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A PERTURBATION PROBLEM WITH TWO SMALL PARAMETERS IN THE FRAMEWORK OF THE HEAT CONDUCTION OF A FIBER REINFORCED BODY

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0. Introduction

We study, using an asymptotic expansion, a problem of perturbation of partial differential equations with two small parameters.

The equations considered are the equations of stationary heat conduction, and we use the terminology of this domain of physics, so the unknown function is sometimes called «temperature».

We consider the problem of heat conduction in a domain Ω, the boundary of which $\partial\Omega$ is smooth. This domain is split up into two subdomains. One of them, $\mathfrak{F}^{e\varepsilon}$, is the union of parallel «fibers» periodically distributed, the period being of the order of a small parameter ε. The radius of these fibers is of the order of e, another small parameter. The second subdomain $\mathfrak{M}^{e\varepsilon}$ is the interior of $\Omega \setminus \mathfrak{F}^{e\varepsilon}$ and is called the «matrix». The conduction coefficient is constant over $\mathfrak{M}^{e\varepsilon}$, and is of the order of $(\varepsilon/e)^2$ in $\mathfrak{F}^{e\varepsilon}$, so the total conductivity of the fibers is equivalent to that of the matrix.

Previously, the study of the limits ($e \to 0$ then $\varepsilon \to 0$) and ($\varepsilon \to 0$ then $e \to 0$) for the problem of elasticity (see [1] or [2]) showed the importance of the relative orders of ε and e; indeed, the two limits $\varepsilon \to 0$ and $e \to 0$ do not commute. The aim of this work is to study the problem of perturbation when the two parameters are both small, and to classify the different «limit problems» according to the relative orders of the parameters.

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* This work was carried out when the two authors were in the Laboratoire de Mécanique Théorique (L.A. 229), Université P. et M. Curie, 4, Pl. Jussieu, 75230 Paris Cedex, France. It constitutes the «Thèse de 3ème cycle» of Mrs. Dinari who is now teaching in Morocco.

The method used is based on a double-scale matched asymptotic expansion. We study the unknown function accurately in the neighbourhood of a fiber by the change of variable $z_\alpha = x_\alpha/e$ ($\alpha = 1, 2$) in the two directions perpendicular to the direction of the fibers, and we build an asymptotic expansion of the unknown function, the terms of which depend on the variables x and z, and are solutions of partial differential equations in z on unbounded domains. The first terms of the expansion being determined, by matching the development with the one obtained for the next fiber, we determine the function (or functions) describing the macroscopic conduction. At this point, essential distinctions have to be made between different relative orders of e and ε. Then the so-called «flux method» used in homogenization theory yields the conduction equation satisfied by that or those functions.

We find three different «limit problems», following the order of magnitude of $\varepsilon^2 |\log e|$ with respect to unity.

If ε and e are such that $\varepsilon^2 |\log e| \gg 1$ or $\varepsilon^2 |\log e| \ll 1$ the limit problem of conduction is a classical one with only one temperature. For the first case, the conductivity in all directions is that of the matrix, the importance of the fibers vanishes entirely; for the second case, the conductivity is increased in the direction of the fibers and remains that of the matrix in the two perpendicular directions. These two results fit those obtained by the study of the limits ($e \to 0$ then $\varepsilon \to 0$) and ($\varepsilon \to 0$ then $e \to 0$).

More interesting is the case when $\varepsilon^2 |\log e| = \lambda$ (λ a positive real number). The limit conduction problem involves two unknown functions, one being the temperature far from the fibers, the other the temperature in the fibers. The derivatives of the last functions in the two directions normal to the fibers do not occur in the limit problem. So, although it is easy to find a variational formulation of this problem, we did not succeed in proving the equivalence between classical and variational formulations, for we lack trace theorems for the space of functions whose only one derivative is square-integrable.

The convergence proof is not considered in this paper for it had not been carried out completely. That mathematical problem is very close to that approached by D. Cioranescu and F. Murat [3]. From another point of view, the subject of the present work reminds the studies of Phan-Thien [7] and [8] on conduction problems and of Russel [9] and Russel and Acrivos [10] on elasticity problems. But in those papers the method is quite different and the main point is the «aspect ratio» of the fibers.

The first section states the problem. The results of Sections 3, 4, 5 are given in Section 2. In the third section we implement the double scale asymptotic method which gives asymptotic expansion matched in the fourth section. The flux method used in the following section yields the conduction equations. The sixth section is devoted to the study of the conduction problem with two temperatures.

1. Statement of the problem

We consider a domain Ω of $\boldsymbol{R}^3$ with a smooth boundary $\partial\Omega$. This domain is composed of two parts. One is the union $\mathfrak{F}^{e\varepsilon}$ of cylinders parallel to the direction $0x_3$ and periodically distributed in the directions x_1 and x_2, the period being homothetic in the ratio ε to a given period $Y =]0, Y_1[\times]0, Y_2[$, ε a small parameter. We denote by $|Y|$ the surface of Y.

The section S^e of each fiber is a disk of radius e (e is the second small parameter, it is supposed to be such that $e \ll \varepsilon$).

The second part of Ω is the interior $\mathfrak{M}^{e\varepsilon}$ of $\Omega \setminus \mathfrak{F}^{e\varepsilon}$ and is called the *matrix*.

