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Algebraic objects
over a small category

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Introduction

The theory of geometric objects has a long history and many results have been obtained up to now. In this paper we treat the theory of particular and abstract geometric objects as a starting point for more general considerations. The purpose of this paper is the introduction of general notions and general results of the theory of particular and abstract geometric objects to the theory of algebraic objects.

In 1968 there appeared an article [32] by A. Zajtz in which the author generalizing the results of the theory of geometric objects obtained so far, introduced the notion of an algebraic object and proved many theorems concerning such objects. Unfortunately the definition of an algebraic object was formulated in such a manner that not every abstract geometric object is an algebraic object. We show this in Section I. Therefore we give here a somewhat modified definition of an algebraic object.

In the theory of geometric objects (as well particular as abstract) the notion of transitive fibre of the object plays a very important role. This notion is no longer valid if we consider algebraic objects over structures more general than a group. In Section II we introduce (by a natural generalization of the notion of a transitive abstract geometric object) the notion of a quasi-transitive algebraic object and, inspired by the theory of abstract machines (cf. [4]), the notion of a quasi-invertible algebraic object. We show that both these concepts closely related. Furthermore, we prove that the quasi-transitive algebraic objects play a similar role in the theory of algebraic objects as the transitive geometric objects in the theory of geometric objects. In this section we also show some imperfections of note [32].

In Section III we introduce the notion of the rank of quasi-transitivity of elements and of quasi-transitive fibres.

J. Gancarzewicz gave in [3] the definition of a commutative algebraic object. We shall deal with such objects in Section IV. In particular we shall discuss some inaccuracies occurring in note [3].

In Section V we give the definition of a regular algebraic object and we prove that regular algebraic objects have properties similar to those of regular abstract geometric objects. A particular geometric object over a group may be regarded as a function defined on a transformation group (cf. [15]). Section VI is based on this idea. In this section we give

the definition of a particular algebraic object (different from that given in [32], [10]) and we give a certain characterization of this concept. We also generalize some results connected with concomitants of a given abstract (a particular, respectively) geometric object obtained by S. Midura and Z. Moszner in [14] and [16].

In Section VII we give the definition of a reduced (with respect to fibre and with respect to structure) algebraic object. The notion of an algebraic object reduced with respect to structure is a generalization of the notion of an effective abstract geometric object.

An algebraic object may be defined also as a function satisfying the translation equation and the identity condition since we know the general solution of the translation equation over a group (cf. [13], [17]), over a groupoid (cf. [27]), and over a small category (cf. [7]). The theory of the translation equation yields some results on algebraic objects. This method is especially convenient for an application to the problem of extensions of algebraic objects⁽¹⁾. (and some other problems). We investigate objects over groupoid in such a way in Section VIII.

I would like to express my best thanks to my teacher Professor Zenon Moszner for his valuable remarks during the preparation of this work.

⁽¹⁾ In this paper we investigate the problem of extending algebraic objects over groupoids with respect to fibre (i.e. we assume that the groupoid is fixed).

The problem of extending algebraic objects over groups with respect to structure (i.e. when the fibre is fixed) has been investigated in [18] and [21].

I. Basic notions

According to [30] we accept the following four definitions:

DEFINITION 1.1. A pair (G, \cdot) , where G is an arbitrary non-empty set and \cdot is an arbitrary binary operation in G (i.e. \cdot is a partial mapping of $G \times G$ into G) will be called a *multiplicative system*. An element $\varepsilon \in G$ will be called a *unity* whenever the following condition for arbitrary $\gamma \in G$ holds:

If $\varepsilon\gamma$ is defined, then $\varepsilon\gamma = \gamma$;

if $\gamma\varepsilon$ is defined, then $\gamma\varepsilon = \gamma$.

The set of unities of (G, \cdot) will be denoted by G_0 .

DEFINITION 1.2. A multiplicative system (G, \cdot) will be called a *small category* if the following axioms are satisfied:

(a) If in the equation

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma, \quad \alpha, \beta, \gamma \in G$$

one of its sides or both the products $\beta\gamma$ and $\alpha\beta$ are defined, then both sides of the equation are defined and the equality holds.

(b) For any $a \in G$ there exist $\varepsilon_l(a), \varepsilon_r(a) \in G_0$ (a *left unity* of a and a *right unity* of a , respectively) such that $\varepsilon_l(a)a$ and $a\varepsilon_r(a)$ are defined.

DEFINITION 1.3. A small category (G, \cdot) will be called a *groupoid*, if in addition to axioms (a), (b) the following condition holds:

(c) For any $a \in G$ there exists an inverse element $a^{-1} \in G$ such that

$$a^{-1}a = \varepsilon_r(a), \quad aa^{-1} = \varepsilon_l(a).$$

DEFINITION 1.4. A groupoid (G, \cdot) will be called a *Brandt groupoid* if the following condition holds:

(d) For any two elements $a, \gamma \in G$ there exists an element $\beta \in G$ such that the products $a\beta$ and $\beta\gamma$ are defined.

Let A be an arbitrary non-empty set and G an arbitrary group. In the set $A \times A \times G$ we define operation \cdot as follows:

The product $(a, b, \alpha)(c, d, \beta)$ is defined iff $b = c$ and then

$$(a, b, \alpha)(c, d, \beta) = (a, d, \alpha\beta).$$

It is easy to verify that the set $A \times A \times G$ with such an operation \cdot is a Brandt groupoid. Such a groupoid will be called a *product groupoid*

(this term has been proposed by A. Zajtz) and it will be also denoted by $A \times A \times G$.

DEFINITION 1.5 (cf. [31]). Let (G_1, \cdot) , (G_2, \cdot) be two multiplicative systems. A function $f: G_1 \rightarrow G_2$ will be called a *homomorphism* if it satisfies the following condition:

If $a\beta$ ($a, \beta \in G_1$) is defined, then $f(a)f(\beta)$ is defined and

$$f(a \cdot \beta) = f(a) \cdot f(\beta).$$

If, in addition, for every $a', \beta' \in G_2$ such that $a'\beta'$ is defined there exist $a \in f^{-1}(\{a'\})$, $\beta \in f^{-1}(\{\beta'\})$ such that $a\beta$ is defined, then homomorphism f will be called a *strong homomorphism*.

If a homomorphism f maps G_1 onto G_2 , then it will be called an *epimorphism*.

An epimorphism which is a strong homomorphism will be called a *strong epimorphism*.

A homomorphism which is a bijection will be called an *isomorphism*.

An isomorphism which is a strong homomorphism will be called a *strong isomorphism*.

Let X_n be an n -dimensional manifold, i.e. a topological space whose every point has a neighbourhood homeomorphic to an open domain of the n -dimensional arithmetical space \mathbf{R}^n .

Let $x_0 \in X_n$ be fixed and let a set Z of local coordinate systems at x_0 be given (i.e., a set of homeomorphisms defined in a neighbourhood of x_0 with values in \mathbf{R}^n) be given. Let G denote the set of all transformations T defined in certain subsets of R and taking its values in R , which can be written in the form

$$T = U_2 \circ U_1^{-1} \quad U_1, U_2 \in Z.$$

Here we understand the composition of mappings as the composition of relations. For $U \in Z$, $T \in G$ we define the composition $T \circ U$ with respect to x_0 in a slightly different way. We shall say that $T \circ U$ exists if and only if $U(x_0)$ belongs to the domain of T . This definition quarantees that if $T \circ U$ exists, then it is a local coordinate system at x_0 . If for every $U \in Z$, $T \in G$ such that $T \circ U$ exists, $T \circ U$ belongs to Z , then Z is called a *local structure at x_0* .

Let Z be a local structure at x_0 . Every function

$$(1.1) \quad \varphi: Z \rightarrow \mathbf{R}^m$$

(R is sometimes replaced by an arbitrary non-empty set) is called an *object at the point x_0* .

An object is called *geometric* whenever there exists a function F defined in a certain subset of the set $\varphi(Z) \times G$, taking its values in $\varphi(Z)$ and satisfying the following two conditions:

(i) For $\omega \in \varphi(Z)$, $T \in G$, $F(\omega, T)$ is defined if and only if there exists $U \in Z$ with $\varphi(U) = \omega$ such that $T \circ U$ is defined.

(ii) For every $U \in Z$, $T \in G$ such that $T \circ U$ is defined we have

$$\varphi(T \circ U) = F(\varphi(U), T).$$

Such a function F is called the *transformation formula* (or *transformation law*) of the object (1.1).

More details on geometric objects can be found in [10], [8], [5], [1].

The class of all geometric objects with the same transformation formula F will be called an *abstract geometric object with the transformation formula F* ⁽²⁾.

Every transformation formula determines uniquely an abstract geometric object with this transformation formula, and conversely, every abstract geometric object has its uniquely determined transformation formula. Thus, we may identify an abstract geometric object with its transformation formula.

In note [32] A. Zajtz has given the definition of an algebraic object. A *left algebraic object* has been defined as a triple (X, G, F) ⁽³⁾, where X is an arbitrary non-empty set, G is a small category and F is a function defined in some subset of $X \times G$ assuming its values in X and satisfying the following axioms:

(A) For arbitrary $x \in X$, $\alpha, \beta \in G$, if $F(x, \beta)$ and $\alpha \cdot \beta$ are defined, then both sides of the following equality

$$(1.2) \quad F(F(x, \beta), \alpha) = F(x, \alpha\beta)$$

are defined and the equality holds.

(B) For arbitrary $x \in X$, $\alpha \in G$, if $F(x, \alpha)$ is defined, then $F(x, \varepsilon_l(\alpha))$ and $F(x, \varepsilon_r(\alpha))$ ($\varepsilon_l(\alpha)$ is the left unity of α and $\varepsilon_r(\alpha)$ is the right unity of α) are defined and

$$F(x, \varepsilon_l(\alpha)) = F(x, \varepsilon_r(\alpha)) = x.$$

A right algebraic object has been defined in a similar way (cf. [32], p. 69). A (left or right) algebraic object defined in this way is not a generalization of the notion of an abstract geometric object (understood as a trans-

⁽²⁾ As far as is known to the author, an abstract geometric object has been so far defined only in the special class of geometric objects called *purely differential geometric objects* (cf. [10], p. 20). In our note an abstract geometric object in this sense will be called *purely differential abstract geometric object*.

⁽³⁾ These notations are different from those used in [32]. We have adopted our notation from [6].

formation formula of any geometric object). It is shown by the following

EXAMPLE 1.1. As a manifold X_n we take an open interval $(-\frac{1}{2}, \frac{1}{2})$ with the natural topology. Let $x_0 \in (-\frac{1}{2}, \frac{1}{2})$ be fixed. As the set Z of coordinate systems at x_0 we take the set of maps of the form:

$$U_k: x \rightarrow x + k \quad \text{for} \quad x \in (-\frac{1}{2}, \frac{1}{2}),$$

where k runs over the set of all integers. Then the set of transformations leading from any coordinate system to another one consists of all local transformations of the form:

$$T_{k,l}(x) = x + l - k \quad \text{for} \quad x \in (k - \frac{1}{2}, k + \frac{1}{2}),$$

where k, l are integers.

This set is obviously a small category. Let us put:

$$\omega = \varphi(U_k) = k.$$

It is clear that φ defines a geometric object. Its transformation formula F is defined as follows:

$F(\omega, T_{k,l})$ is defined iff $\omega = k$, and if $\omega = k$, then $F(\omega, T_{k,l}) = l$.

The function F defined in such way does not satisfy condition (B). Indeed: $F(1, T_{1,2})$ is defined, $T_{2,2}$ is the left unity of $T_{1,2}$, but $F(1, T_{2,2})$ is not defined.

We suggest the following

DEFINITION 1.6. A triple (X, G, F) , where X is a non-empty set, G is a small category and F is a function defined in some subset of $X \times G$ assuming its values in X , will be called a *left algebraic object over G* if F satisfies condition (A) and the following condition

(B') For arbitrary $x \in X$, $\alpha \in G$, if $F(x, \alpha)$ is defined, then $F(x, \varepsilon_r(\alpha))$ is defined and

$$F(x, \varepsilon_r(\alpha)) = x.$$

The set $\pi(D_F)$, where D_F is the domain of F and π is the natural projection of $X \times G$ onto X (i.e. $\pi(x, \alpha) := x$ for $x \in X$, $\alpha \in G$) will be called the *fibre of a left algebraic object (X, G, F)* .

Because we shall deal with left algebraic objects only, we shall leave out the term "left". Furthermore (for convenience), we shall assume that all points of X which do not belong to the fibre of object have been left out — and hence the notation (X, G, F) will mean that X is the fibre of the algebraic object (X, G, F) . We shall often write (\bar{X}, G) instead of (X, G, F) and ax instead of $F(x, \alpha)$, and we shall call ax the *exterior product*.

DEFINITION 1.7 (cf. [32], p. 69, [2], p. 9). An algebraic object (X, G, F) (with fibre A , accordingly to our assumption) is called *complete* if F is defined on the whole of $X \times G$; otherwise the object (X, G, F) is called *non-complete*.

DEFINITION 1.8 (cf. [32], p. 71, [6], p. 276). Let (X, G, F) be an algebraic object and let $\bar{X} \subset X$. If a triple $(\bar{X}, G, f|\bar{X})$ ⁽⁴⁾ is an algebraic object, then this object is called a *subobject* of (X, G, F) .

DEFINITION 1.9 (cf. [32], p. 71). Let (X, G, F) be an algebraic object and let $\bar{X} \subset X$. The set \bar{X} is called *stable* if the following condition holds:

$$F(\bar{X}, G) \subset \bar{X}.$$

It is easy to prove the following

THEOREM 1.1 (cf. [32], p. 72). *Let (X, G, F) be an algebraic object and let $\bar{X} \subset X$. The set \bar{X} is the fibre of a subobject of (X, G, F) iff \bar{X} is a stable subset of X .*

DEFINITION 1.10 (cf. [32], p. 70, [25], p. 186). Let (X, G, F) be an algebraic object. An equivalence relation \mathcal{R} defined on the fibre X is called *conformable with the object* whenever the following condition holds:

If $x\mathcal{R}x'$ and $F(x, a), F(x', a)$ are defined, then $F(x, a)\mathcal{R}F(x', a)$.

Let \mathcal{R} be an equivalence relation conformable with the object (X, G, F) , and let $[x]$ denote the equivalence class of x with respect to \mathcal{R} . We define the function F/\mathcal{R} as follows:

$F/\mathcal{R}([x], a)$ is defined iff there exists $y \in [x]$ (i.e. $y\mathcal{R}x$) such that $F(x, a)$ is defined; and then we put:

$$F/\mathcal{R}([x], a) := [F(y, a)].$$

It can be proved that the function F/\mathcal{R} is well defined (i.e. it is independent of the choice of y) and the triple $(X/\mathcal{R}, G, F/\mathcal{R})$ is an algebraic object. This object is called the *factor-object* of (X, G, F) (modulo \mathcal{R}).

Adapting the notion of the homomorphism of abstract machines defined in [4], p. 51, to the theory of algebraic objects, we introduce the following

DEFINITION 1.11. Let $\Omega_1 = (X, G_1), \Omega_2 = (Y, G_2)$ be two algebraic objects. A pair of mappings $h: X \rightarrow Y, g: G_1 \rightarrow G_2$, where g is a homomorphism of G_1 into G_2 , will be called a *FS-homomorphism* of Ω_1 into Ω_2 if the following condition holds:

If ax ($x \in X, a \in G$) is defined, then $g(a)h(x)$ is defined and

$$h(ax) = g(a)h(x).$$

⁽⁴⁾ $f|\bar{X}$ denotes the restriction of f to the set \bar{X} .

If, in addition, $X = Y$ and h is the identity mapping, then g will be called an *S-homomorphism*; whereas if $G_1 = G_2$ and g is the identity mapping, then h will be called a *homomorphism*.

An FS-homomorphism will be called *strong* if g is a strong homomorphism (of G_1 into G_2) and the following condition holds:

If $g(a)h(x)$ ($x \in X$, $a \in G$) is defined, then there exists $a' \in g^{-1}(\{g(a)\})$, $x' \in h^{-1}(\{h(x)\})$ such that $a'x'$ is defined.

Such an FS-homomorphism (h, g) that h and g are epimorphisms will be called an *FS-epimorphism*. Then the algebraic object Ω_2 will be called an *FS-concomitant* of the algebraic object Ω_1 . If an FS-epimorphism (h, g) is strong, then Ω_2 will be called a *strong FS-concomitant* of Ω_1 .

Such an FS-homomorphism (a strong FS-homomorphism) (h, g) that h is a bijection of X onto Y and g is an isomorphism (a strong isomorphism, respectively) will be called an *FS-isomorphism* (a *strong FS-isomorphism*, respectively). Strong FS-isomorphic objects will be called *FS-equivalent*.

In a similar way we define *S-epimorphism*, *epimorphism*, *S-concomitant*, *concomitant*, *S-isomorphism*, etc.

Algebraic objects Ω_1, Ω_2 will be called *similar* if each of them is a concomitant of the other.

Remark 1.1. In our terminology, the prescripts employed: F-, S-, FS- are abbreviations for: fibre, structure, fibre and structure. And so: an FS-homomorphism is a mapping acting on the fibre and the structure, while S-homomorphism is a mapping acting on the structure only.

II. Quasi-transitive algebraic objects

We start with the following

DEFINITION 2.1 (cf. [10], p. 26, [32], p. 72). An algebraic object (X, G) is called *transitive* if for arbitrary $x, y \in X$ there exists $a \in G$ such that

$$ax = y.$$

We give a generalization of the notion of transitivity.

Let G be an arbitrary non-empty set. By \bar{G} we denote the set of all finite non-empty sequences (called *words*) of elements of G . Elements of \bar{G} will be denoted by $\bar{\alpha}, \bar{\beta}$, etc. or written as $(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)$; $(\beta_m, \beta_{m-1}, \dots, \beta_1)$, etc. In \bar{G} we define an interior product as follows:

For $\bar{\alpha} = (\alpha_n, \dots, \alpha_1)$, $\bar{\beta} = (\beta_m, \dots, \beta_1)$ we put

$$(2.1) \quad \bar{\alpha}\bar{\beta} := (\alpha_n, \dots, \alpha_1, \beta_m, \dots, \beta_1).$$

The number of elements of the sequence $\bar{\alpha}$ will be called the *length* of $\bar{\alpha}$ and it will be denoted by $l(\bar{\alpha})$. We shall identify a one-element word (α) with the element α .

Let (X, G) be an algebraic object. We define an exterior product $\bar{\alpha}x$ ($\bar{\alpha} \in \bar{G}$, $x \in X$) in the following way:

$(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)x$ is defined iff $\alpha_1x, \alpha_2(\alpha_1x), \dots, \alpha_n(\dots \alpha_2(\alpha_1x) \dots)$ are defined; and we put

$$(2.2) \quad (\alpha_n, \alpha_{n-1}, \dots, \alpha_1)x := \alpha_n(\alpha_{n-1}(\dots(\alpha_1x) \dots)).$$

Let $g: G \rightarrow G_1$ be any mapping. We extend g to a mapping (denoted by the same letter g) of the set \bar{G} into \bar{G}_1 by

$$(2.3) \quad g((\alpha_n, \dots, \alpha_1)) := (g(\alpha_n), \dots, g(\alpha_1)).$$

Let now G be a groupoid and let $\bar{\alpha} = (\alpha_n, \alpha_{n-1}, \dots, \alpha_1) \in \bar{G}$. By $(\bar{\alpha})^{-1}$ we denote the element

$$(2.4) \quad \bar{\alpha}^{-1} := (\alpha_1^{-1}, \dots, \alpha_{n-1}^{-1}, \alpha_n^{-1}).$$

THEOREM 2.1. *Let (X, G) be an algebraic object. The exterior product $\bar{\alpha}x$ ($\bar{\alpha} \in \bar{G}$, $x \in X$) has the following properties:*

P.2.1. *$(\beta\bar{\alpha})x$ is defined iff $\beta(\bar{\alpha}x)$ is defined. If $\beta(\bar{\alpha}x)$ is defined, then*

$$(\beta\bar{\alpha})x = \beta(\bar{\alpha}x).$$

P.2.2. *If G is a groupoid and $\bar{\alpha}x$ is defined, then $\bar{\alpha}^{-1}(\bar{\alpha}x)$ is defined and*

$$\bar{\alpha}^{-1}(\bar{\alpha}x) = x.$$

P.2.3. *Let (h, g) be an FS-homomorphism of (X, G) into another object. We have*

If $\bar{\alpha}x$ is defined, then $g(\bar{\alpha})h(x)$ is defined and

$$h(\bar{\alpha}x) = g(\bar{\alpha})h(x).$$

Proof. Property P.2.1 follows immediately from (2.1) and (2.2).

