$, thus $L = \downarrow(V \cup G) = V \cup \downarrow G$. To complete the proof we show that $U \cap L \subset A$, so that (4.1) follows.

First, observe that $W \cap \downarrow G = \emptyset$. Indeed, if some $w \in W$ belonged to $\downarrow G$, there would be $y \in G$ with $w \leq y$ and hence $y \in W$, which is impossible because W and G are disjoint. Next, the fact that $\uparrow G \cap V = \emptyset$ is deduced from $G \cap V = \emptyset$ in a similar way. Finally, we have

$$\begin{aligned} U \cap L &= (\uparrow G \cup W) \cap (V \cup \downarrow G) \\ &= (\uparrow G \cap V) \cup (W \cap V) \cup (\uparrow G \cap \downarrow G) \cup (W \cap \downarrow G) \\ &= \uparrow G \cap \downarrow G = \mathrm{co}(G), \end{aligned}$$

and, since A is convex, also $\mathrm{co}(G) \subset A$, so that $U \cap L \subset A$ as claimed. ■

4.8. Corollary. *Let S be a poset with maximum and minimum. A convex* T_2*-topology τ on S is decomposable if and only if it is* (*weakly*) *locally convex.*

Proof. This follows immediately from Proposition 4.6 and Theorem 4.7. ■

4.9. Corollary. *A strong decomposable topology need not be strongly decomposable.*

Proof. Indeed, the topology τ on the lattice S of Example 4.3 is strong; moreover, it is also T_2 and locally convex. Therefore, by the previous corollary, it is decomposable. On the other hand, since τ is not upper-saturated, it is not strongly decomposable by Corollary 3.8. ■

4.10. Corollary. *The infimum of two strong topologies need not be strong.*

Proof. Consider the same topology τ of the previous corollary; then τ and γ are both strong, but $\tau \wedge \gamma$ is not (otherwise τ would be strongly decomposable by 1.4). ■

5. Topologies on linearly ordered sets

Throughout this section, we denote by L a linearly ordered set. We also use the symbol L^* to indicate the linearly ordered set obtained from L by reversing the order.

5.1. Remark. *Every topology on L is strong.*

Proof. Indeed, every subset of a linearly ordered set is a subsemilattice. ■

Therefore, in this context, characterization of decomposability becomes much easier.

5.2. Proposition. *A topology τ on L is decomposable if and only if it is* (*weakly*) *locally convex and saturated.*

Proof. By Proposition 2.7 the condition is sufficient. Conversely, let τ be a decomposable topology. First, by 2.3, τ is locally convex. Now 5.1 implies that τ is strongly decomposable on L, so 3.8 gives upper saturation. Finally, we get lower saturation applying the same argument with L^* in place of L. ■

Also, when we are concerned with uniqueness of decomposition, things go very much better than in the general case.

5.3. Proposition. *Every decomposable topology on L is uniquely decomposable.*

Proof. This follows from 5.1 and 3.9, considering the two semilattices L and L^*. ∎

We are going to show that, for a locally convex topology on L to be decomposable, it suffices to require a little separation (e.g. the T_1 axiom).

We begin illustrating a peculiar behaviour of convexity: when the order is linear there is some analogy with connectedness.

5.4. Proposition. *Let $\{C_j\}_{j\in J}$ be a family of convex subsets of L such that $C_i \cap C_j \neq \emptyset$ for every $i, j \in J$. Then $\bigcup_{j\in J} C_j$ is convex.*

Proof. Suppose not. Then there are $a, b \in \bigcup_{j\in J} C_j$ and $c \notin \bigcup_{j\in J} C_j$ with $a \le c \le b$. Let $i, j \in J$ be such that $a \in C_i$ and $b \in C_j$. If $x \in C_i \cap C_j$, we have either $c \le x$ or $x \le c$, but this implies $c \in C_i$ or, respectively, $c \in C_j$. So we get a contradiction. ∎

Therefore, given any $A \subset L$ and any $a \in A$ we can construct the convex component of a relative to A as the union of all convex subsets of L containing a and contained in A. We denote it by $C_A(a)$.

Now let τ be a topology on L and suppose that A is τ-closed. What can be said about $C_A(a)$? The next result answers this question.

Recall that a topology τ is *symmetric* if every τ-open set is the union of a collection of τ-closed sets.

5.5. Proposition. *If τ is a weakly locally convex symmetric topology on L, then the τ-closure of any convex set is convex.*

Proof. Let K be a convex subset of L and let $x < y < z$, where $x, z \in \overline{K}$ and $y \in L$. Suppose $y \notin \overline{K}$. Since τ is symmetric, $\overline{\{y\}} \cap \overline{K} = \emptyset$, whence $x \notin \overline{\{y\}}$ and $z \notin \overline{\{y\}}$. By weak local convexity there exist convex (not necessarily open) τ-neighborhoods U of x and W of z such that $y \notin U$ and $y \notin W$. On the other hand, both U and W meet K, so that $a \in U$ and $b \in W$ for suitable $a, b \in K$. As U and W are convex, we cannot have $a \ge y$ nor $b \le y$; thus $a < y < b$, which gives a contradiction. ∎

Similarly one shows the following.

5.6. Proposition. *If τ is a weakly locally convex symmetric topology on L, then τ is convex* (*and hence locally convex*).

Proof. Let K be a convex subset of L and let $x < y < z$, where $x, z \in \tau\text{-int}(K)$ and $y \in L$. Suppose $y \notin \tau\text{-int}(K)$. Since τ is symmetric, we have both $y \notin \overline{\{x\}}$ and $y \notin \overline{\{z\}}$. By weak local convexity there exists a convex (not necessarily open) τ-neighborhood V of y such that $x \notin V$ and $z \notin V$. Now let $v \in V$. As V is convex, we cannot have $v \le x$ nor $v \ge z$; thus $x < v < z$, whence $v \in K$. It follows that $V \subset K$, a contradiction. ∎

We are now ready to state the following result.

5.7. Theorem. *Let τ be a* (*weakly*) *locally convex topology on L. If τ is symmetric* (*in particular*, *if τ is* T_1) *then τ is decomposable.*

Proof. In view of 5.2, we only have to prove saturation; also, we will consider only upper saturation because the argument for lower saturation is perfectly dual. Moreover, by local convexity of τ, we may restrict our attention to convex sets. So let K be a convex τ-open set; we show that $\uparrow K \in \tau$. Since we may assume that $\uparrow K \neq K$ and $\uparrow K \neq L$, it follows that the complement A of K is a non-convex set. Let $a \notin \uparrow K$. Then $a \in A$ and we have $C_A(a) \supset L \setminus \uparrow K$. In fact they coincide because, given any $x \in \uparrow K$ and any $y \in K$ such that $y \leq x$, we have $a < y$ so that $x \in C_A(a)$ would imply $y \in C_A(a)$ as well, which is impossible. Now, as a consequence of Proposition 5.5, $C_A(a)$ is τ-closed. Hence $\uparrow K$ is τ-open, as claimed. ∎

In particular, the order topology τ_0 on L is decomposable; when referring to its (unique) decomposition (π_0, μ_0), we will call π_0 the *upper order topology*, and μ_0 the *lower order topology*. It is easy to see that both π_0 and μ_0 are T_0.

In the final part of this section, we address ourselves to the following question. Suppose we are given a partially ordered set $(S, \leq)$ and we construct a linear order on S, denoted by $\sqsubseteq$, which refines the given partial order; is it possible to do this in such a way that a fixed topology τ, which is decomposable on $(S, \leq)$, turn out to be decomposable also on $(S, \sqsubseteq)$?

We are going to see that the answer is negative, in general.

5.8. Lemma. *Let $(L, \leq)$ be a linearly ordered set with a locally convex T_1-topology τ, and let x be a point which is neither the greatest nor the smallest element of L. Then the subspace $L \setminus \{x\}$ is disconnected.*

Proof. We show that in fact both $\uparrow\{x\} \setminus \{x\}$ and $\downarrow\{x\} \setminus \{x\}$ are τ-open in L. Indeed, denote by A the set $\uparrow\{x\} \setminus \{x\}$ (resp. $\downarrow\{x\} \setminus \{x\}$). Given any $y \in A$ and any convex neighborhood V of y, we have $V \subset A$ and therefore A is open. ∎

5.9. Example. *Let $(S, \leq)$ denote $\mathbb{R}^2$ endowed with the product order and τ be the Euclidean topology. Then τ is decomposable on $(S, \leq)$ but not on $(S, \sqsubseteq)$, where $\sqsubseteq$ is any linear order (which may or may not refine $\leq$).*

Proof. It is easily seen that τ is decomposable on $(S, \leq)$ (see also Example 6.8). On the other hand, if τ were decomposable on $(S, \sqsubseteq)$, by the above lemma all the complements of singletons of S would be disconnected, and this is impossible. ∎

6. Topologies on lattices

We are going to look at the particular case in which the partially ordered set under consideration is a lattice. We are especially interested in products of two (or finitely many) linearly ordered sets.

We will show that there are particular lattices on which every upper complete topology is strong, namely all lattices S in which

$$\forall a, b \in S: \quad \uparrow\{a \wedge b\} \setminus \downarrow\{a \vee b\} \subset \uparrow\{a\} \cup \uparrow\{b\}. \tag{6.1}$$

6.1. Lemma. *Let S be a lattice satisfying (6.1). Given $a_1, a_2 \in S$, let M_1 be a subset of $\downarrow\{a_1\} \setminus \downarrow\{a_2\}$ and M_2 a subset of $\downarrow\{a_2\} \setminus \downarrow\{a_1\}$. Then $\uparrow M_1 \cap \uparrow M_2$ is a subsemilattice of S.*

Proof. Let $x, y \in \uparrow M_1 \cap \uparrow M_2$. There exist $b_1, c_1 \in M_1$ where $b_1 \le x$ and $c_1 \le y$, as well as $b_2, c_2 \in M_2$ such that $b_2 \le x$ and $c_2 \le y$. Thus $b_1 \vee b_2 \le x$ and $c_1 \vee c_2 \le y$, whence $(b_1 \vee b_2) \wedge (c_1 \vee c_2) \le x \wedge y$. So it suffices to show that $(b_1 \vee b_2) \wedge (c_1 \vee c_2) \in \uparrow M_1 \cap \uparrow M_2$.

Now suppose that $(b_1 \vee b_2) \wedge (c_1 \vee c_2) \le b_1 \vee c_1$, hence, *a fortiori*, $b_2 \wedge c_2 \le b_1 \vee c_1 \le a_1$. On the other hand, $a_1 \not\le b_2 \vee c_2$ (otherwise we would get $b_1 \le a_1 \le b_2 \vee c_2 \le a_2$, a contradiction). Therefore $a_1 \in \uparrow\{b_2 \wedge c_2\} \setminus \downarrow\{b_2 \vee c_2\}$ so that $a_1 \in \uparrow\{b_2\} \cup \uparrow\{c_2\}$, by (6.1). It follows that either $b_2 \le a_1$ or $c_2 \le a_1$, but this is impossible.

Thus we have shown that $(b_1 \vee b_2) \wedge (c_1 \vee c_2) \not\le b_1 \vee c_1$; as we clearly have $(b_1 \vee b_2) \wedge (c_1 \vee c_2) \ge b_1 \wedge c_1$ it follows from (6.1) that $(b_1 \vee b_2) \wedge (c_1 \vee c_2)$ is an element of $\uparrow\{b_1\} \cup \uparrow\{c_1\}$, hence of $\uparrow M_1$.

The fact that $(b_1 \vee b_2) \wedge (c_1 \vee c_2) \in \uparrow M_2$ is proved in a similar way. ∎

6.2. Proposition. *Let S be a lattice with the property* (6.1). *If π is an upper complete topology on S then π is strong.*

Proof. Given $x \in S$ and a π-open neighborhood A of x, we have to find a π-open semilattice M such that $x \in M \subset A$. Let $A' = \uparrow(A \cap \downarrow\{x\})$; since π is upper complete, A' is open and contained in A. We may assume that A' is not a semilattice; hence there exist $a_1, a_2 \in A'$ such that $a_1 \wedge a_2$ does not belong to A'.

Let $M_1 = \downarrow\{a_1\} \cap A'$ and $M_2 = \downarrow\{a_2\} \cap A'$. It is easily seen that $x \in \uparrow M_1 \cap \uparrow M_2 \subset \uparrow A' = A'$, where $\uparrow M_1 \cap \uparrow M_2$ is π-open because π is upper complete. Moreover, $M_1 \cap \downarrow\{a_2\} = \emptyset$; indeed if $y \in M_1 \cap \downarrow\{a_2\}$, then $y \le a_1$, $y \le a_2$ and $y \in A'$, so that $a_1 \wedge a_2 \in \uparrow A' = A'$, a contradiction. In a similar way one shows that $M_2 \cap \downarrow\{a_1\} = \emptyset$. Thus the previous lemma applies, which ensures that $\uparrow M_1 \cap \uparrow M_2$ is a semilattice. Therefore we may choose M to be $\uparrow M_1 \cap \uparrow M_2$. ∎

We are going to see that there exists a wide class of lattices satisfying (6.1).

6.3. Example. *Given two linearly ordered sets L' and L'', consider the product $L' \times L''$ endowed with the product ordering. Every sublattice S of $L' \times L''$ satisfies* (6.1).

Proof. Fix $a = (a', a'')$ and $b = (b', b'')$ in S where, without loss of generality, we suppose that $a' \le b'$. Consider any $x = (x', x'') \in \uparrow\{a \wedge b\} \setminus \downarrow\{a \vee b\}$. We show that $x \in \uparrow\{a\} \cup \uparrow\{b\}$. Since this is trivial if $a \le b$, we may assume that $a'' > b''$.

Now, as $x \in \uparrow\{a \wedge b\}$, we have $x' \ge a' \wedge b' = a'$ and $x'' \ge a'' \wedge b'' = b''$. Moreover, $x \notin \downarrow\{a \vee b\}$, so that either $x' > a' \vee b' = b'$ or $x'' > a'' \vee b'' = a''$. In the first case $x > b$; in the second case $x > a$. Therefore $x \in \uparrow\{a\} \cup \uparrow\{b\}$. ∎

We already know that the infimum of two strong topologies need not be strong (see 4.10); in fact, this is true even for two strong upper topologies.

6.4. Example. *Let $C(\mathbb{R})$ be the collection of all continuous functions of $\mathbb{R}$ into itself, with the usual lattice structure. For every $x, r \in \mathbb{R}$, let $F(x, r) = \{f \in C(\mathbb{R}) \mid f(x) > r\}$ and denote by π_1, π_2 the topologies generated by*

$$\{F(x,r) \mid r \in \mathbb{R},\ x \in \mathbb{Q}\}, \quad \{F(x,r) \mid r \in \mathbb{R},\ x \in \mathbb{P}\}$$

respectively, where $\mathbb{Q}$ is the set of rationals and $\mathbb{P}$ is the set of irrationals. Then π_1 and π_2 are strong upper T_0-topologies while $\pi_1 \wedge \pi_2$ is not strong.

Proof. It is apparent that π_1 and π_2 are strong upper topologies. To see that they are T_0 consider $f \neq g$ and let $x \in \mathbb{R}$ be such that $f(x)$ and $g(x)$ are different, say $f(x) < g(x)$; by continuity, there exist $x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{P}$ such that $f(x_1) < g(x_1)$ and $f(x_2) < g(x_2)$. Then $F(x_1, f(x_1))$ and $F(x_2, f(x_2))$ are, respectively, a π_1-neighborhood and a π_2-neighborhood of g not containing f.

