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B. M. SCHEIN

Stationary subsets, stabilizers and transitive representations of semigroups

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PAŃSTWOWE WYDAWNICTWO NAUKOWE

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WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

§ 0. Introduction. This paper is devoted to function semigroups (here "function" means "partial transformation of a set"). The subject of function semigroups and their connections with abstract semigroups is the central theme of the whole theory of semigroups, which is a natural algebraic tool to study systems of functions and transformations closed under superposition. We are interested in fixed points of functions or, to be more precise, in those functions in a function semigroup which have the same fixed point (or which have some fixed point). If F is a function semigroup, then the set of all $f \in F$ which have the same fixed point, say a, is denoted by F_a and the set of all $f \in F$ which have some fixed point is denoted by F_{st}. F_a is called the stabilizer of a in F, F_{st} is called the stationary subset of F. We are interested in abstract algebraic properties of stabilizers and stationary subsets. In other words, what necessary and sufficient properties must a subset H of a semigroup S satisfy in order that there exist an isomorphism P of S onto a function semigroup and $P(H) = P(S)_a$ for some a (or $P(H) = P(S)_{st}$)? This problem is a typical problem in relation algebras (see [15], [16], [19]); by the fundamental theorem on relation algebras, it possesses a solution which may be written in the first order predicate language. This problem is solved in § 2. It turns out that to solve this problem one must consider a natural generalization of the group-theoretic concept of conjugacy. The problem of characterizing the stationary subsets of permutation groups is solved as a corollary to more general semigroup results.

It is wellknown that every representation of an abstract group by permutations or inverse semigroup by univalent functions is a sum of uniquely defined transitive representations. However, this is not true for representations of arbitrary semigroups by functions. In § 3, we study transitive representations of semigroups by functions. There is given a description of all such representations and special types thereof.

In § 4, we consider symmetric representations (i.e., sums of transitive representations) of semigroups by functions. It turns out that one can define a congruence $\varkappa$ on every semigroup S which is the smallest congruence such that the quotient semigroup $S/\varkappa$ has an isomorphic symmetric representation. $\varkappa$ is analogous to the Jacobson radical of a ring. We give characterizations of $\varkappa$ and consider "semisimple" and "radical" semigroups (i.e., those semigroups for which $\varkappa$ is the identity relation or the universal equivalence).

All the results are obtained by the method of determinative pairs devised in [7]. In § 1 we introduce some definitions and state some facts concerning the method of determinative pairs.

We use some symbols of mathematical logic: $\bigwedge$ and $\bigvee$ are the universal and the existential quantifiers; $\wedge$, $\rightarrow$ and $\leftrightarrow$ are conjunction, implication and (logical) equivalence respectively.

§ 1. Main definitions. A *binary relation* on a set A is any subset $\varrho \subset A \times A$. Sometimes we consider binary relations as *multi-valued functions*: $(a_1, a_2) \in \varrho$ means that a_2 is an *image* of a_1 under ϱ, $\varrho\langle a\rangle$ is the set of all images of a under ϱ. The set of all a such that $\varrho\langle a\rangle \neq \emptyset$ is denoted by $pr_1\varrho$. If $pr_1\varrho = A$, then ϱ is called *full*. If $\varrho\langle a\rangle$ contains at most one element for every $a \in A$, then ϱ is called a *function* or a *partial transformation*. If $A \neq \emptyset$, then full functions are called *transformations*. If $\varrho, \sigma \subset A \times A$, then $\sigma \circ \varrho$ is the *product* of ϱ and σ: $(a_1, a_2) \in \sigma \circ \varrho \leftrightarrow (\bigvee a)\,[(a_1, a) \in \varrho \wedge \wedge (a, a_2) \in \sigma]$; $\overset{-1}{\varrho}$ is the *converse* of ϱ: $(a_1, a_2) \in \overset{-1}{\varrho} \leftrightarrow (a_2, a_1) \in \varrho$.

All concepts of the theory of semigroups which are not defined here are defined in [2] or in [5].

$\mathscr{P}_A, \mathscr{F}_A, \mathscr{T}_A$ and $\mathscr{K}_A$ denote respectively the semigroup of all binary relations on a set A and the subsemigroups of $\mathscr{P}_A$ consisting of all functions, all transformations and all univalent (i.e., one-to-one) functions. The operation in these semigroups is the multiplication of binary relations.

A *representation* of a semigroup S by multi-functions (functions, transformations, univalent functions) is any homomorphism of S into a semigroup $\mathscr{P}_A$ (into a semigroup $\mathscr{F}_A, \mathscr{T}_A, \mathscr{K}_A$) for some set A. An isomorphic representation is called *proper* or *faithful*.

Thus, a mapping P: $S \Rightarrow \mathscr{P}_A$ is a representation if $P(st) = P(t) \circ P(s)$ for all $s, t \in S$. The form of the last equality is due to the fact that in the product $P(t) \circ P(s)$ the first factor is $P(s)$, and in the product st the first factor is s.

If P is a representation, then ε_P denotes the *kernel* of P: ε_P is a congruence relation on S and $s \equiv t(\varepsilon_P) \leftrightarrow P(s) = P(t)$. Clearly, P is proper if and only if $\varepsilon_P = \Delta_S$, the identical equivalence. The quotient semigroup S/ε_P is isomorphic to the image $P(S)$ of S under P.

If P is a representation, then $\tau_P = \bigcup_{s \in S} P(s)$ is the *transitivity relation* of P: $(a_1, a_2) \in \tau_P$ means that $(a_1, a_2) \in P(s)$ for some $s \in S$. P is called *transitive* if $\tau_P = A \times A$, the universal equivalence. Define a quasi-order (i.e., reflexive and transitive) relation ζ_P on S by the formula: $s \precsim t(\zeta_P) \leftrightarrow P(s) \subset P(t)$. Clearly, $\varepsilon_P = \zeta_P \cap \overset{-1}{\zeta}_P$ and P is proper if and only if ζ_P is an order relation. ζ_P is called the *fundamental quasi-order* of P.

If F and H are subsets of a semigroup S, then FH is their *global product*, i.e., the set of all products st where $s \in F$ and $t \in H$. We write Hs

instead of $H\{s\}$. A subset H is called *stable* (or a *subsemigroup*) if $HH \subset H$. By definition, $\emptyset$ is a subsemigroup. H is called *quasi-stable* if $s \in H \to s^n \in H$ for all $s \in S$ and all natural n. H is called *reflexive* [22] if $st \in H \to ts \in H$ for all $s, t \in S$. A subsemigroup H is called *left unitary* [3] if $s, st \in H \to t \in H$ for all $s, t \in S$; a subset H is called *strong* [3] if $xv, uv, uy \in H \to xy \in H$. S^I denotes S with an adjoined identity I, $S^1 = S$ whenever S possesses an identity and $S^1 = S^I$ otherwise.

For every subset $H \subset S^1$ one may define an equivalence relation ε_H by the formula: $s \equiv t(\varepsilon_H) \leftrightarrow (\bigwedge x \in S^1)\,[sx \in H \leftrightarrow tx \in H]$ and a subset W_H by the formula: $s \in W_H \leftrightarrow (\bigwedge x \in S^1)\,[sx \notin H]$. Clearly, W_H is a right ideal of S^1 and ε_H is right regular, i.e., $s \equiv t(\varepsilon_H) \to su \equiv tu(\varepsilon_H)$ for all $s, t, u \in S^1$. H is called *right neat* if $W_H = \emptyset$. If $W_H \neq \emptyset$, then W_H is an ε_H-class. Equivalences ε_H were introduced in [3] and studied in [21].

A *determinative pair for a semigroup* S is any pair (ε, W) where ε is a right regular equivalence on S^1 and W is either empty or a right ideal of S^1 and an ε-class. In [7], where the determinative pairs were introduced, we considered a semigroup S with an adjoined identity (even if S had an identity) instead of S^1. This difference is important; however, for our present purposes we can consider S^1.

Let (ε, W) be a determinative pair for S and $\mathfrak{H}$ the set of all ε-classes distinct from W. For every $s \in S$ define a binary relation $P_{(\varepsilon, W)}(s)$ on $\mathfrak{H}$: $(H_1, H_2) \in P_{(\varepsilon, W)}(s) \leftrightarrow H_1 s \subset H_2$. Then $P_{(\varepsilon, W)}$ is a representation of S by functions [7] which is called the *simple representation associated with* (ε, W). According to [13], the simple representation $P_{(\varepsilon_H, W_H)}$ is called *elementary* and denoted by P_H. Here H is an arbitrary subset of S^1.

Let P and Q be representations of a semigroup S by binary relations on sets A and B, and let θ be a bijection of A onto B. θ is called a *similarity* between P and Q (and P, Q are called *similar*) if

$$(a_1, a_2) \in P(s) \leftrightarrow \big(\theta(a_1), \theta(a_2)\big) \in Q(s) \quad \text{for all } s \in S \text{ and } a_1, a_2 \in A.$$

Clearly, the kernels and fundamental quasi-orders of similar representations coincide.

Let $(P_i)_{i \in I}$ be a family of representations of a semigroup S by multi-functions acting on sets $(A_i)_{i \in I}$. Let A be the sum [1] of the family $(A_i)_{i \in I}$, i.e., $A = \bigcup (A_i \times \{i\})_{i \in I}$. Define $\big((a_1, i_1), (a_2, i_2)\big) \in P(s) \leftrightarrow i_1 = i_2 \wedge (a_1, a_2) \in P_{i_1}(s)$ for all $(a_1, i_1), (a_2, i_2) \in A$ and $s \in S$. Then P is a representation of S by multi-functions acting on A. P is called the *sum* of $(P_i)_{i \in I}$ and denoted by $\sum (P_i)_{i \in I}$. $P(s)$ is the multi-function acting on the sum of $(A_i)_{i \in I}$, on each A_i $P(s)$ acts as $P_i(s)$. Clearly, P is a representation by functions (transformations, univalent functions) if so are P_i. The fundamental quasi-order and the kernel of P are given by the formulas:

$$\zeta_{\Sigma(P_i)_{i \in I}} = \bigcap (\zeta_{P_i})_{i \in I}; \qquad \varepsilon_{\Sigma(P_i)_{i \in I}} = \bigcap (\varepsilon_{P_i})_{i \in I}.$$

A representation P is called *symmetric* if τ_P is symmetric, i.e., if $\overset{-1}{\tau_P} = \tau_P$. If $P(s) = \emptyset$ for all $s \in S$, then P is called a *null representation. A representation is symmetric if and only if it is similar to a sum of transitive and null representations* [11].

If $H \subset S$, then H^1 is a subset of S^1: $H^1 = H \cup \{1\}$.

The kernel of an elementary representation P_H is the congruence $\underset{H}{\varepsilon}$ defined by the formula [7]: $s \equiv t(\underset{H}{\varepsilon}) \leftrightarrow (\bigwedge x, y \in S^1)\ [xsy \in H \leftrightarrow xty \in H]$. $\underset{H}{\varepsilon}$ is the largest congruence included in ε_H. A subset $H \subset S$ is called *disjunctive* if $\underset{H}{\varepsilon} = \Delta_S$. If H is right neat, then P_H is a representation by transformations; if H is disjunctive, then P_H is a proper representation.

Let P and Q be representations of a semigroup S by transformations of sets A and B, and let θ be a mapping of A onto B. θ is called a *representative* homomorphism of P onto Q if in order that $(b_1, b_2) \in Q(s)$ it is necessary and sufficient that $(a_1, a_2) \in P(s)$ for some $a_1, a_2 \in A$ such that $\theta(a_1) = b_1$ and $\theta(a_2) = b_2$ [25]. If θ is one-to-one on an element of A (i.e., if there exists an element $a \in A$ such that $\theta(a) = \theta(a_1) \rightarrow a = a_1$ for all $a_1 \in A$), then θ is called a *principal* representative homomorphism. If θ is a (principal) representative homomorphism, then Q is called a *(principal) representative homomorphic image* of P.

A subset H is called *saturated* relative to an equivalence ε if H is a union of ε-classes.

Let ϱ be a binary relation on a set A. An element $a \in A$ is called *fixed* by ϱ if $(a, a) \in \varrho$. If F is a set of binary relations, then F_a denotes the set of all $\varrho \in F$ which fix a, and F_{st} denotes the set of all $\varrho \in F$ which fix some $a \in A$. Thus, $F_{st} = \bigcup (F_a)_{a \in A}$.

A subset H of a semigroup S is called a *multi-stabilizer* if there exists a representation P of S by multi-functions acting on a set A such that $H = \overset{-1}{P}(P(S)_a)$ for some $a \in A$ (i.e., if $s \in H \leftrightarrow (a, a) \in P(s)$). This being the case, H is called a *stabilizer* (*full stabilizer*, *reversible stabilizer*) if P is a representation by functions (transformations, univalent functions). If P is a proper representation, the stabilizer (multi-stabilizer, full or reversive stabilizer) is called *proper*.

A subset H of a semigroup S is called (*multi-*) *stationary* if there exists a representation P of S by (multi-) functions such that $H = \overset{-1}{P}(P(S)_{st})$ (i.e., $s \in H \leftrightarrow P(s)$ has a fixed point). If P is a representation by full (univalent) functions, then H is called *full* (*reversibly*) *stationary*. If P is proper, the corresponding stationary subset is called *proper*.