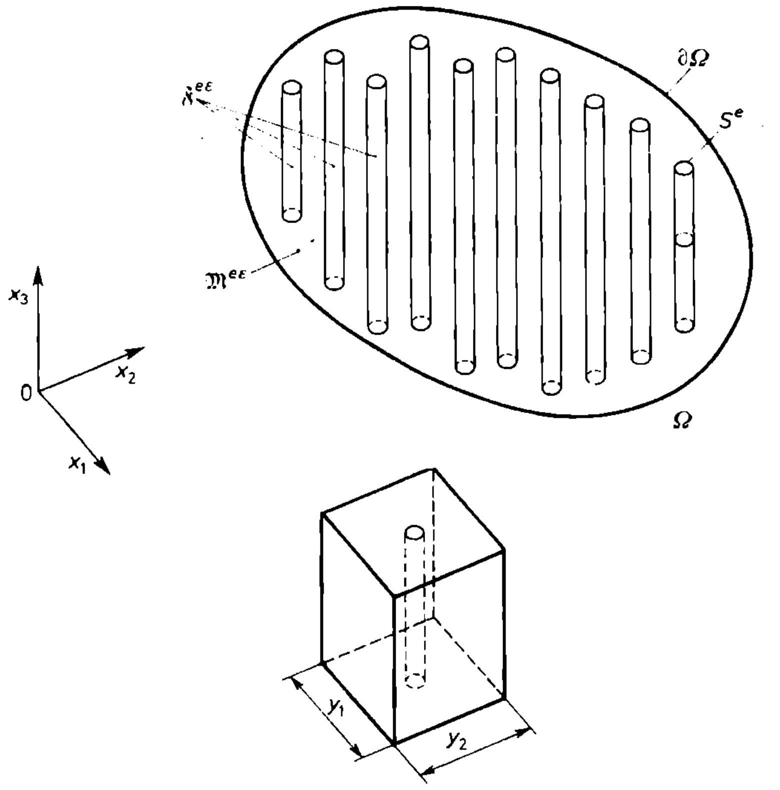

Fig. 1

In this domain Ω, we consider the following problem:
Find $u^{e\varepsilon}$ such that

$$\left.\begin{array}{lll} (1.1) & q_i^{e\varepsilon} = \dfrac{\varepsilon^2}{e^2} k \partial_i u^{e\varepsilon} & \text{in } \mathfrak{F}^{e\varepsilon}, \\ (1.2) & q_i^{e\varepsilon} = K \partial_i u^{e\varepsilon} & \text{in } \mathfrak{M}^{e\varepsilon}, \end{array}\right\} \quad i = 1, 2, 3,$$

(1.3) $\quad \partial_i q_i^{e\varepsilon} + f = 0 \quad$ in Ω,

(1.4) $\quad q_i^{e\varepsilon} n_i$ and $u^{e\varepsilon}$ are continuous on $\partial\mathfrak{M}^{e\varepsilon} \cap \partial\mathfrak{F}^{e\varepsilon}$ (n_i, $i = 1, 2, 3$, are the components of the normal to $\partial\mathfrak{F}^{e\varepsilon}$),

(1.5) $\quad u^{e\varepsilon} = 0 \quad$ on $\partial\Omega$.

In these formulas, k and K are constants, ∂_i denotes the derivative with respect to x_i and we use the convention of repeated indices ($a_i b_i$ denotes $\sum_{i=1}^{3} a_i b_i$).

This is a conduction problem, the conductivity being K in the matrix $\mathfrak{M}^{e\varepsilon}$ and $k\varepsilon^2/e^2$ in the fibers $\mathfrak{F}^{e\varepsilon}$, which are thus very conducting. f is the volumic heat source.

It is easy to prove that this problem has a unique solution and we aim to find the limits of $u^{e\varepsilon}$ when e and ε are small.

2. Results

When e and ε are small, the conduction of the fibered body is governed by equations that depend on the order of $\varepsilon^2|\log e|$ with respect to one.

These equations are:

1° $\varepsilon^2|\log e| \ll 1$:

$$Q_{ij}\frac{\partial^2 U}{\partial x_i\,\partial x_j}+f=0$$

where

$$Q_{ij}=\begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K+k\pi/|Y| \end{pmatrix}$$

and U is the «limit» of $u^{e\varepsilon}$.

2° $\varepsilon^2|\log e| \gg 1$:

$$K\frac{\partial^2 U}{\partial x_i\,\partial x_i}+f=0.$$

3° $\varepsilon^2|\log e| = \lambda$ (λ a real positive number).

The conduction problem is now described by two temperatures U and u:

$$K\frac{\partial^2 U}{\partial x_i\,\partial x_i}+\frac{k\pi}{|Y|}\frac{\partial^2 u}{\partial x_3^2}+f=0,$$

$$\frac{\partial^2 u}{\partial x_3^2}-\frac{2K}{\lambda k}(u-U)=0.$$

3. Asymptotic expansion

In order to study the function $u^{e\varepsilon}$ in the neighbourhood of a fiber, we expand this region by the following change of variables:

$$z_\alpha = x_\alpha/e, \qquad \alpha = 1, 2.$$

Thus, in the variables z_1, z_2, the fiber has a circular section S of radius 1 and it is embedded in a matrix M which is unbounded, for e is small.