We shall prove Property P.2.2 by induction with respect to $l(\bar{\alpha})$. Let $l(\bar{\alpha}) = 1$, i.e. $\bar{\alpha} = \alpha$ (then $\bar{\alpha}^{-1} = \alpha^{-1}$) and let αx be defined. Then, by the definition of an algebraic object, $\alpha^{-1}(\alpha x)$ is defined (as αx and $\alpha^{-1}\alpha$ are defined) and

$$\alpha^{-1}(\alpha x) = (\alpha^{-1}\alpha)x = \varepsilon_r(\alpha)x = x.$$

Now assume that Property P.2.2 holds for words of length n and let $l(\bar{\alpha}) = n+1$. Thus $\bar{\alpha}$ has the form:

$$\bar{\alpha} = \bar{\alpha}_1\alpha, \quad l(\alpha_1) = n,$$

and hence (see (2.4))

$$\bar{\alpha}^{-1} = \alpha^{-1}\bar{\alpha}_1^{-1}.$$

Let $\bar{\alpha}x$ be defined. Then, by P.2.1, $\bar{\alpha}_1(\alpha x)$ is defined and hence by the induction assumption, $\bar{\alpha}_1^{-1}(\bar{\alpha}_1(\alpha x))$ is defined and

$$\bar{\alpha}^{-1}(\bar{\alpha}_1(\alpha x)) = \alpha x.$$

Thus $\alpha^{-1}(\bar{\alpha}_1^{-1}(\bar{\alpha}_1(\alpha x)))$ is defined and

$$\alpha^{-1}(\bar{\alpha}_1^{-1}(\bar{\alpha}_1(\alpha x))) = \alpha^{-1}(\alpha x) = x.$$

Now we obtain from P.2.1 that $(\alpha^{-1}\bar{\alpha}_1^{-1})(\bar{\alpha}_1\alpha x)$ is defined and

$$(\alpha^{-1}\bar{\alpha}^{-1})(\bar{\alpha}_1\alpha x) = x,$$

i.e. $\bar{\alpha}^{-1}(\bar{\alpha}x) = x$. This completes the proof of Property 2.2.

We shall prove Property P.2.3 also by induction with respect to $l(\bar{\alpha})$.

For $l(\bar{\alpha}) = 1$ P.2.3 holds by the definition of an FS-homomorphism. Let us assume that P.2.3 holds for words of length n and let

$$\bar{\alpha} = \bar{\alpha}_1\alpha, \quad l(\bar{\alpha}_1) = n.$$

Suppose that $\bar{\alpha}x$ is defined. Then we obtain, using the induction assumption, P.2.1 and condition (2.3):

$$\begin{aligned} h(\bar{\alpha}x) &= h((\bar{\alpha}_1\alpha)x) = h(\bar{\alpha}_1(\alpha x)) = g(\bar{\alpha}_1)h(\alpha x) = g(\bar{\alpha}_1)(g(\alpha)h(x)) \\ &= g(\bar{\alpha}_1\alpha)h(x) = g(\bar{\alpha})h(x). \end{aligned}$$

Hence, Property P.2.3 is completely proved.

We propose to assume the following

DEFINITION 2.2. An algebraic object (X, G) will be called *quasi-transitive* if for arbitrary $x, y \in X$ there exists $\bar{\alpha} \in \bar{G}$ such that

$$\bar{\alpha}x = y.$$

It is seen that every transitive object is quasi-transitive, but not conversely (suitable examples will be given in the sequel).

On the other hand, if a small category G is a semigroup with unity (and hence for arbitrary elements $\alpha, \beta \in G$ their product $\alpha\beta$ is defined), then by condition (A) of Definition 1.6 quasi-transitivity is equivalent to transitivity.

THEOREM 2.2. *An algebraic object is quasi-transitive iff it does not contain a proper subobject (i.e. a subobject with fibre different from the fibre of the object).*

Proof. It is obvious that the fibre of a quasi-transitive object does not contain a proper stable subset and hence a quasi-transitive object does not contain a proper subobject (see Theorem 1.1). Conversely, let us assume that the object (X, G) does not contain a proper subobject.

Take any $x \in X$. We put:

$$(2.5) \quad \bar{G}x := \{y \in X : \exists_{\bar{a} \in \bar{G}} (\bar{a}x = y)\}.$$

It follows from P.2.1 that $\bar{G}x$ is a stable subset of X . Because (X, G) does not contain a proper subobject, we have $\bar{G}x = X$. Thus for every $y \in X$ there exists $\bar{a} \in \bar{G}$ such that $\bar{a}x = y$. It means that the object (X, G) is quasi-transitive.

Let (X, G) be an algebraic object. We define in the set X the following relation \mathcal{R} :

$$(2.6) \quad x\mathcal{R}y \Leftrightarrow \exists_{\bar{a} \in \bar{G}} \exists_{\bar{\beta} \in \bar{G}} \{(\bar{a}x = y) \wedge (\bar{\beta}y = x)\}.$$

It is seen that \mathcal{R} is an equivalence relation (the reflexivity of \mathcal{R} follows from P.2.1).

In analogy to the theory of abstract machines (cf. [4] p. 55) we shall introduce the notion of a quasi-invertible algebraic object.

DEFINITION 2.3. An algebraic object (X, G) will be called *quasi-invertible* whenever the following condition holds:

If $\bar{a}x = y$ for some $x, y \in X$, $\bar{a} \in \bar{G}$, then there exists $\bar{\beta} \in \bar{G}$ such that $\bar{\beta}y = x$.

If an object (X, G) is quasi-invertible, then the relation \mathcal{R} may be written as follows:

$$(2.7) \quad x\mathcal{R}y \Leftrightarrow \exists_{\bar{a} \in \bar{G}} (\bar{a}x = y).$$

THEOREM 2.3. *An algebraic object (X, G) is quasi-invertible iff it is a disjoint union of quasi-transitive algebraic objects over the same small category G . If the object (X, G) is quasi-invertible, then this decomposition is unique.*

Proof. Let (X, G) be quasi-invertible. Let $X = \bigcup_{s \in S} X_s$ be the decomposition of X into equivalence classes defined by \mathcal{R} (\mathcal{R} is an equivalence relation). We shall show that every pair (X_s, G) with the exterior product in (X, G) restricted to (X_s, G) is an algebraic object. It is sufficient to prove that every X_s is a stable subset of X . Let $x \in X$, $a \in G$ and let ax be defined. Then $ax\mathcal{R}x$ (see (2.7)) and hence $ax \in X_s$. This proves that X_s is a stable subset of X . It follows from (2.7) that the object (X_s, G) is quasi-transitive.

Now we show that the decomposition of (X, G) into the disjoint union of quasi-transitive objects is unique. Let $X = \bigcup_{t \in T} X_t$ be another such decomposition. Then (for fixed t) condition $x, y \in X_t$ implies condition $x\mathcal{R}y$. This means that every set X_t is included in some set X_s . Since

(X_s, G) is quasi-transitive, we obtain by Theorem 2.2 $X_s = X_t$. Thus the decompositions $\{X_s\}$, $s \in S$, and $\{X_t\}$, $t \in T$, are identical.

To complete the proof of Theorem 2.3, let us observe that every quasi-transitive object is quasi-invertible and that the disjoint union of quasi-invertible objects over the same small category G is a quasi-invertible object, too.

From Property P.2.2 we obtain immediately the following

THEOREM 2.4. *Every algebraic object over a groupoid is quasi-invertible.*

DEFINITION 2.4. Let (X, G) be a quasi-invertible algebraic object. The equivalence classes of X with respect to relation \mathcal{R} will be called *quasi-transitive fibres* of the object (X, G) . The quasi-transitive fibre determined by an element x_0 will be denoted by $\bar{G}x_0$.

Now we shall prove a few theorems on the connections between the relation \mathcal{R} and FS-homomorphisms.

THEOREM 2.5. *The relation \mathcal{R} is invariant with respect to FS-homomorphisms, i.e., for any FS-homomorphism (h, g) of the object (X, G) into an object (X_1, G_1) the following condition holds:*

If $x\mathcal{R}y$, then $h(x)\mathcal{R}h(y)$.

Proof. Let $x\mathcal{R}y$. Thus there exist $\bar{\alpha}, \bar{\beta} \in G$ such that

$$\bar{\alpha}x = y, \quad \bar{\beta}y = x.$$

Using P.2.3, we obtain

$$\begin{aligned} h(y) &= h(\bar{\alpha}x) = g(\bar{\alpha})h(x), \\ h(x) &= h(\bar{\beta}y) = g(\bar{\beta})h(y). \end{aligned}$$

Hence $h(x)\mathcal{R}h(y)$.

From Theorem 2.5 results immediately the following

COROLLARY 2.1. *Let (h, g) be an FS-homomorphism of the quasi-invertible object (X, G) into the quasi-invertible object (X_1, G_1) . Then an image through the function h of any quasi-transitive fibre of the object (X, G) is included in some quasi-transitive fibre of the object (X_1, G_1) .*

THEOREM 2.6. *Every FS-concomitant of a quasi-transitive algebraic object is quasi-transitive algebraic object.*

Proof. Let (X, G) be a quasi-transitive algebraic object and let (h, g) be an epimorphism of the object (X, G) onto another object (Y, H) . Let us consider arbitrary elements $x', y' \in Y$. There exists $x, y \in X$ such that

$$(2.8) \quad h(x) = x', \quad h(y) = y'.$$

As the object (X, G) is quasi-transitive, there exists $\bar{\alpha} \in \bar{G}$ such that

$$(2.9) \quad \bar{\alpha}x = y.$$

We obtain from (2.8) and (2.9) by applying P.2.3

$$y' = h(y) = h(\bar{\alpha}x) = g(\bar{\alpha})h(x) = g(\bar{\alpha})x'.$$

This means that the object (Y, H) is quasi-transitive.

THEOREM 2.7. *Let (X, G_1) be a complete algebraic object and let (Y, G_2) be a quasi-transitive algebraic object. Let a pair $h: X \rightarrow Y, g: G_1 \rightarrow G_2$ be an FS-homomorphism of (X, G_1) into (Y, G_2) and let g be an epimorphism (i.e. let $g(G_1) = G_2$). Then (h, g) is an FS-epimorphism.*

Proof. Let us consider a fixed element $x_0 \in X$. We have for arbitrary $\bar{\alpha} \in \bar{G}$

$$h(\bar{\alpha}x_0) = g(\bar{\alpha})h(x_0).$$

If $\bar{\alpha}$ runs over the set \bar{G}_1 , then $g(\bar{\alpha})$ runs over the set G_2 (as g is an epimorphism), and hence (Y, G_2) being quasi-transitive $g(\bar{\alpha})h(x_0)$ runs over the set Y . Thus $h(X) = Y$.

THEOREM 2.8. *Let (X, G) be a quasi-invertible complete algebraic object and let (h, g) be an FS-epimorphism of the object (X, G) onto an object (Y, H) . Then the object (Y, H) is complete, quasi-invertible, and the image through h of a quasi-transitive fibre of the object (X, G) is a quasi-transitive fibre of the object (Y, H) .*

Proof. Completeness of (Y, H) follows immediately from the definition of an FS-epimorphism.

We now prove that (Y, H) is quasi-invertible. Let $x' \in H, y' = \bar{\alpha}'x', \bar{\alpha}' \in \bar{H}$. Take an $x \in X$ such that $h(x) = x'$, and $\bar{\alpha} \in \bar{G}$ such that $g(\bar{\alpha}) = \bar{\alpha}'$. We put

$$(2.10) \quad y = \bar{\alpha}x.$$

Because (X, G) is quasi-invertible, there exists $\bar{\beta} \in \bar{G}$ such that

$$(2.11) \quad \bar{\beta}y = x.$$

We have from (2.10)

$$(2.12) \quad h(y) = h(\bar{\alpha}x) = g(\bar{\alpha})h(x) = \bar{\alpha}'x' = y'.$$

Using (2.11) and (2.12) we obtain

$$x' = h(x) = h(\bar{\beta}y) = g(\bar{\beta})h(y) = g(\bar{\beta})y'.$$

Thus we have proved that (Y, H) is quasi-invertible.

For the proof of the last part of Theorem 2.8 it is sufficient to observe that if in the presented reasoning x' and x are fixed and $\bar{\alpha}'$ runs over the set H , then y' runs over the set $\bar{G}x'$. By condition (2.12) (as $y \in \bar{G}x$) the proof is complete.



The assumption that the object (X, G) is complete is in Theorem 2.8 and in Theorem 2.7 essential. It is shown by the following

EXAMPLE 2.1. Let us put: $X = \{x, y, z\}$, $G = \{(1, 1), (1, 2), (2, 2)\}$. The interior product \cdot in G is defined as in the product groupoid. The exterior product of elements of G by the elements of X is defined as follows:

$$(2.13) \quad (1, 1)x = x,$$

$$(2.14) \quad (1, 1)y = (2, 2)y = y, \quad (1, 2)y = z,$$

$$(2.15) \quad (1, 1)z = (2, 2)z = z, \quad (1, 2)z = y,$$

and in other cases the exterior product is not defined.

It is easy to verify that the pair $\Omega_1 = (X, G)$ with this exterior product is an algebraic object. Now, let us consider the object $\Omega_2 = (X, G)$ with the same fibre X , over the same small category G and with the exterior product defined by (2.13), (2.14), (2.15) and by the following equalities:

$$(1, 2)x = y, \quad (2, 2)x = x.$$

It is seen that Ω_2 is an algebraic object. We take the identity mappings on the set X and on the set G as h and g , respectively. The pair (h, g) is an isomorphism of the object Ω_1 onto Ω_1 ; the object Ω_1 is quasi-invertible, but the object Ω_2 is not quasi-invertible since $(1, 2)x = y$ and there exists no $\bar{a} \in \bar{G}$ such that $\bar{a}y = x$. Now, let us note that the object Ω_2 is quasi-transitive, $(h|\{x\}, g)$ is an FS-homomorphism, g is an epimorphism, but obviously $(h|\{x\}, g)$ is not an FS-epimorphism.

THEOREM 2.9. *If an algebraic object is quasi-invertible, then the relation \mathcal{R} is conformable with this object.*

Proof. Let $x\mathcal{R}y$ and let ax and ay be defined. Then by (2.7) $x\mathcal{R}ax$ and $y\mathcal{R}ay$. Thus $ax\mathcal{R}ay$.

Let (X, G) be a quasi-invertible algebraic object. In view of Theorem 2.6 we can form the factor-object $(X/\mathcal{R}, G)$. The factor-object $(X/\mathcal{R}, G)$ is a concomitant of the object (X, G) (cf. [32], p. 70). The canonical projection may be taken as an epimorphism h of the object (X, G) onto the object $(X/\mathcal{R}, G)$. It is seen that the fibres of the object $(X/\mathcal{R}, G)$ are one-element sets. We shall prove that $(X/\mathcal{R}, G)$ is the "largest" quasi-invertible concomitant of (X, G) which has such property.

THEOREM 2.10. *Let (X, G) be a quasi-invertible algebraic object. Then every quasi-invertible FS-concomitant of the object (X, G) whose fibres are one-element sets, is an FS-concomitant of the object $(X/\mathcal{R}, G)$.*

Proof. Let an object (Y, G_1) be a quasi-invertible FS-concomitant of the object (X, G) and let the fibres of (Y, G_1) be one-element sets. Let (h, g) be an epimorphism of (X, G) onto (Y, G_1) . We put

$$k([x]) := h(x) \quad [x] \in X/\mathcal{R}.$$

We shall show that (g, k) is an epimorphism of $(X/\mathcal{R}, \mathcal{G})$ onto (Y, \mathcal{G}) . First we shall prove that k is well defined. It is sufficient (and it is necessary) to prove that if $x\mathcal{R}y$ then $h(x) = h(y)$. Let us assume that $x\mathcal{R}y'$. Then, in virtue of Theorem 2.5, there holds: $h(x)\mathcal{R}h(y)$. But fibres of (Y, \mathcal{G}) are one-element sets. In consequence, $h(x) = h(y)$. Thus k is well defined.

It is seen that k maps X/\mathcal{R} onto $h(X) = Y$. We have to prove yet that (k, g) is a FS-homomorphism. Let $\alpha[x]$ ($\alpha \in \mathcal{G}$, $[x] \in X/\mathcal{R}$) be defined. It means that there exists $y \in [x]$ (i.e. $y\mathcal{R}x$) such that αy is defined. Then $h(\alpha y)$ is defined and we have:

$$k(\alpha[x]) = k(\alpha y) = h(\alpha y) = g(\alpha)h(y) = g(\alpha)h(x) = g(\alpha)k([x]).$$

Thus (k, g) is an FS-homomorphism.

DEFINITION 2.5 (cf. [32], p. 71, 76). A quasi-invertible algebraic object whose quasi-transitive fibres are one-element sets is called a *scalar*. A concomitant which is a scalar is called a *scalar concomitant*.

Now Theorem 2.10 may be formulated as follows:

THEOREM 2.10. *Let (X, \mathcal{G}) be a quasi-invertible algebraic object. Then every scalar FS-concomitant of the object (X, \mathcal{G}) is an FS-concomitant of the object $(X/\mathcal{R}, \mathcal{G})$.*

THEOREM 2.11. *Let (X, \mathcal{G}) , (Y, \mathcal{H}) be two FS-equivalent algebraic objects. Then (X, \mathcal{G}) is quasi-invertible [quasi-transitive] iff (Y, \mathcal{H}) is quasi-invertible [quasi-transitive].*

We shall omit the easy proof of this theorem.

Remark 2.1. In the above theorem the assumption that the objects (X, \mathcal{G}) and (Y, \mathcal{H}) are FS-equivalent cannot be replaced by the assumption that there exists an FS-isomorphism of (X, \mathcal{G}) onto (Y, \mathcal{H}) and an FS-isomorphism of (Y, \mathcal{H}) onto (X, \mathcal{G}) . This is shown by Example 2.1.

THEOREM 2.12 ⁽⁵⁾. *Let (h, g) be a strong FS-isomorphism of a quasi-invertible object (X, \mathcal{G}) onto a quasi-invertible object (Y, \mathcal{H}) . Then the function h maps the quasi-transitive fibres of the object (X, \mathcal{G}) onto the quasi-transitive fibres of the object (Y, \mathcal{H}) . Thus the function h establishes a one-to-one correspondence between the quasi-transitive fibres of the object (X, \mathcal{G}) and those of (Y, \mathcal{H}) .*

Proof. Let us consider an arbitrary quasi-transitive fibre Xs of the object (X, \mathcal{G}) . It follows from Corollary 2.1 that there exists a quasi-transitive fibre Yp of the object (Y, \mathcal{H}) such that

$$(2.16) \quad h(Xs) \subset Yp.$$

⁽⁵⁾ Theorem 2.12 is a generalization of the theorem of M. Kuczma and M. Kucharzewski for purely differential abstract geometric objects (cf. [9], p. 262, or [10], p. 42).