Thus, we have defined eight types of stabilizers and eight types of stationary subsets of a semigroup. These definitions are not inner; they use some objects (transformations etc.) which lie outside our semigroup.

Inner characterizations of stabilizers and stationary subsets will be given in § 2.

§ 2. Characterizations of stabilizers and stationary subsets.

THEOREM 1. *Let H be a subset of a semigroup S. The following three conditions are equivalent*:

(1) *H is a multi-stabilizer*;

(2) *H is a proper multi-stabilizer*;

(3) *H is a subsemigroup*.

Proof. The implications (2) $\to$ (1) $\to$ (3) follow from the definitions. The only non-trivial part of the theorem states that every subsemigroup is a proper multi-stabilizer of the semigroup.

Let H be a subsemigroup of a semigroup S and let $\hat{\Lambda}$ be the derivative ([9], [12]) of the canonical representation of S by right translations of S^I. Clearly, $\hat{\Lambda}$ is a proper representation by multi-functions. If $s \in H$, then $H^I s = Hs \cup \{s\} \subset H^I$, whence $(H^I, H^I) \in \hat{\Lambda}(s)$. On the other hand, $(H^I, H^I) \in \hat{\Lambda}(s)$ means that $H^I s \subset H^I$, whence $s = Is \in H^I$ and $s \in H$.

THEOREM 2. *Let H be a subset of a semigroup S. The following five conditions are equivalent*:

(1) *H is a stabilizer*;

(2) *H is a proper stabilizer*;

(3) *H is a full stabilizer*;

(4) *H is a proper full stabilizer*;

(5) *H is a left unitary subsemigroup*.

Proof. The implications (4) $\to$ (3) $\to$ (1) and (4) $\to$ (2) $\to$ (1) are evident. If H is a stabilizer, then H is a multi-stabilizer and, by Theorem 1, H is stable. Let $s, st \in H$ and let P be the representation of S by functions such that $H = \overset{-1}{P}\big(P(S)_a\big)$ for some a. Then $(a, a) \in P(s)$ and $(a, a) \in P(st) = P(t) \circ P(s)$. It follows that $a = \big(P(t) \circ P(s)\big)(a) = P(t)\big(P(s)(a)\big) = P(t)(a)$, i.e., $(a, a) \in P(t)$ and $t \in H$. Therefore, (1) implies (5).

It remains to prove that (5) implies (4). Let H be a left unitary subsemigroup of S. Consider the simple representation $P_{(\varepsilon_{H^1}, \emptyset)}$ of S by transformations. If $s \equiv t(\varepsilon_{H^1})$ then $s = s1 \in H^1 \leftrightarrow t = t1 \in H^1$. It follows that H^1 is saturated relative to ε_{H^1}. Let $s, t \in H^1$. By left unitariness and stability of H, $sx \in H^1$ implies $x \in H^1$, which implies $tx \in H^1$. In the same way $tx \in H^1 \to sx \in H^1$, i.e., $s \equiv t(\varepsilon_{H^1})$. It follows that H^1 is an ε_{H^1}-class. We have used the fact that H^1 is a left unitary subsemigroup of S^1. Indeed, if $S = S^1$ and $s \in H$, then $s, s1 \in H$, whence $1 \in H$ and $H = H^1$. If $H = \emptyset$, then $H^1 = \{1\}$, which is a left unitary subsemigroup. If $S \neq S^1$ then the left unitariness and stability of H^1 are verified by direct computation.

Now $H^1 s \subset H^1 \leftrightarrow s \in H^1$ (the proof of this fact was given in the proof of Theorem 1). Thus, $s \in H^1 \leftrightarrow (H^1, H^1) \in P_{(\varepsilon_{H^1}, \emptyset)}(s)$. It follows that H is a full stabilizer of S. Now let P_1 be any proper representation of S by transformations. Then the sum of P_1 and $P_{(\varepsilon_{H^1}, \emptyset)}$ is a proper representation P of S by transformations and H is a stabilizer for P, whence H is a proper full stabilizer.

A subsemigroup H of a semigroup S is called *unitary* if H is left unitary and right unitary (the latter means that $s, ts \in H \to t \in H$).

PROPOSITION 1. *A non-empty subset H of a semigroup S is a unitary strong subsemigroup of S if and only if H^1 is a strong subsemigroup of S^1.*

Proof. Let H be a unitary strong subsemigroup of S. Then H^1 is a unitary subsemigroup of S^1. Let $xv, uv, uy \in H^1$. If $u, v, x, y \in S$ then $xv, uv, uy \in H$ and $xy \in H \subset H^1$ since if one of the elements xv, uv, uy equals 1, then $1 \in S$, $S = S^1$ and, by the unitariness of H, $1 \in H$. Let $x = 1$. Then $v, uv \in H^1$, whence $u \in H^1$. Now $u, uy \in H^1$ implies $xy = y \in H^1$. In the same way $y = 1 \to xy = x \in H^1$. Now let $u = 1$. Then $xv, v \in H^1$, whence $x \in H^1$ and $y = uy \in H^1$. Thus, $xy \in H^1$. In the same way, $v = 1 \to xy \in H^1$. Therefore, H^1 is a strong subsemigroup of S^1.

Now let H^1 be a strong subsemigroup of S^1. If F is a strong subset of a semigroup S and G is a subsemigroup of S, then $F \cap G$ is a strong subset of G [7]. Hence $H = H^1 \cap S$ is a strong subsemigroup of S. Let $s, st \in H$. Then $1 \cdot 1, s1, st \in H$, whence, $t = 1t \in H$ and H is left unitary. In the same way we prove that H is right unitary.

THEOREM 3. *A subset H of a semigroup S is a (proper) reversible stabilizer if and only if H^1 is a strong subsemigroup of S^1 (and S has a proper representation by univalent functions).*

Proof. To prove necessity, let P be a representation of S by univalent functions and $(a, a) \in P(xv) \cap P(uv) \cap P(uy)$. Then $(a, a) \in P(uy) \circ \circ \overline{P(uv)}^{-1} \circ P(xv) = P(y) \circ P(u) \circ \overline{P(u)}^{-1} \circ \overline{P(v)}^{-1} \circ P(v) \circ P(x) \subset P(y) \circ \Delta_A \circ \Delta_A \circ \circ P(x) = P(y) \circ P(x) = P(xy)$. It follows that reversible stabilizers are strong.

Now let H^1 be a strong subsemigroup of S^1. In precisely the same way as in the proof of Theorem 2, we may prove that H is the stabilizer for the elementary representation P_{H^1} of S. It is known [7] that P_{H^1} is a representation by univalent functions, since H^1 is strong. If S has a proper representation P by univalent functions, then H is a stabilizer for the sum of P and P_{H^1}, hence H is a proper reversible stabilizer.

Thus, among the eight types of stabilizers only four types are distinct.

A semigroup S is called a *(left) nil-semigroup* [14] if for every $s \in S$ there exists a natural n such that s^n is a (left) zero of S. Clearly, S is a nil-semigroup if and only if it is a left and right nil-semigroup.

LEMMA 1. *A semigroup possesses no proper (left) unitary subsemigroups if and only if it is a (left) nil-semigroup.*

Proof. Let S be a left nil-semigroup and H be a non-empty left unitary subsemigroup of S. If $s \in H$ then $s^n \in H$ for all n, whence H contains a left zero of S, say, t. For every $u \in S$ $tu = t \in H$, whence $u \in H$ and $H = S$.

Let S be an arbitrary semigroup and $s \in S$. Denote by $\bar{s}$ the set of all $t \in S$ such that $s^m t = s^n$ for some m, n. If $t_1, t_2 \in \bar{s}$, then $s^m t_1 = s^n$ and $s^p t_2 = s^q$ for some natural m, n, p, q. Therefore, $s^{m+p}(t_1 t_2) = s^p(s^m t_1)t_2 = s^p s^n t_2 = s^n(s^p t_2) = s^n s^q = s^{n+q}$ and $t_1 t_2 \in \bar{s}$. Let $t, tu \in \bar{s}$, i.e., $s^m t = s^n$ and $s^p(tu) = s^q$ for some m, n, p, q. Then $s^{p+n}u = s^{p+m}tu = s^{m+q}$, whence $u \in \bar{s}$. Clearly, $s \in \bar{s}$. Thus, $\bar{s}$ is a non-empty left unitary subsemigroup of S. Suppose S has no proper left unitary subsemigroups. Then $\bar{s} = S$ for all $s \in S$, i.e., for every $s, t \in S$ there exist m and n such that $s^m t = s^n$. Let $s = t^p$. Then $t^{mp}t = t^{np}$. Thus, for every $t \in S$ and every natural p there exist m and n such that $t^{mp+1} = t^{np}$. It follows that t is a periodical element. Let b be the period of t (i.e., $t^{q+b} = t^q$ for all sufficiently large q, and b is the smallest positive integer having this property). Then b is a divisor of $(mp+1)-np = (m-n)p+1$. This is impossible if $b \neq 1$ (suppose $p = b!$). Therefore, $b = 1$ and $t^{q+1} = t^q$ for all sufficiently large q. Now let $t^i s = t^j$. Then $t^q s = t^{q+i}s = t^{q+j} = t^q$, i.e., t^q is a left zero of S. Thus, S is a left nil-semigroup.

PROPOSITION 2. *The following three conditions are equivalent for every semigroup S:*

(1) *for every representation P of S by transformations all the transformations from $P(S)$ have a common fixed point;*

(2) *for every proper representation P of S by transformations all the transformations from $P(S)$ have a common fixed point;*

(3) *S is a left nil-semigroup.*

Proof. The implication (1) $\to$ (2) is clear. If (2) is satisfied, then S has no proper left unitary subsemigroups. By Lemma 1, (3) is satisfied. Now let (3) be satisfied and let P be a representation of S by transformations of a set A. If $a \in A$, then $\overset{-1}{P}(P(S)_a)$ is a left unitary subsemigroup of S. If $\overset{-1}{P}(P(S)_a) = S$, then a is a common fixed point for all transformations from $P(S)$. Suppose $\overset{-1}{P}(P(S)_a) = \emptyset$. Then $P(S)_a = \emptyset$. Let s be a left zero of S. Then $s^2 = s$ and $P(s) \circ P(s) = P(s)$. If $(a, a_1) \in P(s)$, then $(a_1, a_1) \in P(s)$ and $P(s) \in P(S)_{a_1} \neq \emptyset$. Thus, a_1 is a common fixed point for all transformations from $P(S)$.

PROPOSITION 3. *For every semigroup S the following two conditions are equivalent:*

(1) *for every proper representation P of S by functions all the functions from $P(S)$ have a common fixed point*;

(2) *S is a left nil-semigroup which is not a nil-semigroup.*

Proof. If (1) is satisfied, then condition (2) of Proposition 2 is satisfied and S is a left nil-semigroup. If S is a nil-semigroup, then S possesses a zero 0. Let P be the simplest representation of S corresponding to the determinative pair $(\Delta_{S^1}, \{0\})$. Then P is proper and $P(0) = \emptyset$; thus, condition (1) is not satisfied. Therefore, (1) implies (2).

Now let (2) be satisfied and let P be a proper representation of S by functions acting on a set A. If $a \in A$, then $\overset{-1}{P}(P(S)_a)$ is a left unitary subsemigroup of S. By Lemma 1, S has no proper left unitary subsemigroups. Thus, either a is a common fixed point for all functions from $P(S)$, or $P(S)_a = \emptyset$. By (2), S possesses two distinct left zeros, say, s and t. Suppose that $P(s) = \emptyset$. Then $P(t) = P(ts) = \emptyset \circ P(t) = \emptyset$ and $s = t$. Therefore, $P(s) \neq \emptyset$. It follows that $(a_1, a_1) \in P(s)$ for some $a_1 \in A$. Therefore, $P(S)_{a_1} = P(S)$.

Corollary. *For every semigroup S there exists such a representation P of S by functions that the functions from $P(S)$ have no common fixed point.*

Proof. Consider a null representation.

A corresponding result for representations by univalent functions will be obtained in § 4.

To characterize stationary subsets of semigroups we need some preliminary definitions. We are going to generalize for arbitrary semigroups the concept of an invariant subset of a group. A subset H of a group G is invariant if $g^{-1}Hg = H$ for all $g \in G$. This definition is not applicable to semigroups and an obvious equivalent $Hg = gH$ for all $g \in G$ does not suit us.

Let S be a semigroup and $s, t \in S$. The element t is called a *conjugate* for s if $sx = xt$ for some $x \in S^1$; t is called a *strong conjugate* for s if $sx = xt$ for some $x \in S^1$ and $xy = s^n$ for some $y \in S$ and natural n. If t is a conjugate for s, we shall write $s \vdash t$. The strong conjugacy relation will be denoted by $\vDash$. Instead of $s \vdash t$ we shall write also $(s, t) \in \beta$; instead of $s \vDash t$ we shall write also $(s, t) \in \beta_0$.