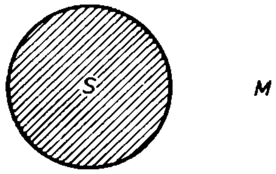

Fig. 2

We look for a double scale expansion of $u^{e\varepsilon}$, the terms of which are functions of $(x_1, x_2, x_3) = x$ and $(z_1, z_2) = z$. The equations (1) have to be modified by replacing the operator ∂_α by $\partial/\partial x_\alpha + (1/e)\,\partial/\partial z_\alpha$, which yields

$$\frac{\partial q_i^{e\varepsilon}}{\partial x_i} + \frac{1}{e}\frac{\partial q_\alpha^{e\varepsilon}}{\partial z_\alpha} + f = 0 \quad \text{in } S \cup M, \tag{3.1}$$

$$\left.\begin{aligned} &q_\alpha^{e\varepsilon} = \frac{\varepsilon^2}{e^2} k \frac{\partial u^{e\varepsilon}}{\partial x_\alpha} + \frac{\varepsilon^2}{e^3} k \frac{\partial u^{e\varepsilon}}{\partial z_\alpha}, \quad \alpha = 1, 2, && (3.2)\\ &q_3^{e\varepsilon} = \frac{\varepsilon^2}{e^2} k \frac{\partial u^{e\varepsilon}}{\partial x_3} && (3.3) \end{aligned}\right\} \quad \text{in } S,$$

$$\left.\begin{aligned} &q_\alpha^{e\varepsilon} = K \frac{\partial u^{e\varepsilon}}{\partial x_\alpha} + \frac{K}{e} \frac{\partial u^{e\varepsilon}}{\partial z_\alpha}, \quad \alpha = 1, 2, && (3.4)\\ &q_3^{e\varepsilon} = K \frac{\partial u^{e\varepsilon}}{\partial x_3} && (3.5) \end{aligned}\right\} \quad \text{in } S,$$

(3.6) $q_\alpha^{e\varepsilon} n_\alpha$ and $u^{e\varepsilon}$ are continuous on ∂S (n_α are the components of the exterior normal to ∂S).

In these formulas, as in the sequel, the Greek indices denote the indices 1 and 2.

We see from (3.2), (3.4) and (3.6) that if we look for an expansion of $u^{e\varepsilon}$ in S in the form

$$u^{e\varepsilon} = g(e, \varepsilon)[u_0(x, z) + e u_1(x, z) + \ldots], \qquad z = (z_1, z_2) \in S, \tag{3.7}$$

we have to look for an expansion of $u^{e\varepsilon}$ in M in the form

$$\begin{aligned} u^{e\varepsilon} = g(e, \varepsilon)\big[&(U_0(x, z) + e U_0^1(x, z) + \ldots) \\ &+ \varepsilon^2 (U_1^0(x, z) + e U_1^1(x, z) + \ldots) + \ldots\big], \qquad z = (z_1, z_2) \in M. \end{aligned} \tag{3.8}$$

In order to distinguish the expansions of $u^{e\varepsilon}$ in S and in M, we use capital U for the terms in M and small u for the terms in S.

Remark 3.1. (i) The forms (3.7) and (3.8) for the expansions of $u^{e\varepsilon}$ are of course «suggested» by the results of the reckoning of the terms, which is done only for the first ones; it is not obvious that the expansion may be continued in this form.

$g(e, \varepsilon)$ is a gauge function which has to be fitted.

(ii) From (3.8) we may see that we have to distinguish between $\varepsilon^2 \ll e$, $\varepsilon^2 \simeq e$ and $\varepsilon^2 \gg e$. The calculus shows that the third case is the most interesting one, its results hold those of the first two. Thus we limit ourselves to the study of the case $\varepsilon^2 \gg e$.

For $\varepsilon^2 \gg e$, the expansion (3.8) may be reorganized as

$$u^{e\varepsilon} = g(e, \varepsilon)[U_0(x, z) + \varepsilon^2 U_1^0(x, z) + eU_0^1(x, z) + \ldots], \quad z = (z_1, z_2) \in M. \tag{3.9}$$

Putting the expansions (3.7) and (3.9) in the equations (3.1)–(3.6) we get

– For $z = (z_1, z_2) \in S$:

$$\Delta_z u_0 = 0, \tag{3.10}$$

$$\Delta_z u_1 + 2\frac{\partial^2 u_0}{\partial x_\alpha \partial z_\alpha} = 0, \tag{3.11}$$

$$\Delta_z u_2 + 2\frac{\partial^2 u_1}{\partial x_\alpha \partial z_\alpha} + \Delta u_0 = 0. \tag{3.12}$$