Since h is a bijection, we obtain from (2.16):

$$(2.17) \quad Xs = h^{-1}(h(Xs)) \subset h^{-1}(Yp).$$

The pair (h^{-1}, g^{-1}) is a strong FS-isomorphism (as (h, g) is a strong FS-isomorphism) of (Y, H) onto (X, G) . According to (2.14) we have

$$h^{-1}(Yp) \cap Xs \neq \emptyset,$$

and hence by applying Corollary 2.1 to the function (h^{-1}, g^{-1}) we obtain

$$(2.18) \quad h^{-1}(Yp) \subset Xs.$$

Conditions (2.17) and (2.18) give $h(Xs) = Yp$.

In paper [32] there are many interesting results and, unfortunately, a few inaccuracies, too. In particular, Lemma 1, Proposition 4 and Proposition 5 of [32] are not true. This is shown in the following

EXAMPLE 2.2. Let $X = \{0, 1, 2, 3\}$ and let G be the product groupoid $A \times A$ where $A = \{a_1, a_2\}$. We put

$$(2.19) \quad h_{a_1}(x) = x, \quad x \in X,$$

$$(2.20) \quad h_{a_2}(x) = x + 1 \pmod{4}, \quad x \in X,$$

$$(2.21) \quad (a_i, a_j)x = h_{a_i}^{-1}h_{a_j}(x), \quad a_i, a_j \in A, x \in X.$$

The pair (X, G) with the exterior product defined by (2.21) is an algebraic object (cf. [27], p. 222).

From (2.19), (2.20) and (2.21) we obtain

$$(2.22) \quad (a_i, a_j)0 = 0, \quad i = 1, 2,$$

$$(2.23) \quad (a_1, a_2)0 = 0h_{a_1}^{-1}h_{a_2}(0) = 1,$$

$$(2.24) \quad (a_2, a_1)0 = h_{a_2}^{-1}h_{a_1}(0) = 3,$$

$$(2.25) \quad (a_1, a_2)1 = h_{a_1}^{-1}h_{a_2}(1) = 2.$$

Conditions (2.22), (2.23) and (2.24) mean that $G\{0\} = \{0, 1, 3\}$. It follows from (2.25) that $2 \in G\{1\}$. Thus we have: $1 \in G\{0\}$, $2 \in G\{1\}$, but $2 \notin G\{0\}$, which shows that Lemma 1 (of [32]) is not true.

As $1 \in G\{0\}$ and $(a_1, a_2)1 = 2 \notin G\{0\}$, it follows that $G\{0\}$ is not a stable subset of X . Thus Proposition 4 of [32] is not true.

It is easy to verify that the object (X, G) has no proper subobjects. As $2 \in G\{0\}$, this object is not transitive. It proves that Proposition 5 is not true, either.

It is seen that if we replace, in the definition of trace, G by \tilde{G} and a by \bar{a} , then Lemma 1 [32] and Proposition 4 [32] are true. Obviously, Proposition 5 of [32] is true if we replace the term "transitive" by the term "quasi-transitive" (see Theorem 2.2).

In note [32] there is yet another inaccuracy. The proof of Theorem 5 ([32], p. 77) is incorrect. From the fact that the object Ω is generated by x_0 it does not follow that for every $x \in \Omega$ there exists $\xi \in G$ such that $x = \xi x_0$ (see e.g. Example 2.1). Thus the mapping $h: \varphi \ni \xi \rightarrow \xi x_0 \in \Omega$ needs not be "onto".

The first part of Theorem 5 of [32] is true. Before we reprove it, we have to recall some definitions from [32]. Let (X, G) be an algebraic object, and let $U \subset X$. The smallest subobject of (X, G) whose fibre contains a set U is called a *subobject generated by U* ([32], p. 72). A subobject generated by one-element set is called *particular* ⁽⁶⁾ ([32], p. 72). An algebraic object φ over G is called *universal* if every particular complete algebraic object over G can be obtained as a concomitant of φ ([32], p. 77).

Now we shall prove the following

THEOREM 2.13. *For every family of algebraic objects over the same small category there exists a universal object.*

Proof. Let all objects of the family in question be over a fixed small category G . We define an algebraic object $\varphi = (\bar{G}, G)$ with the fibre \bar{G} and with the exterior product given by

$\alpha(a_n, \dots, a_1)$ is defined iff αa_n is defined and then

$$(2.26) \quad \alpha(a_n, \dots, a_1) := (\alpha a_n, \dots, a_1).$$

It is seen that φ with the exterior product defined above satisfies Definition 1.6 and hence φ is an algebraic object.

Let us consider an arbitrary complete algebraic object (X, G) generated by a single element x . We put

$$(2.27) \quad h: \bar{G} \ni (a_n, \dots, a_1) \rightarrow (a_n, \dots, a_1)x.$$

As the object (X, G) is complete, h is defined on the whole set \bar{G} . From the fact that x generates (X, G) it follows that h maps \bar{G} onto X . We shall show that h is a homomorphism. Let $\alpha(a_n, \dots, a_1)$ be defined. Then from (2.26) and (2.27) we have

$$h(\alpha(a_n, a_{n-1}, \dots, a_1)) = h((\alpha a_n, a_{n-1}, \dots, a_1)) = (\alpha a_n, a_{n-1}, \dots, a_1)x.$$

Because (X, G) is complete, we obtain by applying P.2.1 and Definition 1.6 (condition A):

$$\begin{aligned} (\alpha a_n, a_{n-1}, \dots, a_1)x &= \alpha a_n [(a_{n-1}, \dots, a_1)x] = \alpha [(a_n(a_{n-1}, \dots, a_1))x] \\ &= \alpha [(a_n, a_{n-1}, \dots, a_1)x] = \alpha h(a_n, a_{n-1}, \dots, a_1). \end{aligned}$$

Thus h is an epimorphism of φ onto (X, G) .

⁽⁶⁾ In the next section we shall use the term "particular object" in another sense.

III. Rank of the quasi-transitivity of algebraic objects

We assume in this section that all algebraic objects considered are quasi-invertible. Thus the term “an algebraic object” means in this section a “quasi-invertible algebraic object”.

We propose to take the following

DEFINITION 3.1. Let $\Omega = (X, G)$ be an algebraic object, X_s its quasi-transitive fibre and let $x \in X_s$. The smallest positive integer n_0 such that for arbitrary $y \in X_s$ there exist $\bar{\alpha}, \bar{\beta} \in \bar{G}$ satisfying the following conditions:

$$\bar{\alpha}x = y, \quad \bar{\beta}y = x, \quad l(\bar{\alpha}) \leq n_0, \quad l(\bar{\beta}) \leq n_0$$

will be called the *rank of quasi-transitivity* of x and it will be denoted by $\text{rqt}_\Omega(x)$, or shortly $\text{rqt}(x)$. If such a positive integer does not exist, then we shall say that the rank of quasi-transitivity of x is infinite.

The maximum $\text{rqt}_\Omega(x)$ over x belonging to a fixed quasi-transitive fibre X_s of the object $\Omega = (X, G)$ (or ∞ if this maximum does not exist) will be called the rank of quasi-transitivity of the fibre X_s and it will be denoted by $\text{rqt}_\Omega(X_s)$, or shortly $\text{rqt}(X_s)$.

For convenience we shall accept the following convention:

$$(3.1) \quad n \cdot \infty = \infty, \quad n \leq \infty, \quad (n \text{ finite or } \infty).$$

We shall prove the following

THEOREM 3.1. Let (h, g) be an FS-epimorphism of the quasi-transitive algebraic object $\Omega_1 = (X, G)$ onto an algebraic object $\Omega_2 = (Y, H)$ ⁽⁷⁾. Then we have

$$(3.2) \quad \text{rqt}_{\Omega_1}(x) \geq \text{rqt}_{\Omega_2}(h(x)), \quad x \in X,$$

$$(3.3) \quad \text{rqt}_{\Omega_1}(X) \geq \text{rqt}_{\Omega_2}(Y).$$

Proof. Let us consider an arbitrary element $x \in X$. If $\text{rqt}_{\Omega_1}(x) = \infty$, then in virtue of (3.1), condition (3.2) holds.

Now assume that $\text{rqt}_{\Omega_1}(x) < \infty$ and consider an arbitrary element $y \in Y$. Let $\tilde{y} \in X$ be such that $h(\tilde{y}) = y$. Because $\text{rqt}_{\Omega_1}(x) < \infty$, there exist $\bar{\alpha}, \bar{\beta} \in \bar{G}$ such that

$$\bar{\alpha}x = \tilde{y}, \quad \bar{\beta}\tilde{y} = x, \quad l(\bar{\alpha}) \leq \text{rqt}_{\Omega_1}(x), \quad l(\bar{\beta}) \leq \text{rqt}_{\Omega_1}(x).$$

Using P.2.3 we obtain

$$(3.4) \quad y = h(\tilde{y}) = h(\bar{\alpha}x) = g(\bar{\alpha})h(x),$$

$$(3.5) \quad h(x) = h(\bar{\beta}\tilde{y}) = g(\bar{\beta})h(\tilde{y}) = g(\bar{\beta})y.$$

⁽⁷⁾ Then the object $\Omega_2 = (Y, H)$ is also quasi-transitive (see Theorem 2.6).

Obviously we have (see (2.3))

$$(3.6) \quad l(\bar{\alpha}) = l(g(\bar{\alpha})), \quad l(\bar{\beta}) = l(g(\bar{\beta})).$$

As y is any element of Y , from conditions (3.4), (3.5) and (3.6) follows (3.2). Condition (3.3) results immediately from (3.2).

THEOREM 3.2. *Let $\Omega_1 = (X, G)$ be a complete algebraic object and let (h, g) be an epimorphism of the object $\Omega_1 = (X, G)$ onto an object $\Omega_2 = (Y, H)$. Then (3.2) holds and we have for any quasi-transitive fibre X_s of the object $\Omega_1 = (X, G)$,*

$$(3.7) \quad \text{rqt}_{\Omega_1}(X_s) \geq \text{rqt}_{\Omega_2}(h(X_s)).$$

Proof. It follows from Theorem 2.8 that for every quasi-transitive fibre X_s of the object Ω_1 , $h(X_s)$ is a quasi-transitive fibre of the object Ω_2 . Thus the pair $(h|X_s, g)$ is an epimorphism of the quasi-transitive object (X_s, G) onto the object $(h(X_s), H)$. Consequently, by applying Theorem 3.1, we obtain (3.2) and (3.7).

The assumption that the object Ω_1 is complete is essential in Theorem 3.2. It is shown by the following

EXAMPLE 3.1. Let us take: $X = \{x, y, z\}$, $G = \{(1, 1), (1, 2), (1, 3), (3, 3)\}$. The interior product in G is defined as in the product groupoid. The exterior product of elements of G by the elements of X is defined as follows:

$$(3.8) \quad (1, 1)x = x,$$

$$(3.9) \quad (1, 1)y = (2, 2)y = y, \quad (1, 2)y = z,$$

$$(3.10) \quad (1, 1)z = (2, 2)z = z, \quad (1, 2)z = y,$$

and in other cases the exterior product is not defined.

It is seen that $\Omega_1 = (X, G)$ with the exterior product so defined is an algebraic object (evidently non-complete and quasi-invertible).

Now, in the same set X we define the exterior product by the elements of the same small category G by conditions: (3.8), (3.9), (3.10) and by:

$$(3.11) \quad (1, 3)x = y, \quad (2, 2)x = x,$$

$$(3.12) \quad (1, 3)y = x, \quad (3, 3)y = y.$$

In other cases the exterior product is not defined.

It is easy to verify that the pair $\Omega_2 = (X, G)$ with the exterior product defined above is a quasi-invertible algebraic object.

The identity mapping of the set X is an isomorphism of the object Ω_1 onto the object Ω_2 . We have

$$\text{rqt}_{\Omega_1}(x) = 1 < 2 = \text{rqt}_{\Omega_2}(x).$$

Thus condition (3.2) does not hold, and hence (3.7) does not hold, either.

Remark 3.1. Let us observe that in the object Ω_2 we have: $x\mathcal{R}y$ (i.e. x and y belong to the same quasi-transitive fibre of Ω_2) and

$$1 = \text{rqt}_{\Omega_2}(x) \neq \text{rqt}_{\Omega_2}(y) = 2.$$

In the next section we shall prove that such a situation is impossible in the class of algebraic objects called commutative.

From Theorem 3.1 and Theorem 2.11 there results directly the following

THEOREM 3.3. *Let (h, g) be a strong FS-isomorphism of an object $\Omega_1 = (X, G)$ onto an object $\Omega_2 = (Y, H)$. Then we have:*

$$\text{rqt}_{\Omega_2}(h(x)) = \text{rqt}_{\Omega_1}(x),$$

$$\text{rqt}_{\Omega_2}(h(Xs)) = \text{rqt}_{\Omega_1}(Xs),$$

where Xs is an arbitrary quasi-transitive fibre of the object Ω_1 .

The assumption that an FS-isomorphism (h, g) is strong is essential in Theorem 3.3. This is shown by Example 3.1.

The ranks of the quasi-transitivity of elements of the same quasi-transitive fibre can be different (see Remark 3.1). But they cannot be arbitrary. Namely, we shall prove:

THEOREM 3.4. *If elements x, y belong to the same quasi-transitive fibre X_s of an algebraic object then the following conditions hold:*

$$(3.13) \quad \text{rqt}(y) \leq 2\text{rqt}(x),$$

$$(3.14) \quad \text{rqt}(x) \leq \text{rqt}(Xs) \leq 2\text{rqt}(x).$$

Proof. If $\text{rqt}(x) = \text{rqt}(y) = \infty$, then in view of (3.1) condition (3.13) holds. Let us now assume that $\text{rqt}(x) < \infty$. Let z be an arbitrary element of the set Xs . Then there exist $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ such that

$$(3.15) \quad l(\bar{\alpha}), l(\bar{\beta}), l(\bar{\gamma}), l(\bar{\delta}) \leq \text{rqt}(x),$$

and

$$(3.16) \quad \bar{\alpha}x = y, \quad \bar{\beta}y = x,$$

$$(3.17) \quad \bar{\gamma}x = z, \quad \bar{\delta}z = x.$$

We obtain from (3.16), (3.17), P.2.1 and (3.15) the following conditions:

$$(3.18) \quad z = \bar{\gamma}x = \bar{\gamma}(\bar{\beta}y) = (\bar{\gamma}\bar{\beta})y, \quad l(\bar{\gamma}\bar{\beta}) \leq 2\text{rqt}(x),$$

$$(3.19) \quad y = \bar{\alpha}x = \bar{\alpha}(\bar{\delta}z) = (\bar{\alpha}\bar{\delta})z, \quad l(\bar{\alpha}\bar{\delta}) \leq 2\text{rqt}(x).$$

Because z is an arbitrary element, conditions (3.18) and (3.19) mean that (3.13) holds. Condition (3.14) follows immediately from (3.13).

Remark 3.2. If we consider algebraic objects over groupoids only, then the definition of $\text{rqt}(x)$ can be simplified. On account of P.2.2, conditions on $\bar{\beta}$ in Definition 3.1 may be omitted.

IV. Commutative algebraic objects

This section is based on note [3]. In particular, the proofs of: Lemma 4.1, Theorem 4.3, Lemma 4.2, Theorem 4.5, Lemma 4.3, and Theorem 4.8, of this section are just slight modifications of the proofs of Lemma 1, Theorem 1, Lemma 2, Theorem 2, Corollary 2 and Theorem 4 of [3], respectively. We give these proofs, because in [3] there are a few essential mis-statements. These mistakes have been discussed in detail in [28].

DEFINITION 4.1 (cf. [3], p. 19). An algebraic object (X, G) is called *commutative* whenever for any $x \in X$ and any $\alpha, \beta \in G$, if $\alpha(\beta x)$ is defined, then $\beta(\alpha x)$ is also defined, and

$$\alpha(\beta x) = \beta(\alpha x).$$

THEOREM 4.1. *An algebraic object (X, G) is commutative iff the following condition (C) holds for $x \in X$, $\bar{\alpha}, \bar{\beta} \in \bar{G}$:*

(C) *If $\bar{\alpha}(\bar{\beta}x)$ is defined, then $\bar{\beta}(\bar{\alpha}x)$ is defined and*

$$\bar{\alpha}(\bar{\beta}x) = \bar{\beta}(\bar{\alpha}x).$$

Proof. We shall prove inductively that if the object (X, G) is commutative, then condition (C) holds. The converse implication is obvious.

Suppose that the object (X, G) is commutative. Let $l(\bar{\alpha}) = 1$, i.e. let $\bar{\alpha} = \alpha$. By induction with respect to $l(\bar{\beta})$ we shall prove that condition (C) holds. If $l(\bar{\beta}) = 1$, then it holds in virtue of Definition 4.1. Now suppose that condition (C) holds for words of length n . Let

$$(4.1) \quad \bar{\beta} = \beta\bar{\beta}_1, \quad l(\bar{\beta}_1) = n,$$

and let $\alpha(\bar{\beta}x) = \alpha(\beta\bar{\beta}_1x)$ be defined. By applying P.2.1 we obtain

$$\alpha(\beta\bar{\beta}_1x) = \alpha(\beta(\bar{\beta}_1x)).$$

It follows from Definition 4.1 that $\beta(\alpha(\bar{\beta}_1x))$ is defined and

$$(4.2) \quad \beta(\alpha(\bar{\beta}_1x)) = \alpha(\beta(\bar{\beta}_1x)).$$

By the inductive assumption $\bar{\beta}_1(\alpha x)$ is defined (as $\alpha(\bar{\beta}_1x)$ is defined) and

$$(4.3) \quad \bar{\beta}_1(\alpha x) = \alpha(\bar{\beta}_1x).$$

It follows from P.2.1, (4.1), (4.3) and (4.2) that $\bar{\beta}(\alpha x)$ is defined and

$$\bar{\beta}(\alpha x) = \beta\bar{\beta}_1(\alpha x) = \beta(\bar{\beta}_1(\alpha x)) = \beta(\alpha(\bar{\beta}_1x)) = \alpha(\beta(\bar{\beta}_1x)) = \alpha(\beta\bar{\beta}_1x) = \alpha(\bar{\beta}x).$$

Thus condition (C) holds for every $\bar{\beta} \in \bar{G}$ (for $l(\bar{\alpha}) = 1$).

Let now $l(\bar{\beta})$ be arbitrarily fixed. It can be easily proved by induction with respect to $l(\bar{\alpha})$ that condition (C) holds for every $\bar{\alpha} \in \bar{G}$.

Before giving the next theorem we shall prove the following

LEMMA 4.1. *Let (X, G) be a commutative algebraic object and let $\bar{a} \in \bar{G}$ be such an element that $\bar{a}x$ is defined for all $x \in X$. Then the mapping*

$$(4.4) \quad h_{\bar{a}}: x \rightarrow \bar{a}x \quad \text{for} \quad x \in X$$

is a strong homomorphism of the object (X, G) into itself.

Proof. The function $h_{\bar{a}}$ is defined on the set X . Let βx be defined. As $\bar{a}(\beta x)$ is defined, by Theorem 4.1, $\beta(\bar{a}x)$ is also defined and

$$\beta(\bar{a}x) = \bar{a}(\beta x).$$

Thus we have

$$h_{\bar{a}}(\beta x) = \bar{a}(\beta x) = \beta(\bar{a}x) = \beta h_{\bar{a}}(x).$$

We have proved that $h_{\bar{a}}$ is a homomorphism. If $\beta h_{\bar{a}}(x)$ is defined (i.e. $\beta(\bar{a}x)$ is defined), then, by Theorem 4.1, $\bar{a}(\beta x)$ is defined and hence βx is defined. This means that $h_{\bar{a}}$ is a strong homomorphism.