It remains to show that $\pi_1 \wedge \pi_2$ is not strong. Let $A = \{f \in C(\mathbb{R}) \mid \exists x \in \mathbb{R}\colon f(x) > 0\}$. The set A is both π_1-open and π_2-open, for if $f(x) > 0$ then there exist $x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{P}$ such that $f(x_1) > 0$ and $f(x_2) > 0$. The function u with constant value 1 clearly belongs to A; we claim that no $(\pi_1 \wedge \pi_2)$-open neighborhood of u contained in A is a semilattice. Indeed, let V be such a neighborhood. Since V is π_1-open, there exist $x_1, \ldots, x_n \in \mathbb{Q}$ and $r_1, \ldots, r_n < 1$ such that $F(x_1, r_1) \cap \ldots \cap F(x_n, r_n) \subset V$. Similarly, since V is π_2-open, there exist $y_1, \ldots, y_n \in \mathbb{P}$ and $s_1, \ldots, s_n < 1$ such that $F(y_1, s_1) \cap \ldots \cap F(y_n, s_n) \subset V$. As $C_1 = \{x_1, \ldots, x_n\}$ and $C_2 = \{y_1, \ldots, y_n\}$ are disjoint closed subsets of $\mathbb{R}$, there are two disjoint open sets $H_1 \supset C_1$ and $H_2 \supset C_2$ and, by Urysohn's lemma, there exist two continuous fuctions $g_1, g_2 : \mathbb{R} \to [0,1]$ such that $g_1(C_1) = \{1\}$, $g_2(C_2) = \{1\}$, $g_1(x) = 0$ for $x \notin H_1$ and $g_2(x) = 0$ for $x \notin H_2$. Hence $g_1 \in F(x_1, r_1) \cap \ldots \cap F(x_n, r_n)$ and $g_2 \in F(y_1, s_1) \cap \ldots \cap F(y_n, s_n)$, so that both g_1 and g_2 are in V. On the other hand, $g_1 \wedge g_2$ is the function with constant value 0 which does not belong to V (not even to A). Therefore V is not a semilattice. ∎

On the other hand, if the lattice S satisfies condition (6.1), then Proposition 6.2 and Remark 1.12(2) imply that the infimum of any collection of strong upper topologies on S is a strong topology. In particular, this is true if S is the product of two linearly ordered sets.

Now, let us suppose that the lattice S is the product of finitely many linearly ordered sets $L_1, \ldots, L_n$, and consider a collection Θ of strong upper T_0-topologies on S. It is easily seen that $\pi = \inf \Theta$ is always T_0, because it is finer than the product topology of $L_1 \times \ldots \times L_n$, where each factor is given the upper order topology. We are going to give suitable sufficient conditions for π to be strong (in the case $n > 2$).

6.5. Lemma. *Let S be the product of a finite collection $\{L_1, \ldots, L_n\}$ of linearly ordered sets. For every subsemilattice W of S which is also an upper set, there exist upper sets $W_1 \subset L_1, \ldots, W_n \subset L_n$ such that $W = W_1 \times \ldots \times W_n$.*

Proof. Denote by $p_1, \ldots, p_n$ the projections of T onto $L_1, \ldots, L_n$ respectively, and let $W_i = p_i(W)$ for each $i = 1, \ldots, n$. One easily sees that every W_i is upper and $W \subset W_1 \times \ldots \times W_n$. To show the reverse inclusion, take any $w = (w_1, \ldots, w_n) \in W_1 \times \ldots \times W_n$. For every i, let $v^{(i)} \in W$ with $p_i(v^{(i)}) = w_i$ and put $v = v^{(1)} \wedge \ldots \wedge v^{(n)}$. Then $v \in W$ and $v \leq w$ so that $w \in W$. ∎

6.6. Lemma. *Let S be as in the previous lemma, and let τ be a strong upper T_0-topology on S; let A be a τ-open set and x a point in A. Then there exist $z \in A$ and a τ-open neighborhood V of x which is a subsemilattice of S and is contained in $\uparrow\{z\}$ (and hence in A).*

Proof. Let W be a τ-open subsemilattice of S with $x \in W \subset A$. By the previous lemma, $W = W_1 \times \ldots \times W_n$, where W_i is an upper subset of L_i for every $i = 1, \ldots, n$. Let

$x = (x_1, \ldots, x_n)$ and denote by $i_1, \ldots, i_h$ (with $i_1 < \ldots < i_h$) the indices in $\{1, \ldots, n\}$ such that x_i is not the least element in W_i.

For every $j = 1, \ldots, h$, take a $w_{i_j} \in W_{i_j}$ with $w_{i_j} < x_{i_j}$. Letting

$$y_j = (x_1, \ldots, x_{i_j-1}, w_{i_j}, x_{i_j+1}, x_n)$$

defines an element of S which is strictly less than x; in particular, every τ-neighborhood of y_j contains x and, since τ is T_0, there exists a τ-open set V_j which contains x and does not contain y_j. Moreover, V_j may be taken to be a semilattice, as τ is strong.

Now define z as $(z_1, \ldots, z_n)$, where z_i is w_{i_j} if $i = i_j$ for some j, and is x_i otherwise; let $V = W \cap V_1 \cap \ldots \cap V_h$, so that v is a τ-open neighborhood of x and is a subsemilattice of S. It remains to show that $V \subset \uparrow\{z\}$. Consider any $v \in V$ and any $i \in \{1, \ldots, n\}$. We have to show that $z_i \leq v_i$. This is clear if x_i is the least element of W_i, since in this case $z_i = x_i$ and $v_i \in W_i$ as $v \in W$. So, assume that $i = i_j$ with $1 \leq j \leq h$, and suppose that z_i (i.e. w_{i_j}) is strictly greater than $v_i = v_{i_j}$. Put $x \wedge v = t$ where $t = (t_1, \ldots, t_n)$. As V is a semilattice, $t \in V$. Moreover, $t_k \leq x_k$ whenever $k \neq i_j$, while $t_{i_j} \leq v_{i_j} < w_{i_j}$, and consequently we have $t < y_j$. Since V is upper, $y_j \in V$; but this is impossible as $V \subset V_j$. ■

6.7. Theorem. *Let S be a lattice which is the product of finitely many linearly ordered sets $L_1, \ldots, L_n$, and let π be the infimum of a collection Θ of strong upper* T_0*-topologies on S. Then π is strong*, *provided that one of the following conditions holds*:

(1) *Θ is* (*at most*) *countable*;

(2) *every chain in S is well-ordered* (*equivalently*: *L_k is well-ordered for each $k = 1, \ldots, n$*).

Proof. Let $\Theta = \{\pi_\alpha\}_{\alpha<\nu}$, where ν is the cardinality of Θ. Without loss of generality we may assume that ν is infinite, and hence identify ν with the set $\{(\xi, \eta) \mid \xi < \nu,\ \eta < \xi\}$ endowed with the lexicographic order. Moreover, let φ be the function of ν onto itself which maps any $(\xi, \eta) \in \nu$ to η. For every $\eta < \nu$ the fiber $\varphi^{-1}(\eta)$ is cofinal in ν.

Now let A be any π-open subset of S, and let a be any point in A. We have to show that there exists a π-open subsemilattice U of S which contains a and is contained in A.

By transfinite induction we can define, for each $\alpha < \nu$, a point $z_\alpha \in A$ and a $\pi_{\varphi(\alpha)}$-open neighborhood V_α of a which is a subsemilattice of $\uparrow\{z_\alpha\}$ and contains each $z_{\alpha'}$ with $\alpha' < \alpha$; thus $V_\alpha \supset V_\beta$ and $z_\alpha \leq z_\beta$ whenever $\alpha > \beta$. This construction is accomplished by making use of the previous lemma. This is done in an obvious way if $\alpha = 0$ or α is a successor ordinal; if α is a limit ordinal—so that we certainly are in case (2)—we apply the lemma to the point

$$x = \inf\{z_{\alpha'} \mid \alpha' < \alpha\} = \min\{z_{\alpha'} \mid \alpha' < \alpha\}.$$

Finally, let $U = \bigcup_{\alpha<\nu} V_\alpha$. Then $a \in U \subset A$, and U is a subsemilattice of S, because it is the union of a chain of subsemilattices. Moreover, for each $\eta < \nu$, we have $\bigcup_{\alpha\in\varphi^{-1}(\eta)} V_\alpha = U$, so that U is π_η-open. Hence U is π-open. ■

The remaining part of the present section is devoted to the construction of some topologies defined on the real plane (endowed with the natural lattice structure given by the componentwise order), which answer a few questions raised in the previous sections.

We say that a topology on a lattice S is *dual-strong* if it strong with respect to the dual lattice S^* (i.e. the lattice obtained from S by reversing the order). A (dual-strong) topology on S is *dual-strongly decomposable* if it has a decomposition consisting of dual-strong topologies.

In the following example we need the fact that every dual-strongly decomposable topology on a lattice is lower-saturated (compare with Corollary 3.8). On the other hand, the notion of dual-strong topology will be of no use in our forthcoming investigation of hypertopologies, as all the most common lower topologies on a hyperspace turn out not to be dual-strong.

6.8. Example. *There exist two metrizable convex topologies on the real plane whose supremum is not convex.*

Proof. For every $(x,y) \in \mathbb{R}^2$, put $\Uparrow(x,y) = \{(x',y') \in \mathbb{R}^2 \mid x' > x,\ y' > y\}$ and $\Downarrow(x,y) = \{(x',y') \in \mathbb{R}^2 \mid x' < x,\ y' < y\}$. The collections $\{\Uparrow(x,y) \mid (x,y) \in \mathbb{R}^2\}$ and $\{\Downarrow(x,y) \mid (x,y) \in \mathbb{R}^2\}$ are bases for two topologies π_1 and μ_1 on $\mathbb{R}^2$; it is easily seen that π_1 is upper and strong, τ_1 is lower and dual-strong, and $\tau = \pi_1 \vee \mu_1$ is the Euclidean topology on $\mathbb{R}^2$.

Let $\pi_2 = \{\emptyset, \uparrow\{(1,1)\}, \uparrow\{(-1,-1)\}, \mathbb{R}^2\}$ and $\mu_2 = \{\emptyset, \downarrow\{(-1,-1)\}, \downarrow\{(1,1)\}, \mathbb{R}^2\}$. Then π_2 is a strong upper topology and μ_2 is a dual-strong lower topology. Putting $\tau' = \tau \vee \pi_2 = (\pi_1 \vee \pi_2) \vee \mu_1$ and $\tau'' = \tau \vee \mu_2 = \pi_1 \vee (\mu_1 \vee \mu_2)$, it follows from Corollary 3.8 and the above observations that both τ' and τ'' are saturated, and hence convex.

Observe that both $\uparrow\{(1,1)\}$ and $\uparrow\{(-1,-1)\}$ are clopen sets with respect to τ', and hence the collection $\{\uparrow\{(1,1)\}, \uparrow\{(-1,-1)\} \setminus \uparrow\{(1,1)\}, \mathbb{R}^2 \setminus \uparrow\{(-1,-1)\}\}$ is a clopen partition of $(\mathbb{R}^2, \tau')$; as it is easily shown that τ and τ' agree on any of these sets, the space $(\mathbb{R}^2, \tau')$ is metrizable, being a disjoint union of metrizable spaces. The proof for τ'' is perfectly symmetric.

Now we are going to show that $\tau' \vee \tau''$ is not convex. The points $(-1,-1)$ and $(1,1)$ are both isolated in $(\mathbb{R}^2, \tau' \vee \tau'')$; hence the two-element set $\{(-1,-1),(1,1)\}$ is $(\tau' \vee \tau'')$-open, and its convex hull is $C = \{(x,y) \in \mathbb{R}^2 \mid |x|,\ |y| \le 1\}$. Consider the point $(-1,1)$ of C. Since $\tau' \vee \tau''$ coincides with the Euclidean topology at that point, it is apparent that no $(\tau' \vee \tau'')$-neighborhoods of $(-1,1)$ can be contained in C. ∎

Observe that, by Proposition 3.9, the topology π_1 defined in the above proof is equal to $\tau \wedge \gamma$, where τ is the Euclidean topology.

6.9. Example. *There exists a* T_0*-topology* π *on* $\mathbb{R}^2$ *which is upper modular but is not upper complete.*

Proof. Let τ, π_1 and μ_1 be as in the preceding proof. Put

$$\pi = \{A \subset \mathbb{R}^2 \mid \uparrow A = A \text{ and } A \setminus \{(0,0)\} \in \tau\}.$$

Clearly, π is a topology, and $\pi \ge \pi_1$ so that π is T_0. To show that π is not upper complete, consider the set $M = \{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 \mid x + y > 0\}$; it is easily seen that M is π-open and yet $\uparrow(M \cap \downarrow\{(0,0)\}) = \uparrow\{(0,0)\} \notin \pi$.

Now we prove that π is upper modular. Let $A \in (\pi \vee \lambda) \wedge \gamma$. Then A is upper and hence it remains to show that $A \setminus \{(0,0)\}$ is τ-open. Fix $a \in A \setminus \{(0,0)\}$. There exists $B \in \pi$

such that $a \in B \cap \downarrow\{a\} \subset A$. Since $(B \setminus \{(0,0)\}) \cap \downarrow\{a\} \in \tau \vee \lambda = \pi_1 \vee \mu_1 \vee \lambda = \pi_1 \vee \lambda$, and π_1 is upper complete (as it is strong), we have $\uparrow((B \setminus \{(0,0)\}) \cap \downarrow\{a\}) \in \pi_1 \leq \tau$, and also $\uparrow((B \setminus \{(0,0)\}) \cap \downarrow\{a\}) \setminus \{(0,0)\} \in \tau$. To complete the proof, it suffices to observe that $a \in \uparrow((B \setminus \{(0,0)\}) \cap \downarrow\{a\}) \setminus \{(0,0)\} \subset \uparrow(B \cap \downarrow\{a\}) \setminus \{(0,0)\} \subset \uparrow A \setminus \{(0,0)\} = A \setminus \{(0,0)\}$. ∎

In the following, the two lines $\{(x,-x) \mid x \in \mathbb{R}\}$ and $\{(x,x) \mid x \in \mathbb{R}\}$ of $\mathbb{R}^2$ will be often referred to as *virtual axes*; clearly, they correspond to a 45° negative rotation of the standard axes. It can be rather helpful, in order to recognize convex sets and convex hulls, envisaging the real plane in such a way that the two virtual axes are, respectively, horizontally and vertically lying.

For every $a, b \in \mathbb{R}$ we denote by $\langle a, b\rangle$ the point $\big(\frac{\sqrt{2}}{2}(a+b), \frac{\sqrt{2}}{2}(-a+b)\big)$ of $\mathbb{R}^2$, so that a and b are the *virtual coordinates* of the above point, i.e. the coordinates with respect to the virtual axes.

Observe that $\langle a', b'\rangle \leq \langle a'', b''\rangle$ if and only if $|a'' - a'| \leq b'' - b'$.