– For $z \in M$:

$$\Delta_z U_0^0 = 0, \tag{3.13}$$

$$\Delta_z U_1^0 = 0. \tag{3.14}$$

– On ∂S:

$$U_0^0 = u_0, \quad U_1^0 = 0, \tag{3.15}$$

$$\frac{\partial u_0}{\partial z_\alpha} n_\alpha = 0, \quad \frac{\partial u_1}{\partial z_\alpha} n_\alpha + \frac{\partial u_0}{\partial x_\alpha} n_\alpha = 0, \tag{3.16}$$

$$\frac{\partial U_0^0}{\partial z_\alpha} n_\alpha = 0; \quad K\frac{\partial U_1^0}{\partial z_\alpha} n_\alpha = k\left(\frac{\partial u_2}{\partial z_\alpha} + \frac{\partial u_1}{\partial x_\alpha}\right) n_\alpha. \tag{3.17}$$

All these equations have to be considered as partial differential equations in $z = (z_1, z_2)$, x being a parameter. Δ_z denotes the two-dimensional laplacian $\partial^2/\partial z_\alpha \partial z_\alpha$ and Δ the three-dimensional one, $\partial^2/\partial x_i \partial x_i$.

Determination of u_0 and u_1

u_0 is a solution of (3.10) and (3.16), thus it is obvious that u_0 does not depend on z:

$$u_0 = u_0(x). \tag{3.18}$$

Then from (3.11) and (3.16) we see that u_1 satisfies

$$\Delta_z u_1 = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial z_\alpha} n_\alpha = -\frac{\partial u_0}{\partial x_\alpha} n_\alpha .$$

As u_0 does not depend on z, the solution of these equations is obviously

$$u_1 = -\frac{\partial u_0}{\partial x_\alpha} z_\alpha + \tilde{u}_1(x). \tag{3.19}$$

Determination of U_0^0 and U_1^0

The equations satisfied by U_0^0, U_1^0 are set on the unbounded domain M.

$U_0^0 = u_0(x)$ is obviously a solution of (3.13), (3.15) and (3.17); from the Holmgren unicity theorem, it is the unique one. Thus

$$U_0^0 = u_0(x) \quad \text{in } M. \tag{3.20}$$

U_1^0 has to satisfy the equations (3.14), (3.15) and (3.17). u_2 is not determined, the second equation of (3.17) may be considered as a Neumann condition for u_2; then from (3.12) and (3.17) we derive the compatibility condition for the existence of u_2:

$$\int_S \left(2\frac{\partial^2 u_1}{\partial z_\alpha \partial x_\alpha} + \Delta u_0\right) dz + \frac{K}{k} \int_{\partial S} \frac{\partial U_1^0}{\partial z_\alpha} n_\alpha \, d\Gamma - \int_{\partial S} \frac{\partial u_1}{\partial x_\alpha} n_\alpha \, d\Gamma = 0$$

which yields

$$\frac{K}{k} \int_{\partial S} \frac{\partial U_1^0}{\partial z_\alpha} n_\alpha \, d\Gamma = -\int_S \left(\frac{\partial^2 u_1}{\partial z_\alpha \partial x_\alpha} + \Delta u_0\right) dz.$$

Now $u_1 = -\dfrac{\partial u_0}{\partial x_\alpha} z_\alpha + \tilde{u}_1(x)$. Thus

$$\frac{K}{k} \int_{\partial S} \frac{\partial U_1^0}{\partial z_\alpha} n_\alpha \, d\Gamma = -\int_S \frac{\partial^2 u_0}{\partial x_3^2} dz.$$

As u_0 does not depend on z, the compatibility condition for the existence of u_2 is

$$\int_{\partial S} \frac{\partial U_1^0}{\partial z_\alpha} n_\alpha \, d\Gamma = -\frac{k}{K} \pi \frac{\partial^2 u_0}{\partial x_3^2}.$$

Therefore U_1^0 has to satisfy the following equations:

$$\begin{aligned} &\Delta_z U_1^0 = 0 \quad \text{in } M, \\ &U_1^0 = 0 \quad \text{on } \partial S, \\ &\int_{\partial S} \frac{\partial U_1^0}{\partial z_\alpha} n_\alpha \, d\Gamma = -\frac{k\pi}{K} \frac{\partial^2 u_0}{\partial x_3^2}. \end{aligned} \tag{3.21}$$

It is shown in the sequel that (3.21) does not determine U_1^0 uniquely but we may state the following proposition.

PROPOSITION 3.1. *The solution of* (3.21) *that is the least singular for large* $|z|$ *is*

$$U_1^0 = -\frac{k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}\log|z|$$

where $|z| = (z_\alpha z_\alpha)^{1/2}$.

Proof. U_1^0 is a harmonic function, so its general form is

$$U_1^0 = a\log r + b + \sum_{\substack{n=-\infty \\ n\neq 0}}^{n=+\infty} r^n(c_n\cos n\theta + d_n\sin n\theta)$$

where a, b, c_n, d_n are constant and r, θ are the polar coordinates of a running point in M.