THEOREM 4.2. *Let (X, G) be a quasi-transitive algebraic object and let $x_0 \in X$ be such an element that for arbitrary $\bar{a}, \bar{\beta} \in \bar{G}$ the following condition holds:*

- (i) *If $\bar{a}(\bar{\beta}x_0)$ is defined then $\bar{\beta}(\bar{a}x_0)$ is defined and $\bar{a}(\bar{\beta}x_0) = \bar{\beta}(\bar{a}x_0)$. Then the object (X, G) is commutative.*

Proof. Let us consider an arbitrary element $x \in X$. There exists $\bar{\gamma} \in \bar{G}$ such that

$$(4.5) \quad x = \bar{\gamma}x_0.$$

Suppose that $\bar{a}(\beta x)$ is defined. Then we obtain in virtue of (4.5), condition (i) and P.2.1:

$$\bar{a}(\beta x) = \bar{a}(\bar{\beta}(\bar{\gamma}x_0)) = \bar{a}(\bar{\gamma}(\bar{\beta}x_0)) = (\bar{a}\bar{\gamma})(\bar{\beta}x_0) = \bar{\beta}((\bar{a}\bar{\gamma})x_0) = \bar{\beta}(\bar{a}(\bar{\gamma}x_0)) = \bar{\beta}(\bar{a}x).$$

Thus condition (C) of Theorem 4.1 holds, and consequently the object (X, G) is commutative.

From Theorem 4.2 results directly the following

COROLLARY 4.1. *Let (X, G) be a quasi-invertible algebraic object and suppose that for every quasi-transitive fibre X_s of (X, G) there exists $x_0 \in X_s$ such that condition (i) of Theorem 4.2 holds. Then the object (X, G) is commutative.*

THEOREM 4.3. *Let (X, G) be a complete algebraic object. The object (X, G) is commutative iff for every $\bar{a} \in \bar{G}$ the mapping $h_{\bar{a}}$ (defined by (4.4)) is a homomorphism.*

Proof. If (X, G) is a complete commutative algebraic object, then, by Lemma 4.1, for every $\bar{a} \in \bar{G}$ $h_{\bar{a}}$ is a homomorphism.

Conversely, if for every $\bar{a} \in \bar{G}$ $h_{\bar{a}}$ is a homomorphism, then we obtain (using the completeness of (X, G)):

$$\alpha(\beta x) = h_{\alpha}(\beta x) = \beta h_{\alpha}(x) = \beta(\alpha x),$$

which means that the object (X, G) is commutative.

Now we shall give a necessary and sufficient condition for a complete algebraic object over a groupoid to be commutative.

Let (X, G) be an algebraic object. We write

$$(4.6) \quad L(x) := \{\bar{a} \in \bar{G} : \bar{a}x = x\} \quad \text{for } x \in X.$$

Let G be a groupoid and let U be any non-empty subset of G . By $\langle U \rangle$ we denote the set of all elements $\bar{\gamma}$ of the form

$$(4.7) \quad \bar{\gamma} = \bar{\alpha}_1^{-1} \bar{\beta}_1^{-1} \bar{\alpha}_1 \bar{\beta}_1 \bar{\alpha}_2^{-1} \bar{\beta}_2^{-1} \bar{\alpha}_2 \bar{\beta}_2 \dots \bar{\alpha}_n^{-1} \bar{\beta}_n^{-1} \bar{\alpha}_n \bar{\beta}_n, \quad \text{where } \bar{\alpha}_i, \bar{\beta}_i \in U.$$

THEOREM 4.4 ⁽⁸⁾. *A complete algebraic object (X, G) over a groupoid G is commutative iff the following condition*

$$(4.8) \quad \langle \bar{G} \rangle \subset L(x)$$

holds for $x \in X$.

Proof. Let G be a groupoid and let (X, G) be a complete algebraic object.

Suppose that the object (X, G) is commutative. Then we have by Theorem 4.1:

$$\bar{\alpha}_1(\bar{\beta}_1 x) = \bar{\beta}_1(\bar{\alpha}_1 x) \quad \text{for } \bar{\alpha}_1, \bar{\beta}_1 \in \bar{G}, x \in X,$$

and consequently, by P.2.2 and P.2.1,

$$(\bar{\alpha}_1^{-1} \bar{\beta}_1^{-1} \bar{\alpha}_1 \bar{\beta}_1)x = x,$$

i.e.

$$(4.9) \quad \bar{\alpha}_1^{-1} \bar{\beta}_1^{-1} \bar{\alpha}_1 \bar{\beta}_1 \in L(x) \quad \text{for } \bar{\alpha}_1, \bar{\beta}_1 \in \bar{G}, x \in X.$$

It is immediately seen that the product (defined by (2.1)) of a finite number of elements of $L(x)$ belongs to $L(x)$. Thus it follows from (4.9) that every element $\bar{\gamma}$ of the form (4.7) belongs to $L(x)$. This means that (4.8) holds.

Conversely, suppose that relation (4.8) holds. We obtain from (4.8) and (4.6):

$$\bar{\alpha}^{-1} \bar{\beta}^{-1} \bar{\alpha} \bar{\beta} x = x \quad \text{for } \bar{\alpha}, \bar{\beta} \in \bar{G}, x \in X,$$

and consequently, by applying P.2.2 and P.2.1, we have

$$\bar{\alpha}(\bar{\beta} x) = \bar{\beta}(\bar{\alpha} x) \quad \text{for } \bar{\alpha}, \bar{\beta} \in \bar{G}, x \in X.$$

⁽⁸⁾ Z. Moszner has proved a similar theorem for algebraic objects over a group (cf. [20], Theorem 3).

Thus, in virtue of Theorem 4.1, the object (X, G) is commutative.

LEMMA 4.2. *Let an algebraic object (X, G) be quasi-transitive and commutative. If $\bar{a} \in \bar{G}$ is such an element that the product $\bar{a}x$ is defined for some $x \in X$, then for all $y \in X$ the product $\bar{a}y$ is defined.*

Proof. Because the object (X, G) is quasi-transitive, for every $y \in X$ there exists $\beta \in \bar{G}$ such that $\beta y = x$. Then $\bar{a}(\beta y)$ is defined and hence, in view of Theorem 4.1, $\bar{a}y$ is defined.

THEOREM 4.5. *If an algebraic object (X, G) is quasi-transitive and commutative, then every homomorphism $h: X \rightarrow X$ of the object (X, G) into itself is of the form $h_{\bar{a}}$ for some $\bar{a} \in \bar{G}$.*

Proof. Let $x_0 \in X$. Then there exists an $\bar{a} \in \bar{G}$ such that $\bar{a}x_0 = h(x_0)$. By Lemma 4.2 $\bar{a}x$ is defined for every $x \in X$, and hence, by Lemma 4.1, $h_{\bar{a}}$ is a homomorphism. Let x be an arbitrary element of X . There exists $\beta \in \bar{G}$ such that $\beta x_0 = x$. Now we have:

$$h(x) = h(\beta x_0) = \beta h(x_0) = \beta h_{\bar{a}}(x_0) = \beta(\bar{a}x_0) = \bar{a}(\beta x_0) = h_{\bar{a}}(\beta x_0) = h_{\bar{a}}(x).$$

Thus $h = h_{\bar{a}}$.

COROLLARY 4.2. *If an algebraic object (X, G) is quasi-transitive and commutative then:*

(a) *every homomorphism $h: X \rightarrow X$ of the object (X, G) into itself is a strong epimorphism;*

(b) *for every $x, y \in X$ there exists a strong epimorphism $h: X \rightarrow X$ of the object (X, G) onto itself such that $h(x) = y$.*

Proof. (a) By Theorem 4.5 and Lemma 4.2, $h = h_{\bar{a}}$ is a strong homomorphism. Let us consider an arbitrary fixed element $x \in X$. For any $y \in X$ there exists $\beta \in \bar{G}$ such that $\beta h_{\bar{a}}(x) = y$. Then we have

$$y = \beta h_{\bar{a}}(x) = \beta(\bar{a}x) = \bar{a}(\beta x) = h_{\bar{a}}(\beta x).$$

Thus every y from X belongs to $h_{\bar{a}}(X)$, i.e. $h_{\bar{a}}$ is an epimorphism.

(b) There exists $\bar{a} \in \bar{G}$ such that $\bar{a}x = y$. By Lemma 4.2, Lemma 4.1 and (a), $h_{\bar{a}}$ is a strong epimorphism. It is obvious that $h_{\bar{a}}(x) = y$.

According to the theory of quasi-algebras (cf. [31]) J. Gancarzewicz has used in [3] the term (not defined) "generator of an algebraic object" in the following sense:

DEFINITION 4.2. Let (X, G) be an algebraic object. A set $U \subset X$ is called a *generator of the algebraic object (X, G)* if:

- (i) U generates the object (X, G) (i.e. $\bar{G}U = X$),
- (ii) any proper subset of U does not generate the object (X, G) .

It is seen that if an object (X, G) is quasi-transitive then every one-element subset of X is a generator and, on the other hand, every generator of this object is a one-element subset of X . If an algebraic object (X, G) is quasi-invertible then, in virtue of Theorem 2.3, the generator

of this object can be characterized as a selector of the family of quasi-transitive fibres of the object (X, G) , i.e., the set which has exactly one common element with every quasi-transitive fibre of the object (X, G) .

Thus, from Corollary 4.1 and Theorem 2.3 we obtain immediately the following

THEOREM 4.6. *If (X, G) is a commutative quasi-invertible algebraic object and U, V are two of its generators, then there exists a strong epimorphism $h: X \rightarrow X$ of the object (X, G) onto itself such that $h(U) = V$.*

From Theorem 3.1 and Corollary 4.1 results the following

THEOREM 4.7. *Let (X, G) be a quasi-transitive commutative algebraic object. Then for every $x, y \in X$*

$$(4.10) \quad \text{rqt}(y) = \text{rqt}(x).$$

Proof. By Corollary 4.1(a) an element y can be written in the form $y = h(x)$, where h is some epimorphism. Hence from (3.2) we have:

$$\text{rqt}(y) \leq \text{rqt}(x).$$

We may interchange in this inequality x and y . This proves that (4.10) holds.

We shall now consider commutative algebraic objects over groupoids. At first we shall prove the following

LEMMA 4.3. *If (X, G) is an algebraic object over a groupoid G and $\bar{a}x = \bar{a}y$ for some $\bar{a} \in \bar{G}$, then $x = y$.*

Proof. If $\bar{a}x = \bar{a}y$, then applying P.2.2 we obtain:

$$x = \bar{a}^{-1}(\bar{a}x) = \bar{a}^{-1}(\bar{a}y) = y.$$

If in Corollary 4.1 G is a groupoid, then by Lemma 4.3 the mapping h is an injection (as by Theorem 4.5 $h = h_{\bar{a}}$ for some $\bar{a} \in \bar{G}$). Thus we have the following

THEOREM 4.8. *If an algebraic object (X, G) over a groupoid G is quasi-transitive and commutative, then:*

(a) *every homomorphism $h: X \rightarrow X$ of the object (X, G) into itself is a strong isomorphism,*

(b) *for every $x, y \in X$ there exists a strong isomorphism $h: X \rightarrow X$ of the object (X, G) onto itself such that $h(x) = y$.*

From Theorem 2.4, Theorem 2.3 and Theorem 4.6 we easily obtain the following

THEOREM 4.9. *If (X, G) is a commutative algebraic object over a groupoid G and U, V are two of its generators, then there exists a strong isomorphism $h: X \rightarrow X$ of the object (X, G) onto itself such that $h(U) = V$.*

For the proof of the next theorem we need the following simple

LEMMA 4.4. *Let (X_1, G) , (X_2, G) and (X_3, G) be algebraic objects, let $h_1: X_1 \rightarrow X_2$ be an isomorphism of the object (X_1, G) onto (X_2, G) and let $h_2: X_2 \rightarrow X_3$ be an isomorphism of (X_2, G) onto (X_3, G) .*

If $h_2 \circ h_1$ is a strong isomorphism (obviously, of the object (X_1, G) onto (X_3, G)), then h_1 and h_2 are strong isomorphisms.

Proof. At first we prove that the isomorphism h_2 is strong. Suppose that $ah_2(x_2)$ ($a \in G$, $x_2 \in X_2$) is defined. Put

$$x_1 := h_1^{-1}(x_2).$$

Then we have

$$(h_2 \circ h_1)(x_1) = h_2(h_1(x_1)) = h_2(x_2),$$

and hence $a[(h_2 \circ h_1)(x)]$ is defined. As the isomorphism $h_2 \circ h_1$ is strong, ax_1 is defined and consequently $h(ax_1) = ah(x_1) = ax_2$ is defined, too. Thus the isomorphism h_2 is strong. The isomorphism h_1 is strong as the composition of two strong isomorphisms $h_2 \circ h_1$ and h_2^{-1} .

Let (X, G) and (Y, G) be two algebraic objects over a groupoid G and let $h: X \rightarrow Y$ be a homomorphism of the object (X, G) into the object (Y, G) . By Theorem 2.4 the object (X, G) and (Y, G) are quasi-invertible and hence, in consequence of Corollary 2.1, h maps quasi-transitive fibres of (X, G) into quasi-transitive fibres of (Y, G) . We define the function $h/\mathcal{R}: X/\mathcal{R} \rightarrow Y/\mathcal{R}$ (the relation \mathcal{R} is defined by (2.8)) as follows:

$$h/\mathcal{R}([x]) := [h(x)], \quad [x] \in X/\mathcal{R}.$$

It is clear that h/\mathcal{R} is well defined.

Furthermore, if h is an epimorphism, then h/\mathcal{R} maps X/\mathcal{R} onto Y/\mathcal{R} .

Now we shall prove the following

THEOREM 4.10. *Let (X, G) and (Y, G) be two commutative algebraic objects over a groupoid G , let $h: X \rightarrow Y$ be an epimorphism of the object (X, G) onto the object (Y, G) and let $p: Y \rightarrow X$ be an epimorphism of the object (Y, G) onto the object (X, G) . If there exists a positive integer n such that $\{(p \circ h)/\mathcal{R}\}^n$ (where $\{(p \circ h)/\mathcal{R}\}^n$ denotes the n -th iterate of the function $(p \circ h)/\mathcal{R}$) is the identity mapping, then h and p are strong isomorphisms (and hence the objects (X, G) and (Y, G) are equivalent).*

Proof. Let $\{(p \circ h)/\mathcal{R}\}^n$ be the identity mapping. Then for every quasi-transitive fibre X_s of the object (X, G) we have:

$$(p \circ h)^n(X_s) \subset X_s,$$

and hence, in virtue of Theorem 4.8, $(p \circ h)^n|_{X_s}$ is a strong isomorphism of the object (X_s, G) onto itself. Thus $(p \circ h)^n$ is a strong isomorphism of the object (X, G) onto itself.

Now by Lemma 4.4 we conclude that h and p are strong isomorphisms.

From Theorem 4.10 follows

COROLLARY 4.3. *Let (X, G) and (Y, G) be two commutative algebraic objects such that sets X/\mathcal{R} and Y/\mathcal{R} are finite. If these objects are similar (i.e., each of them is a concomitant of the other), then they are equivalent.*

Proof. Let h be an epimorphism of (X, G) onto (Y, G) and p be an epimorphism of (Y, G) onto (X, G) . Then $(p \circ h)(X) = X$, and hence $((p \circ h)/\mathcal{R})(X/\mathcal{R}) = X/\mathcal{R}$. Because X/\mathcal{R} is a finite set, $(p \circ h)/\mathcal{R}$ is an element of the permutation group of the set X/\mathcal{R} . Therefore there exists a positive integer n such that $((p \circ h)/\mathcal{R})^n$ is the identity mapping. By applying Theorem 4.10, we obtain that h and p are strong isomorphisms and hence the objects (X, G) and (Y, G) are equivalent.

V. Regular algebraic objects

Let us consider a purely differential abstract geometric object over the group \mathcal{L}_n^s ⁽⁹⁾ with the following transformation formula

$$(5.1) \quad \omega' = F(\omega, L), \quad \omega \in M, \quad L \in \mathcal{L}_n^s,$$

where $M \subset R^n$.

According to [10], we shall denote by M_{ω_0} the transitive fibre of the object (5.1) determined by ω_0 . Obviously, $M_{\omega_0} = F(\omega_0, \mathcal{L}_n^s)$. The unity of the group \mathcal{L}_n^s will be denoted by I .

We shall consider three definitions of regularity of purely differential abstract geometric objects.

DEFINITION 5.1 (cf. [10], p. 43). A purely differential abstract geometric object (5.1) is called *regular* if the following conditions are satisfied:

(a) For every sequence $L_v \in \mathcal{L}_n^s$, $L_v \rightarrow I$, and for every $\omega_0 \in M$ we have

$$\lim_{v \rightarrow \infty} F(\omega_0, L_v) = \omega_0.$$

(b) For every sequence $\omega_v \in M_{\omega_0}$, $\omega_v \rightarrow \omega_0$, there exists a sequence $L_v \in \mathcal{L}_n^s$, $L_v \rightarrow I$ such that $\omega_v = F(\omega_0, L_v)$.

A. Zajtz [33] has introduced the following

DEFINITION 5.2 (cf. [33], p. 44). A purely differential abstract geometric object (5.1) is called *regular* if the following conditions are satisfied:

(a) For every transitive fibre M_{ω_0} the function F is continuous on $M_{\omega_0} \times \mathcal{L}_n^s$.

⁽⁹⁾ For the definition of the group \mathcal{L}_n^s see e.g. [5], [8], [10]. In the group \mathcal{L}_n^s we have natural topology making \mathcal{L}_n^s a topological group.

(b) For every $\omega_0 \in M$ and every open set $A \subset \mathcal{L}_n^s$ the set $F(\omega_0, A)$ is open in M_{ω_0} .

We propose to consider also the following

DEFINITION 5.3. A purely differential abstract geometric object (5.1) is called *regular* if the following conditions are satisfied:

(a) For every $\omega_0 \in M$ and every neighbourhood U of ω_0 in the subspace M_{ω_0} , there exists a neighbourhood V of I such that

$$F(\omega_0, V) \subset U.$$

(b) For every $\omega_0 \in M$ and every neighbourhood V of I there exists a neighbourhood U of ω_0 in the subspace M_{ω_0} such that

$$U \subset F(\omega_0, V).$$

We shall prove the following

THEOREM 5.1. *Definitions 5.2 and 5.3 are equivalent.*

Proof. It is immediately seen that Definition 5.2 implies Definition 5.3.

Let us suppose that the object (5.1) satisfies Definition 5.3. We shall show that then this object satisfies also Definition 5.2.

Let $\omega_0 \in M$, $L_0 \in \mathcal{L}_n^s$ be arbitrarily fixed and let U be any neighbourhood of the element $\omega_1 = F(\omega_0, L_0)$ in the subspace M_{ω_0} . Obviously $M_{\omega_0} = M_{\omega_1}$. In virtue of condition (a) of Definition 5.3, there exists a neighbourhood V of I such that

$$(5.1) \quad F(\omega_1, V) \subset U.$$

For the neighbourhood V there exists a neighbourhood V_1 of L_0 such that

$$(5.2) \quad V_1 \cdot V_1^{-1} \subset V$$

(cf. [23], p. 94).