Lemma 2. *The relations of conjugacy and of strong conjugacy are quasi-stable quasi-order relations and $\beta_0 \subset \beta$.*

Proof. For every $s \in S$ $ss = ss$ and $ss = s^2$, whence $s \vdash s$ and $s \vDash s$. Let $s \vdash t \vdash u$ $(s \vDash t \vDash u)$. Then $sx = xt$ and $ty = yu$ (and $xa = s^n$, $yb = t^m$ for some $a, b \in S$ and natural m, n) for some $x, y \in S^1$. Therefore, $s(xy) = xty = (xy)u$, i.e., $s \vdash u$ (and $xyba = xt^m a = xtt^{m-1}a = sxt^{m-1}a = s(xt)t^{m-2}a = s^2xt^{m-2}a = \ldots = s^{m-1}xta = s^m xa = s^{m+n}$, whence

$s \models u$). Thus, $\vdash$ and $\models$ are reflexive and transitive, i.e., they are quasi-order relations.

If $s \vdash t$, then $sx = xt$ for some $x \in S^1$. Therefore, $s^n x = s^{n-1}sx = s^{n-1}xt = s^{n-2}sxt = \ldots = xt^n$ and $s^n \vdash t^n$. If $xy = s^m$, then $x(ys^{mn-m}) = s^{mn}$, i.e., $x \models y \to x^n \models y^n$. Thus, $\vdash$ and $\models$ are quasi-stable subsets of the semigroup $S \times S$. The inclusion $\beta_0 \subset \beta$ is evident.

A subset H of a semigroup S is called (*weakly*) *invariant* if H contains all (strong) conjugates for all of its elements, i.e., if

$$s \in H \wedge s \vdash t \to t \in H \quad (s \in H \wedge s \models t \to t \in H).$$

We are going to consider invariant subsets which contain all — positive and negative — powers of each of their elements. In the case of groups the statement about negative powers of an element has an evident sense. However, we must define what we mean in the case of arbitrary semigroups.

Let γ and γ_0 be two binary relations on a semigroup S. By definition, $(s, t) \in \gamma$ means that $sxt = x$ for some $x \in S$; $(s, t) \in \gamma_0$ means that $sxt = x$, $xy = s^n$ for some $x, y \in S$ and natural n. γ and γ_0 are called the *inverse conjugacy relation* and the *strong inverse conjugacy relation* respectively. A subset H of a semigroup S is called (*weakly*) *inversely invariant* if it contains all (strong) inverse conjugates for all of its elements, i.e., if

$$s \in H \wedge (s, t) \in \gamma \to t \in H \quad (s \in H \wedge (s, t) \in \gamma_0 \to t \in H).$$

If S is a group, $\beta = \beta_0$ and $\gamma = \gamma_0$, $(s, t) \in \gamma$ means that $s^{-1} \vdash t$. H is inversely invariant if $s^{-1}Hs = H^{-1}$ (here $t \in H^{-1} \leftrightarrow t^{-1} \in H$), i.e., H is both invariant and symmetric, i.e., $H = H^{-1}$. Thus, in the case of a group the invariance follows from the inverse invariance. This does not hold for semigroups. E.g., every subset of a free semigroup is inversely invariant but not every subset is invariant (a subset H of a free semigroup is invariant if and only if H contains all elements obtained from the elements of H by cyclic permutations of letters from which those elements are constructed: $s_1 s_2 \ldots s_n \in H \to s_n s_1 s_2 \ldots s_{n-1} \in H$).

The relations of (strong) conjugacy and inverse conjugacy deserve a more detailed study, which is beyond the scope of the present paper. We mention that γ and γ_0 are quasi-stable, $\overset{2}{\gamma} \subset \beta$, $\overset{2}{\gamma_0} \subset \beta_0$, $\overset{3}{\gamma} \subset \gamma$, $\overset{3}{\gamma_0} \subset \gamma_0$, $\gamma \circ \beta = \beta \circ \gamma = \gamma$ and $\gamma_0 \circ \beta_0 = \beta_0 \circ \gamma_0 = \gamma_0$. Here $\overset{2}{\varrho} = \varrho \circ \varrho$ and $\overset{3}{\varrho} = \varrho \circ \varrho \circ \varrho$. In a left cancellative semigroup β and β_0 are symmetric, i.e., such a semigroup is partitioned into classes of mutually (strongly) conjugate elements. If S is a group, then the congruence on S defined by the commutator of S is precisely the smallest congruence containing

β (i.e., such that conjugate elements are congruent). The smallest semigroup congruence containing β (or β_0) may be called the (*weak*) *commutator* of the semigroup. The quotient semigroup of a semigroup by the (weak) commutator is commutative and has many other properties.

THEOREM 4. *A subset H of a semigroup S is stationary if and only if it is quasi-stable, weakly invariant, weakly inversely invariant and contains every element which has a left zero in H — the latter condition means that*

$$s \in H \wedge st = s \to t \in H.$$

A subset H is proper stationary if and only if it is quasi-stable, weakly invariant, weakly inversely invariant and contains every element having a left zero which is not a zero of S — the latter condition means that

$$st = s \neq 0 \to t \in H.$$

A subset H is full stationary if and only if H^1 is quasi-stable, invariant and inversely invariant. Every full stationary subset is proper full stationary.

Proof. Necessity. Let H be a stationary subset of S, i.e., $H = \overset{-1}{P}\big(P(S)_{st}\big)$ for some representation P of S by functions acting on a set A. If $s \in H$, then $(a, a) \in P(s)$ for some $a \in A$. It follows that $(a, a) \in P(s^n)$ and $s^n \in H$ for all n. Thus, H is quasi-stable. If $st = s$ then $(a, a) \in P(st) = P(t) \circ P(s)$ and $(a, a) \in P(t)$, i.e., $t \in H$. Thus, H contains every element having a left zero in H.

Now let $sx = xt$ and $s \in H^1$ if P is a representation by transformations and let $sx = xt$ and $xy = s^n$, $s \in H$ if P is a representation by functions. Then $(a_1, a_1) \in P(s)$ for some $a_1 \in A$. If $P(x)$ is a transformation ($P(1)$ is Δ_A in the case where $1 \notin S$), then $a_1 \in pr_1 P(x)$. Otherwise, $(a_1, a_1) \in P(s^n) = P(xy) = P(y) \circ P(x)$ and $a_1 \in pr_1 P(x)$. Thus, $(a_1, a_2) \in P(x)$ for some $a_2 \in A$. Therefore, $(a_1, a_2) \in P(x) \circ P(s) = P(sx) = P(xt) = P(t) \circ P(x)$. It follows that $(a_2, a_2) \in P(t)$ and $t \in H$. Thus, H is weakly invariant (H is invariant if P is a representation by transformations).

Now let $sxt = x$ for $x \in S$ and $s \in H$ and, if P is not a representation by transformations, then let $xy = s^n$ for some $y \in S$ and natural n. Then $(a_1, a_1) \in P(s)$ for some $a_1 \in A$. If $P(x)$ is a transformation, then $a_1 \in pr_1 P(x)$. Otherwise, $(a_1, a_1) \in P(s^n) = P(xy) = P(y) \circ P(x)$ and $a_1 \in pr_1 P(x)$. Thus, $(a_1, a_2) \in P(x)$ for some $a_2 \in A$. Hence, $(a_1, a_2) \in P(x) = P(sxt) = P(t) \circ \circ P(x) \circ P(s)$. Since $(a_1, a_2) \in P(x) \circ P(s)$, we obtain $(a_2, a_2) \in P(t)$ and $t \in H$. Therefore, H is weakly inversely invariant (inversely invariant if P is a representation by transformations).

If P is proper, then $P(s) = \emptyset$ implies that $P(s)$ is a zero of $P(S)$ and s is a zero of S. Let $st = s \neq 0$. Then $P(s) \neq \emptyset$, i.e., $(a_1, a_2) \in P(s)$ for some $a_1, a_2 \in A$. Therefore, $(a_1, a_2) \in P(st) = P(t) \circ P(s)$ and $(a_2, a_2) \in P(t)$, whence, $t \in H$.

Sufficiency will be proved in a series of lemmas.

One can easily see that the intersection of any family of left unitary subsemigroups of a semigroup S is a left unitary subsemigroup. Thus, left unitary subsemigroups define a closure operation on the set of subsets of S^1. Let $\langle H\rangle$ denote the smallest left unitary subsemigroup of S^1 containing a subset $H \subset S^1$, $\langle s\rangle = \langle\{s\}\rangle$.

LEMMA 3. *$t \in \langle s\rangle$ if and only if $s^m t = s^n$ for some natural m and n.*

Proof. In the proof of Lemma 1 we have seen that the set $\bar{s}$ of all t such that $s^m t = s^n$ for some n and m is a left unitary subsemigroup containing s. Thus, $\langle s\rangle \subset \bar{s}$. Let H be a left unitary subsemigroup of S^1 and $s \in H$. Then $s^m \in H$ for all m. If $s^m t = s^n$, then $s^m, s^m t \in H$. Thus, $t \in H$ and $\bar{s} \subset H$. Therefore, $\bar{s} \subset \langle s\rangle$ and $\langle s\rangle = \bar{s}$.

Let H be a left unitary subsemigroup of S^1. In the proof of Theorem 2 we have seen that H is a class modulo ε_H. Let η_H denote the smallest right regular equivalence relation on S such that H is an η_H-class.

LEMMA 4. *$s \equiv t(\eta_H)$ if and only if there exist a chain of elements $h_1, h_2, \ldots, h_{2n} \in H$ and a chain of elements $s = x_1, x_2, \ldots, x_{n+1} = t \in S^1$ such that $h_1 x_1 = h_2 x_2$, $h_3 x_2 = h_4 x_3, \ldots, h_{2n-1} x_n = h_{2n} x_{n+1}$.*

Proof. Let η be the binary relation defined as follows: $(s, t) \in \eta$ if and only if there exist chains of elements connecting s and t in the way mentioned in Lemma 4. Clearly, η is reflexive, symmetric and transitive, i.e., η is an equivalence relation. Moreover, η is right regular since all the elements x_i may be simultaneously multiplied by the same element. If $s \in H$, then $h_2 x_2 = h_1 x_1 \in H$, whence $x_2 \in H$. In the same way, $x_2 \in H \to x_3 \in H$ etc. As a result we have $t = x_{n+1} \in H$. Now let $s, t \in H$. Then $1h_1 = h_1 1$, $h_2 1 = 1h_2$, whence $(h_1, h_2) \in \eta$. Therefore, H is an η-class. Let ε be a right regular equivalence relation, let H be an ε-class and $(s, t) \in \eta$. Then $h_{2i} x_{i+1} \equiv h_{2i+1} x_{i+1} (\varepsilon)$ for all $i = 1, 2, \ldots, n-1$. Therefore, $h_1 x_1 \equiv h_{2n} x_{n+1} (\varepsilon)$. Since $s = 1s \equiv h_1 x_1 (\varepsilon)$ and $t = 1t \equiv h_{2n} t (\varepsilon)$, we have $s \equiv t(\varepsilon)$, whence $\eta \subset \varepsilon$. Thus, $\eta = \eta_H$.

LEMMA 5. *Let H be a left unitary subsemigroup of S^1. Then W_H is saturated relative to η_H.*

Proof. Let $s \equiv t(\eta_H)$ and $s \notin W_H$. Then $sx \in H$ for some $x \in S^1$. It follows that $sx \equiv tx(\eta_H)$. Since H is an η_H-class, $tx \in H$ and $t \notin W_H$. Thus, the complement of W_H is saturated relative to η_H and Lemma 5 is proved.

Define $\nu_H = \eta_H \cup (W_H \times W_H)$. Then ν_H is an equivalence relation on S_1, the classes modulo ν_H are W_H (if $W_H \neq \emptyset$) and all those η_H-classes which are not included in W_H. Since η_H is right regular, W_H is a right ideal saturated relative to η_H, we infer that ν_H is right regular and (ν_H, W_H) is a determinative pair for S. The simplest representation of S associated

with this determinative pair will be denoted by Γ_H and called the *principal representation* of S associated with a left unitary subsemigroup H.

LEMMA 6. *$s \equiv t(\eta_{\langle u\rangle})$ if and only if $u^m s = u^n t$ for some natural m and n.*

Proof. If $u^m s = u^n t$, then, by Lemma 4, $s \equiv t(\eta_{\langle u\rangle})$. Now let $s \equiv t(\eta_{\langle u\rangle})$. Use Lemma 4 choosing shortest of the chains of elements mentioned in Lemma 4. For every h_i there exist natural m_i and n_i such that $u^{m_i} h_i = u^{n_i}$. Multiplying these equalities by u several times on the left we obtain the equalities $u^m h_i = u^{p_i}$ where m is the maximal among the numbers $m_1, m_2, \ldots, m_{2n}$. Multiplying all the equalities $h_{2i-1}x_i = h_{2i}x_{i+1}$ by u^m on the left, we obtain $u^{p_{2i-1}}x_i = u^{p_{2i}}x_{i+1}$. If $n > 1$, then the first two equalities imply $u^{p_1+p_3}x_1 = u^{p_2+p_3}x_2 = u^{p_2+p_4}x_3$, i.e., $u^{p_1+p_3}x_1 = u^{p_2+p_4}x_3$ and the first two equalities may be replaced by one equality. This contradicts the minimality of n. Thus, $n = 1$ and $u^{p_1}s = u^{p_2}t$.