6.10. Lemma. *Let $x_0, y', y'' \in \mathbb{R}$, where $y' < y''$, let A be a convex subset of $\mathbb{R}^2$ and suppose that there exists $\varepsilon > 0$ such that, for every $x \in \mathbb{R}$ with $x_0 - \varepsilon < x < x_0$, both $\langle x, y'\rangle$ and $\langle x, y''\rangle$ belong to A. Then $\langle x_0, y\rangle \in A$ for each $y \in]y', y''[$.*

Proof. Fix any $y \in]y', y''[$, put $\delta = \min\{y'' - y, y - y', \varepsilon\}$, and choose x in $]x_0 - \delta, x_0[$. It follows from our assumption that $\langle x, y'\rangle, \langle x, y''\rangle \in A$. Now it is easily seen that $\langle x, y'\rangle \leq \langle x_0, y\rangle \leq \langle x, y''\rangle$, and therefore $\langle x_0, y\rangle \in A$ by convexity. ∎

6.11. Example. *There is a metrizable topology on $\mathbb{R}^2$ which is weakly locally convex but not locally convex.*

Proof. Let us associate with every $\langle x, y\rangle \in \mathbb{R}^2$ a neighborhood base $\mathcal{V}_{\langle x,y\rangle}$ in the following way. Denote by T the set

$$\left\{\left\langle \frac{m}{2^n}, y\right\rangle \,\middle|\, m, n \in \mathbb{N},\ \frac{1}{2^{n+1}} < y < \frac{1}{2^n}\right\}.$$

All the points in $\mathbb{R}^2 \setminus (T \cup \{\langle 0,0\rangle\})$ are isolated. For every $\langle \frac{m}{2^n}, y\rangle \in T$, let $\mathcal{V}_{\langle m/2^n, y\rangle} = \{V^y_{m,n}(\varepsilon) \mid 0 < \varepsilon < 1\}$, where

$$V^y_{m,n}(\varepsilon) = \left\{\left\langle \frac{m}{2^n}, y\right\rangle\right\} \cup \left\{\langle x, y\rangle \,\middle|\, \frac{m+1-\varepsilon}{2^n} < x < \frac{m+1}{2^n}\right\};$$

finally $\mathcal{V}_{\langle 0,0\rangle} = \{W_k \mid k \in \mathbb{N}\}$, where

$$W_k = \left\{\langle x, y\rangle \,\middle|\, |x|,\ |y| < \frac{1}{2^k}\right\}.$$

It is easily observed that this defines a topology τ on $\mathbb{R}^2$. To show that τ is metrizable (and strongly 0-dimensional), put for every $k \in \mathbb{N}$,

$$\mathcal{A}_k = \{W_k\} \cup \left\{V^y_{m,n}\left(\frac{1}{2^k}\right) \,\middle|\, \left\langle \frac{m}{2^n}, y\right\rangle \in T \setminus W_k\right\},\ \mathcal{B}_k = \mathcal{A}_k \cup \left\{\{\langle x, y\rangle\} \,\middle|\, \langle x, y\rangle \in \mathbb{R}^2 \setminus \bigcup \mathcal{A}_k\right\}.$$

Then every $\mathcal{B}_k$ is a τ-open partition of $\mathbb{R}^2$ and the sequence $(\mathcal{B}_k)_{k\in\mathbb{N}}$ is a development of $(\mathbb{R}^2, \tau)$ such that $\mathcal{B}_{k+1}$ refines $\mathcal{B}_k$ for every $h \in \mathbb{N}$.

Note that, for every $\langle x, y\rangle \in T$, the elements of $\mathcal{V}_{\langle x,y\rangle}$ are all convex; hence weak local convexity needs to be checked at the only point $\langle 0, 0\rangle$. To this end, it is sufficient to observe that $\mathcal{V}_{\langle 0,0\rangle}$ is also a neighborhood base for the Euclidean topology, which is (weakly) locally convex at each point.

Now we prove that τ is not locally convex at $\langle 0,0\rangle$. In fact, we are going to see that every τ-open convex subset A of $\mathbb{R}^2$, containing $\langle 0,0\rangle$, is unbounded. Indeed, let A be such a set, and take a $k \in \mathbb{N}$ with $W_k \subset A$. We will show that

$$\forall h \in \mathbb{N}_0\ \forall y \in \,]1/2^{k+1}, 1/2^k[: \quad \langle h/2^k, y\rangle \in A. \tag{6.2}$$

We argue by induction on h. As (6.2) is certainly true if $h = 0$, we assume that it holds for some non-negative integer h and prove it for $h + 1$.

Let $y \in \,]1/2^{k+1}, 1/2^k[$, and choose y' and y'' with $1/2^{k+1} < y' < y < y'' < 1/2^k$. Since A is open and $\langle h/2^k, y'\rangle$, $\langle h/2^k, y''\rangle \in A$, there exist $\varepsilon', \varepsilon'' \in \,]0,1[$ such that both $V^{y'}_{h,k}(\varepsilon')$ and $V^{y''}_{h,k}(\varepsilon'')$ are contained in A; in particular, $\langle x, y'\rangle, \langle x, y''\rangle \in A$ for each $x \in \,](h+1-\widehat{\varepsilon})/2^k, (h+1)/2^k[$, where $\widehat{\varepsilon} = \min\{\varepsilon', \varepsilon''\}$. By Lemma 6.10, it follows that $\langle (h+1)/2^k, y\rangle \in A$. ■

The use of virtual coordinates also helps us to detect a strong decomposition of the Euclidean topology which does not coincide with the standard one (given in the proof of Example 6.8). For every $r \in \mathbb{R}$, let $M_r = \{\langle x, y\rangle \in \mathbb{R}^2 \mid y < r\}$. Then $\mu' = \{M_r \mid r \in \mathbb{R}\}$ is a lower topology on $\mathbb{R}^2$ and, using the same notations as in the proof of Example 6.8, we easily see that $\pi_1 \vee \mu' = \pi_1 \vee \mu_1 = \tau$. On the other hand, it is clear that $\mu' < \mu_1$, since μ_1 is T_0 and μ' is not.

Finding a lower T_0-topology on $\mathbb{R}^2$ which is strictly less than μ_1 and whose supremum with π_1 gives rise to τ is a slightly less easy task, which will be performed in Section 8.

7. The Scott topology

We present here some definitions and results, regarding complete lattices, which are related to the subject of decomposability of topologies, and besides, will be applied in Section 13. For the sake of completeness, we have given all the proofs, even if a number of them can be found, in a somewhat different form, in the book [10].

7.1. Definition. Let S be a complete lattice. The *lim-inf* of a net $(a_j)_{j\in J}$ in S is

$$\liminf_{j\in J} a_j = \sup_{k\in J} \inf_{j\geq k} a_j.$$

We say that $(a_j)_{j\in J}$ *lim-inf-converges* to $x \in S$ if $x \leq \liminf_{j\in J} a_j$.

A subset A of S is *Scott-open* if, for each $a \in A$, every net in S which lim-inf-converges to a is eventually in A. The Scott-open sets actually constitute the family of open sets for a topology on S, called the *Scott topology*, which we denote by σ.

We are going to see that the Scott topology is a particular upper topology. Recall that a subset D of a poset S is *directed* if it is non-empty and directed in the induced order (i.e. for every $x, y \in D$ there exists $z \in D$ such that $z \geq x$ and $z \geq y$).

7.2. Proposition. *A subset U of S is Scott-open if and only if it is an upper set and satisfies the following condition:*

$$\forall D \subset S \text{ directed}: \quad (\sup D \in U \Rightarrow U \cap D \neq \emptyset). \tag{7.1}$$

Proof. Suppose U is Scott-open. Clearly, we may assume that U is non-empty. So, let $x \in U$. For every $y \geq x$, the constant net with value y is lim-inf-convergent to x and therefore $y \in U$. This shows that U is an upper set. Now let D be a directed subset of S and suppose that $\sup D = s \in U$. Then $(d)_{d\in D}$ is a net whose lim-inf is s and hence it must be eventually in U, so that $U \cap D \neq \emptyset$.

Conversely, let U be any (non-empty) upper set satisfying (7.1); let $u \in U$ and let $(a_j)_{j\in J}$ be a net lim-inf-converging to u. We will show that this net is eventually in U. Since U is upper, we have $a = \liminf_{j\in J} a_j \in U$. Now, let $D = \{\inf_{j\geq k} a_j \mid k \in J\}$. Then D is a directed set and $\sup D = a \in U$; by (7.1) this implies that $\inf_{j\geq \widehat{k}} a_j \in U$ for some $\widehat{k} \in J$. As U is an upper set, it follows that $a_j \in U$ for every $j \geq \widehat{k}$. ∎

7.3. Corollary. *A subset F of S is Scott-closed if and only if it is a lower set which is closed under suprema of directed subsets.*

Proof. This is an immediate consequence of the previous proposition. ∎

Now we will show that the Scott topology is upper modular. First we need a preliminary result.

7.4. Lemma. *Let S be a complete lattice. Denote by ω the collection of all subsets V of S such that*

$$\forall D \subset S \text{ directed}: \quad (\sup D \in V \Rightarrow \exists y \in D\colon\ \uparrow\{y\} \cap D \subset V). \tag{7.2}$$

Then:

(1) *ω is a topology*;
(2) $\omega \geq \lambda$;
(3) $\omega \wedge \gamma = \sigma$.

Proof. (1) First, we clearly have $\emptyset \in \omega$ and $S \in \omega$. Now let $V, W \in \omega$ and D be a directed set with $\sup D \in V \cap W$. Then there exist $y, z \in D$ for which $\uparrow\{y\} \cap D \subset V$ and $\uparrow\{z\} \cap D \subset W$. If $x \in D$ is such that $x \geq y$ and $x \geq z$, we have $\uparrow\{x\} \subset \uparrow\{y\} \cap \uparrow\{z\}$ and hence $\uparrow\{x\} \cap D \subset V \cap W$. Therefore $V \cap W \in \omega$. Finally, let $\mathcal{V}$ be a subcollection of ω and D be a directed set with $\sup D \in \bigcup \mathcal{V}$; then $\sup D \in U$ for some $U \in \mathcal{V}$ so that there exists $u \in U$ such that $\uparrow\{u\} \cap D \subset U \subset \bigcup \mathcal{V}$. Hence $\bigcup \mathcal{V} \in \omega$.

(2) Let $V \in \lambda$ and D a directed set with $\sup D \in V$. Then $D \subset \downarrow\{\sup D\} \subset \downarrow V = V$ and therefore $\uparrow\{y\} \cap D \subset D \subset V$ for every $y \in D$; hence $V \in \omega$. As V was arbitrary, we get $\lambda \leq \omega$.

(3) Let $V \in \omega \wedge \gamma$ and D a directed set with $\sup D \in V$. By (7.2) there exists $y \in D$ such that $\uparrow\{y\} \cap D \subset V$ and in particular $D \cap V \neq \emptyset$, so that $V \in \sigma$ by Proposition 7.2. Hence $\omega \wedge \gamma \leq \sigma$.

Conversely, let U be Scott-open. It suffices to show that $U \in \omega$. To this end, consider any directed set D such that $\sup D \in U$; from (7.1) we get $U \cap D \neq \emptyset$. So, let $y \in U \cap D$; we have $\uparrow\{y\} \cap D \subset \uparrow U = U$ and hence $U \in \omega$, as required. ∎

7.5. Proposition. *The Scott topology is upper modular.*

Proof. It suffices to show that $(\sigma \vee \lambda) \wedge \gamma \leq \sigma$. Now from 7.4(2) and 7.4(3) it follows that $(\sigma \vee \lambda) \wedge \gamma \leq \omega \wedge \gamma = \sigma$. ∎

We conclude this section by recalling that it is still an open question to characterize those complete lattices on which the Scott topology is strong (see [14, Prob. 527(i)]).

8. Uniqueness of decomposition

In this section we present some uniqueness and non-uniqueness results for decomposition of topologies. Roughly speaking, we are going to see that, for a decomposable topology on a poset, being uniquely decomposable is an exception, while having several different decompositions is the rule.

We will be able to say a little more in the particular case of hypertopologies (see §14).

The *lower Vietoris topology* ([1]) on a poset S (which is simply called "lower topology" in [10]) is generated by the family $\{S \setminus \uparrow\{x\} \mid x \in S\}$, and will be denoted by V^-.

Given a topology τ on S and $x, y \in S$, letting

$$\tau(x,y) = \{A \in \tau \mid x \notin A \text{ or } y \in A\}$$

defines another topology on S (which is clearly coarser than τ).

8.1. Proposition. *For a topology τ on a poset S, the following are equivalent:*

(1) *for each $x \in S$, the τ-closure of $\{x\}$ is $\uparrow\{x\}$;*

(2) $\mathsf{V}^- \le \tau \le \lambda$;

(3) *τ is a lower topology such that, whenever $x \not\le y$, there exists a τ-neighborhood of y which does not contain x.*

(4) *τ is a lower topology such that $\pi \vee \tau$ is T_1, whenever π is an upper T_0-topology.*

Proof. (1)⇒(2). Since each set of the form $\uparrow\{x\}$ is τ-closed, we have $\mathsf{V}^- \le \tau$. On the other hand, if C is a τ-closed subset of S then, for each $c \in C$, the τ-closure of the singleton of c, i.e. $\uparrow\{c\}$, must be contained in C. It follows that C is upper, and therefore $\tau \le \lambda$.

(2)⇒(3). If $x \not\le y$ then $S \setminus \uparrow\{x\}$ is a V^--open (hence τ-open) neighborhood of y which does not contain x.

(3)⇒(1). Since τ is lower, the τ-closure of $\{x\}$ must contain $\uparrow\{x\}$; it remains to show that $\uparrow\{x\}$ is τ-closed. Let $y \notin \uparrow\{x\}$. This means that $x \not\le y$, so there exists a τ-neighborhood U of y (where we may assume $\downarrow U = U$) such that $x \notin U$, hence $\uparrow\{x\} \cap U = \emptyset$.

(3)⇒(4). Fix a point $x_0 \in S$; for each $s \ne x_0$ we have to find a $(\pi \vee \tau)$-neighborhood of x_0 not containing s. If $s \not\le x_0$ there is a τ-open set which does the job. Now, let $s \le x_0$. Every π-neighborhood of s contains x_0, but π is T_0, and hence there is a π-neighborhood of x_0 which avoids s.

(4)⇒(3). Let $x, y \in S$ with $x \not\le y$, and suppose that τ is a lower topology with the property that every τ-neighborhood of y contains x. One immediately sees that $\gamma(y,x)$ is an upper T_0-topology with the same property, and therefore $\gamma(y,x) \vee \tau$ cannot be T_1. ∎

([1]) The *upper Vietoris topology* will be introduced in Section 10 but might be defined more generally, in a lattice S with greatest element 1, as the topology generated by all subsets of the form $\{c \in S \mid c \vee d = 1\}$, where d runs over S. For more details, we refer to [17].

8.2. Definition. A (lower) topology satisfying the equivalent conditions above is called *lower compatible*.

One could also consider *upper compatible* topologies: they are precisely the "order compatible" topologies as defined in [7], and this motivates our terminology.

8.3. Definition. Let S be a poset, and consider two elements $x, y \in S$ with $x \not\leq y$. Given an upper topology π on S, we say that π *separates x from y* if there exists $h \in S$ such that $\uparrow\{h\}$ is a (not necessarily open) π-neighborhood of x which does not contain y.

In particular, γ separates any x from any y, provided that $x \not\leq y$. Also, observe that:

(1) If π separates x from y, and π' is an upper topology finer than π, then π' also separates x from y.

(2) If π separates x from y, and $x' \geq x$, then π separates x' from y.

(3) If π separates x from y, and $y' \leq y$, then π separates x from y'.

8.4. Theorem. *Let μ be a lower compatible topology on a poset S. If $x, y \in S$ are incomparable then $\mu(x, y)$ is a T_0-topology, strictly coarser than μ, such that for every upper topology π which separates x from y we have*

$$\pi \vee \mu(x, y) = \pi \vee \mu. \tag{8.1}$$

Moreover, μ and $\mu(x, y)$ agree on every μ-closed set which does not contain y.

Proof. Since $x \not\geq y$, the set $S \setminus \uparrow\{y\}$ is μ-open but not $\mu(x, y)$-open; thus $\mu(x, y) < \mu$.