We obtain from (5.1) and (5.2):

$$(5.3) \quad F[F(\omega_1, V_1^{-1}), V_1] = F(\omega_1, V_1 V_1^{-1}) \subset F(\omega_1, V) \subset U.$$

We have also:

$$(5.4) \quad F(\omega_1, V_1^{-1}) = F[F(\omega_1, L_0^{-1}), V_1^{-1} L_0] = F(\omega_0, V_1^{-1} L_0).$$

But $V_1^{-1} L_0$ is a neighbourhood of I (cf. [23], p. 94). Thus, in virtue of condition (b) of Definition 5.3, there exists a neighbourhood U_1 of ω_0 in the subspace M_{ω_0} such that

$$(5.5) \quad U_1 \subset F(\omega_0, V_1^{-1} L_0).$$

We get from (5.5), (5.4) and (5.3):

$$F(U_1, V_1) \subset F[F(\omega_0, V_1^{-1} L_0), V_1] = F[F(\omega_1, V_1^{-1}), V_1] \subset U.$$

Thus the function F is continuous in (ω_0, L_0) . Because ω_0 and L_0 are fixed arbitrarily, this means that condition (a) of Definition 5.2 is satisfied.

Now we shall prove that F satisfies condition (b) of Definition 5.2. Let $\omega_0 \in M$ be arbitrarily fixed and let $A \subset \mathcal{L}_n^s$ be any open set. Let us consider an arbitrary element $\omega_1 \in F(\omega_0, A)$, i.e. element of the form

$$\omega_1 = F(\omega_0, L_0), \quad L_0 \in A.$$

Let V be any neighbourhood of L_0 such that

$$(5.6) \quad V \subset A.$$

We have:

$$(5.7) \quad F(\omega_0, V) = F[F(\omega_1, L_0^{-1}), V] = F(\omega_1, VL_0^{-1}).$$

But VL_0^{-1} is a neighbourhood of I . Thus, in view of condition (b) of Definition 5.3, there exists a neighbourhood U of ω_1 in the subspace M_{ω_0} such that

$$(5.8) \quad U \subset F(\omega_1, VL_0^{-1}).$$

We obtain from (5.8), (5.7) and (5.6):

$$U \subset F(\omega_1, VL_0^{-1}) = F(\omega_0, V) \subset F(\omega_0, A).$$

This means that the set $F(\omega_0, A)$ is open.

As a generalization of Definition 5.1 and Definition 5.3 we shall introduce the notion of a regular algebraic object.

Let \mathcal{G} be a small category and let G_0 be the set of all unities of \mathcal{G} . By \bar{G}_0 we shall denote the set of all finite non-empty sequences of elements of G_0 . We recall that if (X, \mathcal{G}) is a quasi-invertible algebraic object, then by $\bar{G}x_0$ we denote the quasi-transitive fibre (of this object) determined by x_0 .

DEFINITION 5.4. Let (X, \mathcal{G}) be a complete quasi-invertible algebraic object and let X and \mathcal{G} be topological spaces with convergence, i.e. L^* -Fréchet topological spaces (cf. [12], p. 83). The object (X, \mathcal{G}) will be called *regular* if the following conditions are satisfied:

(a) For every sequence $\bar{a}_r \in \bar{G}$, $\bar{a}_r \rightarrow \bar{a} \in \bar{G}_0$ and for every $x_0 \in X$ we have

$$\lim_{r \rightarrow \infty} \bar{a}_r x_0 = x_0.$$

(b) For every sequence $x_r \in \bar{G}x_0$, $x_r \rightarrow x_0$ there exists a sequence $\bar{a}_r \in \bar{G}$, $\bar{a}_r \rightarrow \bar{a} \in \bar{G}_0$, such that $x_r = \bar{a}_r x_0$.

DEFINITION 5.5. Let (X, \mathcal{G}) be a complete quasi-invertible algebraic object and let X and \bar{G} be topological spaces with topologies defined by open sets. The object (X, \mathcal{G}) will be called *regular* if the following conditions are satisfied:

(a) For every $x_0 \in X$, $\bar{\varepsilon} \in \bar{G}_0$ and every neighbourhood U of the element x_0 in the subspace $\bar{G}x_0$, there exists a neighbourhood V of the element $\bar{\varepsilon}$ such that

$$Vx_0 \subset U.$$

(b) For every $x_0 \in X$, $\bar{\varepsilon} \in \bar{G}_0$ and every neighbourhood V of the element $\bar{\varepsilon}$, there exists a neighbourhood U of the element x_0 in the subspace $\bar{G}x_0$, such that

$$U \subset Vx_0.$$

The notion of regularity of an algebraic object is defined only for complete quasi-invertible algebraic objects. Therefore we shall write shortly "a regular algebraic object" instead of "a complete quasi-invertible regular algebraic object". The extension of the function $g: G \rightarrow G_1$ defined by (2.3) will be denoted in this section by \bar{g} .

We shall show that Definitions 5.4 and 5.5 are generalizations of Definitions 5.1 and 5.3, respectively. For this purpose, we shall consider some special topology in \bar{G} . Accordingly to Definitions 5.1 and 5.3 we distinguish two cases relative to the kind of topology considered.

Let G be a small category and also be a topological space, i.e., an L^* -Fréchet topological space or a topological space with topology defined by open sets. We introduce a topology in \bar{G} in the following way:

(1) We treat G^n as the product of topological spaces (cf. [12], p. 86, [5], p. 64),

(2) We treat \bar{G} as the disjoint union of the topological spaces G^n (cf. [5], p. 62).

It means that if G is an L^* -Fréchet topological space, then a sequence $\bar{a}_\nu \in \bar{G}$ converges to $\bar{a}_0 \in G^{n_0}$ iff there exists a positive integer ν_0 such that $\bar{a}_\nu \in G^{n_0}$ for $\nu > \nu_0$ and the sequence \bar{a}_ν , $\nu > \nu_0$ is convergent to \bar{a}_0 in G^{n_0} . The topology in \bar{G} defined by (1) and (2) is said to be *induced from the topology in G* .

We accept the following

DEFINITION 5.6. A semigroup G which is also a topological space is called a *topological semigroup* if the function $G \times G \ni (x, y) \rightarrow xy$ is continuous.

It is immediately seen that if a semigroup G is an L^* -Fréchet topological space, then G is a topological semigroup iff the following condition holds:

For every convergent sequences $\alpha_\nu, \beta_\nu \in G$ the sequence $\alpha_\nu \beta_\nu$ is convergent and

$$(5.9) \quad \lim_{\nu \rightarrow \infty} \alpha_\nu \beta_\nu = \lim_{\nu \rightarrow \infty} \alpha_\nu \lim_{\nu \rightarrow \infty} \beta_\nu.$$

Similarly, if a semigroup G is a topological space with the topology given by open sets, then G is a topological semigroup iff the following condition holds:

For every two elements $\alpha, \beta \in G$ and every neighbourhood U of $\alpha\beta$ there exist neighbourhoods U_1, U_2 of α, β , respectively, such that

$$U_1 U_2 \subset U.$$

THEOREM 5.2. *Let G be a topological semigroup with unity (with L^* -Fréchet topology), let X be an L^* -Fréchet topological space and let (X, G) be a quasi-invertible algebraic object⁽¹⁰⁾. Let the object (X, G) be regular in the sense of Definition 5.1, where the group \mathcal{L}_n^s is replaced by the semigroup G . Then the object (X, G) with the topology in \bar{G} induced from the topology in G , is regular in the sense of Definition 5.4.*

Proof. First we shall show that condition (a) of Definition 5.4 is satisfied. Let us consider an arbitrary $x_0 \in X$ and an arbitrary sequence $\bar{\alpha}_v \in \bar{G}$ such that

$$(5.10) \quad \lim_{v \rightarrow \infty} \bar{\alpha}_v = \bar{\varepsilon} \in \bar{G}_0.$$

Obviously $\bar{\varepsilon}$ has the form:

$$(5.11) \quad \bar{\varepsilon} = (I, \dots, I),$$

k times

where I denotes the unity of G .

As the topology in \bar{G} is induced from the topology in G , it follows from (5.10) and (5.11) that there exists a positive integer ν_0 such that $\bar{\alpha}_v$ for $v > \nu_0$ has the form:

$$(5.12) \quad \bar{\alpha}_v = (\alpha_k^v, \dots, \alpha_1^v)$$

and

$$(5.13) \quad \lim_{v \rightarrow \infty} \alpha_i^v = I \quad \text{for } i = 1, 2, \dots, k.$$

From the definition of an algebraic object (condition (A)) we have

$$(5.14) \quad (\alpha_k^v, \dots, \alpha_1^v)x_0 = (\alpha_k^v \dots \alpha_1^v)x_0.$$

G being a topological semigroup, from (5.9) and (5.13) we obtain:

$$(5.15) \quad \lim_{v \rightarrow \infty} (\alpha_k^v \dots \alpha_1^v) = \lim_{v \rightarrow \infty} \alpha_k^v \dots \lim_{v \rightarrow \infty} \alpha_1^v = I \dots I = I.$$

The object (X, G) is regular in the sense of Definition 5.1. Hence, it follows from (5.15) that

$$(5.16) \quad \lim_{v \rightarrow \infty} [(\alpha_k^v \dots \alpha_1^v)x_0] = x_0.$$

⁽¹⁰⁾ Obviously the object (X, G) is complete.

We get from (5.12), (5.14) and (5.16):

$$\lim_{v \rightarrow \infty} [\bar{\alpha}_v x_0] = \lim_{v \rightarrow \infty} [(a_k^v \dots a_1^v) x_0] = x_0.$$

This means that condition (a) of Definition 5.4 is satisfied. It is obvious that condition (b) of this definition is also satisfied.

THEOREM 5.3. *Let G be a topological semigroup with unity (with topology defined by open sets), let X be a topological space with topology defined by open sets and let (X, G) be a quasi-invertible algebraic object. Let the object (X, G) be regular in the sense of Definition 5.3, where the group \mathcal{L}_n^s is replaced by the semigroup G . Then the object (X, G) with the topology in \bar{G} induced from the topology in G is regular in the sense of Definition 5.5.*

Proof. The proof of this theorem can be obtained by a natural modification of the proof of Theorem 5.2.

It is immediately seen from Theorems 5.2 and 5.3 that Definitions 5.4 and 5.5 are generalizations of Definitions 5.1 and 5.3, respectively.

M. Kucharzewski and M. Kuczma have proved (cf. [9], p. 264, Theorem 2) that if two regular purely differential abstract geometric objects are equivalent, then their transitive fibres are homeomorphic. We shall generalize this result.

THEOREM 5.4. *Let (X, G) and (Y, H) be regular algebraic objects (in the sense of Definition 5.4 or Definition 5.5), let $g: G \rightarrow H$ be a homomorphism of G into H such that for every unity ε of G , $g(\varepsilon)$ is a unity of H , and let the pair (h, g) ($h: X \rightarrow Y$) be an FS-homomorphism of the object (X, G) into the object (Y, H) . If the function \bar{g} (defined by (2.1)) is continuous, then the restriction of h to any quasi-transitive fibre of the object (X, G) is continuous.*

Proof. The proof will be realized in two cases: (1) the object (X, G) is regular in the sense of Definition 5.4, (2) the object (X, G) is regular in the sense of Definition 5.5.

(1) Let X_s be any quasi-transitive fibre of the object (X, G) . Let us consider an arbitrary element $x_0 \in X_s$ and an arbitrary sequence $x_v \in X_s$, $x_v \rightarrow x_0$. In virtue of condition (b) of Definition 5.4, there exists a sequence $\bar{\alpha}_v \in \bar{G}$, $\bar{\alpha}_v \rightarrow \bar{\varepsilon} \in \bar{G}_0$ such that

$$(5.17) \quad x_v = \bar{\alpha}_v x_0.$$

Since $g(\varepsilon)$ is a unity of H , for $\varepsilon \in G_0$, we have:

$$(5.18) \quad \bar{g}(\bar{\varepsilon}) \in H_0 \quad \text{for} \quad \bar{\varepsilon} \in \bar{G}_0,$$

where H_0 denotes the set of all non-empty sequences of unities of H .

It follows from the continuity of g that

$$(5.19) \quad \lim_{v \rightarrow \infty} \bar{g}(\bar{\alpha}_v) = \bar{g}(\bar{\varepsilon}).$$

By applying condition (a) of Definition 5.4 we obtain from (5.17), P.2.3, (5.18) and (5.19):

$$\lim_{\nu \rightarrow \infty} h(x_\nu) = \lim_{\nu \rightarrow \infty} (\bar{g}(\bar{a}_\nu)h(x_0)) = h(x_0).$$

This means that the function $h|X_g$ is continuous.

(2) Let us consider an arbitrary element $x_0 \in X$ and an arbitrary neighbourhood U of the element $h(x_0)$ in the subspace $\bar{H}h(x_0)$ ($\bar{H}h(x_0)$ denotes the quasi-transitive fibre of the object (Y, H) determined by $h(x_0)$). Let $\bar{\varepsilon} \in \bar{G}_0$. Then $\bar{g}(\bar{\varepsilon}) \in H_0$ (by assumption, $g(\varepsilon) \in H_0$ for $\varepsilon \in G_0$) and in view of condition (a) of Definition 5.5, there exists a neighbourhood V of the element $\bar{g}(\bar{\varepsilon})$ such that

$$(5.20) \quad Vh(x_0) \subset U.$$

It follows from the continuity of \bar{g} that there exists a neighbourhood V_1 of the element $\bar{\varepsilon}$ such that

$$(5.21) \quad g(V_1) \subset V.$$

By applying P.2.3 we obtain from (5.21) and (5.20):

$$(5.22) \quad h(V_1x_0) = \bar{g}(V_1)h(x_0) \subset Vh(x_0) \subset U.$$

In virtue of condition (b) of Definition 5.5, there exists a neighbourhood U_1 of the element x_0 in the subspace $\bar{G}x_0$ such that

$$U_1 \subset V_1x_0.$$

This condition and (5.22) give:

$$h(U_1) \subset h(V_1x_0) \subset U.$$

Thus the function $h|\bar{G}x_0$ is continuous.

Let us consider an arbitrary strong FS-isomorphism (h, g) of an algebraic object (X, G) onto an algebraic object (Y, H) . Then (h^{-1}, g^{-1}) is obviously a strong FS-isomorphism of the object (Y, H) onto the object (X, G) . Furthermore, we then have for $\varepsilon \in G$ (the isomorphism $g: G \rightarrow H$ being strong):

$g(\varepsilon)$ is a unity of H iff ε is a unity of G .

According to the above remarks, we obtain from Theorem 2.11 and Theorem 5.4 the following

COROLLARY 5.1. *Let (X, G) and (Y, H) be regular algebraic objects and let (h, g) , $h: X \rightarrow Y$, $g: G \rightarrow H$ be a strong FS-isomorphism of the object (X, G) onto the object (Y, H) .*

If the function \bar{g} is a homeomorphism (of \bar{G} onto \bar{H}), then the restriction of h to any quasi-transitive fibre of the object (X, G) is a homeomorphism of that quasi-transitive fibre onto a suitable quasi-transitive fibre of the object (Y, H) .

For further considerations we accept the following

DEFINITION 5.6. Algebraic objects (X, G) and (Y, H) with any topologies in \bar{G} and \bar{H} will be called *topologically homologous*, if the small categories G and H are identical and the topological spaces \bar{G} and \bar{H} are identical.

Let (X, G) and (Y, G) be topologically homologous algebraic objects and let $h: X \rightarrow Y$ be a homomorphism of the object (X, G) into (Y, G) . Then the pair (h, id_G) (id_G denotes the identity mapping on the set G) is an FS-homomorphism of the object (X, G) into (Y, G) and the mapping id_G is obviously a homeomorphism.

Using this remark we obtain from Theorem 5.4 the following

COROLLARY 5.2. Let (X, G) and (Y, G) be topologically homologous regular algebraic objects and let $h: X \rightarrow Y$ be a homomorphism of the object (X, G) into the object (Y, G) . Then the restriction of the function h to any quasi-transitive fibre of the object (X, G) is continuous.

Similarly, from Corollary 5.1 we obtain the following

COROLLARY 5.3. Let (X, G) and (Y, G) be topologically homologous regular algebraic objects and let $h: X \rightarrow Y$ be a strong isomorphism of the object (X, G) onto the object (Y, G) . Then the restriction of the the function h to any quasi-transitive fibre of the object (X, G) is a homeomorphism of that quasi-transitive fibre onto a suitable quasi-transitive fibre of the object (Y, G) .

VI. Particular algebraic objects

The symbol $f: X \overset{\circ}{\rightarrow} Y$ will denote a function defined in some subset of the set X and assuming values in the set Y .

A geometric object can be defined as a function defined on a transformation group (or, more generally, on a pseudogroup of transformations) (cf. [15], p. 324). Looking at geometric objects from such a point of view we give the following

DEFINITION 6.1. Let X be an arbitrary non-empty set and let G be a small category. A mapping $f: \bar{G} \overset{\circ}{\rightarrow} X$ will be called a *particular algebraic object* (or, shortly, a *particular object*) if there exist an algebraic object (X, G) with fibre included in X ⁽¹¹⁾ and $x_0 \in X$ such that for every $a \in G$ the following conditions hold:

⁽¹¹⁾ In Section I we have accepted the convention that (X, G) denotes an algebraic object with fibre X . Thus, in the case where (contrary to this convention) the symbol (X, G) denotes an algebraic object whose fibre may be different from X (obviously included in X), we shall always write "an algebraic object (X, G) with fibre included in X " instead of "an algebraic object (X, G) ". This is the situation occurring in Definition 6.1.

$f(\bar{\alpha})$ is defined iff $\bar{\alpha}x_0$ is defined;
if $\bar{\alpha}x_0$ is defined, then

$$(6.1) \quad f(\bar{\alpha}) = \bar{\alpha}x_0.$$

If f is a particular algebraic object, then the range of f , i.e. the set $f(D_f)$, will be called the *fibre* of f and it will be denoted by \mathfrak{A}_f .

A particular algebraic object f satisfying the conditions of Definition 6.1 will be called a *particular algebraic object of the algebraic object* (X, G) . We shall also say that this particular object is generated by x_0 .

We shall prove the following

THEOREM 6.1. *Let X be a non-empty set and let G be a small category. A mapping $f: \bar{G} \rightarrow X$ is a particular algebraic object iff the following conditions are satisfied:*

(A) *If $f(a_{n+1}, a_n, \dots, a_1)$ is defined, then $f(a_n, \dots, a_1)$ is defined.*

(B) *If $f(a_n, \dots, a_1)$ and $f(\beta_m, \dots, \beta_1)$ are defined and $f(a_n, \dots, a_1) = f(\beta_m, \dots, \beta_1)$ then for arbitrary $\alpha \in G$ we have:*

*$f(\alpha, a_n, \dots, a_1)$ is defined iff $f(\alpha, \beta_m, \dots, \beta_1)$ is defined,
if $f(\alpha, \beta_m, \dots, \beta_1)$ is defined, then*

$$f(\alpha, a_n, \dots, a_1) = f(\alpha, \beta_m, \dots, \beta_1).$$

(C) *If $f(\alpha, a_n, \dots, a_1)$ and $\beta\alpha$ are defined, then $f(\beta, \alpha, a_n, \dots, a_1)$ and $f(\beta\alpha, a_n, \dots, a_1)$ are defined and*

$$f(\beta, \alpha, a_n, \dots, a_1) = f(\beta\alpha, a_n, \dots, a_1).$$

(D) *$f(\bar{G}_0)$ is a one-element set (\bar{G}_0 is the set of all finite non-empty sequences of unities of G).*

(E) *If $f(a_n, a_{n-1}, \dots, a_1)$ is defined, then $f(\varepsilon_r(a_n), a_{n-1}, \dots, a_1)$ is defined and for $n \geq 2$*

$$f(\varepsilon_r(a_n), a_{n-1}, \dots, a_1) = f(a_{n-1}, \dots, a_1)$$

($\varepsilon_r(a_n)$ denotes the right unity of a_n).

