DISSERTATIONES MATHEMATICAE 340 (1995)

INTRODUCING TO DIFFERENT ASPECTS OF DIFFERENTIABILITY

Verba docent, exempla trahunt

What leads students to some particular interests in sciences? No doubt, masters-teachers, provided that they appear at the right time and place.

Lagrange's book [7] was in common use for more than half a century. In the late thirties of the 19th century, a young student in Bonn made in a letter some observations concerning the contents of that book (cf. Kennedy [6]). These observations eventually might imply some results which would belong to Nonstandard Analysis. However, at that time there was no outstanding mathematician at the Bonn University. Our student left his mathematical investigations and became interested in philosophy and economy. A science-fiction story remains to be written about our world in the 20th century under the assumption that this student wrote his Ph.D. dissertation in Mathematics, since his name was: Karl Marx.

Those of us who entered the Mathematics Faculty of the Warsaw University in 1950 had lectures of Calculus delivered by Stanisław Mazur. Mazur's style of lectures, his logical presentation of ideas, his beautiful, calligraphic writing on the blackboard with his refrain:

$$\textit{For every } \varepsilon > 0 \textit{ there exists a } \delta > 0 \ \ldots$$

were, indeed, fascinating.

Derivatives and *derivation* (in one variable) were, in a sense, well located in the presented theory. That was not so with *differentials* which seemed to be somewhat artificial, introduced for unknown reasons. The motivation became clear when we

---

1991 *Mathematics Subject Classification*: 11F25, 11F60, 12Hxx, 13Nxx, 14F10, 16S32, 16W25, 17B40, 22Axx, 26A24, 26A27, 26A33, 26A36, 26D10, 28A15, 30F30, 32C38, 32K15, 34XX, 35XX, 44XX, 45J05, 45K05, 46E50, 46G05, 46J15, 46L57, 47B47, 47E05, 47F05, 476xx, 47N20, 49J15, 49J20, 49J24, 49J50, 49K15, 49K20, 49K24, 49L05, 49L10, 49L15, 53XX, 57Rxx, 60H10, 60H15, 65D25, 65Lxx, 65Mxx, 65Nxx, 65P05, 90D25, 93B29, 93C15, 93C20, 93E05. The listed sections are the only sections which contain in their titles the words *derivation* or *differential* in an explicit form.

started experiments in the Physics Laboratory. It was then obligatory to estimat the errors made. This was a discovery: the knowledge of the calculus of differentials is most helpful in estimating errors in a proper way!

Mechanics was taught *before* Ordinary Differential Equations. In order to understand the lectures, we both, Rolewicz and myself, learned something from classical handbooks on that subject. Then I bought the just published book *Operational Calculus* by Jan Mikusiński (cf. [8]) which showed us a *different aspect* of the theory of linear differential equations with scalar coefficients. Again a fascination! A lot of algebra instead of *epsilons* and *deltas*...

After three years of studies I became an assistant of Witold Pogorzelski (my future M.A. and Ph.D. father) at Technical University of Warsaw. I then understood quickly that my university background in mathematics, in particular, in Analysis, is not sufficient in order to teach mechanical engineers. I learned that Calculus is, indeed, a calculus. I also learned something else. Every student passing examinations repeated after Pogorzelski:

The operation of integration is, in a sense, inverse to the operation of differentiation.

This idea, coming from Leibniz from the very beginning of Calculus was a basis for the so-called *Symbolic Calculus*. Leibniz wrote in 1710 about the symbolic calculus:

> When I was eighteen years old [i.e. in 1664—D. P.-R.] writing the book "De Arte Combinatoria", which I have published 2 years later, I found a leading idea, a PRODIGIOUS MYSTERY OF GENUINE ANALYSIS... I think nobody had observed this, since otherwise, had anyone observed it, he would have abandoned all to deal with it, as man could hardly achieve more... This, at least, permits us to argue with little effort, putting symbols instead of matters to carry out through the imagination.

(For this and the following quotations, cf. for instance, [10].) From a letter to de l'Hospital:

> ...Once this project is realized, there will only remain to manhood the desire of happiness... One of the secrets of analysis is ... the art of a clever use of useful signs.

Since Leibniz's time the signs

$$\Delta, \quad \sum, \quad \int, \quad d \to \frac{d}{dx} = D$$

have been treated as algebraic operations by such mathematicians as Lagrange, Arbogast, Poisson, d'Alembert and many others.

Augustin Cauchy, a former officer of artillery, established a military order in Mathematical Analysis, introducing precise definitions of limits and convergence. His ideas became obligatory for several decades, killing some previous ideas in their germ. Nevertheless, even Cauchy wrote in 1821, not very consistently trying to explain that:

> As to methods, I have sought to make them as rigorous as those of geometry, so as never to have recourse to justification drawn from the generality of algebra.