To prove that $\mu(x, y)$ is T_0, consider two distinct elements $a, b \in S$, and suppose first that $a \not\geq x$. If $a \not\geq b$ there exists a μ-open set V containing a but not b and hence $V \setminus \uparrow\{x\}$ is a $\mu(x, y)$-open neighborhood of a not containing b, while if $a > b$ then $b \not\geq x$ and we may use the same argument with b in place of a. It remains to consider the case in which both $a \geq x$ and $b \geq x$. Since $y \not\geq x$, we have $y \not\geq a$ and $y \not\geq b$. We may assume $a \not\geq b$, so that a has a μ-open neighborhood V which avoids b. Then $V' = V \cup (S \setminus \uparrow\{a, b\})$ is another μ-open set containing a but not b; moreover, V' is $\mu(x, y)$-open, as $y \in U$.

Now let C be any μ-closed set with $y \notin C$. To see that μ and $\mu(x, y)$ induce the same topology on C, we consider a μ-open set W. There is a suitable $\mu(x, y)$-open set W' such that $W' \cap G = W \cap G$: indeed, it suffices to put $W' = W \cup (S \setminus C)$.

We complete the proof by showing that $\mu \leq \pi \vee \mu(x, y)$, whence the equality (8.1) follows at once. Let $s \in S$ and let M be a μ-open neighborhood of s. We have to find a $(\pi \vee \mu(x, y))$-neighborhood N of s which is contained in M. Two cases are possible:

Case 1: $s \not\geq x$. Just let $N = M \setminus \uparrow\{x\}$. This is a μ-open set which does not contain x, and therefore it is $\mu(x, y)$-open (hence, *a fortiori*, $(\pi \vee \mu(x, y))$-open).

Case 2: $s \geq x$. In this case π separates s from y, i.e. there exists $h \in S$ such that $\uparrow\{h\}$ is a π-neighborhood of s which does not contain y. This means that $y \not\geq h$, and thus there exists a μ-open neighborhood L of y which does not contain h (hence $L \cap \uparrow\{h\} = \emptyset$, as L is lower). The μ-open set $L \cup M$ is in fact $\mu(x, y)$-open, because it contains y, so it is a $\mu(x, y)$-neighborhood of s. Therefore $N = \uparrow\{h\} \cap (L \cup M)$ is a $(\pi \vee \mu(x, y))$-neighborhood of s; moreover, $N \subset M$, since our construction implies that $N = \uparrow\{h\} \cap M$. ∎

8.5. Corollary. *For a poset S the following are equivalent*:

(1) *the discrete topology* $\mathbf{1}$ *on S is uniquely decomposable*;
(2) *every decomposable topology on S is uniquely decomposable*;
(3) *S is linearly ordered.*

Proof. As (3)⇒(2) has already been proved in 5.3, and (2)⇒(1) is obvious, it remains to show that (1)⇒(3).

Suppose S is not linearly ordered: there exist two incomparable elements $x, y \in S$. Since λ is lower compatible and γ separates x from y, the previous theorem gives us a lower topology $\lambda(x, y) \neq \lambda$ such that $\mathbf{1} = \gamma \vee \lambda = \gamma \vee \lambda(x, y)$. ■

We apply Theorem 8.4 to the topological product of a collection of linearly ordered sets, each endowed with the order topology. We first consider the case of two linearly ordered sets.

8.6. Lemma. *Let $(L_1, \leq_1), (L_2, \leq_2)$ be two linearly ordered sets, both having at least two elements. Let (L, τ) be the topological product of (L_1, τ_1) and (L_2, τ_2), where τ_i is the order topology relative to $\leq_i$ for $i = 1, 2$. Then τ has two different strong decompositions both consisting of* T_0*-topologies.*

Proof. If L_1 and L_2 both have exactly two elements, then τ is the discrete topology on L and we may simply use Corollary 8.5. Thus we may suppose that there exist $a_1, b_1, c_1 \in L_1$ with $a_1 <_1 b_1 <_1 c_1$; moreover, fix $z_2, w_2 \in L_2$ with $z_2 <_2 w_2$.

For $i = 1, 2$ put, for every $l \in L_i$, $\Uparrow l = \{l' \in L_i \mid l' >_i l\}$ and $\Downarrow l = \{l' \in L_i \mid l' <_i l\}$; let π_i, μ_i be the topologies on L_i generated by $\{\Uparrow l \mid l \in L_i\}$ and $\{\Downarrow l \mid l \in L_i\}$, respectively.

It is easily seen that $\tau_i = \pi_i \vee \mu_i$ for $i = 1, 2$, which implies for general reasons that $\tau = \pi \vee \mu$, where π and μ are the topological products, respectively, of π_1, π_2 and of μ_1, μ_2. Observe also that π is a strong topology.

Consider the incomparable elements $x = (c_1, z_2)$ and $y = (a_1, w_2)$ of L. If z_2 is the minimum of L_2, then $x \in (\Uparrow b_1) \times L_2 \subset \uparrow\{(b_1, z_2)\}$, and the last set fails to contain y. If, on the contrary, there exists $l_2 \in L_2$ with $l_2 <_2 z_2$, then $x \in (\Uparrow b_1) \times (\Uparrow l_2) \subset \uparrow\{(b_1, l_2)\}$ and again the last set does not contain y. Thus in any case π separates x from y.

On the other hand, for every $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in L with $u \not\leq v$, there exists a π-open V containing v but not u: indeed, supposing for example $u_1 > v_1$, we may put $V = (\Downarrow u_1) \times L_2$.

Thus the lemma follows from Theorem 8.4. ■

8.7. Lemma. *Let σ_1, σ_2 be topologies on a set X, and let τ be a topology on a set Y. If $\sigma_1 \not\leq \sigma_2$, then $\sigma_1 \times \tau \not\leq \sigma_2 \times \tau$.* ■

8.8. Proposition. *Let the lattice L be the product of a family $\{L_i\}_{i \in I}$ of linearly ordered sets, and let $\tau = \prod_{i \in I} \tau_i$, where τ_i is the order topology on L_i. If for some $i' \neq i''$ both $L_{i'}$ and $L_{i''}$ have more than one element, then τ is not uniquely decomposable* (*even in the realm of* T_0*-topologies*).

Proof. For every $i \in I$ let, as in Lemma 8.6, π_i and μ_i be the topologies on L_i generated, respectively, by $\{\Uparrow l \mid l \in L_i\}$ and $\{\Downarrow l \mid l \in L_i\}$, so that $\tau_i = \pi_i \vee \mu_i$. Then the topologies $\widetilde{\pi}$, $\widetilde{\mu}$ and $\widetilde{\tau}$ on $\prod_{i \in I \setminus \{i', i''\}} L_i$, respectively defined as the topological products of all topologies π_i, μ_i and τ_i with $i \in I \setminus \{i', i''\}$, are such that $\widetilde{\pi} \vee \widetilde{\mu} = \widetilde{\tau}$; also, $\widetilde{\pi}$ is easily seen to be strong.

Write $\prod_{i \in I}(L_i, \tau_i)$, up to homeomorphisms, as

$$(L_{i'}, \tau_{i'}) \times (L_{i''}, \tau_{i''}) \times \Big(\prod_{i \in I \setminus \{i', i''\}} (L_i, \tau_i) \Big).$$

Since $(L_{i'}, \tau_{i'}) \times (L_{i''}, \tau_{i''})$ has a double strong decomposition by Lemma 8.6 and $\widetilde{\tau}$ is strongly decomposable, it follows from Lemma 8.7 that τ has two different strong decompositions (consisting of T_0-topologies). ∎

In the sequel we will assume that S is a semilattice. Let us note that, in this case, as γ is a strong topology, Corollary 8.5 also holds if we replace decomposability with strong decomposability (and unicity is meant in the realm of strong decompositions).

8.9. Definition. Let τ be a topology on S. We say that $D \subset S$ is *τ-upper dense* in S if it is dense with respect to $\tau \vee \gamma$, i.e. for every $x \in S$ and every τ-neighborhood W of x there exists $y \geq x$ with $y \in W \cap D$. Equivalently, each (basic) τ-open U has the property that $\downarrow(U \cap D) = \downarrow U$.

8.10. Proposition. (1) *If D is τ-upper dense in S and τ' is coarser than τ then D is also τ'-upper dense.*

(2) *A necessary condition for a $D \subset S$ to be τ-upper dense in S is being cofinal in S and, if τ is an upper topology, this condition is also sufficient.*

(3) *If D is a subsemilattice of S which is both τ-upper dense and ϑ-upper dense in S, where τ and ϑ are locally convex topologies on S, then D is also $(\tau \vee \vartheta)$-upper dense.*

(4) *If D is a cofinal subsemilattice of S, and (π, μ) is a decomposition of a topology τ on S, then D is τ-upper dense if and only if it is μ-upper dense.*

Proof. (1) This is obvious.

(2) Necessity is clear. Conversely, let D be cofinal in S. Then, as is easily seen, D is dense with respect to γ. Now, if τ is upper, we have $\gamma = \tau \vee \gamma$.

(3) It suffices to show that $U \subset \downarrow(U \cap D)$ whenever U is a basic $(\tau \vee \vartheta)$-open set. So we may assume that $U = V \cap W$ where $V \in \tau$, $W \in \vartheta$, and both are convex.

For every $x \in U$, since $x \in V$, we have $y \in V \cap D$ for some $y \geq x$ and, since $x \in W$, we have $z \in W \cap D$ for some $z \geq x$. By convexity $u = y \wedge z$ belongs to V and W, hence to U; moreover, $u \in D$ as well, as D is a semilattice. Therefore $x \in \downarrow(U \cap D)$, as required.

(4) Apply (1), (2) and (3). ∎

We can now give a result which, in view of 3.9, ensures some kind of uniqueness for strong decompositions.

8.11. Theorem. *Let (π, μ') and (π, μ'') be two decompositions of the same topology τ on a semilattice S; suppose that μ' and μ'' agree on a subsemilattice D of S. Then $\mu' = \mu''$, provided that one of the following conditions holds*:

(1) *D is τ-upper dense in S*;

(2) *S has a greatest element*, *denoted by* 1, *and D is τ-upper dense in $S \setminus \{1\}$.*

Proof. In case (1), we may denote by 1 an element not in S, put $S' = S \cup \{1\}$, and think of 1 as the greatest element of S'; thus condition (2) is satisfied, with S' in place of S. Note that this does not cause any trouble with topologies: no matter how the τ-neighborhoods of 1 are defined, they are always π-neighborhoods, because a neighborhood of 1 with respect to either μ' or μ'' must be the whole of S'.

So, suppose condition (2) holds. It suffices to show that $\mu' \le \mu''$, because one can prove the reverse inequality in exactly the same way. Let V' be a μ'-open set, which we may clearly assume to be different from S, so that $V' \subset S \setminus \{1\}$. By our hypothesis, there exists a μ''-open set V'' such that $V' \cap D = V'' \cap D$. Since D is τ-upper dense in $S \setminus \{1\}$ we have

$$V' = \downarrow V' = \downarrow(V' \cap D) = \downarrow(V'' \cap D). \tag{8.2}$$

Now, if $V'' \neq S$, the last set in (8.2) is just V'', thus it is μ''-open.

If $V'' = S$ then $V'' = \downarrow(V'' \cap D) \cup \{1\}$, so that $V'' = V' \cup \{1\}$ by (8.2) and therefore $V' = S \setminus \{1\}$. Since V' is a τ-open set, for each $x \in V'$ there exist a π-open set P_x and a μ''-open set M_x such that $x \in P_x \cap M_x \subset V'$. Observe that $1 \in P_x$, so that $1 \notin M_x$. Hence $V' = \bigcup_{x \in V'} M_x$, which is μ''-open. ■

9. Hyperspace topologies

From this section on, we consider a topological or metric space X, and the lattice we deal with is the collection $\mathcal{C}(X)$ of all closed subsets of X ordered by *reverse inclusion*; in some cases, we also consider $\mathcal{C}_0(X) = \mathcal{C}(X) \setminus \{\emptyset\}$. Note that the binary operation, when $\mathcal{C}(X)$ or $\mathcal{C}_0(X)$ is considered as a semilattice, is the binary set union (because of the reverse inclusion ordering).

The lattice $\mathcal{C}(X)$ is called the *hyperspace* of X, and the topologies defined on $\mathcal{C}(X)$ (or on $\mathcal{C}_0(X)$) are called *hyperspace topologies* or, more briefly, *hypertopologies*. In this context, X will also be called the *underlying space*.

If X is a T_1-space, a topology τ on $\mathcal{C}(X)$ is *admissible* [13] if the mapping $\iota : x \mapsto \{x\}$ is an embedding of X into the space $(\mathcal{C}(X), \tau)$.

Recall that an *Urysohn space* is a topological space in which any two distinct points have disjoint closed neighborhoods.

9.1. Proposition. *Let X be an Urysohn space and let τ be a convex locally convex admissible T_1-topology on $\mathcal{C}_0(X)$ such that every singleton has a τ-neighborhood base consisting of open upper sets. Then τ is upper-regular.*

Proof. Let $\mathcal{T}$ be the set of all singletons of X; we have to prove, according to Proposition 4.5, that

$$\forall C \in \mathcal{C}_0(X) \setminus \mathcal{T}\ \exists \mathcal{U}, \mathcal{V} \in \tau\colon \quad \mathcal{T} \subset \mathcal{U}, \quad C \in \mathcal{V}, \quad \mathcal{U} \cap \mathcal{V} = \emptyset. \tag{9.1}$$

First observe that the subspace $\mathcal{T}$ of $\mathcal{C}_0(X)$ is Urysohn, as it is homeomorphic to X, and therefore for every $\{x\}, \{y\} \in \mathcal{T}$ with $x \neq y$ there exist two τ-open subcollections $\mathcal{W}_1$ and $\mathcal{W}_2$ such that $\{x\} \in \mathcal{W}_1$, $\{y\} \in \mathcal{W}_2$ and $\tau\text{-cl}(\mathcal{W}_1 \cap \mathcal{T}) \cap \tau\text{-cl}(\mathcal{W}_2 \cap \mathcal{T}) \cap \mathcal{T} = \emptyset$, and hence τ-cl$(\mathcal{W}_1) \cap \tau$-cl$(\mathcal{W}_2) \cap \mathcal{T} = \emptyset$ (because $\mathcal{W}_1, \mathcal{W}_2$ are upper and every element of $\mathcal{T}$ has a local base consisting of upper sets).

Now, let $C \in \mathcal{C}_0(X) \setminus \mathcal{T}$. We consider two cases.

CASE 1: $C = X$. Fix two distinct points x_1 and x_2 in X, and let A_1 and A_2 be open neighborhoods of x_1 and x_2 respectively such that $\overline{A}_1 \cap \overline{A}_2 = \emptyset$. Let $\mathcal{V}$ be a convex (not necessarily open) τ-neighborhood of X in $\mathcal{C}_0(X)$ such that $X \setminus A_1 \notin \mathcal{V}$ and $X \setminus A_2 \notin \mathcal{V}$. As $\mathcal{V}$ is a lower set, we must have

$$\forall D \in \mathcal{V}: \quad D \cap A_1 \neq \emptyset, \quad D \cap A_2 \neq \emptyset. \tag{9.2}$$

We claim that for each $z \in X$ there exists a τ-neighborhood $\mathcal{U}_z$ of $\{z\}$ such that $\mathcal{U}_z \cap \mathcal{V} = \emptyset$ so that $\mathcal{U} = \bigcup_{z \in X} \mathcal{U}_z$ and $\mathcal{V}$ are the required τ-open neighborhoods of $\mathcal{T}$ and X respectively.