Assuming that differentiation, integration and summation commute we get

$$\int_{\partial S}\frac{\partial U_1^0}{\partial z_\alpha}n_\alpha\,d\Gamma = a\int_0^{2\pi}d\theta + \sum_{\substack{n=-\infty \\ n\neq 0}}^{n=+\infty}\left[c_n\int_0^{2\pi}\cos n\theta\,d\theta + d_n\int_0^{2\pi}\sin n\theta\,d\theta\right] = 2\pi a.$$

Therefore

$$a = -\frac{k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}.$$

The Dirichlet condition on ∂S gives

$$0 = b + \sum_{n=1}^{+\infty}[(c_n + c_{-n})\cos n\theta + (d_n - d_{-n})\sin n\theta],$$

and the unicity of the Fourier series yields

$$b = c_n + c_{-n} = d_n - d_{-n} = 0.$$

Thus the general form of U_1^0 is

$$U_1^0 = -\frac{k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}\log|z| + \sum_{n=1}^{\infty}[c_n(r^n - r^{-n})\cos n\theta + d_n(r^n + r^{-n})\sin n\theta].$$

If c_n or d_n is not zero, then U_1^0 is $O(r^n)$ for large r, and so the most regular solution of (3.21) is such that $c_n = d_n = 0$ for every n.

Result of the expansion

In the neighbourhood of a fiber we may write (for $\varepsilon^2 \gg e$):

$$u^{e\varepsilon} = g(e,\varepsilon)\left[u_0(x) + e\left(-\frac{\partial u_0}{\partial x_\alpha}z_\alpha + \tilde{u}_1(x)\right) + \dots\right] \quad \text{for } z\in S, \tag{3.22}$$

$$u^{e\varepsilon} = g(e,\varepsilon)\left[u_0(x) + \varepsilon^2\left(-\frac{k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}\log|z|\right) + \dots\right] \quad \text{for } z\in M. \tag{3.23}$$

4. Matching of the expansion

The development (3.23) holds true for z belonging to M which is an expanded neighbourhood of a fiber. In order to be able to construct an expansion of $u^{e\varepsilon}$ in the whole Ω, we have to match two expansions of $u^{e\varepsilon}$ valid for two neighbouring fibers. The matching has to be carried out at a distance of the order of ε from the two fibers, i.e. for $|z| \simeq \varepsilon/e$. Therefore we require that the greatest order of the terms of the expansion (3.23) should be one for $|z| \simeq \varepsilon/e$. The most significant term is then called $U(x)$, it represents the temperature far from the fibers; $u(x)$ denotes the temperature in the fibers, it is the term of order 1 in the expansion (3.22).

For $z = \varepsilon/e$ in (3.23) we get

$$u^{e\varepsilon} = U(x) + \ldots,$$

$$U(x) = g(e, \varepsilon)\left(u_0(x) - \varepsilon^2 |\log e| \frac{k}{2K} \frac{\partial^2 u_0}{\partial x_3^2}\right). \tag{4.1}$$

Remark 4.1. The matching justifies the choice of U_1^0 of the preceding section. Indeed, if for U_1^0 we had kept terms such as $c_n r^n \cos n\theta$ then the function $U(x)$ would have depended on θ and the matching would not have been possible.

The function $U(x)$ of (4.1) has to be of order 1; hence three cases are to be considered, according to the order of $\varepsilon^2 |\log e|$:

1° $\varepsilon^2 |\log e| \ll 1$. We then take $g(e, \varepsilon) = 1$ and get

$$U(x) = u_0(x) \tag{4.2}$$

and from (3.22)

$$u(x) = u_0(x). \tag{4.3}$$

In this case, the temperature u in the fibers is equal to the temperature U far from them.

2° $\varepsilon^2 |\log e| \gg 1$. We choose $g(e, \varepsilon) = 1/\varepsilon^2 |\log e|$ and find

$$U(x) = -\frac{k}{2K} \frac{\partial^2 u_0}{\partial x_3^2}, \qquad u(x) = 0.$$

This seems to show that when $\varepsilon^2 |\log e| \gg 1$, the temperature in the fibers $u(x)$ tends to zero with e and ε; this is obviously untrue. For $f = 0$ and suitable boundary conditions on $\partial\Omega$ we may have $u^{e\varepsilon}$ constant all over and whatever e and ε may be.

Therefore the expansions (3.22) and (3.23) have to be modified for $\varepsilon^2 |\log e| \gg 1$.

It is easy to prove that the expansions

$$(4.4)\qquad u^{e\varepsilon} = C+\frac{1}{\varepsilon^2|\log e|}[u_0(x)+\ldots] \quad \text{in } S,$$

$$(4.5)\qquad u^{e\varepsilon} = C+\frac{1}{\varepsilon^2|\log e|}\left[u_0(x)-\varepsilon^2\frac{k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}\log|z|+\ldots\right] \quad \text{in } M$$

are correct, C being a real constant number. Then the two temperatures U and u are:

$$(4.6)\qquad U(x) = C-\frac{k}{2K}\frac{\partial^2 u_0}{\partial x_3^2},$$

$$(4.7)\qquad u(x) = C.$$

3° $\varepsilon^2|\log e| = \lambda$ (λ a real positive number). We take $g(e, \varepsilon) = 1$ and

$$(4.8)\qquad U(x) = u_0(x)-\lambda\frac{k}{2K}\frac{\partial^2 u_0}{\partial x_3^2},$$

$$(4.9)\qquad u(x) = u_0(x).$$

Thus when $\varepsilon^2|\log e|$ is equal to λ, the temperatures U and u are related by the equation

$$(4.10)\qquad U-u = -\lambda\frac{k}{2K}\frac{\partial^2 u}{\partial x_3^2}.$$

Now the asymptotic expansions are matched and the relations between the temperatures U and u are known. We note that these relations are quite different according to the order of $\varepsilon^2|\log e|$ with respect to one.