(F) *If $(\varepsilon, \varepsilon_n, \dots, \varepsilon_1) \in \bar{G}_0$ and $f(\varepsilon, \varepsilon_n, \dots, \varepsilon_1)$ is defined, then $f(\varepsilon)$ is defined ⁽¹²⁾.*

Proof. Let a mapping $f: \bar{G} \rightarrow X$ be a particular algebraic object, i.e. suppose that there exist an algebraic object (X, G) with fibre included in X and $x_0 \in X$ such that (6.1) holds. It follows immediately from the definition of the exterior product $\bar{\alpha}x$ (see Section II) that f satisfies condition (A) of Definition 6.1.

⁽¹²⁾ The author thinks that conditions (A)–(F) are independent of each other, but he has no suitable examples to prove this hypothesis.

For the proof of condition (B), suppose that $(\alpha_n, \dots, \alpha_1)x_0$ and $(\beta_m, \dots, \beta_1)x_0$ are defined and that

$$(6.2) \quad (\alpha_n, \dots, \alpha_1)x_0 = (\beta_m, \dots, \beta_1)x_0.$$

If $(\alpha, \alpha_n, \dots, \alpha_1)x_0$ is defined, then we obtain from P.2.1 and (6.2):

$$\begin{aligned} (\alpha, \alpha_n, \dots, \alpha_1)x_0 &= \alpha((\alpha_n, \dots, \alpha_1)x_0) = \alpha((\beta_m, \dots, \beta_1)x_0) \\ &= (\alpha, \beta_m, \dots, \beta_1)x_0. \end{aligned}$$

This proves that condition (B) holds.

Condition (C) follows simply from condition (A) of the definition of an algebraic object and from condition (2.2).

Now we shall prove condition (D). Let $(\varepsilon_n, \dots, \varepsilon_1)x_0$ ($\varepsilon_i \in G_0$) be defined. Then it follows from (2.2) that $\varepsilon_1 x_0$ is defined. Because $\varepsilon_r(\varepsilon_1) = \varepsilon_1$, we have, by the definition of an algebraic object:

$$\varepsilon_1 x_0 = x_0.$$

It is easy to prove (by induction with respect to n) that

$$(\varepsilon_n, \dots, \varepsilon_1)x_0 = x_0.$$

Thus condition (D) holds.

For the proof of condition (E), assume that $(\alpha, \alpha_n, \dots, \alpha_1)x_0$ is defined. We have from (2.2):

$$(\alpha, \alpha_n, \dots, \alpha_1)x_0 = \alpha((\alpha_n, \dots, \alpha_1)x_0).$$

Thus, by condition (B') of the definition of an algebraic object, $\varepsilon_r(\alpha)((\alpha_n, \dots, \alpha_1)x_0)$ is defined and

$$\varepsilon_r(\alpha)((\alpha_n, \dots, \alpha_1)x_0) = (\alpha_n, \dots, \alpha_1)x_0.$$

From this equality and from (2.1) it follows simply that condition (E) holds.

For the proof of condition (F), suppose that $(\varepsilon, \varepsilon_n, \dots, \varepsilon_1) \in \bar{G}_0$ and $(\varepsilon, \varepsilon_n, \dots, \varepsilon_1)x_0$ is defined. Then $\varepsilon((\varepsilon_n, \dots, \varepsilon_1)x_0)$ is defined and (see the proof of condition (D)) $(\varepsilon_n, \dots, \varepsilon_1)x_0 = x_0$. Thus εx_0 is defined.

The proof of the first part of Theorem 6.1 is complete.

Now assume that the mapping $f: \bar{G} \rightarrow X$ satisfies conditions (A)–(F). Let us consider the pair (X, \mathcal{G}) . We define the exterior product ax , $a \in \mathcal{G}$, $x \in X$ in the following way:

ax is defined iff there exists a sequence $(\alpha_n, \dots, \alpha_1)$ such that $f(\alpha_n, \dots, \alpha_1) = x$ and $f(\alpha, \alpha_n, \dots, \alpha_1)$ is defined. Then we put:

$$(6.3) \quad ax = f(\alpha, \alpha_n, \dots, \alpha_1).$$

It follows from condition (B) that this exterior product is well defined, i.e. it is independent of the choice of the sequence (a_n, \dots, a_1) . Let $x \in X$ be of the form

$$(6.4) \quad x = f(a_n, \dots, a_1).$$

and let ax and βa be defined. From (C) and (6.3) it follows that $f(\beta, a, a_n, \dots, a_1)$ and $f(\beta a, a_n, \dots, a_1)$ are defined and

$$f(\beta, a, a_n, \dots, a_1) = f(\beta a, a_n, \dots, a_1).$$

This means that $\beta(ax)$ and $(\beta a)x$ are defined and $\beta(ax) = (\beta a)x$. Thus condition (A) of Definition 1.6 is satisfied.

Let us assume (for the proof of condition (B') of Definition 1.6) that $x \in X$ is of the form (6.4) and ax is defined, i.e. $f(a, a_n, \dots, a_1)$ is defined. It follows from (E) that $f(\varepsilon_r(a), a_n, \dots, a_1)$ is defined and

$$f(\varepsilon_r(a), a_n, \dots, a_1) = f(a_n, \dots, a_1).$$

Thus $\varepsilon_r(a)x$ is defined and

$$\varepsilon_r(a)x = f(\varepsilon_r(a), a_n, \dots, a_1) = f(a_n, \dots, a_1) = x.$$

We have proved that the pair (X, G) with the exterior product ax defined by (6.3) is an algebraic object.

We have yet to prove that there exists $x_0 \in X$ such that relation (6.1) holds. We put

$$(6.5) \quad x_0 = f(\bar{G}_0).$$

Suppose that $f(a_n, \dots, a_1)$ is defined. Then it follows from (A) that $f(a_{n-1}, \dots, a_1), \dots, f(a_1)$ are defined. Now, using (E), we obtain that $f(\varepsilon_r(a_1))$ is defined and hence, on account of (D) and (6.5), we have

$$f(\varepsilon_r(a_1)) = x_0.$$

This equality and condition (C) give (according to the definition of the exterior product ax):

$$(6.6) \quad f(a_1) = f(a_1, \varepsilon_r(a_1)) = a_1 f(\varepsilon_r(a_1)) = a_1 x_0.$$

From (6.6) and (6.3) we obtain (as $f(a_2, a_1)$ is defined):

$$f(a_2, a_1) = a_2 f(a_1) = a_2 (a_1 x_0) = (a_2, a_1) x_0,$$

and by the inductive continuation

$$f(a_n, \dots, a_1) = (a_n, \dots, a_1) x_0.$$

Conversely, let us suppose that $(a_n, \dots, a_1) x_0$ is defined. Thus $a_1 x_0$ and consequently $\varepsilon_r(a_1) x_0$ is defined. Let us consider a sequence $(\varepsilon_n, \dots, \varepsilon_1) \in \bar{G}_0$ such that $f(\varepsilon_n, \dots, \varepsilon_1)$ is defined and hence such that

$$x_0 = f(\varepsilon_n, \dots, \varepsilon_1).$$

Using (6.3) we obtain:

$$x_0 = \varepsilon_r(a_1)x_0 = \varepsilon_r(a_1)f(\varepsilon_n, \dots, \varepsilon_1) = f(\varepsilon_r(a_1), \varepsilon_n, \dots, \varepsilon_1).$$

From this equality by applying (F) and (D) it follows that $f(\varepsilon_r(a_1))$ is defined and

$$f(\varepsilon_r(a_1)) = x_0.$$

As $f(\varepsilon_r(a_1))$ is defined, then (see (C)) $f(a_1, \varepsilon_r(a_1))$ and $f(a_1)$ are defined. Now using (6.3) we obtain:

$$f(a_1) = f(a_1, \varepsilon_r(a_1)) = a_1 f(\varepsilon_r(a_1)) = a_1 x_0.$$

Because $(a_2, a_1)x_0$ is defined, we have

$$(a_2, a_1)x_0 = a_2(a_1x_0) = a_2f(a_1) = f(a_2, a_1).$$

Thus $f(a_2, a_1)$ is defined. By continuation of this reasoning we obtain that $f(a_n, \dots, a_1)$ is defined.

Remark 6.1. If a small category G is a semigroup (obviously with unity) and (X, G) is an algebraic object with fibre included in X , then for every sequence $(a_n, \dots, a_1) \in \bar{G}$ and every x_0 belonging to the fibre of the object (X, G) , the product $(a_n, \dots, a_1)x_0$ is defined and $(a_n, \dots, a_1)x_0 = (a_n \dots a_1)x_0$. Thus in this case we may replace in Definition 6.1 \bar{a} by a and \bar{G} by G . In such a way we obtain an equivalent definition of a particular algebraic object. Evidently, conditions (A)–(F) may be then replaced by simpler conditions.

Let us accept the following

DEFINITION 6.2. Let $f_1: \bar{G}_1 \xrightarrow{\circ} X_1, f_2: \bar{G}_2 \xrightarrow{\circ} X_2$ be two particular algebraic objects with fibres $\mathbb{C}f_1, \mathbb{C}f_2$ respectively. A pair of mappings $h: \mathbb{C}f_1 \rightarrow \mathbb{C}f_2, g: G_1 \rightarrow G_2$, where g is a homomorphism of G_1 into G_2 , will be called an *FS-homomorphism* of the object f_1 into the object f_2 if the following condition holds for $\bar{a} \in \bar{G}_1$:

If $f_1(\bar{a})$ is defined, then $f_2(g(\bar{a}))$ is defined, and

$$(6.7) \quad h(f_1(\bar{a})) = f_2(g(\bar{a})).$$

If, in addition, small categories G_1 and G_2 are identical and g is the identity mapping, then h will be called a *homomorphism*.

An FS-homomorphism (h, g) such that h maps $\mathbb{C}f_1$ onto $\mathbb{C}f_2$ and g is an epimorphism, will be called an *FS-epimorphism*. The object f_2 will be then called an *FS-concomitant* of the object f_1 . An FS-homomorphism (h, g) such that h is a bijection of $\mathbb{C}f_1$ onto $\mathbb{C}f_2$, and g is a strong isomorphism of G_1 onto G_2 , will be called a *strong FS-isomorphism* if the following condition holds:

If $f_2(g(\bar{\alpha}))$ is defined, then $f_1(\bar{\alpha})$ is defined.

In this case objects f_1 and f_2 will be called *FS-equivalent*.

Between the notion of homomorphism of algebraic objects and that of homomorphism of particular algebraic objects there is a strong relationship. We shall show this in the next three theorems.

THEOREM 6.2. *Let (X, G_1) and (Y, G_2) be two algebraic objects. Let $g: G_1 \rightarrow G_2$ be a homomorphism [a strong homomorphism] of G_1 into [onto] G_2 and let $h: X \rightarrow Y$ be any mapping [a bijection of X onto Y].*

The pair (h, g) is an FS-homomorphism [a strong FS-isomorphism] of the object (X, G_1) into [onto] the object (Y, G_2) iff for every $x \in X$ the pair $(h|_{\bar{G}_1 x}, g)$ is an FS-homomorphism [a strong FS-isomorphism] of the particular algebraic object generated by x into [onto] the particular algebraic object generated by $h(x)$.

Proof. Let $x \in X$ be arbitrarily fixed. Let us denote particular objects generated by x and $h(x)$ by f_1 and f_2 , respectively. Then, in virtue of Definition 6.1, we have:

$$(6.8) \quad f_1(\bar{\alpha}) = \bar{\alpha}x, \quad f_2(\bar{\alpha}) = \bar{\alpha}h(x),$$

where, obviously, $f_1(\bar{\alpha})$ is defined iff $\bar{\alpha}x$ is defined and, similarly, $f_2(\bar{\alpha})$ is defined iff $\bar{\alpha}h(x)$ is defined.

Suppose that the pair (h, g) is an FS-homomorphism of the object (X, G_1) into the object (Y, G_2) . Let $f_1(\bar{\alpha})$ be defined. Then by (6.8) and P.2.3 we have:

$$(6.9) \quad h(f_1(\bar{\alpha})) = h(\bar{\alpha}x) = g(\bar{\alpha})h(x) = f_2(g(\bar{\alpha})).$$

Thus the pair $(h|_{\bar{G}_1 x}, g)$ is an FS-homomorphism of the particular object f_1 into the particular object f_2 .

Let us assume additionally that (h, g) is a strong FS-isomorphism of the object (X, G_1) onto the object (Y, G_2) . Then $h|_{\bar{G}_1 x}$ is a bijection of $\bar{G}_1 x = \mathcal{C}f_1$ onto $\bar{G}_2 h(x) = \mathcal{C}f_2$. Furthermore, if $f_2(g(\bar{\alpha}))$ is defined, then by (6.8) $g(\bar{\alpha})h(x)$ is defined and consequently $\bar{\alpha}x$ is defined, i.e. $f_1(\bar{\alpha})$ is defined. This means that $(h|_{\bar{G}_1 x}, g)$ is a strong FS-isomorphism of the particular object f_1 onto the particular object f_2 .

Suppose, conversely, that for every $x \in X$ the pair $(h|_{\bar{G}_1 x}, g)$ is an FS-homomorphism of the particular object f_1 (generated by x) into the particular object f_2 (generated by $h(x)$). Let us consider arbitrary $\bar{\alpha} \in \bar{G}_1$, $x \in X$ such that $\bar{\alpha}x$ is defined. Then by (6.8) $f_1(\bar{\alpha})$ is defined, which implies that $f_2(g(\bar{\alpha}))$ is defined, and

$$h(f_1(\bar{\alpha})) = f_2(g(\bar{\alpha})).$$

Using (6.8) and this equality we obtain:

$$h(\bar{\alpha}x) = h(f_1(\bar{\alpha})) = f_2(g(\bar{\alpha})) = g(\bar{\alpha})h(x).$$

Hence (h, g) is an FS-homomorphism of the object (X, G_1) into the object (Y, G_2) .

Finally, let us assume that h is a bijection (of X onto Y) and that for every $x \in X$ the pair $(h|\bar{G}_1 x, g)$ is a strong FS-isomorphism of the particular object f_1 onto the particular object f_2 . Suppose that $g(\bar{\alpha})h(x)$ is defined. Then by (6.8) $f_2 g(\bar{\alpha})$ is defined. Consequently $f_1(\bar{\alpha})$ is defined, i.e. $\bar{\alpha}x$ is defined. This proves that (h, g) is a strong FS-isomorphism of the object (X, G_1) onto the object (Y, G_2) .

Remark 6.2. An analogue of Theorem 6.2 for epimorphism is not valid. It can be shown by Example 3.1. In this example the identity mapping of the set X is an isomorphism of the object Ω_1 onto the object Ω_2 . However, the identity mapping on the set $\bar{G}y = \{y, z\}$ is not an epimorphism of the particular object generated by the element y in Ω_1 onto the particular object generated by the element y in Ω_2 .

THEOREM 6.3. *Let (X, G_1) , (Y, G_2) be two algebraic objects and let the object (X, G_1) be quasi-transitive. Let $g: G_1 \rightarrow G_2$ be a homomorphism [a strong isomorphism] of G_1 into [onto] G_2 and let $h: X \rightarrow Y$ be any mapping [a bijection] of X into [onto] Y . If there exists $x_0 \in X$ such that the pair (h, g) is an FS-homomorphism [a strong FS-isomorphism] of the particular object generated by x_0 into [onto] the particular object generated by $h(x_0)$, then (h, g) is an FS-homomorphism [a strong FS-isomorphism] of the object (X, G_1) into [onto] the object (Y, G_2) .*

Proof. Let us denote by f_1, f_2 the particular objects generated by x_0 and $h(x_0)$, respectively. Then, obviously, condition (6.8) holds for $x = x_0$. Suppose that (h, g) is an FS-homomorphism of the particular object f_1 into the particular object f_2 . Let us consider an arbitrary $x \in X$. There exists $\bar{\beta} \in \bar{G}_1$ such that

$$(6.10) \quad x = \bar{\beta}x_0.$$

We obtain from (6.10), (6.8) and (6.7):

$$(6.11) \quad h(x) = h(\bar{\beta}x_0) = h(f_1(\bar{\beta})) = f_2(g(\bar{\beta})) = g(\bar{\beta})h(x_0).$$

Now suppose that $\bar{\alpha}x$ is defined. Then we get using P.2.1, (6.7), (6.8), (6.10) and (6.11):

$$\begin{aligned} h(\bar{\alpha}x) &= h(\bar{\alpha}(\bar{\beta}x_0)) = h((\bar{\alpha}\bar{\beta})x_0) = h(f_1(\bar{\alpha}\bar{\beta})) = f_2(g(\bar{\alpha}\bar{\beta})) \\ &= g(\bar{\alpha}\bar{\beta})h(x_0) = (g(\bar{\alpha})g(\bar{\beta}))h(x_0) = g(\bar{\alpha})(g(\bar{\beta})h(x_0)) = g(\bar{\alpha})h(x). \end{aligned}$$

Thus the pair (h, g) is an FS-homomorphism of the object (X, G_1) into the object (Y, G_2) .

Assume additionally that h is a bijection of X onto Y and (h, g) is a strong FS-isomorphism of the particular object f_1 onto the particular object f_2 . We shall show that (h, g) is a strong FS-isomorphism of the object (X, G_1) onto the object (Y, G_2) . It is enough to prove that the

FS-homomorphism (h, g) is strong. Suppose that x is of the form (6.10) and $g(\bar{\alpha})h(x)$ is defined. Then $g(\bar{\alpha})(g(\bar{\beta})h(x_0))$ is defined, and thus $g(\bar{\alpha}\bar{\beta})h(x_0)$ is defined. This means, in virtue of (6.8), that $f_2(g(\bar{\alpha}\bar{\beta}))$ is defined. Then, because (h, g) is a strong FS-isomorphism of f_1 onto f_2 , $f_1(\bar{\alpha}\bar{\beta})$ is defined, i.e. (see (6.8)) $(\bar{\alpha}\bar{\beta})x_0$ is defined. Consequently $\bar{\alpha}(\bar{\beta}x_0)$ is defined and hence, by (6.10), $\bar{\alpha}x$ is defined. Thus the FS-homomorphism (h, g) is strong, as asserted.

From Theorem 6.3 and Theorem 2.3 immediately follows

THEOREM 6.4. *Let (X, G_1) be a quasi-invertible algebraic object, (Y, G_2) an arbitrary algebraic object, let $g: G_1 \rightarrow G_2$ be a homomorphism [a strong isomorphism of G_1 onto G_2] and $h: X \rightarrow Y$ any mapping [a bijection of X onto Y].*

If for every quasi-transitive fibre X_s of the object (X, G_1) there exists $x_0 \in X$ such that the pair (h, g) is an FS-homomorphism [a strong FS-isomorphism] of the particular algebraic object generated by x_0 into [onto] the particular algebraic object generated by $h(x_0)$, then (h, g) is an FS-homomorphism [a strong FS-isomorphism] of the algebraic object (X, G_1) into [onto] the algebraic object (Y, G_2) .

We shall now prove a few theorems which can be regarded (on account of Remark 6.1) as generalizations of the results obtained by S. Midura and Z. Moszner in [14] and [16].

Let (X, G) be an algebraic object and let $x \in X$ be fixed. We denote

$$L(x) := \{\bar{\alpha} \in \bar{G} : \bar{\alpha}x = x\}.$$

THEOREM 6.5. *Let (X, G_1) and (Y, G_2) be algebraic objects and let a pair $h: X \rightarrow Y$, $g: G_1 \rightarrow G_2$ be an FS-homomorphism of the object (X, G_1) into the object (Y, G_2) .*

Then for any $x \in X$ we have

$$g(L(x)) \subset L(h(x)).$$

Proof. Let $x \in X$ be arbitrarily fixed and let $\bar{\alpha} \in L(x)$, i.e. let $\bar{\alpha}x = x$. From this equality it follows (by applying P.2.3) that $g(\bar{\alpha})h(x)$ is defined and

$$g(\bar{\alpha})h(x) = h(\bar{\alpha}x) = h(x).$$

Thus $g(\bar{\alpha}) \in L(h(x))$.