Indeed, given $z \in X$, since some open neighborhood A of z is disjoint from A_1 or A_2, we must have either $\iota(A) \cap \iota(A_1) = \emptyset$ or $\iota(A) \cap \iota(A_2) = \emptyset$, where ι is the injection of X into $\mathcal{C}_0(X)$. Now $\iota(A)$ is an open subspace of $\mathcal{T}$, hence $\iota(A) = \mathcal{A} \cap \mathcal{T}$ for a suitable τ-open $\mathcal{A} \subset \mathcal{C}_0(X)$; as $\mathcal{A}$ is a τ-neighborhood of $\{z\}$, there must be a τ-neighborhood $\mathcal{U}_z$ of $\{z\}$ which is an upper set and is contained in $\mathcal{A}$.

It is easy to see that $\mathcal{U}_z$ is disjoint from $\mathcal{V}$. Suppose, on the contrary, that there is some D in $\mathcal{U}_z \cap \mathcal{V}$. Since $D \in \mathcal{V}$, we have $D \cap A_1 \neq \emptyset$ and $D \cap A_2 \neq \emptyset$; on the other hand, D is also in $\mathcal{U}_z$, which is upper, and therefore $\iota(D) \subset \mathcal{U}_z$. It follows that $\mathcal{U}_z$ meets both $\iota(A_1)$ and $\iota(A_2)$. Since $\mathcal{U}_z \subset \mathcal{A}$ we have, for $i = 1, 2$,

$$\iota(A) \cap \iota(A_i) = \mathcal{A} \cap \iota(A_i) \supset \mathcal{U}_z \cap \iota(A_i) \neq \emptyset,$$

a contradiction.

CASE 2: $C \neq X$. First, we know from the preceding case that there exists a convex (in fact lower) τ-neighborhood $\mathcal{M}$ of X such that no $\{z\} \in \mathcal{M}$ is τ-adherent to $\mathcal{M}$. Fix two distinct points x' and x'' in C. There exist τ-open neighborhoods $\mathcal{W}'$ and $\mathcal{W}''$ of $\{x'\}$ and $\{x''\}$ respectively such that τ-cl$(\mathcal{W}') \cap \tau$-cl$(\mathcal{W}'') \cap \mathcal{T} = \emptyset$. For every $\{z\} \in \mathcal{T}$, let $\mathcal{W}_z$ be an upper τ-neighborhood of $\{z\}$ such that either $\mathcal{W}_z \cap \mathcal{W}' = \emptyset$ or $\mathcal{W}_z \cap \mathcal{W}'' = \emptyset$; we may also assume that $\mathcal{W}_z$ is disjoint from $\mathcal{M}$. Letting $\mathcal{U} = \bigcup_{z \in X} \mathcal{W}_z$ defines a (not necessarily open) τ-neighborhood of $\mathcal{T}$.

Now consider the sets $\mathcal{N}' = \mathrm{co}(\mathcal{W}' \cup \mathcal{M})$ and $\mathcal{N}'' = \mathrm{co}(\mathcal{W}'' \cup \mathcal{M})$. Both are τ-neighborhoods of C (since $\{x'\} \in \mathcal{W}'$, $X \in \mathcal{M}$ and $\{x'\} \subset C \subset X$, we have $C \in \mathrm{co}(\mathcal{W}' \cup \mathcal{M}) = \mathcal{N}'$, which is open because τ is convex; similarly for $\mathcal{N}''$). We claim that $\mathcal{V} = \mathcal{N}' \cap \mathcal{N}''$ is disjoint from $\mathcal{U}$, i.e. $\mathcal{N}' \cap \mathcal{N}'' \cap \mathcal{W}_z = \emptyset$ for each $z \in X$.

Indeed, suppose that for some $z \in X$ there exists $D \in \mathcal{N}' \cap \mathcal{N}'' \cap \mathcal{W}_z$. As $D \in \mathrm{co}(\mathcal{W}' \cup \mathcal{M})$, there are D_1' and D_2' in $\mathcal{W}' \cup \mathcal{M}$ with $D_1' \subset D \subset D_2'$. Now D_1' cannot belong to $\mathcal{M}$ (otherwise, as $\mathcal{M}$ is lower, we would also have $D \in \mathcal{M}$, which is impossible because $D \in \mathcal{W}_z$ and $\mathcal{W}_z \cap \mathcal{M} = \emptyset$), hence $D_1' \in \mathcal{W}'$. As $\mathcal{W}_z$ is upper, we also have $D_1' \in \mathcal{W}_z$ so that $\mathcal{W}' \cap \mathcal{W}_z \neq \emptyset$. In a similar manner one shows that $\mathcal{W}'' \cap \mathcal{W}_z \neq \emptyset$ as well. Thus we have a contradiction. ∎

9.2. COROLLARY. *Let X be an Urysohn space and let τ be a convex locally convex admissible* T_2*-topology on $\mathcal{C}_0(X)$ such that every singleton has a τ-neighborhood base consisting of open upper sets. Then τ is decomposable.*

Proof. This follows from the previous proposition and Theorem 4.7. ■

We are going to give another sufficient condition for a hypertopology to be upper-regular on $\mathcal{C}_0(X)$.

9.3. Lemma. *Let X be a T_2-space and let τ be a convex locally convex topology on $\mathcal{C}_0(X)$. Then the set $\mathcal{T}$ of all singletons of X is τ-closed.*

Proof. Suppose, on the contrary, that there exists some $C \notin \mathcal{T}$ such that every τ-neighborhood of C intersects $\mathcal{T}$. We have two cases.

Case 1: $C \neq X$. Fix $x \in C$; as $\{x\} \neq C$, there exist τ-open sets $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$, with $\mathcal{V}_1 \cap \mathcal{V}_2 = \mathcal{V}_2 \cap \mathcal{V}_3 = \emptyset$, such that $\{x\} \in \mathcal{V}_1$, $C \in \mathcal{V}_2$ and $X \in \mathcal{V}_3$. Now $\mathrm{co}(\mathcal{V}_1 \cup \mathcal{V}_3)$ is a τ-open neighborhood of C, hence so is $\mathcal{W} = \mathcal{V}_2 \cap \mathrm{co}(\mathcal{V}_1 \cup \mathcal{V}_3)$; therefore $\{y\} \in \mathcal{W}$ for a suitable $y \in X$, and in fact $\{y\} \in \mathcal{V}_1 \cup \mathcal{V}_3$. But this is impossible because $\mathcal{V}_2 \cap (\mathcal{V}_1 \cup \mathcal{V}_3) = \emptyset$.

Case 2: $C = X$. Fix $x \in X$ and let V be a non-dense open set containing x. There exist convex τ-neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of X such that $\overline{V} \notin \mathcal{V}_1$ and $X \setminus V \notin \mathcal{V}_2$. It follows that $\mathcal{V}_1 \cap \mathcal{V}_2$ does not contain any singleton, contrary to the assumption that X is τ-adherent to $\mathcal{T}$. ■

9.4. Proposition. *Let X be a T_2-space and let τ be a convex locally convex regular topology on $\mathcal{C}_0(X)$ such that every singleton has a τ-neighborhood base consisting of open upper sets. Then τ is upper-regular.*

Proof. Since τ is regular, and by the previous lemma the set of all singletons of X is closed, condition 4.5(2) is satisfied. ■

9.5. Corollary. *Let X be a T_2-space and let τ be a convex locally convex T_3-topology on $\mathcal{C}_0(X)$ such that every singleton has a τ-neighborhood base consisting of open upper sets. Then τ is decomposable.*

Proof. Apply Theorem 4.7. ■

Observe that, by Remark 1.12(2), the topology in Example 6.4 is upper complete but is not strong. Now we give an example of a hypertopology with the same properties.

9.6. Example. *Let X be a T_2-topological space having a non-empty open subset without isolated points. Define a topology π on $\mathcal{C}(X)$ as follows: given any $C \in \mathcal{C}(X)$ the basic π-neighborhoods of C have the form*

$$\mathcal{V}(A_1, \ldots, A_n) = \{F \in \mathcal{C}(X) \mid F \not\supset A_i \text{ for all } i = 1, \ldots, n\},$$

where $A_1, \ldots, A_n$ are non-empty, open and disjoint from C. Then π is an upper complete topology which is not strong.

Proof. First, it is easily seen that the $\mathcal{V}(A_1, \ldots, A_n)$ are indeed a basic system of neighborhoods (in fact open neighborhoods) and that the topology π is upper.

Next we show that π is not strong. Let A be a non-empty open set without isolated points; let C be the complement of A, so that $\mathcal{V}(A)$ is a π-neighborhood of C. We claim that no π-neighborhood $\mathcal{U}$ of C contained in $\mathcal{V}(A)$ is a semilattice. Indeed, let $A_1, \ldots, A_n$ be non-empty open sets, disjoint from C, such that $\mathcal{V}(A_1, \ldots, A_n) \subset \mathcal{U}$. Since each A_i is infinite, we can pick $x_i, y_i \in A_i$ for every $i \in \{1, \ldots, n\}$ in such a way that the points $x_1, \dots, x_n, y_1, \dots, y_n$ are all distinct. Then there exist open neighborhoods $G_1, \dots, G_n, H_1, \dots, H_n$ of $x_1, \dots, x_n, y_1, \dots, y_n$, respectively, such that $G_i \cap H_j = \emptyset$ for every $i, j \in \{1, \dots, n\}$. The closed sets $E = X \setminus (\bigcup_{i=1}^n G_i)$ and $F = X \setminus (\bigcup_{i=1}^n H_i)$ are elements of $\mathcal{V}(A_1, \dots, A_n)$, hence of $\mathcal{U}$, while $E \cup F = X \notin \mathcal{V}(A)$.

It remains to prove that π is upper complete. Since a base for $\pi \vee \lambda$ consists of all subcollections of $\mathcal{C}(X)$ of the form

$$\mathcal{W}(A_1, \dots, A_n; C) = \mathcal{V}(A_1, \dots, A_n) \cap \downarrow\{C\}$$

where $C \in \mathcal{C}(X)$ and $A_1, \dots, A_n$ are non-empty open sets disjoint from C, it will be enough to show that the upper set of a collection of this form is π-open; in fact, we have

$$\uparrow \mathcal{W}(A_1, \dots, A_n; C) = \mathcal{V}(A_1, \dots, A_n).$$

Indeed, let $F \in \mathcal{V}(A_1, \dots, A_n)$; then $C \cup F \in \downarrow\{C\}$. On the other hand, for each $i = 1, \dots, n$, as $C \cap A_i = \emptyset$ and $F \not\supset A_i$, we also have $C \cup F \not\supset A_i$, so that $C \cup F \in \mathcal{V}(A_1, \dots, A_n)$; therefore $\mathcal{V}(A_1, \dots, A_n) \subset \uparrow \mathcal{W}(A_1, \dots, A_n; C)$. The reverse inclusion is obvious. ■

10. The Vietoris topology

Given an open subset G of X, let G^+ and G^- denote the collections of all $C \in \mathcal{C}(X)$ such that $C \subset G$ and $C \cap G \neq \emptyset$ respectively. Since G^+ is an upper set in the lattice $\mathcal{C}(X)$, the topology V^+, generated on $\mathcal{C}(X)$ by all the sets of this form where G runs through all open subsets of X, is an upper topology and is called the *upper Vietoris topology* (or Tikhonov topology). Similarly, the sets G^- generate the *lower Vietoris topology*, V^-, which we have already encountered in Section 8.

The *Vietoris topology* is $\mathsf{V} = \mathsf{V}^+ \vee \mathsf{V}^-$; hence it is automatically decomposable. In fact, V is strongly decomposable, as we are going to see.

10.1. Remark. *The topology* V^+ *is strong.*

Proof. Indeed, one immediately sees that $A \cup B$ is a member of G^+ whenever both A and B are. ■

10.2. Corollary. $\mathsf{V} \wedge \gamma = \mathsf{V}^+$.

Proof. Apply Proposition 3.9. ■

On the other hand, V^- is not lower complete in general, not even lower modular, as the following result shows.

10.3. Proposition. *Let X be a non-discrete* T_2*-space. Then the topology* $(\mathsf{V}^- \vee \gamma) \wedge \lambda$ *on* $\mathcal{C}_0(X)$ *is strictly finer than* V^-.

Proof. Let x be a non-isolated point of X. There exists a net $(x_j)_{j \in J}$ which converges to x in X, with $x_j \neq x$ for every $j \in J$. Now consider the set $\mathcal{U} = \mathcal{C}_0(X) \setminus \{\{x_j\} \mid j \in J\}$; it is clearly a lower set which is not V^--open. We show that $\mathcal{U}$ is $(\mathsf{V}^- \vee \gamma)$-open. Indeed, we take an arbitrary $C \in \mathcal{U}$ and exhibit a suitable $(\mathsf{V}^- \vee \gamma)$-neighborhood of C which is contained in $\mathcal{U}$. There are two possibilities.

CASE 1: C has at least two points. Let a and b be two distinct points of C and let A and B be disjoint open sets containing a and b, respectively. Then $C \in A^- \cap B^- \subset \mathcal{U}$.

CASE 2: $C = \{z\}$. We can take $\{\{z\}\}$ itself as a neighborhood of $\{z\}$ because $\{\{z\}\} = \uparrow\{\{z\}\}$. ■

Nevertheless, a sort of dual of Corollary 10.2 still holds.

10.4. PROPOSITION. *If the topology of X is symmetric* (*in particular*, *if X is* T_1) *then* $\mathsf{V} \wedge \lambda = \mathsf{V}^-$ *on* $\mathcal{C}(X)$.

PROOF. In fact we show that $\downarrow\mathcal{U} \in \mathsf{V}^-$ for every $\mathcal{U} \in \mathsf{V}$; further, we may restrict ourselves to those $\mathcal{U}$ which belong to a suitable base β.

Take as β the set of all subcollections of the form

$$G^+ \cap G_1^- \cap G_2^- \cap \ldots \cap G_n^-, \tag{10.1}$$

where $G, G_1, \ldots, G_n$ are open subsets of X and $\bigcup_{i=1}^n G_i \subset G$. It is well known (and very easy to check) that β is indeed a base for V.

So, let $\mathcal{U} \in \beta$; then $\mathcal{U}$ has the form (10.1), and we may assume that all the G_i's are non-empty. We claim that $\downarrow\mathcal{U} = G_1^- \cap \ldots \cap G_n^- = \mathcal{V} \in \mathsf{V}^-$, where of course only the inclusion $\downarrow\mathcal{U} \supset \mathcal{V}$ is non-trivial.

Consider any $C \in \mathcal{V}$. For each $i = 1, \ldots, n$ it is possible to take $x_i \in G_i \cap C$; since the topology of X is symmetric, we have $F = \bigcup_{i=1}^n \overline{\{x_i\}} \subset G$, that is, $F \in G^+$. On the other hand, it is clear that $F \in \mathcal{V}$, as well; hence $F \in \mathcal{U}$. Now $F \subset C$, and therefore $C \in \downarrow\mathcal{U}$. As C was arbitrary, the conclusion follows. ■

Observe that V is upper-saturated, because V^+ is strong. Also, in the proof of the above proposition we have shown that V is lower-saturated; thus, it is convex.

It is easily seen that the Vietoris topology is upper-regular on $\mathcal{C}(X)$, as $\emptyset$ is isolated. The situation is less trivial on $\mathcal{C}_0(X)$.