Now let H be quasi-stable, weakly invariant and weakly inversely invariant, and let it contain right identities of its elements (i.e., $s \in H \wedge st = s \to t \in H$). Consider the representation $P = \sum (\Gamma_{\langle s\rangle})_{s\in H}$ of S. If $s \in H$ then $\langle s\rangle s \subset \langle s\rangle$, i.e., $\langle s\rangle$ is a fixed point of $\Gamma_{\langle s\rangle}(s)$. Thus, $P(s)$ has a fixed point and $s \in \overset{-1}{P}\big(P(S)_{st}\big)$.

Suppose that $s \in \overset{-1}{P}\big(P(S)_{st}\big)$, i.e., that $P(s)$ has a fixed point. Hence, there exists an $h \in H$ such that $\Gamma_{\langle h\rangle}(s)$ has a fixed point, say, H_1, i.e., $H_1 s \subset H_1$. If $s_0 \in H_1$, then $s_0 s \equiv s_0(\nu_{\langle h\rangle})$. Since $\nu_{\langle h\rangle} \subset \eta_{\langle h\rangle}$, we obtain $s_0 s \equiv s_0(\eta_{\langle h\rangle})$ or, by Lemma 5, $h^p s_0 s = h^q s_0$ for some natural p and q. Now $s_0 \notin W_{\langle h\rangle}$, whence $s_0 s_1 \in \langle h\rangle$ for some $s_1 \in S^1$. By Lemma 2, $h^m s_0 s_1 = h^n$ for some natural m and n.

To prove $s \in H$ we need some further lemmas.

LEMMA 7. *Weakly invariant subsets are reflexive.*

Proof. If H is weakly invariant and $st \in H$, then $(st)s = s(ts)$ and $st = st$, whence $st \vDash ts$ and $st \in H$.

LEMMA 8. *A subset H of S is quasi-stable, weakly invariant and weakly inversely invariant, and contains right identities of its elements (i.e., $s \in H \wedge st = s \to t \in H$) if and only if for all natural m, n, p, q and all $h, s, s_0, s_1 \in S$*

$$(*) \qquad h \in H \wedge h^p s_0 s = h^q s_0 \wedge h^m s_0 s_1 = h^n \to s \in H.$$

Proof. Necessity. Let the antecedent of $(*)$ be true. Consider three possible cases:

Case 1. Let $p < q$. Multiplying the equalities in the antecedent of $(*)$ by h on the left a sufficient number of times, we obtain $p = m$. Repeating this process, we may obtain for n a multiple of $q-p$, i.e., $n = k(q-p)$. This process does not alter the equality $p = m$ and the dif-

ference $q-p$. Then $h^{q-p}(h^p s_0) = (h^p s_0)s$ and $(h^p s_0)s_1 = (h^{q-p})^k$, i.e., $h^{q-p} \models s$. Now $h \in H \to h^{q-p} \in H \to s \in H$, whence $s \in H$.

Case 2. Let $p = q$. As in Case 1, we may consider $p = m$. Multiplying $h^p s_0 s = h^p s_0$ by s_1 on the left, we obtain $s_1 h^p s_0 s = s_1 h^p s_0$. Now $h^n \in H$, i.e., $h^p s_0 s_1 \in H$. Since H is reflexive by Lemma 7, $s_1 h^p s_0 \in H$. The element $s_1 h_p s_0$ is a left zero for s. Thus, $s \in H$.

Case 3. Let $p > q$. We may consider $m = p$ and n to be a multiple of $p-q$: $n = k(p-q)$. Then $h^{p-q}(h^q s_0)s = h^p s_0 s = h^q s_0$ and $(h^q s_0)s_1 = (h^{p-q})^k$, i.e., $(h^{p-q}, s) \in \gamma_0$. Since H is quasi-stable and weakly inversely invariant, $h^{p-q} \in H$ and $s \in H$.

Sufficiency. Let (*) be true. Suppose that $h \in H$, $s = h^k$, $s_0 = h$, $p+k = q$, $m = 1$, $s_1 = h$ and $n = 3$. Then, by (*), $h^k \in H$, i.e., H is quasi-stable.

Now let $sx = xt$, $xy = s^k$ and $s \in H$, let $s_0 = x$ if $x \in S$, $s_1 = y$, and let m, n, p, q be such that $s^p xt = s^q x \wedge s^m xy = s^n$. By (*), $t \in H$. If $x = 1$, then $s = t$ and $t \in H$. Thus, H is weakly invariant.

Now let $hxs = x$ and $xy = h^k$. Then the antecedent of (*) is true for $s_0 = x$, $s_1 = y$, $m = q = 1$, $p = 2$ and $n = k+1$. Therefore, $s \in H$ and H is weakly inversely invariant.

Suppose that $st = s \in H$. Then the antecedent of (*) is true if h, s_0, s and s_1 are replaced by s, s, t and s respectively and if $p = q$, $m+2 = n$. Therefore, $t \in H$ and H contains all right identities of its elements. Lemma 8 is proved.

Returning now to the argument preceding Lemma 7, we see that $s \in H$, whence, $H = \overset{-1}{P}\big(P(S)'_{st}\big)$, i.e., H is a stationary subset.

Now suppose that H satisfies the condition $st = s \neq 0 \to t \in H$, which is stronger than $s \in H \wedge st = s \to t \in H$. Let Λ_0 be the canonical representation of S by the right translations of $S^1 \setminus \{0\}$ (i.e., Λ_0 is the simplest representation determined by the determinative pair $(\Delta_{S^1}, \{0\})$). Consider the sum of P and Λ_0. Clearly $(P+\Lambda_0)(s)$ has a fixed point if and only if either $P(s)$ has (i.e., $s \in H$) or $\Lambda_0(s)$ has (i.e., $xs = x \neq 0$ for some $x \in S^1$). Thus, $(P+\Lambda_0)(s)$ has a fixed point precisely when $s \in H$. Thus, H is a proper stabilizer.

LEMMA 9. *A subset H^1 of S^1 is quasi-stable, invariant and inversely invariant if and only if for all natural p and q and for all h, s_0, $s \in S$*

$$(**) \qquad h \in H \wedge h^p s_0 s = h^q s_0 \to s \in H.$$

Proof. Necessity. Let the antecedent of (**) be true. If $p \leqslant q$, then $h^{q-p}(h^p s_0) = (h^p s_0)s$, i.e., $h^{q-p} \vdash s$. Since H^1 is quasi-stable and invariant, $h^{q-p} \in H$ and $s \in H$. If $p > q$, then $h^{p-q}(h^q s_0)s = h^q s_0$, i.e., $(h^{p-q}, s) \in \gamma$. It follows that $s \in H$.

Sufficiency. Let (**) be true. If $h \in H$, $s_0 = h$, $s = h^k$ and $p+1+k = q+1$, then $h^k \in H$, i.e., H is quasi-stable. Suppose that $hs_0 = s_0 s$ and $h \in H$. If $p = 1$ and $q = 2$, we infer, by (**), that $s \in H$, i.e., H is invariant.

Suppose $h \in H$ and $hs_0 s = s_0$. If $p = q+1$, then, by (**), $s \in H$ and H is inversely invariant.

Lemma 9 is proved.

Consider the representation P, which is the sum of Λ (here $\Lambda = P_{(\Delta_{S^1}, \emptyset)}$, i.e., Λ is the canonical representation of S by right translations of S^1) and of the family $(P_{(\eta_{\langle h \rangle}, \emptyset)})_{h \in H}$ of representations. Clearly, P is a proper representation of S by transformations. Let H^1 be quasi-stable, invariant and inversely invariant. If $s \in H$, then $\langle s \rangle$ is a fixed point of $P_{(\eta_{\langle s \rangle}, \emptyset)}(s)$, whence $P(s)$ has a fixed point. Suppose that $P(s)$ has a fixed point. Then either $\Lambda(s)$ has a fixed point or $P_{(\eta_{\langle h \rangle}, \emptyset)}(s)$ has a fixed point, say, H_1, for some $h \in H$. In the first case, $xs = x$ for some $x \in S^1$. If $x = 1$, then $s = 1 \in S$; thus, $x \in S$. It follows that $1 \vdash s$. Since $1 \in H^1$ and H^1 is invariant, $s \in H$. In the second case, $s_0 s \equiv s_0 (\eta_{\langle h \rangle})$ or, by Lemma 6, $h^p s_0 s = h^q s_0$. By Lemma 9, $s \in H$. Thus, $H = \overset{-1}{P}\big(P(S)_{st}\big)$ and H is a proper full stationary subset of S.

Theorem 4 is proved.

COROLLARY. *A subset H of a semigroup S is stationary if and only if H satisfies condition (*). H is full stationary if and only if H satisfies condition (**).*

PROPOSITION 4. *For every semigroup S the following three conditions are equivalent:*

(1) *for every representation P of S by transformations all the transformations from $P(S)$ have fixed points;*

(2) *for every proper representation P of S by transformations all the transformations from $P(S)$ have fixed points;*

(3) *for every element of S there exists a left zero, i.e.,*

$$(\bigwedge s)(\bigvee t)[ts = t].$$

Proof. The implication (1) $\to$ (2) is clear, the implication (2) $\to$ (3) follows from the consideration of the case $P = \Lambda$. Let (3) be satisfied and let H be a full stationary subset of S. If $ts = t$, then $1 \vdash s$. Since $1 \in H^1$, $s \in H^1$ for all $s \in S$, i.e., $H^1 = S^1$. Thus, S is the only full stationary subset of S, i.e., (1) is true.

Proposition 4 has been proved by E. S. Ljapin [5].

PROPOSITION 5. *For every semigroup S the following two conditions are equivalent:*

(1) *for every proper representation P of S by functions all the functions from $P(S)$ have fixed points*;

(2) *every element of S has a left zero, but S is a semigroup without zero.*

Proof. (1) means that S is the only proper stationary subset of S. By Proposition 4, (1) implies $(\bigwedge s)(\bigvee t)\,[ts = t]$. If S has 0, then $\Lambda_0(0) = \emptyset$, i.e., $\Lambda_0(0)$ has no fixed point. Hence, (1) $\to$ (2).

Let (2) be true and H be a proper stationary subset of S, then $ts = t \to s \in H$, by Theorem 4. Therefore, $H = S$ and (1) is true,

PROPOSITION 6. *For every semigroup S the following two conditions are equivalent*:

(1) *for every non-null representation P of S by functions all the functions from $P(S)$ have fixed points*;

(2) *S is simple (i.e., S has no proper ideals) and every element of S has a left zero.*

Proof. (1) means that every homomorphic image of S satisfies condition (1) of Proposition 5. It follows from Proposition 5 that every element of S has a left zero and homomorphic images of S are semigroups without zero, whence S is simple. Now (2) means that all homomorphic images of S satisfy condition (2) of Proposition 5, whence (1) is satisfied.

PROPOSITION 7. *For every semigroup S the following two properties are equivalent*:

(1) *for all proper representations P of S by functions all nonempty functions from $P(S)$ have fixed points*;

(2) *every non-zero element of S has a left zero which is not a zero of S, i.e.,* $(\bigwedge s)(\bigvee t)\,[s \neq 0 \to ts = t \neq 0]$.

Proof. Consider Λ_0 as P in order to prove the implication (1) $\to$ (2).

Now let (2) be true. Suppose H to be a proper stationary subset of S. If $0 \in H$, then, by (*), $H = S$. If $0 \notin H$ then, by Theorem 4 and condition 2, $H = S \setminus \{0\}$. It follows that (1) is true.

Remark. If S has 0, then there exists a proper representation of S by functions which maps 0 onto $\emptyset$ (e.g., Λ_0 is such a representation).

PROPOSITION 8. *The following two conditions are equivalent for every semigroup S*:

(1) *for every representation P of S by functions all non-empty functions from $P(S)$ have fixed points*;

(2) *for every element $s \in S$ there exists a left zero (i.e., such an element $t \in S$ that $ts = t$) which generates the same principal ideal as s.*

Proof. Condition (1) implies condition (1) of Proposition 7, whence implies condition (2) of Proposition 7. Let I be an ideal of S, and let $s \notin I$

and S/I be the Rees quotient semigroup of S by I. Then s has a left zero which is not included in I. It follows that $(s) \subset (t)$ for this left zero t (here (s) is the principal ideal generated by s). But $ts = t$ implies $(t) \subset (s)$, whence $(s) = (t)$. Thus, (1) implies (2).

Now let (2) be satisfied. Then (2) is satisfied for every homomorphic image of S. It follows that every homomorphic image of S satisfies condition (2) of Proposition 7, whence (1) is true.

The axioms for stationariness, proper stationariness and full stationariness of a subset are infinite in number (since the quasi-stability of H is equivalent to the infinity of axioms $s \in H \to s^n \in H$ for each natural n). Clearly, invariance, inverse invariance and other conditions considered in Theorem 4 may be written as formulas of the lower predicate calculus, i.e., as elementary axioms.