The equations governing U and u are determined in the following section.

5. Flux method

This method was developed for homogenization theory; for more details, see [11] and [12]. The idea of the method is the following:

Let D_ε be any subdomain of Ω composed of a number of entire cells c_p^ε (see Fig. 3 and notations). As ε is small, such domain D_ε may approach any «smooth» subdomain D of Ω.

We integrate the equation (1.3) in D_ε:

$$\int_{D_\varepsilon}(\partial_i q_i^{e\varepsilon}+f)\,dx = 0,$$

and then integrate by parts:

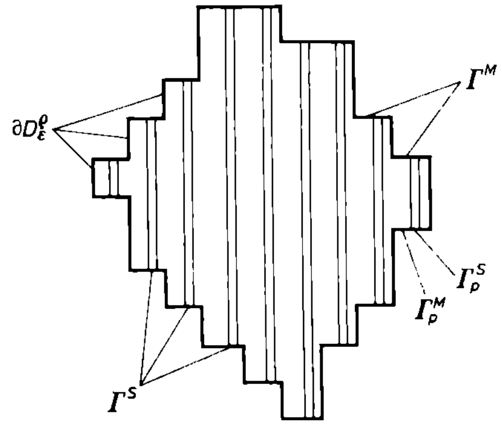

Fig. 3

$$\int_{\partial D_\varepsilon} q_i^{e\varepsilon} n_i \, d\Gamma + \int_{D_\varepsilon} f \, dx = 0 \tag{5.1}$$

where $q^{e\varepsilon}$ is given by (1.1) and (1.2).

We have to distinguish between different parts of ∂D_ε. Let $\partial D_\varepsilon^\varrho$ be the part of ∂D_ε with a normal $n = (n_1, n_2, 0)$ perpendicular to the direction $0x_3$ of the fibers.

Let Γ^S and Γ^M be the parts of ∂D_ε with a normal $n = (0, 0, \pm 1)$ parallel to the fibers and corresponding respectively to the fibers and to the matrix; and let $\Gamma = \Gamma^S \cup \Gamma^M$.

With this notation, (5.1) may be written:

$$\int_{\partial D_\varepsilon^\varrho} K \frac{\partial u^{e\varepsilon}}{\partial x_\alpha} n_\alpha \, d\Gamma + \int_{\Gamma^S} \frac{\varepsilon^2}{e^2} k \frac{\partial u^{e\varepsilon}}{\partial x_3} n_3 \, d\Gamma + \int_{\Gamma^M} K \frac{\partial u^{e\varepsilon}}{\partial x_3} n_3 \, d\Gamma + \int_{D_\varepsilon} f \, dx = 0. \tag{5.2}$$

Now we use the expansions of $u^{e\varepsilon}$ in this equation.

All the points of D_ε^ϱ are far from the fibers, so the first term of $u^{e\varepsilon}$ is here equal to $U(x)$.

The points of Γ^S are points of some fibers, thus $u^{e\varepsilon}$ is here of the order of $u(x)$ and the following proposition holds true (it is justified in the sequel).

Proposition 5.1. *Up to a term $o(1)$ (very much smaller than one) we may write*

$$\int_{\Gamma^S} \frac{\varepsilon^2}{e^2} \frac{\partial u^{e\varepsilon}}{\partial x_3} n_3 \, d\Gamma = \int_\Gamma \frac{\pi}{|Y|} \frac{\partial u}{\partial x_3} n_3 \, d\Gamma + o(1).$$

Some points of Γ^M are near a fiber, other ones are far from them but the following proposition holds true.

Proposition 5.2. *Up to a term $o(1)$, we have*

$$\int_{\Gamma^M} \frac{\partial u^{e\varepsilon}}{\partial x_3} n_3 \, d\Gamma = \int_\Gamma \frac{\partial U}{\partial x_3} n_3 \, d\Gamma + o(1).$$

With these two propositions, (5.2) yields

$$\int_{\partial D_\varepsilon^Q} K \frac{\partial U}{\partial x_\alpha} n_\alpha \, d\Gamma + \int_\Gamma \left(\frac{k\pi}{|Y|} \frac{\partial u}{\partial x_3} + K \frac{\partial U}{\partial x_3} \right) n_3 \, d\Gamma + \int_{D_\varepsilon} f \, dx = o(1).$$

Integrating by parts we get

$$\int_{D_\varepsilon} \left(K \frac{\partial^2 U}{\partial x_i \, \partial x_i} + \frac{k\pi}{|Y|} \frac{\partial^2 u}{\partial x_3^2} + f \right) dx = o(1). \tag{5.3}$$

Now we use the following lemma proved in [11] and [12].

LEMMA 5.1. *If $h(x)$ is a regular function such that*

$$\int_{D_\varepsilon} h(x) \, dx = 0$$

for any domain D_ε composed of cells c_p^ε then $h(x)$ is of order $O(\varepsilon)$.