10.5. PROPOSITION. *Let X be a* T_2*-space. Then the Vietoris topology on* $\mathcal{C}_0(X)$ *is upper-regular if and only if X is an Urysohn space.*

PROOF. The "if" part follows at once from Proposition 9.1 (it is well known and easily seen that the Vietoris topology is admissible). Conversely, suppose that X is not Urysohn and let x, y be two points whose neighborhoods always have some common adherent point. Let $\mathcal{U}$ be any V-open set containing the set $\mathcal{T}$ of all singletons, and let $\mathcal{V}$ be any V-neighborhood of the element $C = \{x, y\}$ of $\mathcal{C}_0(X)$. There are two open neighborhoods A, B of x and y respectively, such that $A^- \cap B^- \cap (A \cup B)^+ \subset \mathcal{V}$ and, taking a common adherent point of A and B, say z, there exists an open neighborhood G of z such that $G^+ \subset \mathcal{U}$. Given any $a \in G \cap A$ and any $b \in G \cap B$ the closed set $F = \{a, b\}$ is an element of $G^+ \cap (A^- \cap B^- \cap (A \cup B)^+)$ and hence, *a fortiori*, $\mathcal{U} \cap \mathcal{V} \neq \emptyset$. Therefore condition 4.5(2) is not satisfied, so that V is not upper-regular. ■

If $\mathcal{G}$ is a collection of subsets of the space X, we will use the symbol $\mathcal{G}^-$ to denote $\bigcap_{G \in \mathcal{G}} G^-$. With this convention, a base for V^- consists of all $\mathcal{G}^-$, where $\mathcal{G}$ is a finite collection of open subsets of X. This is why V is sometimes called the *finite topology.* If we take all $\mathcal{G}^-$ where $\mathcal{G}$ is a locally finite collection of open subsets of X we obtain a base for a topology $\mathsf{L}^- \geq \mathsf{V}^-$ on $\mathcal{C}(X)$, the *lower locally finite topology.* The *locally finite topology* [1] is $\mathsf{L} = \mathsf{V}^+ \vee \mathsf{L}^-$.

10.6. PROPOSITION. *Let X be a symmetric space. The topologies L and L^- on $\mathcal{C}(X)$ satisfy the following*:

(1) $\mathsf{L} \wedge \gamma = \mathsf{V}^+$;
(2) $\mathsf{L} \wedge \lambda = \mathsf{L}^-$.

PROOF. The first equality holds since V^+ is strong. To prove the second, observe that the union of a locally finite collection of closed sets is closed, and modify accordingly the argument used in 10.4. ■

11. The Hausdorff metric topology

Throughout this section we denote by (X, d), or simply X when d is understood, a metric space.

Given $x \in X$ and $A \subset X$, the distance from x to A is

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}$$

and the *excess* of E over F, where $E, F \subset X$, is

$$e(E, F) = \sup\{d(e, F) \mid e \in E\}$$

where the sup is taken in the lattice $[0, +\infty]$, so that $e(\emptyset, E) = 0$ for every $E \subset X$.

The *Hausdorff distance* between two sets $E, F \subset X$ is

$$H(E, F) = \max\{e(E, F),\ e(F, E)\}.$$

When restricted to $\mathcal{C}(X)$, the function H is a metric—in fact a "generalized" or "extended" metric, because it can take the value $+\infty$; the induced topology is called the *Hausdorff metric topology*, and denoted by H (sometimes H_d, where the subscript indicates dependence on the metric d of X).

Thus, for every $C \in \mathcal{C}(X)$, an H-open neighborhood base at C is given by the sets

$$\mathcal{S}(C, \varepsilon) = \{F \in \mathcal{C}(X) \mid H(C, F) < \varepsilon\}$$

where ε runs through the positive numbers.

Similarly, the sets of the form

$$\mathcal{S}^+(C, \varepsilon) = \{F \in \mathcal{C}(X) \mid e(F, C) < \varepsilon\}$$
$$(\text{resp. } \mathcal{S}^-(C, \varepsilon) = \{F \in \mathcal{C}(X) \mid e(C, F) < \varepsilon\}),$$

for every $\varepsilon > 0$, constitute a local base at $C \in \mathcal{C}(X)$ for an upper topology H^+ (resp. a lower topology H^-), which is called the *upper* (*lower*) *Hausdorff topology.*

Since $\mathcal{S}(C, \varepsilon) = \mathcal{S}^+(C, \varepsilon) \cap \mathcal{S}^-(C, \varepsilon)$ for every $C \in \mathcal{C}(X)$ and every $\varepsilon > 0$, we have $\mathsf{H} = \mathsf{H}^+ \vee \mathsf{H}^-$. Hence the Hausdorff metric topology is decomposable. In fact, H is strongly decomposable, as we are going to see.

11.1. REMARK. *The topology H^+ is strong.*

Proof. Indeed, it is easily seen that $e(E\cup F, C) = \max\{e(E,C), e(F,C)\}$. Therefore $E \cup F$ is a member of $\mathcal{S}^+(C,\varepsilon)$ whenever both E and F are. ■

As a consequence we have $\mathsf{H}\wedge\gamma = \mathsf{H}^+$, but even more is true.

11.2. Proposition. *For every $C \in \mathcal{C}(X)$ and every $\varepsilon > 0$, we have*

$$\uparrow\mathcal{S}(C,\varepsilon) = \mathcal{S}^+(C,\varepsilon) \quad \text{and} \quad \downarrow\mathcal{S}(C,\varepsilon) = \mathcal{S}^-(C,\varepsilon). \tag{11.1}$$

Proof. The first equality in (11.1) follows from Lemma 3.4, where the semilattice is $\mathcal{C}(X)$, and we take $\mathcal{S}(C,\varepsilon)$ as K, $\mathcal{S}^+(C,\varepsilon)$ as T and $\mathcal{S}^-(C,\varepsilon)$ as U.

For the second equality, we only have to prove that $\mathcal{S}^-(C,\varepsilon) \subset \downarrow\mathcal{S}(C,\varepsilon)$, as the reverse inclusion is obvious. If $C = \emptyset$ the result is trivial because $\mathcal{S}(C,\varepsilon) = \{\emptyset\}$ and therefore $\downarrow\mathcal{S}(C,\varepsilon) = \mathcal{C}(X)$.

Now suppose $C \neq \emptyset$; choose F arbitrarily in $\mathcal{S}^-(C,\varepsilon)$ and let δ be a positive number less than ε such that $e(C,F) < \delta$. For every $x \in C$, since $d(x,F) < \delta$, there exists $f(x) \in F$ with $d(x,f(x)) < \delta$. Let A be the closure of $\{f(x) \mid x \in C\}$. Given any $a \in A$ and a positive $\eta < \varepsilon - \delta$ there exists a suitable $c \in C$ such that $d(a,f(c)) < \eta$ and we have

$$d(a,C) \le d(a,c) \le d(a,f(c)) + d(c,f(c)) < \delta + \eta;$$

since a is arbitrary we get $e(A,C) \le \delta + \eta < \varepsilon$. On the other hand, for every $x \in C$,

$$d(x,A) \le d(x,f(x)) < \delta$$

so that $e(C,A) \le \delta < \varepsilon$. It follows that $H(A,C) < \varepsilon$, which means that $A \in \mathcal{S}(C,\varepsilon)$. Since $F \supset A$, we get $F \in \downarrow\mathcal{S}(C,\varepsilon)$ and we conclude that $\mathcal{S}^-(C,\varepsilon) \subset \downarrow\mathcal{S}(C,\varepsilon)$. ■

11.3. Corollary. $\mathsf{H}\wedge\gamma = \mathsf{H}^+$ *and* $\mathsf{H}\wedge\lambda = \mathsf{H}^-$.

Like V^-, we see that also H^- is not lower modular. We recall that H^- is always finer than V^- (and H^+ is coarser than V^+, while H and V are generally incomparable; see [9]).

11.4. Proposition. *Let X be a non-discrete metric space. The topology $(\mathsf{H}^-\vee\gamma)\wedge\lambda$ on $\mathcal{C}(X)$ is strictly finer than H^-.*

Proof. As $\mathsf{V}^- \le \mathsf{H}^-$, we also have $(\mathsf{V}^-\vee\gamma)\wedge\lambda \le (\mathsf{H}^-\vee\gamma)\wedge\lambda$. Thus it suffices to observe that in fact the set $\mathcal{U}$ constructed in the proof of Proposition 10.3 is not H^--open. ■

In connection with the previous result we note that, for some metric spaces (such as $\mathbb{R}$), even $(\mathsf{V}^+\vee\mathsf{H}^-)\wedge\lambda$ is strictly finer than H^-.

11.5. Proposition. *Let (X,d) be a metric space. Suppose that there exist two subsets $A = \{a_n \mid n \in \mathbb{N}\}$ and $B = \{b_{i,j} \mid i,j \in \mathbb{N}\}$ of X such that*

(1) $d(a_{n'}, a_{n''}) \ge 2|n' - n''|$;
(2) $1/(i+1) \le d(a_j, b_{i,j}) < 1/i$ *for every $i,j \in \mathbb{N}$.*

Then $(\mathsf{V}^+\vee\mathsf{H}^-)\wedge\lambda > \mathsf{H}^-$ on $\mathcal{C}(X)$.

Proof. Observe that, by (1) and (2), the set $B_i = \{b_{i,j} \mid j \in \mathbb{N}\}$ is unbounded for every $i \in \mathbb{N}$, and the sequence $(B_i)_{i\in\mathbb{N}}$ is H^--convergent to A.

For each $i \in \mathbb{N}$, let $A_i = \{x \in X \mid 1/(i+1) \le d(x,A) < 1/i\}$. We have $B_i \subset A_i$, hence A_i is unbounded and $(A_i)_{i\in\mathbb{N}}$ is H^--convergent to A as well. Thus the set

$$\uparrow(\{A_i \mid i \in \mathbb{N}\} \cup \{S \subset A \mid \mathrm{diam}(S) < +\infty\})$$

is not H^--closed in $\mathcal{C}(X)$; nevertheless, we claim that its complement $\mathcal{G}$ is $(\mathsf{V}^+ \vee \mathsf{H}^-)$-open.

Take $C \in \mathcal{G}$. First suppose that $C \subset A$ (so that $\mathrm{diam}(C) = +\infty$). Fix a point $p \in X$ and denote by W the set of all $x \in X$ such that $d(x,A) < 1/(d(x,p)+1)$. It turns out that W is an open set containing A (hence C) and therefore $\mathcal{U} = W^+ \cap \mathcal{S}^-(C,1)$ is a $(\mathsf{V}^+ \vee \mathsf{H}^-)$-neighborhood of C. We show that $\mathcal{U} \subset \mathcal{G}$.

Suppose not, and let $D \in \mathcal{U} \setminus \mathcal{G}$. As $e(C,D) < 1$ and C is unbounded, D is unbounded as well. Thus $D \subset A_{\widehat{\imath}}$ for some $\widehat{\imath} \in \mathbb{N}$, and there exists $\overline{x} \in D$ with $d(\overline{x},p) \ge \widehat{\imath}$. Since $D \subset W$ we get $d(\overline{x},A) < 1/(d(\overline{x},p)+1) \le 1/(\widehat{\imath}+1)$, which is impossible.

Next suppose that $e(C,A) > 1$. Let $\eta = e(C,A) - 1$. Then $\mathcal{S}^-(C,\eta) \subset \mathcal{G}$. Indeed, for every $E \in \mathcal{S}^-(C,\eta)$, we have

$$e(E,A) = \eta + e(E,A) - \eta > e(C,E) + e(E,A) - \eta \ge e(C,A) - \eta = 1.$$

It remains to consider the case where $C \not\subset A$ and $e(C,A) \le 1$. Let $h = \min\{i \in \mathbb{N} \mid \exists x \in C\colon\ d(x,A) > 1/(i+1)\}$ and let $x' \in C$ with $d(x',A) > 1/(h+1)$. We have $e(C,A) \le 1/h$, so that $x' \in A_h$. Since $C \not\subset A_h$, there exists $x'' \in C$ such that $d(x'',A) < 1/(h+1)$. We will show that $\mathcal{S}^-(C,\varepsilon) \subset \mathcal{G}$, where $\varepsilon = \min\{d(x',A) - 1/(h+1),\, 1/(h+1) - d(x'',A)\}$.

Given any $F \in \mathcal{S}^-(C,\varepsilon)$, let $y', y'' \in F$ be such that $d(x',y') < \varepsilon$ and $d(x'',y'') < \varepsilon$; then

$$d(y',A) = \varepsilon + d(y',A) - \varepsilon > d(x',y') + d(y',A) - \varepsilon \ge d(x',A) - \varepsilon \ge \frac{1}{h+1} \tag{11.2}$$

and

$$d(y'',A) \le d(y'',x'') + d(x'',A) < \varepsilon + d(x'',A) \le \frac{1}{h+1}. \tag{11.3}$$

By (11.2) we can have neither $F \subset A$ nor $F \subset A_i$ for every $i > h$, while (11.3) implies that $F \not\subset A_i$ also for $i \le h$. Hence $F \in \mathcal{G}$. ∎

12. The proximal topology

Given a metric space (X,d), the *proximal topology* on $\mathcal{C}(X)$ (also called *d-proximal* where dependence on d is to be emphasized) is defined as $\mathsf{D} = \mathsf{H}^+ \vee \mathsf{V}^-$, so that it is a strongly decomposable topology which is always coarser than both V and H. This section is concerned also with the topology $\mathsf{V} \wedge \mathsf{H}$, the intersection of the Hausdorff and Vietoris topologies, which still has no name and will be denoted by N. Clearly, $\mathsf{D} \le \mathsf{N}$.

12.1. Remark. $\mathsf{D} \wedge \gamma = \mathsf{H}^+$ *and* $\mathsf{D} \wedge \lambda = \mathsf{V}^-$.

Proof. Indeed, $\mathsf{H}^+ \le \mathsf{D} \wedge \gamma \le \mathsf{H} \wedge \gamma = \mathsf{H}^+$. Similarly, $\mathsf{V}^- \le \mathsf{D} \wedge \lambda \le \mathsf{V} \wedge \lambda = \mathsf{V}^-$. ∎

Thus D behaves like one reasonably expects. And what about N?

12.2. Remark. $\mathsf{N} \wedge \gamma = \mathsf{H}^+$ *and* $\mathsf{N} \wedge \lambda = \mathsf{V}^-$.

Proof. As in 12.1. ∎

In order to derive non-trivial consequences from 12.2, we have to make sure that D and N are different, in general.

Let us recall that a metric space X is said to have the *UC property* if the gap between two disjoint subsets of X is always non-zero. We will use the term *N-space* for a metric space which is not totally bounded and does not have the UC property.

12.3. Proposition. *Let X be a metric space. The following are equivalent*:

(1) *X is an N-space*;
(2) *the topology N on $\mathcal{C}(X)$ is not locally convex*;
(3) *the topology N on $\mathcal{C}(X)$ is non-decomposable*;
(4) *the topology N on $\mathcal{C}(X)$ is strictly finer than D.*

Proof. The equivalence between (1) and (4) is proved in [12]. Now, since V and H are saturated, N is saturated too; hence it is decomposable if and only if it is locally convex, so that (2)⇔(3). Finally, 12.1 and 12.2 imply, by Remark 1.4, that N is decomposable if and only if it coincides with D; therefore (3)⇔(4). ■

12.4. Corollary. *The infimum of two decomposable topologies need not be decomposable.*

Proof. Let X be an N-space. The topologies V and H on $\mathcal{C}(X)$ are decomposable, while $\mathsf{N} = \mathsf{V} \wedge \mathsf{H}$ is not, by the previous proposition. ■

12.5. Corollary. *A convex topology need not be locally convex.*

Proof. Let X be an N-space. By 12.3, the topology N is not locally convex. On the other hand, N is convex because it is saturated. ■

12.6. Corollary. *Let X be an N-space. Then V^+, V^-, H^+ and H^- generate a non-distributive sublattice Λ of the lattice of all topologies on $\mathcal{C}(X)$*; *moreover*, *Λ consists of exactly ten elements.*

Proof. Suppose Λ is distributive; we have

$$\begin{aligned}\mathsf{V} \wedge \mathsf{H} &= (\mathsf{V}^+ \vee \mathsf{V}^-) \wedge (\mathsf{H}^+ \vee \mathsf{H}^-)\\ &= (\mathsf{V}^+ \wedge \mathsf{H}^+) \vee (\mathsf{V}^+ \wedge \mathsf{H}^-) \vee (\mathsf{V}^- \wedge \mathsf{H}^+) \vee (\mathsf{V}^- \wedge \mathsf{H}^-)\\ &= \mathsf{H}^+ \vee \mathbf{0} \vee \mathbf{0} \vee \mathsf{V}^- = \mathsf{H}^+ \vee \mathsf{V}^-,\end{aligned}$$

where $\mathbf{0}$ denotes the trivial topology. Thus $\mathsf{N} = \mathsf{D}$, which is impossible by 12.3.