THEOREM 5. *Stationary subsets, proper stationary subsets and full stationary subsets cannot be characterized by finite systems of elementary axioms.*

Proof. Let F_X be the free semigroup over an alphabet X, and let the elements of F_X be words over X. Let $H_p \subset F_X$, $\alpha \in H_p$ if and only if p is not a divisor of the length of α (here p is a prime). An equality of the form $\alpha\beta\gamma = \beta$ is impossible since the length of $\alpha\beta\gamma$ is more than the length of β. Thus, the inverse conjugacy relation on F_X is empty, i.e., every subset of F_X is inversely invariant and weakly inversely invariant. If $\alpha\gamma = \gamma\beta$, then α and β have the same length. It follows that H_p is invariant (and weakly invariant). Let $\alpha \in H_p$. Then $\alpha^n \in H_p$ if and only if n is not a multiple of p. Denote by A_p the formula $(\bigwedge s)[s \in H \to s^p \in H]$. Clearly, H is quasi-stable if and only if H satisfies A_p for all prime p. H_p satisfies A_q for all prime $q \neq p$. Thus, A_p does not follows from all the other axioms characterizing (proper) invariance or full invariance. It follows that the system of axioms characterizing invariance (proper invariance, full invariance) is not equivalent to any finite subsystem of itself. Theorem 5 follows from this fact and Gödel's Completeness Theorem for the lower predicate calculus.

PROPOSITION 9. *Invariance*: $sx = xt \wedge s \in H \to t \in H$, *inverse invariance*: $sxt = x \wedge s \in H \to t \in H$, *and quasi-stability*: $s \in H \to s^p \in H$ *for each prime p constitute a system of independent elementary axioms characterizing full stationary subsets.*

Proof. In the proof of Theorem 5 we have seen that A_p does not follow from all the other axioms mentioned in Proposition 9. Now let $x, y \in X$ and let $H \subset F_X$ consist of all the words having the form $(xy)^n$. Clearly, H is quasi-stable, i.e., satisfies A_p for all prime p. H is inversely invariant (as every subset of F_X). However, H is not invariant, since $xy \vdash yx$ but $yx \notin H$. Let Z be the additive group of integers and N be the

set of natural numbers. Clearly, N is quasi-stable and invariant but not inversely invariant.

One can define binary, ternary etc. stationary relations, e.g., a binary relation α on a semigroup S is (proper, full) stationary if there exists a (proper) representation P of S by (full) functions such that $(s, t) \in \alpha$ if and only if the functions $P(s)$ and $P(t)$ have a common fixed point. The methods developed here to solve the problem of the characterization of stationary subsets permit the characterization of these stationary relations. This program has been effectuated by V. S. Garvackiĭ.

Now we consider reversibly stationary subsets of semigroups.

Let ζ be a quasi-order relation on a semigroup S^1. ζ is called *stable* if $(s_1, t_1) \in \zeta$ and $(s_2, t_2) \in \zeta$ imply $(s_1 s_2, t_1 t_2) \in \zeta$. ζ is called *steady* if $(z, xv) \in \zeta \wedge (z, uv) \in \zeta \wedge (z, uy) \in \zeta \rightarrow (z, xy) \in \zeta$. The universal quasi-order relation $S^1 \times S^1$ is steady. The intersection of all steady quasi-orders of S^1 is a steady quasi-order which is denoted by $\hat{\zeta}$ and called the *strong quasi-order* of S^1 [6], [7]. Instead of $(s, t) \in \hat{\zeta}$ we shall write also $s \prec\!\!\!\sim t$. The strong quasi-order of a semigroup is stable [6], [7]. The strong quasi-order may be defined in an inner way: $s \prec\!\!\!\sim t$ if and only if there exists a finite chain of elements of S^1 which connect s and t in a simple prescribed way [6]. The symmetric part of $\hat{\zeta}$ is denoted by $\hat{\varepsilon} = \hat{\zeta} \cap \overset{-1}{\hat{\zeta}}$ and called the *strong equivalence* on S^1. A semigroup S is isomorphic to a semigroup of univalent functions if and only if $\hat{\varepsilon}$ is identical (or, which is the same, $\hat{\zeta}$ is an order relation) [7]. This condition is equivalent to an infinite system of simple elementary axioms [6].

The intersection of any family of strong subsemigroups is again a strong subsemigroup. Thus, strong subsemigroups determine a closure operation on the set of subsets of S^1. If $H \subset S^1$, then $[H]$ denotes the smallest strong subsemigroup of S^1 which contains H; $[s] = [\{s\}]$. Semigroups $[s]$ will play the same role as $\langle s \rangle$ in the proof of Theorem 4. We consider $[s]$ in more detail.

LEMMA 10. *$t \in [s]$ if and only if $s^m \prec\!\!\!\sim s^n t s^n$ for some natural m and n. One may consider only the case where $m > n$.*

Proof. The set of all t such that $s \prec\!\!\!\sim t$ is the smallest strong subset containing s [6], [7]. Therefore, $[s]$ is majorantly closed, i.e., $t \in [s] \wedge t \prec\!\!\!\sim u \rightarrow u \in [s]$. Since $s^m \in [s]$ for all m, $s^m \prec\!\!\!\sim s^n t s^n$ implies $s^n t s^n \in [s]$. By proposition 1, $[s]$ is unitary, whence $t s^n \in [s]$ and $t \in [s]$. Let H be the set of all t for which there exist m and n such that $m > n$, $s^m \prec\!\!\!\sim s^n t s^n$. We have proved that $H \subset [s]$. Clearly, $s \in H$, since $s^3 \prec\!\!\!\sim sss$. Let $t, u \in H$, i.e., $s^a \prec\!\!\!\sim s^b t s^b$ and $s^c \prec\!\!\!\sim s^d u s^d$ for some natural a, b, c, d. Multiplying one of these inequalities by s several times, we obtain $a = c$. Hence, we may suppose at once that $s^a \prec\!\!\!\sim s^b t s^c$ and $s^a \prec\!\!\!\sim s^d u s^e$ where b, c, d, e are less

than a. If $a > c+d$, we may reduce our inequalities to the form $s^{2a-d-c} \prec s^b ts^{a-d}$, $s^{2a-d-c} \prec s^{a-c}us^c$, multiplying our inequalities by s several times. If $a < c+d$, then in the same way we obtain $s^{c+d} \prec s^{b+c+d-a}ts^c$, $s^{c+d} \prec s^d us^{c+c+d-a}$. Thus, we may consider only the case where $a = c+d$ since other cases may be reduced to this one. Since $\hat{\zeta}$ is steady, $s^a \prec s^b ts^c$, $s^a \prec s^d s^c$, $s^a \prec s^d us^c$ imply $s^a \prec s^b tus^c$ and $tu \in H$. Thus, H is stable.

Now let $xv, uv, uy \in H$. In this case one can choose natural numbers a, b, c, d, p, q, r in such a way that $s^p \prec s^a xvs^b$, $s^q \prec s^c uvs^b$, $s^r \prec s^c uys^d$. Multiplying the first inequality by s on the left and the second one on the right by s several times, we obtain $p = r > q$. Let $p-q = n$. Then $s^{p+n} \prec s^a xvs^{b+n}$, $s^{p+n} \prec s^{c+n}uvs^{b+n}$, $s^{p+n} \prec s^{c+n}uys^d$. By the steadiness of $\hat{\zeta}$, $s^{p+n} \prec s^a xys^d$ and $xy \in H$. Thus, H is a strong subsemigroup of S^1 containing s. Hence $[s] \subset H$ and $H = [s]$.

THEOREM 6. *A subset H of a semigroup S is reversibly stationary if and only if for all natural m and n and all $s, t, u, v \in S$*

$$(***) \qquad s \in H \wedge s^m \prec uv \wedge s^n \prec utv \to t \in H.$$

H is a proper reversibly stationary subset if and only if H satisfies (***) *and the condition*

$$(****) \qquad 0 \neq s \prec uv \wedge s \prec utv \to t \in H$$

and the semigroup S has a proper representation by univalent functions (i.e., $\hat{\varepsilon} = \Delta_{S^1}$).

Proof. Necessity. Let P be a representation of S by univalent functions acting on a set A and $H = \overset{-1}{P}\big(P(S)_{st}\big)$. Define $s \prec t(\zeta_P) \leftrightarrow P(s) \subset P(t)$, i.e., ζ_P is the fundamental quasi-order of P. Then ζ_P is a steady quasi-order [13], whence $\hat{\zeta} \subset \zeta_P$. Thus, $s \prec t \to P(s) \subset P(t)$. Let the antecedent of (***) be true. Then $(a_1, a_1) \in P(s)$ for some $a_1 \in A$. Therefore, $(a_1, a_1) \in P(s^m) \subset P(uv) = P(v) \circ P(u)$ and $(a_1, a_1) \in P(s^n) \subset P(utv) = P(v) \circ \circ P(t) \circ P(u)$. It follows that $(a_1, a_2) \in P(u)$ and $(a_2, a_1) \in P(v)$ for some $a_2 \in A$. Since $P(u)$ and $P(v)$ are univalent, $(a_2, a_2) \in P(t)$, whence $t \in H$ and (***) is satisfied.

Now let P be a proper representation and let the antecedent of (****) be true. Then $P(s) \neq \emptyset$, i.e., $(a_1, a_2) \in P(s)$ for some $a_1, a_2 \in A$. Thence, $(a_1, a_2) \in P(uv)$, i.e., $(a_1, a) \in P(u)$ and $(a, a_2) \in P(v)$ for some $a \in A$. Since $(a_1, a_2) \in P(utv) = P(v) \circ P(t) \circ P(u)$, $(a, a) \in P(t)$ and $t \in H$.

Sufficiency. Let H be a subset of S satisfying (***) and let $P = \sum (P_{[h]})_{h \in H}$ be a representation of S by univalent functions (by definition the sum of the empty family of representations is the null representation on the empty set). If $h \in H$, then $[h]h \subset [h]$, i.e., $[h]$ is a fixed point of $P_{[h]}(h)$. Therefore, $P(h)$ has a fixed point.

Conversely, let $P(t)$ have a fixed point. This means that $P_{[s]}(t)$ has a fixed point, say, H_1, for some $s \in H$. But $H_1 = [s]^{\cdot}.x$ for some $x \in S^1$ [7], i.e., $y \in H_1 \to yx \in [s]$. If $y \in H_1$, then $yx \in [s]$ and $yt \in H_1$, i.e., $ytx \in [s]$. By Lemma 10, $s^m \prec s^p yxs^p$ and $s^n \prec s^q ytxs^q$. Multiplying one of these inequalities by s several times, we obtain $p = q$. Let $u = s^p y$ and $v = xs^p$. Then, by (***), $t \in H$. It follows that H is a reversibly stationary subset.

Now let (****) be satisfied and let $\hat{s}$ be the smallest strong subset of S^1 containing s. Let Γ be the sum of all elementary representations $P_{\hat{s}}$ for $s \in S$, $s \neq 0$. Then $\varepsilon_\Gamma = \hat{\varepsilon}$ [7]. If $\hat{\varepsilon} = \Delta_{S^1}$, then Γ is a proper representation of S by univalent functions. Suppose that $\Gamma(s)$ has a fixed point. This means that $P_{\hat{t}}(s)$ has a fixed point, say, H_1, for some $t \neq 0$. Then $H_1 = \hat{t}^{\cdot}.x$ for some $x \in S^1$ and $y \in H_1$ implies $yx \in \hat{t}$ and $ys \in H_1$, i.e., $ysx \in \hat{t}$. But $x \dot{\in} \hat{y}$ means $y \prec x$ [7]; thence, $t \prec yx$ and $t \prec ysx$. By (****), $s \in H$.

Let P_0 be the sum of P and Γ. Then $P(s)$ has a fixed point if and only if either $P(s)$ or $\Gamma(s)$ has one, which means that $s \in H$. Thus, H is a proper reversibly stationary subset.

COROLLARY 1. *A subset H of an inverse semigroup S is a (proper) reversibly stationary subset if and only if H contains inverse subsemigroups of S generated by each element of H, (H contains all non-zero idempotents of S) and $sts^{-1} \in H \to t \in H$.*

Proof. Let P be a (proper) representation of S by univalent functions. If $P(s)$ has a fixed point, then every element of the inverse subsemigroup generated by $P(s)$ has the same fixed point. If $(a, a) \in P(sts^{-1})$, then $(a, a_1) \in P(s)$ for some $a_1 \in A$, whence $(a_1, a_1) \in P(t)$. If P is proper, then $P(s)$ is a non-empty identical function on a subset of A for every idempotent $s \in S$, $s \neq 0$; thus, $P(s)$ has fixed points.

To prove the sufficiency, suppose that H satisfies the conditions of our Corollary and the antecedent of (***) is satisfied. Then H is majorantly closed [8], i.e., $(s \prec t \wedge s \in H) \to (t \in H)$. Indeed, if $s \prec t$ and $s \in H$, then $s = ti$ for an idempotent i. Then $(ti)t(ti)^{-1} = (ti)^2(ti)^{-1} \in H$ and, by our supposition, $t \in H$. If $s \in H$, then $s^n s^{-m} \in H$. Since $s^n s^{-m} \prec utvv^{-1}u^{-1} \prec utu^{-1}$, $t \in H$. Therefore, (***) is satisfied. By Theorem 6, H is a reversibly stationary subset. If H contains all non-zero idempotents and $0 \neq s \prec uv \wedge s \prec utv$, then $ss^{-1} \in H$, $ss^{-1} \prec utvv^{-1}u^{-1} \prec utu^{-1}$, $utu^{-1} \in H$ and $t \in H$, i.e., (****) is satisfied and H is proper.