Then (5.3) yields

$$K \frac{\partial^2 U}{\partial x_i \, \partial x_i} + \frac{k\pi}{|Y|} \frac{\partial^2 u}{\partial x_3 \, \partial x_3} + f = 0 \quad \text{in } \Omega. \tag{5.4}$$

This equation takes different forms according to the order of $\varepsilon^2 |\log e|$ with respect to one.

1° $\varepsilon^2 |\log e| \ll 1$. From (4.2) and (4.3), U and u are equal, thus (5.4) becomes

$$Q_{ij} \frac{\partial^2 U}{\partial x_i \, \partial x_j} + f = 0$$

where

$$(Q_{ij}) = \begin{pmatrix} K & 0 & 0 \\ K & K & 0 \\ 0 & 0 & K + k\pi/|Y| \end{pmatrix}.$$

This equation governing the temperature U in Ω is a classical heat conduction equation; the conductivity in the direction of the fibers is raised up by $k\pi/|Y|$. This result is the same as the one obtained in the limit ($\varepsilon \to 0$ then $e \to 0$).

2° $\varepsilon^2 |\log e| \gg 1$. u is a constant (4.5), thus (5.4) becomes

$$K \frac{\partial^2 U}{\partial x_i \, \partial x_i} + f = 0.$$

This is the same result as in the limit ($e \to 0$ then $\varepsilon \to 0$). The fibers do not conduct heat and the conductivity of the fibered body is that of the matrix.

3° $\varepsilon^2 |\log e| = \lambda$. U and u are related by (4.10). These two functions are now solutions of the system

$$K \frac{\partial^2 U}{\partial x_i \, \partial x_i} + \frac{k\pi}{|Y|} \frac{\partial^2 u}{\partial x_3^2} + f = 0,$$
(5.5)
$$\frac{\partial^2 u}{\partial x_3^2} - \frac{2K}{\lambda k}(u - U) = 0.$$

This is a nonstandard conduction problem with two temperatures U and u, U being the temperature far from the fibers, u the temperature in the fibers. This problem is partly studied in the following section.

Justification of Proposition 5.1

We want to justify that

$$\frac{\varepsilon^2}{e^2} \int_{\Gamma^S} \frac{\partial u^{e\varepsilon}}{\partial x_3} n_3 \, d\Gamma = \frac{\pi}{|Y|} \int_{\Gamma} \frac{\partial u}{\partial x_3} n_3 \, d\Gamma + o(1).$$

Γ is the part of ∂D_ε where the normal is parallel to the fibers. If we number the fibers of D_ε with p varying from 1 to N, we may write $\Gamma = \bigcup_{p=1}^{N} \Gamma_p$, where Γ_p is the part of Γ corresponding to the cell c_p^ε. We define Γ_p^S and Γ_p^M to be the parts of Γ_p corresponding to the fibers and to the matrix in the same way. Then

$$\int_{\Gamma^S} \frac{\varepsilon^2}{e^2} \frac{\partial u^{e\varepsilon}}{\partial x_3} n_3 \, d\Gamma = \sum_{p=1}^{N} \int_{\Gamma_p^S} \frac{\varepsilon^2}{e^2} \frac{\partial u^{e\varepsilon}}{\partial x_3} n_3 \, d\Gamma.$$

Now, in the fiber p, $u^{e\varepsilon} = u(x) + o(1)$, therefore

$$\frac{\varepsilon^2}{e^2} \int_{\Gamma^S} \frac{\partial u^{e\varepsilon}}{\partial x_3} n_3 \, d\Gamma = \sum_{p=1}^{N} \frac{\varepsilon^2}{e^2} \int_{\Gamma_p^S} \frac{\partial u}{\partial x_3} n_3 \, d\Gamma + \sum_{p=1}^{N} \frac{\varepsilon^2}{e^2} \int_{\Gamma_p^S} o(1) \, d\Gamma.$$

$u(x)$ is almost constant on Γ_p^S (it is a function of x independent of z). Hence we may write

$$\sum_{p=1}^{N} \frac{\varepsilon^2}{e^2} \int_{\Gamma_p^S} \frac{\partial u}{\partial x_3} n_3 \, d\Gamma \simeq \sum_{p=1}^{N} \frac{\pi}{|Y|} \int_{\Gamma_p} \frac{\partial u}{\partial x_3} n_3 \, d\Gamma = \frac{\pi}{|Y|} \int_{\Gamma} \frac{\partial u}{\partial x_3} n_3 \, d\Gamma,$$

for

$$\int_{\Gamma_p^S} d\Gamma = e^2 \pi = \frac{e^2}{\varepsilon^2} \frac{\pi}{|Y|} \int_{\Gamma_p} d\Gamma.$$

In a similar way,

$$\sum_{p=1}^{N} \int_{\Gamma_p^S} \frac{\varepsilon^2}{e^2} o(1) \, d\Gamma = o(1).$$

Therefore

$$\frac{\varepsilon^2}{e^2}\int_{\Gamma^S}\frac{\partial u^{e\varepsilon}}{\partial x_3}n_3\,d\Gamma=\frac{\pi}{|Y|}\int_{\Gamma}\frac{\partial u}{\partial x_3}n_3\,d\Gamma+o(1).$$

Proposition 5.1 is justified.