The elements of Λ are: $\mathbf{0}$, V, V^+, V^-, H, H^+, H^-, D, N and $\mathsf{T} = \mathsf{V} \vee \mathsf{H}$. They are all distinct, and Λ does not have other elements; the easy verification of these facts is left to the reader. ■

13. The Kuratowski convergence

Given a net $(A_j)_{j\in J}$ in $\mathcal{C}(X)$, define its *Kuratowski upper limit* as follows:

$$\operatorname*{Ls}_{j\in J} A_j = \bigcap_{k\in J} \overline{\bigcup_{j\geq k} A_j} = \{x \in X \mid \forall \text{ neighborhood } V \text{ of } x\ \forall k \in J\ \exists j \geq k\colon\ V \cap A_j \neq \emptyset\}.$$

We say that $(A_j)_{j\in J}$ is K^+-convergent to some $A\in\mathcal{C}(X)$ if $\operatorname{Ls}_{j\in J} A_j \subset A$. This is the *upper Kuratowski convergence*; it is not a topological convergence, but it is “upper” in the sense that the sets which are closed under K^+-limits of nets are lower. These form the collection of closed sets of a topology $\tau\mathsf{K}^+$, the *topologization* of K^+, which is “coarser” than K^+ in the sense that all nets which are K^+-convergent are also $\tau\mathsf{K}^+$-convergent.

Taking into account the ordering of the (complete) lattice $\mathcal{C}(X)$, one readily sees that the Kuratowski upper limit is nothing else but the lim-inf as defined in 7.1 and that, therefore, upper Kuratowski convergence is precisely lim-inf-convergence in $\mathcal{C}(X)$.

Moreover, $\tau\mathsf{K}^+$ is the counterpart of the Scott topology (in particular, it is upper).

13.1. Proposition. *The topologization of the upper Kuratowski convergence coincides with the Scott topology of the complete lattice $\mathcal{C}(X)$.*

Proof. See [18]. ∎

A net $(A_j)_{j\in J}$ in $\mathcal{C}(X)$ is K-convergent to A if it is both K^+-convergent and V^--convergent to A. This is *Kuratowski convergence*, which is non-topological as well—thus it has its own topologization, $\tau\mathsf{K}$.

13.2. Proposition. *Let X be a topological space. Then $\tau\mathsf{K}\wedge\gamma = \tau\mathsf{K}^+$ on $\mathcal{C}(X)$.*

Proof. It suffices to prove that $\tau\mathsf{K}\wedge\gamma \le \tau\mathsf{K}^+$. So, let $\mathcal{F}$ be a subcollection of $\mathcal{C}(X)$ which is $\tau\mathsf{K}$-closed and such that $\downarrow\mathcal{F}=\mathcal{F}$; we show that, given any net $(A_j)_{j\in J}$ in $\mathcal{F}$, we have $\operatorname{Ls}_{j\in J} A_j \in \mathcal{F}$ (so that $C\in\mathcal{F}$ whenever $\operatorname{Ls}_{j\in J} A_j \subset C$). Indeed, denote by A the Kuratowski upper limit of $(A_j)_{j\in J}$ and let $B_j = A_j\cup A$ for every $j\in J$. Then $(B_j)_{j\in J}$ is a net in $\mathcal{F}$ and $\operatorname{Ls}_{j\in J} B_j = A$. On the other hand, $(B_j)_{j\in J}$ is also V^--convergent to A, as $A\subset B_j$ for every $j\in J$. Hence this net K-converges to A, and therefore $A\in\mathcal{F}$. ∎

13.3. Proposition. *Let X be a regular space. Then $\tau\mathsf{K}\wedge\lambda = \mathsf{V}^-$ on $\mathcal{C}(X)$.*

Proof. It is easily seen that $\tau\mathsf{K}\wedge\lambda \ge \mathsf{V}^-$ in general. Now if X is regular then convergence with respect to the Vietoris topology implies Kuratowski convergence on $\mathcal{C}(X)$. In fact, given any net $(A_j)_{j\in J}$ which V^+-converges to some $A\in\mathcal{C}(X)$, we show that $\operatorname{Ls}_{j\in J} A_j\subset A$. Indeed, consider an open set G containing A. By our assumption we have $A_j\subset G$ eventually, so that $\operatorname{Ls}_{j\in J} A_j \subset \overline{G}$; as G was arbitrary, it follows that

$$\operatorname*{Ls}_{j\in J} A_j \subset \bigcap\{\overline{G} \mid G \text{ open, } G\supset A\},$$

and the latter set equals A, since X is regular.

We conclude that $\tau\mathsf{K}\le\mathsf{V}$ whence $\tau\mathsf{K}\wedge\lambda \le \mathsf{V}\wedge\lambda = \mathsf{V}^-$, by 10.4 (obviously, every regular space is symmetric). ∎

From the preceding results and Remark 1.4 it follows that if X is regular, then $\tau\mathsf{K}$ is decomposable on $\mathcal{C}(X)$ if and only if $\tau\mathsf{K} = \tau\mathsf{K}^+\vee\mathsf{V}^-$, and this equality in general fails.

13.4. Proposition. *If X is an infinite-dimensional Banach space, then on $\mathcal{C}(X)$ we have $\tau\mathsf{K} > \tau\mathsf{K}^+\vee\mathsf{V}^-$ and therefore $\tau\mathsf{K}$ is not decomposable.*

Proof. See [4, Example 4.5]. ∎

The *co-compact topology* C on $\mathcal{C}(X)$ is generated by the collections of the form G^+, where G is the complement of a compact set. It is clearly a strong (upper) topology. The *Fell topology* is defined as $\mathsf{F} = \mathsf{C} \vee \mathsf{V}^-$ (and hence it is strongly decomposable).

It is well known that $\mathsf{C} \le \tau\mathsf{K}^+$ and that if X is locally compact we have equality. Following [8], we say that X is *consonant* if $\tau\mathsf{K}^+ = \mathsf{C}$ on $\mathcal{C}(X)$ (so that every locally compact space is consonant). We are going to give a new characterization of consonant Hausdorff spaces.

13.5. Lemma. *Let X be a T_2-space. Consider a non-empty collection $\mathcal{A}$ of open subsets of X with the following properties:*

(1) *if $A \in \mathcal{A}$ and B is an open set containing A, then $B \in \mathcal{A}$;*

(2) *if $A \in \mathcal{A}$ and $\mathcal{V}$ is an open cover of A, then there exists a finite $\mathcal{U} \subset \mathcal{V}$ such that $\bigcup \mathcal{U} \in \mathcal{A}$;*

(3) *if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$.*

Then there exists a compact subset K of X such that $\mathcal{A}$ is the collection of all open sets containing K.

Proof. Take $K = \bigcap \mathcal{A}$. We first show that K is closed. Let $x \notin K$. There exists some $A \in \mathcal{A}$ such that $x \notin A$. For every $a \in A$, consider two disjoint open sets U_a and V_a where $x \in U_a$ and $a \in V_a$. Since $\{V_a \mid a \in A\}$ is an open cover of A, it follows from (2) that, for suitable $a_1, \ldots, a_n \in A$, we have $B = \bigcup_{i=1}^n V_{a_i} \in \mathcal{A}$. Letting $U = \bigcap_{i=1}^n U_{a_i}$ we get an open neighborhood of x which is disjoint from B and hence from K.

Next we show that K is compact. Let $\mathcal{W}$ be a collection of open sets whose union contains K; as K is closed, the collection $\mathcal{W}_0 = \mathcal{W} \cup \{X \setminus K\}$ is an open cover of X. Since $X \in \mathcal{A}$ by (1), it follows from (2) that the union of some finite subcollection $\mathcal{Z}_0$ of $\mathcal{W}_0$ belongs to $\mathcal{A}$. Hence $\mathcal{Z} = \mathcal{Z}_0 \setminus \{X \setminus K\}$ is a finite subcollection of $\mathcal{W}$ whose union contains K.

Since it is clear that each member of $\mathcal{A}$ is an open set containing K, we complete the proof by showing that the collection $\mathcal{G}$ of all open sets containing K and not belonging to $\mathcal{A}$ is empty.

Suppose not, and let $\mathcal{H}$ be a non-empty chain in $\mathcal{G}$. If $\bigcup \mathcal{H} \notin \mathcal{G}$ then $\bigcup \mathcal{H} \in \mathcal{A}$ and, by (2), there exists $H \in \mathcal{H}$ such that $H \in \mathcal{A}$, which is impossible. Hence $\mathcal{G}$ satisfies the assumptions of Zorn's lemma and therefore has a maximal element G. Also, $G \neq X$, since $X \in \mathcal{A}$ by (1). Now, there is no $x \in X$ such that $X \setminus \{x\} = G$. Otherwise, by (1), each member of $\mathcal{A}$ would contain x and hence $x \in K$, contrary to the fact that $G \supset K$. Thus it is possible to take two distinct points outside G which have disjoint open neighborhoods V' and V''; as G is maximal, both $A' = G \cup V'$ and $A'' = G \cup V''$ belong to $\mathcal{A}$, but $A' \cap A'' = G \notin \mathcal{A}$, and this contradicts (3). ∎

Note that the above lemma is equivalent to a result (stated without proof) in [8, Prop. 2.2].

13.6. Theorem. *Let X be a T_2-space. Then X is consonant if and only if $\tau\mathsf{K}^+$ is a strong topology on $\mathcal{C}(X)$.*

PROOF. If X is consonant then $\tau\mathsf{K}^+ = \mathsf{C}$, which is a strong topology; thus it remains to prove the converse.

Consider the collection $\mathcal{O}(X)$ of all open subsets of X with the usual (inclusion) ordering. $\mathcal{O}(X)$ is a complete lattice, which is isomorphic to $\mathcal{C}(X)$ via complementation. We denote by $\mathcal{F}^*$ the image of any $\mathcal{F} \subset \mathcal{C}(X)$ under this isomorphism, so that $\mathcal{F}^*$ is the collection of the complements of all members of $\mathcal{F}$.

As $\tau\mathsf{K}^+$ is always finer than C, it suffices to show that every $\tau\mathsf{K}^+$-open subsemilattice of $\mathcal{C}(X)$ must be of the form G^+, where G is the complement of a compact set.

So, take any $\tau\mathsf{K}^+$-open $\mathcal{F} \subset \mathcal{C}(X)$. It follows from 13.1 and 7.2 that $\mathcal{A} = \mathcal{F}^*$ satisfies properties (1) and (2) of the previous lemma; if in addition $\mathcal{F}$ is a semilattice, then $\mathcal{A}$ satisfies (3) as well. Hence $\mathcal{A}$ coincides with the collection of all open sets containing a suitable compact set K, and therefore $\mathcal{F} = G^+$, where $G = X \setminus K$. ∎

Even if $\tau\mathsf{K}^+$ is not a strong topology in general, it is always upper modular.

13.7. PROPOSITION. *$(\tau\mathsf{K}^+ \vee \lambda) \wedge \gamma = \tau\mathsf{K}^+$.*

PROOF. Straightforward consequence of 13.1 and 7.5. ∎

If X is a metrizable space, we will denote by $\mathcal{M}(X)$ the set of all compatible metrics, i.e. the metrics which induce the topology of X. Given any $d \in \mathcal{M}(X)$, the *upper Wijsman topology*, W_d^+, corresponding to d is generated by all the subcollections of $\mathcal{C}(X)$ of the form

$$\mathcal{A}_d^+(a, \varepsilon) = \{C \in \mathcal{C}(X) \mid d(a, C) > \varepsilon\}$$

where a is a point of X, ε is a positive number and we put $d(a, \emptyset) = +\infty$ for every $a \in X$.

The *Wijsman topology* is defined as $\mathsf{W}_d = \mathsf{W}_d^+ \vee \mathsf{V}^-$.

In connection with these two topologies, let us define also B_d^+ and B_d. The *upper ball topology*, B_d^+, is generated by all collections of the form G^+ where G is the complement of a closed d-ball S (i.e. $S = \overline{S}_d(a, \varepsilon) = \{x \in X \mid d(a, x) \le \varepsilon\}$, for some $a \in X$ and $\varepsilon > 0$). The *ball topology* is $\mathsf{B}_d = \mathsf{B}_d^+ \vee \mathsf{V}^-$.

The Wijsman and ball topologies are distinct, in general; conditions under which they coincide can be found in [11].

It is easy to see that W_d^+ is coarser than H_d^+, hence than V^+, and therefore $\mathsf{W}_d \le \mathsf{V}$. Similarly, $\mathsf{B}_d^+ \le \mathsf{V}^+$, thus $\mathsf{B}_d \le \mathsf{V}$. It is easy to see as well that $\mathsf{B}_d^+ \ge \mathsf{W}_d^+ \ge \tau\mathsf{K}^+$ and $\mathsf{B}_d \ge \mathsf{W}_d \ge \tau\mathsf{K}$.

13.8. REMARK. *The topologies W_d^+ and B_d^+ are strong, so that $\mathsf{W}_d \wedge \gamma = \mathsf{W}_d^+$ and $\mathsf{B}_d \wedge \gamma = \mathsf{B}_d^+$. Moreover, $\mathsf{W}_d \wedge \lambda = \mathsf{V}^-$ and $\mathsf{B}_d \wedge \lambda = \mathsf{V}^-$.*

PROOF. As $d(a, C' \cup C'') = \min\{d(a, C'), d(a, C'')\}$ for every $a \in X$ and $C', C'' \in \mathcal{C}(X)$, it follows that W_d^+ is strong. Moreover, we have $\mathsf{V}^- \le \mathsf{W}_d \wedge \lambda \le \mathsf{V} \wedge \lambda = \mathsf{V}^-$.

The proof for the ball topology is even easier. ∎

Observe that the result of [5, Prop. 1(ii)] can be immediately proved by using the fact that the upper Wijsman topology is strong.

13.9. PROPOSITION. *The infimum of a collection of strong upper hypertopologies need not be strong.*

Proof. Let X be the metrizable non-consonant space constructed in [15, Example 9.1]. Then every W_d^+ is a strong upper topology while $\inf_{d\in\mathcal{M}(X)}\mathsf{W}_d^+ = \tau\mathsf{K}^+$ [4, Theorem 6.4], which is not strong by Theorem 13.6. ∎

14. Uniqueness of decomposition for hypertopologies

We are going to see that most of the hyperspace topologies introduced in the previous sections are not uniquely decomposable, even if we restrict ourselves to strong decompositions.

First, we apply the results of Section 8. To this end, let us note that all the lower topologies we deal with are lower compatible.

Moreover, an upper topology T^+ on $\mathcal{C}(X)$ separates A from B if and only if there exist a T^+-open neighborhood $\mathcal{W}$ of A in $\mathcal{C}(X)$ and an open set H such that $F\cap H=\emptyset$ for every $F\in\mathcal{W}$, while $B\cap H\neq\emptyset$. Observe further that, in case X is a T_1-space, T^+ separates A from B if and only if it separates A from $\{y\}$, for some $y\in B$.