COROLLARY 2. *If H is a (proper) stationary subset of an inverse semigroup, then $H \cap H^{-1}$ is a (proper) reversibly stationary subset. A (proper) stationary subset H of an inverse semigroup is a (proper) reversibly stationary subset if and only if $H = H^{-1}$.*

Proof. Let P be a (proper) representation of an inverse semigroup S by functions and $H = \overset{-1}{P}\big(P(S)_{st}\big)$. Let $\tilde{P}$ be the biunivalent of P [26]

i.e., $\tilde{P}(s) = P(s) \cap \overline{P(s^{-1})}$. If P is proper, then $\tilde{P}$ is proper [26]. Clearly, $\tilde{P}$ is a representation by univalent functions and $H \cap H^{-1} = \overset{-1}{\tilde{P}}\big(\tilde{P}(S)_{st}\big)$. If $H = H^{-1}$, then H is a (proper) reversibly stationary subset. If H is reversibly stationary, then H is stationary and, by Corollary 1, $H = H^{-1}$.

Now we could easily characterize inverse semigroups whose every (proper) representation P by univalent functions has the following property: for every $s \in S$, $(s \neq 0)$ $P(s)$ has a fixed point. Every non-zero element of such inverse semigroups has an idempotent minorant distinct from zero.

COROLLARY 3. *Let S be a group and $H \subset S$. The following six conditions are equivalent:*

(1) *H is stationary;*

(2) *H is proper stationary;*

(3) *H is full stationary;*

(4) *H is reversibly stationary;*

(5) *H is proper reversibly stationary;*

(6) *H is quasi-stable, invariant and symmetric, i.e., H contains cyclic subgroups of S generated by the elements of H and H is invariant.*

Proof. Evident.

PROPOSITION 10. *Multi-stationary subsets are quasi-stable and reflexive.*

Proof. Let P be a representation of a semigroup S by multi-functions acting on a set A. If $(a, a) \in P(s)$, then $(a, a) \in P(s^n)$ for all n. If $(a, a) \in P(st) = P(t) \circ P(s)$, then $(a, a_1) \in P(s)$ and $(a_1, a) \in P(t)$ for some $a_1 \in A$. Therefore, $(a_1, a_1) \in P(s) \circ P(t) = P(ts)$.

The problem of finding a characterization of multi-stationary and proper multi-stationary subsets is still open. It is not known whether quasi-stability and reflexivity are sufficient for a subset to be multi-stationary.

§ 3. Transitive representations. A subset H of a semigroup S is a stabilizer of S relative to a representation P of S by functions acting on a set A if $H = \overset{-1}{P}\big(P(S)_a\big)$ for some $a \in A$.

A subsemigroup F of S^1 is left unitary if and only if $F = H^1$ where H is a left unitary subsemigroup of S.

THEOREM 7. *Let P be a transitive representation of a semigroup S by functions and let H be a stabilizer relative to P. Then P is a principal representatively homomorphic image of the principal representation Γ_{H^1} and the elementary representation P_{H^1} is a principal representatively homomorphic image of P. The elementary representation P_{H^1} and the principal repre-*

sentation Γ_{H^1} are both transitive and H is a stabilizer relative to both of these representations.

Proof. Let $H = \overset{-1}{P}(P(S)_a)$. Then, by Theorem 2, H^1 is a left unitary subsemigroup of S^1. In the proof of Theorem 2 we have seen that H^1 is an ε_{H^1}-class and $H^1 s \subset H^1 \leftrightarrow s \in H$. Let H_1 be an ε_{H^1}-class distinct from W_{H^1} and $x \in H_1$. Then $xy \in H^1$ for some $y \in S$, i.e., $H_1 y \subset H^1$. Let H_2 be an ε_{H^1}-class and $z \in H_2$. Then $1z \in H_2$, i.e., $H^1 z \subset H_2$ and $H_1 yz \subset H_2$. Thus, the elementary representation P_{H^1} is transitive for every left unitary subsemigroup H of S. P is similar to a simple representation of the form $P_{(\varepsilon, W)}$ where $s \equiv t(\varepsilon) \leftrightarrow P(s)\langle a\rangle = P(t)\langle a\rangle$ and $s \in W \leftrightarrow a \notin pr_1 P(s)$ [11]. Let $\mathfrak{H} = (S^1/\varepsilon_{H^1}) \setminus \{W_{H^1}\}$, i.e., $\mathfrak{H}$ is the set on which act functions $P_{H^1}(s)$ for $s \in S$. Now ε_{H^1} is the largest right regular equivalence having a class H^1 [21], whence $\varepsilon \subset \varepsilon_{H^1}$. If $s \in W$, then $a \notin pr_1 P(s)$, i.e., $(a, a_1) \in P(s)$. Since P is transitive, $(a_1, a) \in P(t)$ for some $t \in S$; hence, $(a, a) \in P(st)$ and $st \in H^1$. Therefore, $s \notin W_{H^1}$. If $s \in W_{H^1}$, then $sx \notin H^1$ for some $x \in S^1$ and $(a, a) \in P(sx)$. Therefore, $a \in pr_1 P(s)$ and $s \notin W$. Thus, $W = W_{H^1}$. Define a mapping θ of A onto $\mathfrak{H}$: $\theta(a_1) = H_1$ if and only if the ε-class containing all $s \in S^1$ such that $(a, a_1) \in P(s)$ is included in H_1. Since $\theta(a) = H^1$ and the ε-class consisting of all s such that $(a, a) \in P(s)$ is H^1, we infer that θ is univalent at a. Let $(H_1, H_2) \in P_{H^1}(s)$ for some $H_1, H_2 \in \mathfrak{H}$, i.e., $H_1 s \subset H_2$. Since $H_1 \neq W$, there exists an ε-class H_0 such that $H_0 \subset H_1$. Let $H_0 s \subset H_{00}$ where H_{00} is an ε-class. Since $H_{00} \subset H_2$, $H_{00} \neq W$. Let $(a, a_1) \in P(s)$ for $s \in H_0$ and $(a, a_2) \in P(t)$ for $t \in H_{00}$. Then $\theta(a_1) = H_1$, $\theta(a_2) = H_2$ and $(a_1, a_2) \in P(s)$, since $H_0 s \subset H_{00}$. It follows that θ is a principal representative homomorphism of P onto P_{H^1}.

Since η_{H^1} is the smallest right regular equivalence with a class H^1, and W_{H^1} is an ε-class, $\nu_{H^1} \subset \varepsilon$. One can easily verify that the canonical map of S^1/ν_{H^1} onto S^1/ε restricted to ν_{H^1}-classes distinct from W_{H^1} is a univalent (at H^1) representative homomorphism of Γ_{H^1} onto $P_{(\varepsilon, W)}$. Thus, there exists a principal representative homomorphism of Γ_{H^1} onto P. Since H^1 is an ν_{H^1}-class, H is a stabilizer relative to Γ_{H^1}. The proof of the transitivity of Γ_{H^1} is the same as in the case of the principal representation P_{H^1}.

COROLLARY 1. *Let P be a transitive representation of a semigroup S by transformations. Then every stabilizer relative to P is a right neat left unitary subsemigroup of S and there exist principal representative homomorphisms of Γ_{H^1} onto P and P onto P_{H^1} for every stabilizer H relative to P.*

Proof. If H is a stabilizer of S relative to P and P is similar to $P_{(\varepsilon, \emptyset)}$, then H is a stabilizer relative to $P_{(\varepsilon, \emptyset)}$. Since $W_{H^1} = \emptyset$ (see the proof of the previous theorem), H^1 is right neat in S^1 and H is right neat in S.

Let $\underset{H}{\nu}$ denote the kernel of the representation Γ_{H^1}. According to

[7], $s \equiv t(\underset{H}{\nu}) \leftrightarrow (\bigwedge x \in S^1)\,[xs \equiv xt(\nu_{H^1})]$. If Q is a representatively homomorphic image of P, then $\varepsilon_P \subset \varepsilon_Q$. It proves the first part of

COROLLARY 2. *Let P be a transitive representation of a semigroup S by functions and H is a stabilizer relative to P. Then $\underset{H}{\nu} \subset \varepsilon_P \subset \underset{H}{\varepsilon}$. Moreover, $\varepsilon_P = \bigcap \underset{H}{\varepsilon}$ where the intersection is taken for all stabilizers H relative to P.*

Proof. Clearly, $\varepsilon_P \subset \bigcap \underset{H}{\varepsilon}$. Let $(s, t) \in \bigcap \underset{H}{\varepsilon}$; then, for all H and all $x, y \in S^1$, $xsy \in H \leftrightarrow xty \in H$ or, which is the same, for all $a \in A$ and all $x, y \in S^1$, $(a, a) \in P(xsy) \leftrightarrow (a, a) \in P(xty)$. Let $(a_1, a_2) \in P(s)$ and $y = 1$. Since P is transitive, there exists an $x \in S$ such that $(a_2, a_1) \in P(x)$. Then $(a_2, a_2) \in P(xsy)$, whence $(a_2, a_2) \in P(xty) = P(xt) = P(t) \circ P(x)$ and $(a_1, a_2) \in P(t)$. Therefore, $P(s) \subset P(t)$. In the same way $P(t) \subset P(s)$, i.e., $P(s) = P(t)$ ant $s \equiv t(\varepsilon_P)$.

COROLLARY 3. *A semigroup S has a proper transitive representation by functions if and only if S contains a left unitary subsemigroup H such that $\underset{H}{\nu} = \Delta_S$. If S has a disjunctive left unitary subsemigroup, then S has a proper transitive representation by functions.*

Proof. If $\underset{H}{\nu} = \Delta_S$, then Γ_{H^1} is a proper transitive representation of S. If S has a proper transitive representation P by functions and H is a stabilizer relative to P, then $\underset{H}{\nu} \subset \varepsilon_P = \Delta_S$. If H is a disjunctive left unitary subsemigroup, then P_{H^1} is a proper transitive representation of S.

In the same way we may prove

COROLLARY 4. *A semigroup S has a proper transitive representation by transformations if and only if S contains a right neat left unitary subsemigroup H such that $\underset{H}{\nu} = \Delta_S$. If S contains a right neat left unitary disjunctive subsemigroup, then S has a proper transitive representation by transformations.*

A result inessentially different from Corollary 4 has been obtained in [24].

THEOREM 8. *Every transitive representation of a semigroup S by univalent functions is similar to an elementary representation of S associated with a strong subsemigroup of S^1. Every representation associated with a strong subsemigroup of S^1 is a transitive representation of S by univalent functions.*

Proof. Let P be a transitive representation of S by univalent functions acting on a set A, $a \in A$ and $H = \overset{-1}{P}(P(S)_a)$. By Theorem 3, H^1 is a strong subsemigroup of S^1. All ε_{H^1}-classes distinct from W_{H^1} have the form $H^1 \cdot x$ for $x \in S^1$. Let P be similar to a simple representation $P_{(\varepsilon, W)}$ where

ε and W are chosen as in the proof of Theorem 7. We have proved that $\varepsilon \subset \varepsilon_{H^1}$ and $W = W_{H^1}$. Let H_1 be an ε_{H^1}-class distinct from W. Then $H_1 = H^{1\cdot}.x$. If $s \in H_1$, then $a \in pr_1 P(s)$, i.e., $(a, a_1) \in P(s)$ for some $a_1 \in A$. Since $sx \in H^1$, $(a, a) \in P(sx) = P(x) \circ P(s)$ and $(a_1, a) \in P(x)$. If $h \in H_1$, then $hx \in H^1$, $(a, a) \in P(hx) = P(x) \circ P(h)$ and $(a, a_1) \in P(h)$ since $P(x)$ is univalent. It follows that $P(s)(a) = P(h)(a)$, i.e., $s \equiv h(\varepsilon)$. Thus, $\varepsilon_{H^1} \subset \varepsilon$ and $P_{(\varepsilon, W)} = P_{H^1}$. Conversely, if H^1 is a strong subsemigroup of S^1, then P_{H^1} is a representation by univalent functions [7] which is transitive by Theorem 7.

COROLLARY. *A semigroup has a proper transitive representation by univalent functions if and only if it contains a disjunctive strong unitary subsemigroup.*

Proof. It follows from Theorem 8 and Proposition 1.

Theorem 8 and Corollary are included in [2], where they are referred to [7]. However, these results are not contained in [7] explicitly: they easily follow from the results of [7] concerning determinative pairs and strong subsets.

Applying Theorem 8 to inverse semigroups, we may easily develop a theory of representations of inverse semigroups by univalent functions (see [8], where such a theory is exposed).