If (h, g) is a strong FS-isomorphism of an algebraic object (X, G_1) onto an algebraic object (Y, G_2) , then (h^{-1}, g^{-1}) is a strong FS-isomorphism of (Y, G_2) onto (X, G_1) . Thus from Theorem 6.5 results immediately the following

THEOREM 6.6. *Let (X, G_1) and (Y, G_2) be two algebraic objects and let (h, g) be a strong FS-isomorphism of the object (X, G_1) onto the object (Y, G_2) .*

Then for any $x \in X$ we have

$$g(L(x)) = L(h(x)).$$

The next theorem is preceded by the following

LEMMA 6.1. *Let $g: G_1 \rightarrow G_2$ be a homomorphism of the groupoid G_1 into the groupoid G_2 . Then for every unity ε of the groupoid G_1 the element $g(\varepsilon)$ is a unity of the groupoid G_2 .*

Proof. First we shall prove that an element a of a groupoid is a unity iff a satisfies the following equation

$$(6.12) \quad aa = a.$$

It is seen that every unity of a groupoid satisfies (6.12). Suppose that an element a of a groupoid satisfies (6.12). Then we have

$$aa = a = \varepsilon_l(a)a.$$

Multiplying this equality by a^{-1} we obtain

$$a = \varepsilon_l(a),$$

which means that a is a unity.

Now, let us consider an arbitrary unity ε of the groupoid G_1 . We have:

$$\varepsilon\varepsilon = \varepsilon.$$

From this equality we obtain (g being a homomorphism):

$$g(\varepsilon)g(\varepsilon) = g(\varepsilon).$$

Thus $g(\varepsilon)$ is a unity of the groupoid G_2 .

THEOREM 6.7. *Let (X, G_1) and (Y, G_2) be two algebraic objects over groupoid G_1, G_2 , respectively; suppose that the object (X, G_1) is quasi-transitive and that (Y, G_2) is complete. Let $g: G_1 \rightarrow G_2$ be a homomorphism of the groupoid G_1 into the groupoid G_2 . If there exist $x_0 \in X$ and $y_0 \in Y$ such that*

$$(6.13) \quad g(L(x_0)) \subset L(y_0),$$

then there exists a mapping $h: X \rightarrow Y$ such that $h(x_0) = y_0$ and the pair (h, g) is an FS-homomorphism of the object (X, G_1) into the object (Y, G_2) .

Proof. Let $x \in X$. There exists $\bar{\alpha} \in \bar{G}$ such that

$$(6.14) \quad x = \bar{\alpha}x_0.$$

We put:

$$(6.15) \quad h(x) := g(\bar{\alpha})y_0.$$

We shall prove that the mapping h is well defined. Suppose that

$$\bar{\alpha}x_0 = \bar{\beta}x_0.$$

From this equality we obtain, by applying P.2.2 and P.2.1

$$x_0 = \beta^{-1}(\beta x_0) = \beta^{-1}(\bar{\alpha} x_0) = (\beta^{-1}\bar{\alpha})x_0.$$

This means that $\beta^{-1}\bar{\alpha} \in L(x_0)$. Thus by (6.12) $g(\beta^{-1}\bar{\alpha}) \in L(y_0)$, i.e. $[g(\beta^{-1}\bar{\alpha})]y_0 = y_0$. From this equality we obtain:

$$y_0 = g[(\beta^{-1}\bar{\alpha})]y_0 = [g(\beta^{-1})g(\bar{\alpha})]y_0 = (g(\beta))^{-1}(g(\bar{\alpha})y_0),$$

and hence, since $[g(\beta)]^{-1} = g(\beta)$, we have in view of P.2.2:

$$g(\beta)y_0 = g(\bar{\alpha})y_0.$$

This proves that h is well defined. We shall show that $h(x_0) = y_0$. Because X is the fibre of the object (X, G) and $x_0 \in X$, there exists $\bar{\alpha} \in \bar{G}$ such that $\bar{\alpha}x_0$ is defined. Then $\varepsilon_r(\alpha)x_0$ is defined and

$$(6.16) \quad \varepsilon_r(\alpha)x_0 = x_0.$$

By Lemma 6.1, $g(\varepsilon_r(\alpha))$ is the unity of G_2 . Thus from (6.14), (6.15) and (6.16) we have:

$$h(x_0) = h(\varepsilon_r(\alpha)x_0) = g(\varepsilon_r(\alpha))y_0 = y_0.$$

We have to prove yet that (h, g) is an FS-homomorphism. Let us consider an arbitrary element $x \in X$, written in the form (6.14), and let αx ($\alpha \in G$) be defined. Then using (6.15) and P.2.1 we obtain:

$$h(\alpha x) = h((\alpha\bar{\alpha})x_0) = g(\alpha\bar{\alpha})y_0 = g(\alpha)[g(\bar{\alpha})y_0] = g(\alpha)h(x).$$

This proves that (h, g) is an FS-homomorphism.

THEOREM 6.8. *Let (X, G_1) and (Y, G_2) be two complete quasi-transitive algebraic objects over groupoids G_1, G_2 , respectively, and let $g: G_1 \rightarrow G_2$ be a strong isomorphism of the groupoid G_1 onto the groupoid G_2 .*

If there exist $x_0 \in X$ and $y_0 \in Y$ such that

$$(6.17) \quad g(L(x_0)) = L(y_0),$$

then there exists a bijection $h: X \rightleftarrows Y$ such that $h(x_0) = y_0$ and the pair (h, g) is a strong FS-isomorphism of the object (X, G_1) onto the object (Y, G_2) .

Proof. We define the function h as in Theorem 6.7. It follows from Theorem 6.7 and Theorem 2.7 that the pair (h, g) is an FS-epimorphism of the object (X, G_1) onto the object (Y, G_2) . Because the object (X, G_1) is complete, this FS-epimorphism is obviously strong. Thus, we have only to show that the mapping h is an injection. Let us consider two arbitrary elements: $x = \bar{\alpha}x_0$, $y = \beta x_0$. Suppose that $h(x) = h(y)$. Then

$$g(\bar{\alpha})y_0 = h(x) = h(y) = g(\beta)y_0,$$

and hence

$$y_0 = [g(\beta)]^{-1}(g(\bar{\alpha})y_0) = g(\beta^{-1})(g(\bar{\alpha})y_0) = g(\beta^{-1}\bar{\alpha})y_0.$$

This means that $g(\beta^{-1}\bar{\alpha}) \in L(y_0)$. Thus by (6.17) we have: $\beta^{-1}\bar{\alpha} \in L(x_0)$, i.e. $\beta^{-1}\bar{\alpha}x_0 = x_0$, which implies

$$y = \beta x_0 = \bar{\alpha}x_0 = x.$$

This proves that the mapping h is an injection.

From Theorem 6.7, Theorem 2.3 and Theorem 2.4 results directly the following

THEOREM 6.9. *Let (X, G_1) and (Y, G_2) be two algebraic objects over groupoids G_1, G_2 , respectively, and let the object (Y, G_2) be complete. Let $g: G_1 \rightarrow G_2$ be a homomorphism of the groupoid G_1 into the groupoid G_2 . If for every quasi-transitive fibre X_s of (X, G_1) there exist $x_0^s \in X_s$ and $y_0^s \in Y$ such that:*

$$g(L(x_0^s)) \subset L(y_0^s),$$

then there exists a mapping $h: X \rightarrow Y$ such that $h(x_0^s) = y_0^s$ (for every s) and the pair (h, g) is an FS-homomorphism of the object (X, G_1) into the object (Y, G_2) .

We obtain (in a similar way as Theorem 6.9) from Theorem 6.8 the following

THEOREM 6.10. *Let (X, G_1) and (Y, G_2) be two complete algebraic objects over groupoids G_1, G_2 , respectively, and let $g: G_1 \rightarrow G_2$ be a strong isomorphism of the groupoid G_1 onto the groupoid G_2 . Let $\overline{X|\mathcal{R}} = \overline{Y|\mathcal{R}}$ (i.e. let the sets $X|\mathcal{R}$ and $Y|\mathcal{R}$ have the same cardinality) and suppose that there exists a bijection $f: X|\mathcal{R} \rightleftharpoons Y|\mathcal{R}$ of $X|\mathcal{R}$ onto $Y|\mathcal{R}$ such that for every quasi-transitive fibre $X_s \in X|\mathcal{R}$ there exist $x_0^s \in X_s$ and $y_0^s \in f(X_s)$ satisfying the condition*

$$g(L(x_0^s)) = L(y_0^s).$$

Under these assumptions there exists a bijection $h: X \rightleftharpoons Y$ such that $h(x_0^s) = y_0^s$ and the pair (h, g) is a strong FS-isomorphism of the object (X, G_1) onto the object (Y, G_2) .

On account of Theorem 6.2, Theorem 6.3 and Theorem 6.4 we may formulate Theorem 6.5, Theorem 6.6, Theorem 6.7 and Theorem 6.8 in terms of particular algebraic objects. In such terms the set $L(x)$ may be written as follows:

$$L(x) = \{\bar{\alpha} \in \bar{G}: f(\bar{\alpha}) = x\},$$

where $f: \bar{G} \rightarrow X$ is a particular algebraic object and x belongs to the fibre of the object f .

VII. Reduced algebraic objects

The new notions in this section are borrowed from the theory of abstract machines (see e.g. [4], [26], [29]). Obviously, they have been suitably modified so as to suit the theory of algebraic objects.

Let (X, G) be an algebraic object. We define a relation F on the set X . We shall write xFy whenever the following conditions are satisfied for every $a \in G$:

ax is defined iff ay is defined; if ax is defined, then $ax = ay$.

If an object (X, G) is complete, then the relation F may be written as follows:

$$xFy \quad \text{iff} \quad ax = ay \text{ for every } a \in G.$$

DEFINITION 7.1. An algebraic object (X, G) will be called *reduced with respect to the fibre* (or shortly *F-reduced*), if for every $x, y \in X$ the following condition holds:

If xFy , then $x = y$.

It is seen that the relation F is an equivalence and is conformable with the object. Hence it determines the factor-object $(X/F, G)$. This object is obviously F-reduced. Thus the object $(X/F, G)$ is an F-reduced concomitant of the object (X, G) . We shall show that it is the largest F-reduced FS-concomitant of the object (X, G) .

THEOREM 7.1. *Every F-reduced FS-concomitant of the object (X, G) is an FS-concomitant of the object $(X/F, G)$.*

Proof. Let an algebraic object (Y, G_1) be F-reduced and let a pair $h: X \rightarrow Y$, $g: G \rightarrow G_1$ be an FS-epimorphism of the object (X, G) onto the object (Y, G_1) . We define the following mapping h/F :

$$h/F: X/F \ni [x] \rightarrow h(x).$$

We shall prove that h/F is well defined. Suppose that xFy . There exists $a \in G$ such that ax is defined (since X is the fibre of the object (X, G)). Then ay is defined and

$$ax = ay.$$

This equality gives:

$$g(a)h(x) = h(ax) = h(ay) = g(a)h(y).$$

Because the object (Y, G_1) is F-reduced, we hence obtain

$$h(x) = h(y).$$

Thus h/F is well defined.

Let $a[x]$ be defined. It means that there exists $y \in [x]$ such that ay is defined. Then we have:

$$\begin{aligned} h/F(a[x]) &= h/F(a[y]) = h/F([ay]) = h(ay) \\ &= g(a)h(y) = g(a)h/F([y]) = g(a)h/F([x]). \end{aligned}$$

This shows that the pair $(h/F, g)$ is an FS-homomorphism of the object $(X/F, G)$ into the object (Y, G_1) . Obviously h/F maps X/F onto Y (because h maps X onto Y).

We shall now consider a reduction of an algebraic object (X, G) with respect to the structure G . Let (X, G) be an algebraic object. We define a relation S on the set G . We shall write $aS\beta$ whenever the following conditions are satisfied for every $x \in X$:

ax is defined iff βx is defined; if ax is defined, then $ax = \beta x$.

DEFINITION 7.2. An algebraic object (X, G) will be called *reduced with respect to the structure* (or shortly *S-reduced*), if for every $a, \beta \in G$ the following condition holds:

If $aS\beta$, then $a = \beta$.

If a small category G is a group, then the notion of S-reduced algebraic object is equivalent to the notion of an effective algebraic object ⁽¹³⁾.

THEOREM 7.2. If an algebraic object (X, G) is complete, then the relation S is a congruence relation ⁽¹⁴⁾ in the category G .

Proof. Let us consider elements $a_1, a_2, \beta_1, \beta_2 \in G$ such that $a_1 S \beta_1$, $a_2 S \beta_2$ and $a_1 a_2, \beta_1 \beta_2$ are defined. Then for every $x \in X$ we have:

$$(a_1 a_2)x = a_1(a_2 x) = a_1(\beta_2 x) = \beta_1(\beta_2 x) = (\beta_1 \beta_2)x.$$

Thus $a_1 a_2 S \beta_1 \beta_2$.

Remark 7.1. We can define an algebraic object a little more generally than in Definition 1.6. We need not assume that G is a small category. We may replace a small category G by an arbitrary multiplicative system. Then condition (B') of Definition 1.6 may be replaced by the following:

If ax is defined and $\varepsilon_r(a)$ is a right unity of a , then $\varepsilon_r(a)x$ is defined and $\varepsilon_r(a)x = x$.

⁽¹³⁾ An abstract geometric object (X, G) over a group G is called *effective*, if for $a \in G$, the condition $ax = x$ for all $x \in X$ yields that a is the unity of the group G (cf. [6], p. 272).

⁽¹⁴⁾ An equivalence relation " \sim " in a multiplicative system G is called a *congruence relation* if it satisfies the following condition (cf. [31], p. 12): For any $a_1, \beta_1, a_2, \beta_2 \in G$, if $a_1 \sim \beta_1, a_2 \sim \beta_2$ and $a_1 a_2$ and $\beta_1 \beta_2$ are defined, then $a_1 a_2 \sim \beta_1 \beta_2$.

Let (X, G) be a complete algebraic object. Then (according to Theorem 7.2) we can form the factor-multiplicative system G/S ⁽¹⁵⁾. Let us consider the pair $(X, G/S)$ with the exterior product defined in the following way:

$$(7.1) \quad [a]x := ax \quad [a] \in G/S, x \in X.$$

It is obvious that the exterior product $[a]x$ is well defined. The pair $(X, G/S)$ with the exterior product defined by (7.1) may be regarded as an algebraic object (see Remark 7.1). Obviously, the object $(X, G/S)$ is S-reduced. Furthermore, it is an S-concomitant of the object (X, G) .

THEOREM 7.3. *Let (X, G) be a complete algebraic object. Then every S-reduced FS-concomitant of the object (X, G) is a concomitant of the object $(X, G/S)$.*

Proof. Let (Y, G_1) be a S-reduced algebraic object and let a pair $h: X \rightarrow Y$, $g: G \rightarrow G_1$ be an FS-epimorphism of the object (X, G) onto the object (Y, G_1) . We put:

$$g/S: G/S \ni [a] \rightarrow g(a).$$

We shall prove that g/S is well defined. Suppose that $a_1 S a_2$. Then for every $x \in X$ we have (the object (X, G) being complete):

$$(7.2) \quad g(a_1)h(x) = h(a_1x) = h(a_2x) = g(a_2)h(x).$$

If x runs over the set X , then $h(x)$ runs over the set Y . Furthermore, the object (Y, G_1) is S-reduced. Thus from (7.2) we obtain:

$$g(a_1) = g(a_2).$$

This means that g/S is well defined. Now we verify that g/S is a homomorphism of G/S into G_1 . Let $[a]$ $[\beta]$ be defined. Since for fixed $[a]$ and $[\beta]$ the product $[a]$ $[\beta]$ is independent of the choice of a and β , we may assume that $a\beta$ is defined. Then we have:

$$g/S([a][\beta]) = g([a\beta]) = g(a\beta) = g(a)g(\beta) = g/S([a])g/S([\beta]).$$

Thus g/S is a homomorphism. Obviously, g/S maps G/S onto G . Furthermore, we have for $x \in X$, $a \in G$:

$$h([a]x) = h(ax) = g(a)h(x) = g/S([a])h(x).$$

Hence $(h, g/S)$ is an FS-epimorphism of the object (X, G) onto the object (Y, G_1) .

⁽¹⁵⁾ We define multiplication \cdot in the set G/S in the following way: $[a]$ $[\beta]$ is defined iff there exist $a_1 \in [a]$, $\beta_1 \in [\beta]$ such that $a_1\beta_1$ is defined, and, if this holds, we put:

$$[a] [\beta] := [a_1\beta_1].$$

Let $\Omega = (X, G)$ be a complete algebraic object. We can apply to the object Ω the operation of S-reduction and so we obtain the object $\Omega/S = (X, G/S)$. Now we can apply to the object Ω/S the operation of F-reduction, obtaining a new object, which we denote by $\Omega/S/F$. On the other hand, we can first apply to the object Ω the operation of F-reduction, obtaining the object Ω/F , and next we can apply to the object Ω/F the operation of S-reduction, obtaining the object $\Omega/F/S$. Thus, there is the question, if objects $\Omega/S/F$ and $\Omega/F/S$ are identical. The answer to this question is positive. Let $\alpha, \beta \in G$ be fixed. It is seen that if for every $x \in X$ we have

$$\alpha x = \beta x,$$

then for every $[x] \in X/F$

$$\alpha[x] = \beta[x].$$

This means that the relation S considered with respect to the object Ω , and the relation S considered with respect to the object Ω/F , are identical. Similarly, the relation F considered with respect to the object Ω , and the relation F considered with respect to the object Ω/S , are identical relations. Thus objects $\Omega/F/S$ and $\Omega/S/F$ have the same fibre X/F , and the same structure G/S . It is easy to verify that the exterior products in these algebraic objects are identical, too.

Thus we have shown that the operations of F-reduction and of S-reduction are commutative.

VIII. Translation equation and algebraic objects over groupoids

Let X ⁽¹⁶⁾ be an arbitrary set and G an arbitrary small category. We say that a function $F: X \times G \rightarrow X$ ⁽¹⁷⁾ satisfies the translation equation, if it satisfies condition (A) of Definition 1.6. The functional equation (1.2) is called the *translation equation*. If a function $F: X \times G \rightarrow X$ satisfies condition (B') of Definition 1.6, we say that F satisfies the *identity condition*. Thus an algebraic object can be defined as a function $F: X \times G \rightarrow X$ (X is an arbitrary set and G is an arbitrary small category) satisfying the translation equation and the identity condition.

In the case when G is a group, the general solution of the translation equation under some additional assumption was given by S. Łojasiewicz in [13], and the general solution without any additional assumption

⁽¹⁶⁾ X is sometimes called a *basic set*.

⁽¹⁷⁾ Analogously to Section VI, the symbol $F: X \times G \rightarrow X$ denotes a function defined in a subset of the set $X \times G$ and assuming values in the set X .

was given by Z. Moszner in [17], [19] (in a way independent from [13]). The general solution of the translation equation over a groupoid was given by J. Tabor in [27], and over a category by A. Krupińska in [7].

In this section we deal with the translation equation over groupoids only.

Now we quote four theorems, which play a very important role in further considerations.

THEOREM 8.1 (cf. [22], p. 11). *Every Brandt groupoid is strongly isomorphic to some product groupoid.*

THEOREM 8.2 (cf. [30]). *A multiplicative system is a groupoid iff it is a disjoint union of Brandt groupoids.*

In view of Theorem 8.1, instead of arbitrary Brandt groupoids, we may consider product groupoids only.