Polish mathematicians, Jan Śniadecki and Józef Hoene-Wroński, working by the end of the 18th century and in the first decades of the 19th century, had some observable contributions in creation of the framework of Mathematical Analysis (cf. Boyer [3]).

The old approach was forgotten for many years. The Operational Calculus was born a hundred years ago in the book *Electromagnetic Theory* of an engineer Oliver Heaviside. This theory, in view of its applications in engineering, later split up into at least two directions: an algebraic part, leading to different operational calculi, and the theory of integral transforms, in particular, the Laplace transform. Therefore engineering students knew more about this subject than university students. (Some more references about operational calculi and algebraic methods can be found, for instance, in [9] and [10].)

Lectures on Variational Calculus delivered by Krzysztof Maurin showed us that the crucial point of this calculus is derivation in infinite-dimensional spaces.

At that time there was no course of Functional Analysis in Warsaw University. However, Stanisław Mazur organized a student seminar in Functional Analysis. We then learned about Fréchet and Gateaux differentials in Banach spaces. Years later we got to know that there exist scores of different differentials in topological linear spaces (cf. [1] and [2]). In the next decades, together with the development of optimization theory, a real bush of various differentials and subdifferentials has grown (and is still growing).

Six years of Witold Pogorzelski's Seminar in Partial Differential Equations, Integral Equations and their Applications, together with six years of Jan Mikusiński's Seminar in Generalized Functions and Operational Calculus created a good background for the subsequent development of Algebraic Analysis. Essentially different in spirit, essentially different from Mazur's seminar, all three together formed an unrepeatable triangle...

Since the space of Mikusiński operators is not metrizable, and so are some spaces of generalized functions, some new problems of convergence appeared, which have been and are still being investigated. One can only regret that these problems are neglected by topologists, despite their evident importance in applications.

On the other hand, Kaplansky's book [5] on *Differential Algebras* again showed some new aspects of differentiability. For instance, it is shown there that there exist ordinary differential equations which cannot be solved in quadratures (i.e. by a finite number of integrations).

The results of Singer and Wermer [12], Sinclair [11] and their followers opened a new field for investigations, since they show that a "good" derivation cannot be a continuous operator.

Clearly, the above remarks do not exhaust all items from the Mathematics Subject Classification given above. This is only a short personal review of some roads leading to the subject of our conference. I think everybody has his own road. Our main goal is to discuss different aspects together. We hope that this may imply a kind of unification of various approaches in the future.

Danuta Przeworska-Rolewicz

References

[1] V. I. Averbukh and O. G. Smolyanov, *The theory of differentiation in linear topological spaces*, Uspekhi Mat. Nauk 22 (6) (1967), 201–258 (in Russian).

[2] —, —, *Various definitions of derivatives in topological linear spaces*, ibid. 23 (4) (1968), 67–113 (in Russian).

[3] C. B. Boyer, *The History of the Calculus and its Conceptual Development* (*The Concept of Calculus*), Dover, New York, 1949.

[4] C. G. Fraser, *The calculus of algebraic analysis*: *some observations on mathematical analysis in the 18th century*, Arch. Hist. Exact Sci. 39 (1988), 317–335.

[5] I. Kaplansky, *An Introduction to Differential Algebra*, Paris, 1957.

[6] H. C. Kennedy, *Karl Marx and the foundations of differential calculus*, Historia Math. 4 (1977), 303–318.

[7] J. P. Lagrange, *Théorie des Fonctions Analytiques contenant les Principes du Calcul Différentiel*, *dégagés de toute considération d'infiniment petits*, *d'évanouissans de limites et de fluxions*, *et réduit à l'analyse algébrique de quantités finies*, 2nd revised and enlarged edition, Courcier, Paris, 1813 (1st ed., 1797).

[8] J. Mikusiński. *Operational Calculus*, Monografie Mat. 30, PWN–Polish Scientific Publishers, Warszawa, 1953 (1st Polish ed.).

[9] D. Przeworska-Rolewicz, *Algebraic Analysis*, PWN–Polish Scientific Publishers and D. Reidel, Warszawa–Dordrecht, 1988.

[10] —, *Prodigious Mystery of Genuine Analysis*, in: Proc. Conf. Generalized Functions and Convergence, To the memory of Professor Jan Mikusiński, World Sci., Singapore, 1990, 51–58.

[11] A. H. Sinclair, *Continuous derivations on Banach algebras*, Proc. Amer. Math. Soc. 20 (1969), 166–170.

[12] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. 129 (1955), 260–264.