§ 4. Symmetric representations and radicals. There exists an analogy between the representations of semigroups and modules over rings. Every right module A over a ring R may be considered as a representation of R by endomorphisms of the abelian group of A. If $r \in R$, then the transformation of A: $\gamma_r(a) = ar$ is an endomorphism of the abelian group A, and the correspondence $\Gamma(r) = \gamma_r$ is a homomorphism (i.e., a representation) of R into the ring of all endomorphisms of an abelian group. Every representation of R by endomorphisms of an abelian group may be considered as an R-module. This fact permits us to see a close connection between modules over rings and representations of semigroups. A semigroup is represented by transformations of a set; a ring is a semigroup endowed with a structure of an abelian group; the corresponding representations are by transformations of sets endowed with a structure of an abelian group.

If A is a right R-module, then A has a distinct element 0 which is fixed by all γ_r. Suppose P is a representation of a semigroup S by functions acting on a set B. Let 0 be an element which does not belong to B and $B_0 = B \cup \{0\}$. Define $P_0(s) = P(s) \cup \big((pr_1 P(s))' \times \{0\}\big)$ where C' is the complementation of a subset $C \subset B_0$ in B_0. Thus, $P_0(s)$ is a transformation of B_0 which coincides with $P(s)$ on $pr_1 P(s)$ and transforms all other elements into 0. P_0 is a representation of S and $\varepsilon_P = \varepsilon_{P_0}$ [25]. Conversely,

let P be a representation of S by transformations of a set B and let $0 \in B$ be a common fixed point for all $P(s)$, $s \in S$. Define $P^-(s) = P(s) \cap (B^- \times B^-)$, where $B^- = B \setminus \{0\}$. Then $P^-(s)$ is a function ($P^-(0)$ is $P(s)$ restricted to B^-) and P^- is a representation of S by functions acting on B^-. Moreover, $(P^-)_0 = P$.

In particular, if A is an R-module, then Γ^- is a representation of the multiplicative semigroup of R by functions acting on A. Thus, all the concepts of the theory of representations of semigroups by functions may be transferred to modules.

This interconnection between the modules and the representations of semigroups may be useful for both theories. Here we exploit only some consequences of such an interconnection.

Quite clearly, transitive representations of semigroups by functions correspond to primitive modules: an R-module A is primitive ifand only if the representation Γ^- is transitive. Congruences on rings are in a natural one-to-one correspondence with ideals. However, this does not hold for semigroups: not every congruence on a semigroup possesses a class which is an ideal and, generally speaking, every ideal is a class modulo many distinct congruences. Thus, the theory of semigroup congruences cannot be reduced to the theory of ideals. To preserve the analogy between rings and semigroups we must keep this circumstance in mind.

The concept of radical is very important in the ring theory. Many familiar radicals are defined in terms of some special modules over rings. The typical situation is the following: let K be a class of modules (naturally, K must satisfy certain conditions) and R be a ring. Let M be an R-module in K. The kernel of M is the ideal $I \subset R$, $r \in I \leftrightarrow \gamma_r = \gamma_0$ (thus, I defines just the congruence ε_{r-} on R). Let I be the intersection of all the kernels of all R-modules belonging to K. Then I is the radical of R defined by the class K. E.g., if K is the class of primitive modules, the radical obtained is the Jacobson radical of the ring.

Let E be the set of conguences on a semigroup S, $\varepsilon \in E$ if and only if ε is a kernel of a transitive representation of S by functions. Write $\varkappa = \bigcap E$. Then $\varkappa$ is the semigroup analog of the Jacobson radical. We call $\varkappa$ the *Jacobson radical of* S. Moreover, let Q be the set of all fundamental quasi-orders on S associated with transitive representations of S by functions and $\lambda = \bigcap Q$. We call λ the *Jacobson quasi-order* of S. Clearly, $\varkappa = \lambda \cap \overset{-1}{\lambda}$. The concept of the Jacobson quasi-order can be introduced for rings but we are not aware of any attempts to do this.

Let K be the class of all R-modules M satisfying the property: $mr = nr \neq 0 \to m = n$ for all $r \in R$, $m, n \in M$. The radical defined by K is called the compressive radical [22]. Let $\bar{Q}$ be the set of all fundamental orders on a semigroup S associated with all representations of S by uni-

valent functions (clearly, representations by univalent functions correspond to the class K of modules). We call $\hat{\zeta} = \bigcap \bar{Q}$ the *compressive quasi-order* of S. The intersection $\hat{\varepsilon}$ of all kernels of all representations of S by univalent functions is the analog of the ring compressive radical. $\hat{\varepsilon}$ may be called the *compressive radical* of S. By [7], [13], $\hat{\zeta}$ and $\hat{\varepsilon}$ coincide with the strong quasi-order and strong equivalence of S.

In the same way, the intersection $\hat{\varkappa}$ of all the kernels of transitive representations of S by univalent functions corresponds to the corpoidal radical [22] of a ring. We call $\hat{\varkappa}$ the *corpoidal radical* of S. The *corpoidal quasi-order* of S is the intersection $\hat{\lambda}$ of all the fundamental quasi-orders for all transitive representations of S by univalent functions.

Thus, we have introduced three types of radical congruences and radical quasi-orders for semigroups. Let α be a radical and β a quasi-order corresponding to α on a semigroup S. S is called *semisimple* (or *α-semisimple*) if $\alpha = \Delta_S$. Clearly, S is α-semisimple if and only if β is an order relation. S is called a *radical* semigroup (or an *α-radical* semigroup) if $\alpha = S \times S$.

Using the results of [7] and [11], we conclude that $S/\varkappa$ is the maximal homomorphic image of S having a proper symmetric representation by functions, $S/\hat{\varepsilon}$ is the maximal homomorphic image of S having a proper representation by univalent functions and $S/\hat{\varkappa}$ is the maximal homomorphic image of S having a proper symmetric representation by univalent functions. In particular, S has a proper (symmetric) representation by univalent functions if and only if S is ($\hat{\varkappa}$-) $\hat{\varepsilon}$-semisimple.

A short summary of ideas and results in this direction may be found in [18].

Explicit formulas for $\hat{\zeta}$ and $\hat{\varepsilon}$ are given in [6]. Here we give such formulas for λ, $\varkappa$, $\hat{\lambda}$ and $\hat{\varkappa}$.

THEOREM 9. *Let λ be the Jacobson quasi-order of a semigroup S. The following five statements are equivalent for all $s, t \in S$:*

(1) $s \prec t(\lambda)$;

(2) *for every $x \in S^1$ there exist natural numbers m and n such that* $(sx)^m s = (sx)^n t$;

(3) *for every $x \in S$ there exist natural numbers m and n such that* $(sx)^m s = (sx)^n t$;

(4) *for every $x \in S^1$ there exists a natural number n such that* $(sx)^n s = (sx)^n t$;

(5) *for every $x \in S$ there exists a natural number n such that* $(sx)^n s = (sx)^n t$.

Proof. Let $s \prec t(\lambda)$. Then $s \prec t(\zeta_P)$ for every transitive representation P of S by functions. In particular, this is true for $P_{\langle h \rangle}$, $h \in S$. Since ζ_P is stable, $xs \prec xt(\zeta_P)$ for all $x \in S^1$. By [13], $xs \in \langle h \rangle \to xt \in \langle h \rangle$. If $h = xs$,

we obtain $xt \in \langle xs \rangle$. By Lemma 3, $(xs)^p xt = (xs)^q$ for some p and q. Multiplying this equality by s on the left, we obtain $(sx)^{p+1} t = s(xs)^p xt = s(xs)^q = (sx)^q s$, i.e., the implication (1) $\to$ (2) is verified. The implication (2) $\to$ (3) is evident as well as the implication (4) $\to$ (5).

Let (2) be true. For every $x \in S^1$ there exist p and q such that $(sx)^p s = (sx)^q t$. Substitute here $x(sx)^{a-1}$ for x (a is a natural number). Then $(sx)^{an} s = \big(sx(sx)^{a-1}\big)^n s = \big(sx(sx)^{a-1}\big)^m t = (sx)^{am} t$ for some m and n. Multiplying the last equality by $(sx)^a$ on the left several times, we obtain $ma \geqslant q$. Then $(sx)^{an+1} = (sx)^{an} sx = (sx)^{am} tx = (sx)^{am-q}(sx)^q tx = (sx)^{am-q}(sx)^p sx = (sx)^{am-q+p+1}$. If $an+1 = am-q+p+1$, then $q-p = a(m-n)$; hence, $q-p$ is divided by any natural number a. Thus, $q = p$. If $an+1 \neq am-q+p+1$, then the element sx is periodic. Let u and v be the smallest numbers such that $(sx)^{u+v} = (sx)^u$. Then the difference $(an+1)-(am-q+p+1) = a(n-m)+q-p$ is divided by v for any a. Suppose that $a = v$: then $q-p$ is divided by v. It follows that $(sx)^{u+p} t = (sx)^{u+q} t = (sx)^{u+p} s$. Therefore, (2) implies (4). In the same way (3) implies (5).

Now suppose (5) to be true and P to be a symmetric representation of S by functions. Let $(a_1, a_2) \in P(s)$. Since P is symmetric, there exists an $x \in S$ such that $(a_2, a_1) \in P(x)$, whence $(a_1, a_1) \in P(sx)$. Therefore, $(a_1, a_2) \in P(s) \circ P\big((sx)^n\big) = P(t) \circ P\big((sx)^n\big)$. Since $(a_1, a_1) \in P\big((sx)^n\big)$, $(a_1, a_2) \in P(t)$. Thus, $P(s) \subset P(t)$ and $(s, t) \in \zeta_P$. This is true for every symmetric representation P, whence $(s, t) \in \lambda$. Thus, (5) implies (1). Theorem 9 is proved.

COROLLARY 1. *Let $\varkappa$ be the Jacobson radical of a semigroup S. The following five statements are equivalent for all $s, t \in S$:*

(1) $s \equiv t(\varkappa)$;

(2) *for every $x \in S^1$ there exist natural numbers m, n, p, q such that $(sx)^m s = (sx)^n t$ and $(tx)^p s = (tx)^q t$;*

(3) *for every $x \in S$ there exist natural numbers m, n, p, q such that $(sx)^m s = (sx)^n t$ and $(tx)^p s = (tx)^q t$;*

(4) *for every $x \in S^1$ there exists a natural number n such that $(sx)^n s = (sx)^n t$ and $(tx)^n s = (tx)^n t$;*

(5) *for every $x \in S$ there exists a natural number n such that $(sx)^n s = (sx)^n t$ and $(tx)^n s = (tx)^n t$.*

The equivalence (1) $\leftrightarrow$ (2) in Corollary 1 was proved in [20], the equivalence (1) $\leftrightarrow$ (4) was proved in [4] by other methods.

COROLLARY 2. *A semigroup S is Jacobson semi-simple if and only if $(\bigwedge x)(\bigvee n)\,[(sx)^n s = (sx)^n t \wedge (tx)^n s = (tx)^n t] \to s = t$ where n is a variable over the set of natural numbers and x is a variable either over S^1 or over S.*

COROLLARY 3. *A semigroup S is Jacobson radical if and only if S is a left nil-semigroup.*

Proof. If S is a left nil-semigroup, then for all $s, t, x \in S$ there exists a natural n such that $(sx)^n$ and $(tx)^n$ are left zeros of S. By Corollary 2, $s \equiv t(\varkappa)$, i.e., S is radical.

Suppose that S is a radical semigroup. If H is a proper left unitary subsemigroup of S, then P_H is a non-trivial transitive representation of S by functions. Since $\varkappa \subset \varepsilon_{P_H} =_{H} \varepsilon \neq S \times S$, S cannot be radical. Thus, S contains no proper left unitary subsemigroups. By Lemma 1, S is a left nil-semigroup.

Recall that S is Jacobson semisimple if and only if S has a proper symmetric representation by functions.

COROLLARY 4. *Every left cancellative semigroup is semisimple.*

By Corollary 3, a left zero semigroup L containing precisely two distinct elements is radical. Thus, right cancellative semigroups need not be semisimple.

COROLLARY 5. *If S is a regular semigroup having no subsemigroups isomorphic to L, then S is semisimple.*

Proof. Let $\bar{s}$ denote an inverse element for $s \in S$. Suppose that $s \equiv t(\varkappa)$ and $s^2 = s$. Substitute $x = 1$ in condition (4) of Corollary 1. Then $s = (s1)^n s = (s1)^n t = st$. Now substitute $x = \bar{t}$. Then $t = (t\bar{t})^n t = (t\bar{t})^n s = t\bar{t}s$ since $t\bar{t}$ is an idempotent. If follows that $t^2 = t\bar{t}st = t\bar{t}s = t$, i.e., t is an idempotent. Supposing that $\bar{t} = t$, we obtain $t = tts = ts$. If $s \neq t$, then $\{s, t\}$ is a subsemigroup of S isomorphic to L. Therefore, $s = t$. By Theorem 7.38 of [2], $\varkappa = \Delta_S$ and S is semisimple.

Note that a regular semisimple semigroup may well contain L as a subsemigroup (consider as an example the full transformation semigroup $\mathcal{T}_A$ over a set A containing at least three distinct elements).