THEOREM 8.3 (cf. [27], p. 222). *The general solution of the translation equation in a product groupoid $A \times A \times G$, for a basic set X , is the family of functions F which can be obtained in the following manner:*

(a) *To every $a \in A$ we choose arbitrarily a set X_a and \bar{X}_a such that $\bar{X}_a \subset X_a \subset X$, and the sets \bar{X}_a have the same cardinality (for $a \in A$).*

(b) *To every $a \in A$ we construct arbitrarily a function f_a transforming X_a onto \bar{X}_a such that $f_a(x) = x$ for $x \in \bar{X}_a$.*

(c) *Let a_0 be any fixed element of A . To every $a \in A$ we construct arbitrarily a one-to-one function h_a transforming \bar{X}_a onto \bar{X}_{a_0} .*

(d) *We choose an arbitrary function H defined on $\bar{X}_{a_0} \times G$ and satisfying the translation equation.*

(e) *We put:*

$$(8.1) \quad F(x, a, b, \alpha) = h^{-1}H(h_b f_b(x), \alpha) \quad \text{for } x \in X_b.$$

It has been proved in [27] that we may assume (without loss of generality) that the function H satisfies the identity condition. Next, we shall assume (for convenience) that the function H in formula (8.1) satisfies identity condition, i.e. that:

$$H(x, 1) = x \quad \text{for } x \in X_{a_0}.$$

THEOREM 8.4 (cf. [27] p. 228). *The general solution of the translation equation in a groupoid E for a basic set X , is the family of functions F which can be obtained in the following manner:*

Let $E = \bigcup_{s \in S} E_s$, where every E_s is a Brandt groupoid and $E_{s_1} \cap E_{s_2} = \emptyset$ for $s_1 \neq s_2$ (see Theorem 8.2).

(a) *For every $s \in S$ we choose an arbitrary function $F_s: X \times E_s \rightarrow X$ satisfying the translation equation.*

(b) *We put*

$$(8.2) \quad F = \bigcup_{s \in S} F_s.$$

It is easy to verify that the function F given by formula (8.1) satisfies identity condition iff every function f_a is the identity function (on the set X_a). Thus, if F satisfies identity condition, then for every $a \in A$ we have $\bar{X}_a = X_a$.

It is seen, also, that the function F given by formula (8.2) satisfies the identity condition iff every function F_s satisfies it.

It has been proved in note [27] that, assuming the definition of an algebraic object given by A. Zajtz in [32], every algebraic object over a Brandt groupoid is complete, and every algebraic object over a groupoid can be extended to a complete algebraic object over the same fibre. If we accept Definition 1.6, this is no more true, as is shown by the following

EXAMPLE 8.1. Let X be the set of positive integers and let G be the product groupoid $A \times A$, where $A = \{a, b\}$. We put:

$$\begin{aligned} h_a(x) &= x+1 & \text{for } x \in X, \\ h_b(x) &= x & \text{for } x \geq 2. \end{aligned}$$

Substituting these functions into formula (8.1) (and taking as f_a, f_b the identity functions on suitable sets) we obtain a function F satisfying the translation equation and the identity condition, i.e., we obtain an algebraic object. The set X is the fibre of this object, but $F(1, a, b)$ is not defined. Thus this object is not complete. Now we shall show that this object cannot be extended to any algebraic object with the same fibre X . We have:

$$\begin{aligned} F(x, b, b) &= x & \text{for } x \geq 2, \\ F(x, a, a) &= x & \text{for } x \in X, \\ F(x, b, a) &= h_b^{-1}(h_a(x)) = h_b^{-1}(x+1) = x+1 & \text{for } x \in X, \\ F(x, a, b) &= h_a^{-1}(h_b(x)) = h_a^{-1}(x) = x-1 & \text{for } x \geq 2. \end{aligned}$$

Let us suppose that the function F can be extended to a function \tilde{F} defined on $X \times (A \times A)$ and satisfying the translation equation and the identity condition. Let us denote:

$$\tilde{F}(1, a, b) = y.$$

Then we have

$$(8.3) \quad \tilde{F}(y, b, a) = \tilde{F}[\tilde{F}(1, a, b), b, a] = \tilde{F}(1, b, b) = 1.$$

On the other hand (since \tilde{F} is an extension of F) we obtain

$$\tilde{F}(y, b, a) = F(y, b, a) = y+1 > 1,$$

contrary to (8.3).

We shall prove the following

THEOREM 8.5. *A function $F: X \times A \times A \times G \rightarrow X$ satisfying the translation equation and the identity condition can be extended to a function $\tilde{F}: X \times A \times A \times G \rightarrow X$ (defined on $X \times A \times A \times G$) satisfying the translation equation and the identity condition iff for every $a, b \in A$ we have*

$$(8.4) \quad \overline{X \setminus X_a} = \overline{X \setminus X_b} \quad (1^8).$$

Proof. Let us assume that a function $F: X \times A \times A \times G \rightarrow X$ satisfies the translation equation, the identity condition and condition (8.4). Then the function F can be written in the form

$$(8.5) \quad F(x, a, b, a) = h_a^{-1} H(h_b(x), a) \quad \text{for } x \in X_b,$$

where H satisfies the identity condition and functions h_a, H are given by construction (a)–(e).

By (8.4) for every $a \in A$ there exists a bijection $h_a^*: X \setminus X_a \rightarrow X \setminus X_{a_0}$ of the set $X \setminus X_a$ onto the set $X \setminus X_{a_0}$. We put:

$$\begin{aligned} \tilde{h}(x) &:= \begin{cases} h_a(x) & \text{for } x \in X_a, \\ h_a^*(x) & \text{for } x \in X \setminus X_a; \end{cases} \\ \tilde{H}(x, a) &:= \begin{cases} H(x, a) & \text{for } x \in X_{a_0}, a \in G, \\ x & \text{for } x \in X \setminus X_{a_0}, a \in G; \end{cases} \\ \tilde{F}(x, a, b, a) &:= \tilde{h}_a^{-1} \tilde{H}(\tilde{h}_b(x), a). \end{aligned}$$

It is immediately seen that the function \tilde{F} is defined on the whole set $X \times A \times A \times G$ and that it satisfies the translation equation and the identity condition.

Let us now assume that the function $F: X \times A \times A \times G \rightarrow X$ of the form (8.5) is given, $\tilde{F}: X \times A \times A \times G \rightarrow X$ is an extension of the function F and \tilde{F} satisfies the translation equation and the identity condition. Then the function \tilde{F} can be written in the form:

$$\tilde{F}(x, a, b, a) = \tilde{h}_a^{-1} \tilde{H}(\tilde{h}_b(x), a),$$

where \tilde{h}_a is a bijection of X onto X , and $\tilde{H}: X \times G \rightarrow X$ satisfies the translation equation and the identity condition. Thus we have for $x \in X_a$:

$$(8.6) \quad \tilde{h}_a^{-1} \tilde{h}_b(x) = \tilde{F}(x, a, b, 1) = F(x, y, b, 1) = h_a^{-1} h_b(x).$$

Furthermore, $h_a^{-1} h_b$ is a bijection of X_b onto X_a , and $\tilde{h}_a^{-1} \tilde{h}_b$ is a bijection of X onto itself. Hence, it follows from (8.6) that $\tilde{h}_b^{-1} \tilde{h}_a$ maps injectively the set $X \setminus X_b$ onto the set $X \setminus X_a$. This means that (8.4) holds.

(1⁸) The set X_a can be (independently of the construction (a)–(e)) defined, also, as follows: $X_a := \{x \in X : \exists_{b \in A, a \in G} (F(x, b, a, a) \text{ is defined})\}$. The symbol \bar{C} denotes the cardinality of the set C .

According to Theorem 8.1 and Theorem 8.2, every groupoid E can be written in the following form:

$$(8.7) \quad E = \bigcup_{s \in S} A_s \times A_s \times G_s,$$

where $A_s \times A_s \times G_s$ is some product groupoid (for every $s \in S$).

We obtain easily from Theorem 8.4 and Theorem 8.5 the following

THEOREM 8.6. *Let X be an arbitrary set and E an arbitrary groupoid written in the form (8.7).*

A function $F: X \times E \rightarrow X$ satisfying the translation equation and the identity condition can be extended to a function $\tilde{F}: X \times E \rightarrow X$ (defined on $X \times E$) satisfying the translation equation and the identity condition iff for every $s \in S$ and every $a_s, b_s \in A_s$ we have

$$(8.8) \quad \overline{X \setminus X_{a_s}} = \overline{X \setminus X_{b_s}}.$$

Theorem 8.6 implies the following

COROLLARY 8.1. *Let X be an arbitrary set, E an arbitrary groupoid, $F: X \times E \rightarrow X$ a function satisfying the translation equation and the identity condition, and let a set $Y \supset X$ satisfy the condition*

$$(8.9) \quad \overline{Y \setminus X} = \overline{Y}.$$

Then the function F can be extended to a function $\tilde{F}: Y \times E \rightarrow Y$ defined on the set $Y \times E$, satisfying the translation equation and the identity condition.

Proof. Let the groupoid E be written in the form (8.7). Then by (8.9) we have for arbitrary $s \in S$, $a_s, b_s \in A_s$:

$$(8.10) \quad \overline{Y \setminus X_{b_s}} \leq \overline{Y} = \overline{Y \setminus X} \leq \overline{Y \setminus X_{a_s}},$$

and similarly,

$$(8.11) \quad \overline{Y \setminus X_{a_s}} \leq \overline{Y \setminus X_{b_s}}.$$

Inequalities (8.10) and (8.11) give (for $s \in S$, $a_s, b_s \in A_s$):

$$\overline{Y \setminus X_{a_s}} = \overline{Y \setminus X_{b_s}}.$$

Now, by applying Theorem 8.6, we obtain the assertion.

Now we prove the following

LEMMA 8.1 ⁽¹⁹⁾. *For every set X there exists a set Y such that*

$$X \subset Y \quad \text{and} \quad \overline{Y \setminus X} = \overline{Y}.$$

⁽¹⁹⁾ This proof has been given by A. Grzaślewicz.

Proof. Let X_1, X_2 be infinite sets such that $\overline{X_1} = \overline{X_2}$, $X_1 \cap X_2 = \emptyset$ and $X \subset X_1$. Let

$$Y = X_1 \cup X_2.$$

We have

$$\overline{Y} = \overline{X_1} + \overline{X_2} = \overline{X_2} \leq \overline{(X_1 \cup X_2) \setminus X} = \overline{Y \setminus X},$$

and consequently $\overline{Y} = \overline{Y \setminus X}$.

Of course $X \subset Y$. Thus (8.12) holds.

By Corollary 8.1 and Lemma 8.1 we obtain

THEOREM 8.7. *Let E be a groupoid. For every set X there exists a set Y such that every solution $F: X \times E \rightarrow X$ of the translation equation satisfying the identity condition can be extended to a solution $F: Y \times E \rightarrow Y$ (defined on the set $Y \times E$) of the translation equation satisfying the identity condition.*

Theorem 8.5, Theorem 8.6, Theorem 8.7 and Corollary 8.1 can be obviously written in terms of algebraic objects. For example, Theorem 8.7 can be formulated in terms of algebraic objects as follows:

THEOREM 8.7'. *For every set X there exists a set Y such that every algebraic object (X, E) over a groupoid E with fibre contained in X can be extended to a complete algebraic object (Y, E) with fibre Y .*

For a function $F: X \times E \rightarrow X$ we shall denote by D_F^1 ⁽²⁰⁾ the projection of the domain of F onto the set X .

Let E be a groupoid and let $F: X \times E \rightarrow X$ be a solution of the translation equation. In the set D_F^1 we define the following relations T_F, \mathcal{R}_F :

We write $xT_F y$ if there exists $a \in E$ such that

$$F(x, a) = y.$$

We write $x\mathcal{R}_F y$ if there exists a sequence $x = x_1, \dots, x_n = y$ such that $x_i T_F x_{i+1}$, for $i = 1, \dots, n-1$.

If for all $x, y \in D_F^1$ we have $xT_F y$ [$x\mathcal{R}_F y$], then we say that the solution F is transitive [quasi-transitive, respectively].

We shall prove the following

THEOREM 8.8. *Let a function $F: X \times A \times A \times G \rightarrow X$ be a solution of the translation equation written in the form (8.1).*

Then $xT_F y$ iff there exist elements $a, b \in A$ such that $h_b f_b(x) T_H h_a(y)$.

Proof. Condition $xT_F y$ holds (by definition) iff there exists $(a, b, \alpha) \in A \times A \times G$ such that

$$(8.12) \quad h_a^{-1} H(h_b f_b(x), \alpha) = y.$$

⁽²⁰⁾ If $F: X \times E \rightarrow X$ satisfies the translation equation and the identity condition, then D_F^1 is the fibre of the object given by F (see Def. 1.6).

Relation (8.12) may be written in the equivalent form:

$$H(h_b f_b(x), \alpha) = h_a(y).$$

This completes the proof.

Theorem 8.8 implies directly:

COROLLARY 8.2. *Let a function $F: X \times A \times A \times G \xrightarrow{\circ} X$ be written in the form (8.1).*

If the function H is transitive (on the set $X_{a_0} \times G$, of course), then the function F is also transitive.

We shall prove a certain necessary condition and then a sufficient condition for the transitivity of solutions of the translation equation in an arbitrary groupoid.

THEOREM 8.9. *Let E be a groupoid and let $F: X \times E \xrightarrow{\circ} X$ be a transitive function written in the form (8.2).*

Then for every $x, y \in D_F^1$ there exists $s \in S$ such that $x, y \in D_{F_s}^1$.

Proof. Let $x, y \in D_F^1$. Since the function F is transitive, there exists $a \in E$ such that $F(x, a) = y$. Obviously, $a \in E_s$ for some $s \in S$. In virtue of Theorem 8.4 $x \in D_{F_s}^1$. Because $F(x, a)$ is defined (and, clearly $\varepsilon_l(a)$ is defined), it follows that $F(y, \varepsilon_l(a)) = F[F(x, a), \varepsilon_l(a)]$ is defined. But $\varepsilon_l(a) \in E_s$. Thus by Theorem 8.4 $y \in D_{F_s}^1$.

THEOREM 8.10. *Let E be a groupoid and let $F: X \times E \xrightarrow{\circ} X$ be a function written in the form (8.2). Let us assume that there exists a subset $\tilde{S} \subset S$ such that*

(a) *for every $s \in \tilde{S}$ the function F_s is transitive,*

(b) *for every $x, y \in D_F^1$ there exists $s \in \tilde{S}$ such that $x, y \in D_{F_s}^1$.*

Then the function F is transitive.

Proof. The proof is immediate.

The condition formulated in Theorem 8.10 is for the transitivity of F a sufficient condition only (it is not a necessary condition). This is shown by the following

EXAMPLE 8.2. We put

$$X = \{x, y\},$$

$$E_1 = A \times A, \quad A = \{a_1, a_2\},$$

$$E_2 = B \times B, \quad B = \{b_1, b_2\},$$

where E_1, E_2 are regarded as product groupoids. We define functions F_1, F_2 as follows:

$$F_1(x, (a_i, a_j)) = F_1(y, (a_i, a_j)) = y \quad \text{for } i, j = 1, 2,$$

$$F_2(x, (b_i, b_j)) = F_2(y, (b_i, b_j)) = x \quad \text{for } i, j = 1, 2.$$

It is easy to verify that F_1 and F_2 satisfy the translation equation. The functions F_1 and F_2 are not transitive, whereas the function $F = F_1 \cup F_2$ is transitive.

It is seen from the general solution of the translation equation in a group (cf. [17] or [19]) that every transitive solution of the translation equation in a group satisfies the identity condition. This is not true in a Brandt groupoid. Let us look at the following.

EXAMPLE 8.3. Consider the product groupoid $(\mathbf{R} \setminus \{0\}) \times (\mathbf{R} \setminus \{0\})$, where \mathbf{R} denotes the set of real numbers. Let \mathbf{R}^+ and \mathbf{R}^- denote the set of positive real numbers and the set of negative real numbers, respectively. We put

$$\begin{aligned} X &= \mathbf{R} \setminus \{0\}, \\ X_a &= \mathbf{R} \setminus \{0\} && \text{for } a \in \mathbf{R} \setminus \{0\}, \\ \bar{X}_a &= \mathbf{R}^+ && \text{for } a > 0, \\ \bar{X}_a &= \mathbf{R}^- && \text{for } a < 0, \\ f_a(x) &= |x| && \text{for } a > 0, x \in \mathbf{R} \setminus \{0\}, \\ f_a(x) &= -|x| && \text{for } a < 0, x \in \mathbf{R} \setminus \{0\}, \\ h_a(x) &= ax && \text{for } a \in \mathbf{R} \setminus \{0\}, x \in \bar{X}_a, \\ F(x, a, b) &= h_a^{-1} h_b f_b(x) && \text{for } a, b \in \mathbf{R} \setminus \{0\}, x \in X_b. \end{aligned}$$

In virtue of Theorem 8.3, the function F so defined satisfies the translation equation. We have for $a > 0$, $b > 0$

$$(8.13) \quad F(x, a, b) = h_a^{-1} h_b(|x|) = h_a^{-1}(|bx|) = \frac{1}{a} |bx| = \left| \frac{b}{a} x \right|,$$

and for $a < 0$, $b < 0$:

$$(8.14) \quad F(x, a, b) = h_a^{-1} h_b(-|x|) = h_a^{-1}(-b|x|) = -\frac{b}{a} |x| = -\left| \frac{b}{a} x \right|.$$

It follows from (8.13) and (8.14) that if $x \in X$ is fixed, and a, b run over the set $\mathbf{R} \setminus \{0\}$, then the values of F run over the set X . This means that F is transitive. But we have for $a > 0$:

$$F(x, a, a) = f_a(x) = |x|.$$

Thus F does not satisfy the identity condition.

Now we shall give some necessary and some sufficient condition for the quasi-transitivity of solutions of the translation equation.

THEOREM 8.11. *Let E be a groupoid and let $F: X \times E \rightarrow X$ be a function written in the form (8.2), where every function F_s is quasi-transitive.*

The function F is quasi-transitive iff for every $x, y \in D_F^1$ there exists a sequence $s_1, s_2, \dots, s_n \in \mathcal{S}$ such that

$$(8.15) \quad D_{F_{s_i}} \cap D_{F_{s_{i+1}}} \neq \emptyset \quad \text{for } i = 1, 2, \dots, n-1.$$

$$x \in D_{F_{s_i}}, \quad y \in D_{F_{s_n}}.$$

Proof. Suppose that the function F is quasi-transitive. Let $x, y \in D_F^1$. There exist $a_1, a_2, \dots, a_n \in E$, $x = x_1, x_2, \dots, x_n \in D_F^1$ such that $F(x, a_1) = x_2$, $F(x_2, a_2) = x_3, \dots, F(x_n, a_n) = y$. Let $s_i \in \mathcal{S}$ be an index such that $a_i \in E_{s_i}$. Then we have (see the proof of Theorem 8.9):

$$x_1, x_2 \in D_{F_{s_1}}^1; \quad x_2, x_3 \in D_{F_{s_2}}^1; \quad \dots; \quad x_n, y \in D_{F_{s_n}}^1.$$

This means that (8.15) holds.

The proof of the converse implication is obvious.

THEOREM 8.12. *Let E be a groupoid and let $F: X \times E \rightarrow X$ be a function written in the form (8.2). Let us assume that there exists a subset $\tilde{\mathcal{S}} \subset \mathcal{S}$ such that:*

- (a) *for every $s \in \tilde{\mathcal{S}}$ the function F_s is transitive,*
- (b) *for every $x, y \in D_F^1$ there exists a sequence $s_1, s_2, \dots, s_n \in \tilde{\mathcal{S}}$ such that*

$$D_{F_{s_i}}^1 \cap D_{F_{s_{i+1}}}^1 \neq \emptyset \quad \text{for } i = 1, 2, \dots, n-1,$$

$$x \in D_{F_{s_1}}^1, \quad y \in D_{F_{s_n}}^1.$$

Proof. The easy proof of this theorem is omitted.

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