14.1. Proposition. (1) *If G and H are two disjoint open subsets of X (with $H\neq\emptyset$), then V^+ separates every $A\in G^+$ from every $B\in H^-$.*

(2) *If X is metrizable then, for every $d\in\mathcal{M}(X)$, the topologies W_d^+, B_d^+ and H_d^+ separate any $A\in\mathcal{C}(X)$ from any $B\not\subset A$.*

(3) *If X is T_2, then C separates A from B provided that there exists a point $y\in B\setminus A$ which has a compact neighborhood.*

Proof. (1) This is clear.

(2) Let y be a point of B which is not in A. Given any $d\in\mathcal{M}(X)$, let $H=S_d(y,\varepsilon)$, where $\varepsilon<d(y,A)$. Then, taking $\mathcal{W}=\mathcal{A}_d^+(y,\varepsilon)$, we have $A\in\mathcal{W}$ and $F\cap H=\emptyset$ for every $F\in\mathcal{W}$. Hence, W_d^+ separates A from B. The same is true for B_d^+ and H_d^+, as they are finer than W_d^+.

(3) Let K be a compact neighborhood of y disjoint from A, and denote the complement and the interior of K by G and H, respectively. Then $\mathcal{W}=G^+$ is a C-open neighborhood of A in $\mathcal{C}(X)$ such that $F\cap H=\emptyset$ for every $F\in\mathcal{W}$, while $B\cap H\ni y$. ∎

Recall that a topological space X is said to be *irreducible* if any two non-empty open subsets of X have non-empty intersection. Thus, if a T_1-space X is not irreducible then, by Proposition 14.1(1), there exist $x,y\in X$ such that V^+ separates $\{x\}$ from $\{y\}$.

14.2. Theorem. *Let X be a T_2-space and T^+ an upper topology on $\mathcal{C}(X)$ for which there exist two points $x,y\in X$ such that T^+ separates $\{x\}$ from $\{y\}$.*

(1) *If n is a natural number such that X has more than $n+1$ points, then there exists a (lower) topology $\mu<\mathsf{V}^-$ on $\mathcal{C}(X)$ such that $\mathsf{T}^+\vee\mu=\mathsf{T}^+\vee\mathsf{V}^-$ and μ coincides with V^- on the collection $\mathcal{E}_n(X)$ of all subsets of X having at most n elements.*

(2) *If X is metrizable and d is a non-totally bounded compatible metric on X, then there exists a (lower) topology $\mu<\mathsf{H}_d^-$ on $\mathcal{C}(X)$ such that $\mathsf{T}^+\vee\mu=\mathsf{T}^+\vee\mathsf{H}^-$ and μ coincides with H_d^- on the collection $\mathcal{TB}(X,d)$ of all closed totally bounded subsets of (X,d).*

(3) *If X is collectionwise normal and non-countably compact, then there exists a (lower) topology $\mu < \mathsf{L}^-$ on $\mathcal{C}(X)$ such that $\mathsf{T}^+ \vee \mu = \mathsf{T}^+ \vee \mathsf{L}^-$ and μ coincides with L^- on the collection $\mathcal{CK}(X)$ of all closed countably compact subsets of X (in particular, on the collection $\mathcal{K}(X)$ of all compact subsets of X).*

Proof. (1) Let $w_1, \ldots, w_n, w_{n+1}$ be distinct points of X, with $w_{n+1} = y$ and $x \neq w_1, \ldots, w_n$. If we can prove that $\mathcal{C}_n(X) = \mathcal{C}(X) \setminus \mathcal{E}_n(X)$ is V^--open, then we will put $\mu = \mathsf{V}^-(\{x\}, C)$ with $C = \{w_1, \ldots, w_n, w_{n+1}\}$, and the result will follow from Theorem 8.4. Let D be any element of $\mathcal{C}_n(X)$. Then there exist distinct points $w'_1, \ldots, w'_{n+1} \in D$ and (as X is T_2) pairwise disjoint neighborhoods $W_1, \ldots, W_{n+1}$ of $w'_1, \ldots, w'_{n+1}$, respectively. Thus $W_1^- \cap \ldots \cap W_{n+1}^-$ is a V^--neighborhood of D which is entirely contained in $\mathcal{C}_n(X)$.

(2) Since X is not totally bounded, there exists an infinite $D \subset X$, that we can suppose non-containing x, which is ε-uniformly discrete for some $\varepsilon > 0$; then $C = D \cup \{y\}$ is δ-uniformly discrete for $\delta = \min\{\varepsilon, d(y, D \setminus \{y\})\}$, and hence $C \notin \mathcal{TB}(X, d)$. Again, what only remains to prove is that $\mathcal{C}(X) \setminus \mathcal{TB}(X, d)$ is H_d^--open.

Let $M \in \mathcal{C}(X)$ and $r > 0$ be such that M cannot be covered by a finite number of open balls of radius r. It is easily shown that no element of $\mathcal{S}_d^-(M, r/2)$ can be covered by a finite number of open balls of radius $r/2$.

(3) Let $\mathcal{A}$ be a countable open cover of X which admits no finite subcover. Fix $A \in \mathcal{A}$ such that $x \in A$. The set $C = (X \setminus A) \cup \{y\}$ is closed and non-countably compact. We now prove that $\mathcal{C}(X) \setminus \mathcal{CK}(X)$ is L^--open.

If C is a closed non-countably compact subset of X, then there exists a closed discrete (countably) infinite subset D of C. By collectionwise normality, there exists a discrete family $\mathcal{B} = \{B_x\}_{x \in D}$ of open subsets of X such that $x \in B_x$ for every $x \in D$. Then every member of $\mathcal{B}^- = \bigcap_{x \in D} B_x^-$ contains a closed discrete infinite subset, and therefore is not countably compact. ■

Let X be a T_2-space with infinitely many points. In view of Theorem 14.2(1), one may ask whether, given an upper topology T^+ on $\mathcal{C}(X)$, a suitable lower topology $\mu' < \mathsf{V}^-$ can be defined in such a way that:

(1) $\mathsf{T}^+ \vee \mu' = \mathsf{T}^+ \vee \mathsf{V}^-$;

(2) μ' agrees with V^- on the collection of all finite sets.

The answer is negative, as we are going to see.

14.3. Theorem. *Let X be a T_1-space and T^+ an upper topology on $\mathcal{C}(X)$.*

(1) *If μ is a lower topology on $\mathcal{C}(X)$ which agrees with V^- on the collection $\mathcal{F}_0$ of all non-empty finite subsets of X and is such that $\mathsf{T}^+ \vee \mu = \mathsf{T}^+ \vee \mathsf{V}^-$, then $\mu = \mathsf{V}^-$.*

(2) *If X is metrizable and $d \in \mathcal{M}(X)$, and μ is a lower topology on $\mathcal{C}(X)$ which agrees with H_d^- on the collection $\mathcal{UD}_0$ of all non-empty uniformly discrete subsets of X and is such that $\mathsf{T}^+ \vee \mu = \mathsf{T}^+ \vee \mathsf{H}_d^-$, then $\mu = \mathsf{H}_d^-$.*

(3) *If μ is a lower topology on $\mathcal{C}(X)$ which agrees with L^- on the collection $\mathcal{CD}_0$ of all non-empty closed discrete subsets of X and is such that $\mathsf{T}^+ \vee \mu = \mathsf{T}^+ \vee \mathsf{L}^-$, then $\mu = \mathsf{L}^-$.*

Proof. We apply Theorem 8.11 and, in view of 8.10(4), it suffices to show that $\mathcal{F}_0$ is V^--upper dense, $\mathcal{UD}_0$ is H_d^--upper dense, and $\mathcal{CD}_0$ is L^--upper dense. The first and third fact are obvious; to prove the second one, observe that, given $C \in \mathcal{C}_0(X)$ and $\varepsilon > 0$, by Zorn's lemma there exists a maximal $(\varepsilon/2)$-uniformly discrete subset D of C, which is clearly non-empty. It turns out that $D \in \mathcal{S}_d^-(C, \varepsilon)$. ∎

We turn now to some cases of decompositions to which Theorem 14.2 does not apply. The first question deals with the upper Vietoris topology on the hyperspace of an irreducible (T_1-)space. It turns out that 14.2(1) and 14.2(3) hold for every $\mathsf{T}^+ \geq \mathsf{V}^+$, even if T^+ fails to separate $\{x\}$ from $\{y\}$, for every $x, y \in X$.

14.4. Proposition. *Let x be a point of a T_1-space X, and let $C \subset X$ be a closed non-empty set which does not contain x. If T^+ is an upper topology finer than V^+ then* $\mathsf{T}^+ \vee \mathsf{V}^-(\{x\}, C) = \mathsf{T}^+ \vee \mathsf{V}^-$.

Proof. We prove that $\mathsf{V}^- \leq \mathsf{T}^+ \vee \mathsf{V}^-(\{x\}, C)$. Let $\mathcal{A} \in \mathsf{V}^-$ and consider any $A \in \mathcal{A}$; we may assume that A is non-empty (as $\emptyset$ is isolated in V^+). If $A \neq \{x\}$, let $\mathcal{U} = \mathcal{A} \cap (X \setminus \{x\})^-$. Then $A \in \mathcal{U} \subset \mathcal{A}$, and $\mathcal{U} \in \mathsf{V}^-(\{x\}, C)$. If $A = \{x\}$, let $U_1, \ldots, U_n$ be open neighborhoods of x such that $U_1^- \cap \ldots \cap U_n^- \subset \mathcal{A}$. Then $X^- \cap (\bigcap_{i=1}^n U_n)^+$ is a $(\mathsf{T}^+ \vee \mathsf{V}^-(\{x\}, C))$-neighborhood of $\{x\}$ which is contained in $\mathcal{A}$. ∎

As already claimed, the above result holds as well with L^- in place of V^-. Indeed, if X is not irreducible we may apply Theorem 14.2(3), and if X is irreducible then $\mathsf{V}^- = \mathsf{L}^-$ as every locally finite collection of open sets is in fact finite.

On the other hand, as we will see at the end of this section, there exists a large class of irreducible T_1-spaces whose hyperspace admits a (lower) topology $\mu > \mathsf{V}^-$ such that for every lower topology μ' with $\mathsf{V}^+ \vee \mu = \mathsf{V}^+ \vee \mu'$ we have $\mu = \mu'$.

Now we are going to consider the Fell topology when every compact subset of the underlying space has empty interior. In view of 14.1(3), Theorem 14.2(1) does not apply to this case.

Recall that a *regular open* subset of a topological space X is an open set $U \subset X$ such that $\operatorname{int}(\operatorname{cl}(U)) = U$. Observe that an open set $A \subset X$ is a regular open set if and only if we have $\operatorname{int}(B \setminus A) \neq \emptyset$ for every open set $B \subset X$ such that $B \setminus A \neq \emptyset$.

14.5. Lemma. *Let S be an open subset of a T_1-space X and consider a family*

$$\{A_{i,j} \mid i \in I,\ j \in J_i\}$$

of open subsets of X, where the index set J_i is finite for each $i \in I$. Then the inclusion

$$\bigcup_{i \in I} \Big(\bigcap_{j \in J_i} A_{i,j}^- \Big) \subset S^-$$

holds in $\mathcal{C}(X)$ if and only if, for every $i \in I$, there exists $j \in J_i$ such that $A_{i,j} \subset S$.

Proof. First suppose that for every $i \in I$ there exists $j \in J_i$ such that $A_{i,j} \subset S$. Take an arbitrary $C \in \bigcup_{i \in I}(\bigcap_{j \in J_i} A_{i,j}^-)$ and let $h \in I$ be such that $C \in \bigcap_{j \in J_h} A_{h,j}^-$; in particular, $C \in A_{h,k}^-$, where k is an element of J_h for which $A_{h,k} \subset S$. Fix $x \in A_{h,k} \cap C$. We have $x \in S$, hence $C \in S^-$.

Conversely, suppose that there exists $\widehat{\imath} \in I$ such that $A_{\widehat{\imath},j} \not\subset S$ for every $j \in J_{\widehat{\imath}}$. If we pick $x_j \in A_{\widehat{\imath},j} \setminus S$ for each $j \in J_{\widehat{\imath}}$, then $C = \{x_j \mid j \in J_{\widehat{\imath}}\}$ is a closed subset of X, and we have $C \notin S^-$ while $C \in \bigcap_{j \in J_{\widehat{\imath}}} A_{\widehat{\imath},j}^- \subset \bigcup_{i \in I}(\bigcap_{j \in J_i} A_{i,j}^-)$. ∎

14.6. Proposition. *Let X be a T_2-space whose compact subsets have empty interior. If the regular open subsets form a base for the topology of X (in particular, if X is regular), then the Fell topology on $\mathcal{C}(X)$ (or on $\mathcal{C}_0(X)$) has a unique strong decomposition.*

Proof. If (π, μ) is a strong decomposition of F, then by 3.9 we have $\pi = \mathsf{C}$; moreover, as $\mathsf{F} \le \mathsf{V}$, it follows from 10.4 that $\mathsf{F} \wedge \lambda = \mathsf{V}^-$ and therefore μ must be coarser than V^-.

Now argue by contradiction, and suppose that $\mu \ne \mathsf{V}^-$. Thus there must be an open set $S \in X$ and a non-empty closed set M intersecting S such that S^- is not a μ-neighborhood of M, and our hypotheses allow us to assume that S is regular open.

Let $\mathcal{U}$ be a basic F-neighborhood of M of the form $(X \setminus K)^+ \cap \mathcal{W}$, where K is a (possibly empty) compact subset of X which is disjoint from M and $\mathcal{W}$ is a μ-open neighborhood of M. As we have seen, $\mathcal{W}$ is V^--open, so that

$$\mathcal{W} = \bigcup_{i\in I} \Big(\bigcap_{j\in J_i} A^-_{i,j} \Big)$$

where each $A_{i,j}$ is an open subset of X and the set J_i is finite for every $i \in I$. Since $\mathcal{W} \not\subset S^-$, Lemma 14.5 implies that, for some $h \in I$ and each $j \in J_h$, we have $A_{h,j} \not\subset S$. Since S is regular open, it follows that $B_j = \mathrm{int}(A_{h,j} \setminus S) \ne \emptyset$ for every $j \in J_h$; thus selecting a point x_j from each $B_j \setminus K$ (which is possible because K has empty interior) and letting

$$D = \{x_j \mid j \in J_h\}$$

defines a closed subset of X such that $D \cap K = \emptyset$ and $D \in \bigcap_{j\in J_h} A^-_{h,j} \subset \bigcup_{i\in I}(\bigcap_{j\in J_i} A^-_{i,j}) = \mathcal{W}$. Therefore $D \in \mathcal{U}$. On the other hand, $D \notin S^-$ hence $\mathcal{U} \not\subset S^-$.

As $\mathcal{U}$ was arbitrary, S^- is not an F-neighborhood of M, a contradiction. ∎

The above argument also allows us to obtain the results of unique decomposition announced after Proposition 14.4.

Let Y be any regular space for which all compact sets have empty interior (for example, the rational or the irrational line), and X be the topological space having the same underlying set of Y, and whose open sets are the complements of the compact subspaces of Y. Then

$$\mu = \{A^- \mid A \text{ is open in } Y\}$$

is a topology on $\mathcal{C}(X)$ which is (strictly) finer than V^-, and it follows from the same argument of the previous proof that for every lower topology μ' on $\mathcal{C}(X)$ such that $\mathsf{V}^+ \vee \mu = \mathsf{V}^+ \vee \mu'$, we have $\mu = \mu'$ (to prove that $\mu' \le \mu$ use the fact that the upper Vietoris topology on $\mathcal{C}(X)$ coincides with the restriction to $\mathcal{C}(X)$ of the co-compact topology on $\mathcal{C}(Y)$).

Observe that the same result can be obtained by supposing Y to be a regular space without isolated points, and giving X the cofinite topology on the underlying set of Y.

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