COROLLARY 6. *An idempotent semigroup S is semisimple if and only if $(\bigwedge x)\,[xs = xsxt \wedge xt = xtxs] \to s = t$. In particular, every restrictive semigroup (i.e., an idempotent semigroup satisfying the identity $xyz = yxz$) is semisimple.*

Proof. If S is restrictive, then $(\bigwedge x)\,[xs = xsxt \wedge xt = xtxs]$ means that $s = st$ and $t = ts$. Therefore, $s = st = stt = tst = tt = t$.

A *band of semigroups* is a pair (S, ε) where S is a semigroup and ε is a congruence on S such that the quotient semigroup S/ε is idempotent (which means that all the ε-classes are subsemigroups of S). A band is called *semisimple* if S/ε is semisimple. If all ε-classes are semisimple semigroups, (S, ε) is called a *band of semisimple semigroups*.

COROLLARY 7. *If (S, ε) is a semisimple band of semisimple semigroups, then S is a semisimple semigroup.*

Proof. Let (S, ε) be a semisimple band of semisimple semigroups and $s \equiv t(\varepsilon)$. Denoting the Jacobson radical of S/ε by $\bar{\varkappa}$ and the images of s, t in S/ε by $\bar{s}$ and $\bar{t}$, we see, by Corollary 1, that $\bar{s} \equiv \bar{t}(\bar{\varkappa})$, i.e., $\bar{s} = \bar{t}$ or, which is the same, $s \equiv t(\varepsilon)$. Denote the Jacobson radical of the semigroup $\varepsilon\langle s\rangle$ by $\varkappa_s$. Then, again by Corollary 1, $s \equiv t(\varkappa_s)$, i.e., $s = t$. Therefore, S is semisimple.

Let ε be a congruence on a semigroup S. The Jacobson radical of ε is the smallest congruence $\varkappa(\varepsilon)$ on S such that $\varepsilon \subset \varkappa(\varepsilon)$ and $S/\varkappa(\varepsilon)$ is semisimple. In particular, $\varkappa = \varkappa(\Delta_S)$.

COROLLARY 8. *Let ε be a congruence on a semigroup. Then $s \equiv t(\varkappa(\varepsilon))$ if and only if for every $x \in S$ there exists a natural number n such that $(sx)^n s \equiv (sx)^n t(\varepsilon)$ and $(tx)^n s \equiv (tx)^n t(\varepsilon)$.*

Proof. Apply Corollary 1 to the semigroup S/ε.

THEOREM 10. *Let $\hat{\lambda}$ be the corpoidal quasi-order of a semigroup S and let $\hat{\zeta}$ be the strong quasi-order of S. The following three statements are equivalent for all $s, t \in S$:*

(1) $s \prec t(\hat{\lambda})$;

(2) *for all $x \in S^1$ there exist natural numbers n such that $(sx)^{2n+1} \prec (sx)^n tx(sx)^n(\hat{\zeta})$;*

(3) *for all $x \in S$ there exist natural numbers n such that $(sx)^{2n+1} \prec (sx)^n tx(sx)^n(\hat{\zeta})$.*

Proof. Let $s \prec t(\hat{\lambda})$. Then $s \prec t(\zeta_P)$ for every symmetric representation P by univalent functions. Suppose that $P = P_{[h]}$ for $h \in S$. Then $sxy \in [h] \to txy \in [h]$ for every $x, y \in S^1$. If $h = sxy$, then $txy \in [sxy]$ and, by Lemma 10, $(sxy)^m \prec (sxy)^n txy(sxy)^n$ where $\prec$ stays for $\hat{\zeta}$. Condition (2) of our theorem follows from this fact and from

LEMMA 11. *Let $tx \in [sx]$ for all $x \in \{1, s, s^2, \ldots\}$. Then $s^{2n+1} \prec s^n ts^n$ for some natural n.*

Proof. There exists a representation P of S by univalent functions such that $s \prec t \leftrightarrow P(s) \subset P(t)$ for all $s, t \in S$ [13]. This fact and formula (6) of [13] imply $s^{m+n} \prec t \wedge s^m \prec t \to s^{m+n} \prec s^m$. Using this formula and supposing that $s^{m+n} \prec t \wedge s^m \prec t$ we obtain $s^{m+n} \prec s^m$, whence $s^{m+n} \prec s^{m+n}1$, $s^{m+n} \prec s^m 1$ and $s^{m+n} \prec s^m s^n$. Since $\prec$ is a steady quasi-order, $s^{m+n} \prec s^{m+n}s^n = s^{m+2n}$. But $s^{m+n} \prec s^m$ implies $s^{m+2n} \prec s^{m+n}$. Therefore, the following formula is true: $s^{m+n} \prec t \wedge s^m \prec t \to s^{m+n} \equiv s^{m+2n}(\hat{\varepsilon})$.

If s is an element of finite order, then $s^{p+q} = s^p$ for some natural p and q. The smallest possible q is called the period of s. If the image of s in the quotient semigroup $S/\hat{\varepsilon}$ is of finite order, then let q be its period. The symbol $\equiv$ stays for $\hat{\varepsilon}$. If q exists, let a be a multiple of q and

$a > |k-2l-1|$ where $s^k \prec s^l t s^l$. If $x = s^{a-1}$, then there exist natural m and n such that $s^{am} = (ss^{a-1})^m \prec (ss^{a-1})^n t s^{a-1} (ss^{a-1})^n = s^{an} t s^{an+a-1}$. Multiplying this inequality by s^a on the left and on the right several times, we may make $n \geqslant l$. Multiplying the inequality $s^k \prec s^l t s^l$ by s several times we obtain $s^{k+2an+a-1-2l} \prec s^{an} t s^{an+a-1}$. Two cases are possible.

Case 1. Let $am = k+2an+a-1-2l$, i.e., $a(m-2n-1) = k-2l-1$. This equality is possible only if $m-2n-1 = 0$ (since a is an arbitrary multiple of q or, if q does not exist, a is an arbitrary natural number). Thus, $k-2l-1 = 0$ and $s^{2l+1} \prec s^l t s^l$.

Case 2. Let $am \neq k+2an+a-1-2l$. By the formula which we have just proved, q exists and the difference $k+2an+a-1-2l-am = k-2l-1+a(2n+1-m)$ is a multiple of q. It follows that $k-2l-1$ is a multiple of q. Multiplying the inequality $s^k \prec s^l t s^l$ by s several times, we obtain $l \geqslant p$. Then $s^k \equiv s^{2l+1}$ and $s^{2l+1} \prec s^l t s^l$.

Lemma 11 is proved.

This Lemma shows that (1) $\to$ (2) in Theorem 10. The implication (2) $\to$ (3) is obvious. To prove (3) $\to$ (1) suppose that (3) is fulfilled and let P be a symmetric representation of S by univalent functions over a set A. If $(a_1, a_2) \in P(s)$, then $(a_2, a_1) \in P(x)$ for some $x \in S$. Thus, $(a_1, a_1) \in P(sx)$. The fundamental quasi-order ζ_P is steady [13]; hence, $\zeta \subset \zeta_P$. Therefore, $(a_1, a_1) \in P\big((sx)^{2n+1}\big) \subset P\big((sx)^n tx (sx)^n\big) = P\big((sx)^n\big) \circ P(tx) \circ P\big((sx)^n\big)$. Since $P\big((sx)^n\big)$ is a univalent function which contains the pair (a_1, a_1), $(a_1, a_1) \in P(tx) = P(x) \circ P(t)$. It follows that $(a_1, a_2) \in P(t)$ and $P(s) \subset P(t)$, i.e., $(s, t) \in \zeta_P$. Since it is true for every symmetric representation P by univalent functions, $s \prec t(\hat{\lambda})$.

COROLLARY 1. *Let $\hat{\varkappa}$ be the corpoidal radical of a semigroup S and $\hat{\zeta}$ the strong quasi-order of S. The following three statements are equivalent for all $s, t \in S$:*

(1) $s \equiv t(\hat{\varkappa})$;

(2) *for every $x \in S^1$ there exists a natural number n such that $(sx)^{2n+1} \prec (sx)^n tx (sx)^n$ and $(tx)^{2n+1} \prec (tx)^n sx (tx)^n (\hat{\zeta})$;*

(3) *for every $x \in S$ there exists a natural number n such that $(sx)^{2n+1} \prec (sx)^n tx (sx)^n$ and $(tx)^{2n+1} \prec (tx)^n sx (tx)^n (\hat{\zeta})$.*

COROLLARY 2. *A semigroup S is corpoidal semisimple if and only if the existence of a natural n for every $x \in S$ such that $(sx)^{2n+1} \prec (sx)^n tx (sx)^n$ and $(tx)^{2n+1} \prec (tx)^n sx (tx)^n$ implies $s = t$.*

COROLLARY 3. *Every cancellative semigroup is corpoidal semisimple.*

Proof. If S is cancellative, then $\{s\}$ is a strong subset of S^1 for every $s \in S^1$. It follows that $\hat{\zeta} = \Delta_S$. By Corollary 1, $s \prec t(\hat{\lambda})$ is equivalent to $sx = tx$ or to $s = t$. Therefore, $\hat{\varkappa} = \Delta_S$.

COROLLARY 4. *A regular semigroup is corpoidal semisimple if and only if it is an inverse semigroup.*

Proof. Let S be a corpoidal semisimple regular semigroup. Since $\hat{\varepsilon} \subset \hat{\varkappa}$, we infer that $\hat{\varepsilon} = \Delta_S$, i.e., that S is embeddable into an inverse semigroup [6], [7]. It follows that S is an inverse semigroup.

On the other hand, every inverse semigroup has a proper symmetric representation by univalent functions (in fact, every representation of an inverse semigroup by univalent functions is symmetric [8]). Thus, inverse semigroups are corpoidal semisimple.

COROLLARY 5. *An idempotent semigroup is corpoidal semisimple if and only if it is commutative, i.e., if it is a semilattice.*

Proof. Idempotent semigroups are regular. Apply Corollary 4.

A band of semigroups (S, ε) is called a *semilattice of semigroups* if S/ε is a semilattice.

COROLLARY 6. *If (S, ε) is a semilattice of corpoidal semisimple semigroups, then S is corpoidal semisimple.*

Proof. Suppose $s \equiv t(\hat{\varkappa})$ for $s, t \in S$. Elementary formulas expressing the fact that $x \prec y(\hat{\zeta})$ [6] show that homomorphisms of semigroups are isotone relative to the strong quasi-orders of the semigroups considered. If $\bar{x}$ is the image of $x \in S$ under the canonical homomorphism of S onto S/ε, then, by Corollary 1, $\bar{s} \equiv \bar{t}(\hat{\varkappa})$ in S/ε, i.e., $\bar{s} = \bar{t}$ and $s \equiv t(\varepsilon)$. Since $\varepsilon\langle s\rangle$ is corpoidal semisimple and $s \equiv t(\hat{\varkappa})$, we infer that $s \equiv t(\hat{\varkappa}_s)$ where $\hat{\varkappa}$ is the corpoidal radical of $\varepsilon\langle s\rangle$. Thus, $s = t$ and S is corpoidal semisimple.

COROLLARY 7. *A semigroup S is corpoidal radical if and only if the quotient semigroup $S/\hat{\varepsilon}$ of S by the strong equivalence relation $\hat{\varepsilon}$ is a nil-semigroup.*

Proof. If S^1 has proper strong subsemigroups, then S has transitive representations with non-trivial kernels by univalent functions. This being the case, S cannot be corpoidal radical. Therefore, if S is corpoidal radical, then $[s] = S^1$ for all $s \in S$. By Lemma 10, for every s and $t \in S$ there exist m and n such that $s^m \prec s^n t s^n$. If $s = t^a$, then $t^{am} \prec t^{2an+1}$. If $a \neq 1$, then $am \neq 2an+1$. In the proof of Lemma 11 we have seen that this implies that the image $\bar{t}$ of t in $S/\hat{\varepsilon}$ is an element of finite order q which is a divisor of the difference $am-(2an+1) = a(m-2n)-1$. If $a = q$, then the latter equality is possible only when $q = 1$. Thus, $q = 1$, i.e., $t^{p+1} \equiv t^p(\hat{\varepsilon})$. We will denote t^p by t_0. Multiplying the inequality $s^m \prec s^n t s^n$ by s several times, we obtain $s_0 \prec s_0 t s_0$ and, since $\prec$ is steady, $s_0 \prec 1 s_0$, $s_0 \prec s_0 s_0$, $s_0 \prec s_0 t s_0$ imply $s_0 \prec t s_0$. It follows that $t s_0 \prec t t s_0$. Now the inequalities $t s_0 \prec 1 t s_0$, $t s_0 \prec t t s_0$, $t s_0 \prec t s_0$ imply that $t s_0 \prec s_0$. Thus, $s_0 \equiv t s_0(\hat{\varepsilon})$. In the same way we may prove that $s_0 \equiv s_0 t(\hat{\varepsilon})$, i.e., $\bar{s}_0$ is the zero of $S/\hat{\varepsilon}$ and $S/\hat{\varepsilon}$ is a nil-semigroup.

If $S/\hat{\varepsilon}$ is a nil-semigroup and the image of s^n is the zero of $S/\hat{\varepsilon}$, then $s^n \prec s^n t s^n$ for all $s, t \in S$, i.e., $s \equiv t(\hat{\varkappa})$ and S is corpoidal radical.

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