Justification of Proposition 5.2

We want to justify that

$$\int_{\Gamma^M}\frac{\partial u^{e\varepsilon}}{\partial x_3}n_3\,d\Gamma=\int_{\Gamma}\frac{\partial U}{\partial x_3}n_3\,d\Gamma+o(1).$$

We may write

$$\int_{\Gamma^M}\frac{\partial u^{e\varepsilon}}{\partial x_3}n_3\,d\Gamma=\sum_{p=1}^{N}\int_{\Gamma_p^M}\frac{\partial u^{e\varepsilon}}{\partial x_3}n_3\,d\Gamma. \tag{5.6}$$

On each of Γ_p^M we may use the expansion of $u^{e\varepsilon}$, but before we have to express this expansion in terms of $U(x)$ for the different cases.

1° $\varepsilon^2|\log e|\ll 1$. From (3.23) and (4.2) we have

$$u^{e\varepsilon}=U+o(1)\quad\text{in each }\Gamma_p^M.$$

Indeed, the following term of the expansion is very much smaller than U, for $|z|$ is at most equal to ε/e and $\varepsilon^2\log|z|$ is very much smaller than one.

Thus (5.6) becomes

$$\int_{\Gamma^M}\frac{\partial u^{e\varepsilon}}{\partial x_3}n_3\,d\Gamma=\sum_{p=1}^{N}\int_{\Gamma_p^M}\frac{\partial U}{\partial x_3}n_3\,d\Gamma+o(1).$$

As e is very small, the measure of Γ^S is very small, and so

$$\int_{\Gamma^M}\frac{\partial u^{e\varepsilon}}{\partial x_3}n_3\,d\Gamma=\int_{\Gamma}\frac{\partial U}{\partial x_3}n_3\,d\Gamma+o(1).$$

The proposition is justified for $\varepsilon^2|\log e|\ll 1$.

2° $\varepsilon^2|\log e|\gg 1$ and 3° $\varepsilon^2|\log e|=\lambda$. For these two cases we may write

$$\frac{\partial u^{e\varepsilon}}{\partial x_3}=\frac{\partial U}{\partial x_3}+A\frac{\partial^3 u_0}{\partial x_3^3}\frac{\log|x-x_p|}{\log e}+\ldots\quad\text{in each }\Gamma_p^M \tag{5.7}$$

where x_p is the center of the section Γ_p^S and

$$A=\frac{k}{2K}\quad\text{if }\varepsilon^2|\log e|\gg 1,$$

$$A=\frac{\lambda k}{2K}\quad\text{if }\varepsilon^2|\log e|=\lambda.$$

Indeed, if $\varepsilon^2|\log e| \gg 1$, from (4.5) and (4.6) we may write (in Γ_p^M)

$$u^{e\varepsilon} = U(x)+\frac{k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}+\frac{1}{\log e}\frac{k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}\log\left|\frac{x-x_p}{e}\right|+o(1)$$

$$= U(x)+\frac{k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}\frac{\log|x-x_p|}{\log e}+o(1).$$

The term $u_0/(\varepsilon^2|\log e|)$ is always very much smaller than one, whereas $(\log|x-x_p|)/\log e$ may be of the order of one (for $|x-x_p| \simeq e$).

If $\varepsilon^2|\log e| = \lambda$, from (4.8) and (3.23) we get (in Γ_p^M)

$$u^{e\varepsilon} = U(x)+\frac{\lambda k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}-\frac{k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}\varepsilon^2\log\left|\frac{x-x_p}{e}\right|+o(1)$$

$$= U(x)+\frac{\lambda k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}\left[1+\frac{1}{\log e}\log\left|\frac{x-x_p}{e}\right|\right]+o(1)$$

$$= U(x)+\frac{\lambda k}{2K}\frac{\partial^2 u_0}{\partial x_3^2}\frac{\log|x-x_p|}{\log e}+o(1).$$

The relation (5.7) is thus settled for $\varepsilon^2|\log e| \gg 1$ and for $\varepsilon^2|\log e| = \lambda$. Now from (5.6) and (5.7) we have

$$\int_{\Gamma^M}\frac{\partial u^{e\varepsilon}}{\partial x_3}n_3\,d\Gamma = \int_{\Gamma^M}\frac{\partial U}{\partial x_3}n_3\,d\Gamma + A\sum_{p=1}^{N}\int_{\Gamma_p^M}\frac{\partial^3 u_0}{\partial x_3^3}n_3\frac{\log|x-x_p|}{\log e}d\Gamma+o(1). \tag{5.8}$$

In order to end the justification of the proposition, we just have to prove that

$$\sum_{p=1}^{N}\int_{\Gamma_p^M}\frac{\partial^3 u_0}{\partial x_3^3}n_3\frac{\log|x-x_p|}{\log e}d\Gamma = o(1).$$

u_0 is a function of x, hence almost constant on each Γ_p^M. Therefore

$$\sum_{p=1}^{N}\int_{\Gamma_p^M}\frac{\partial^3 u_0}{\partial x_3^3}n_3\frac{\log|x-x_p|}{\log e}d\Gamma \simeq \sum_{p=1}^{N}\frac{\partial^3 u_0}{\partial x_3^3}n_3\int_{\Gamma_p^M}\frac{\log|x-x_p|}{\log e}d\Gamma.$$

It is obvious that all the integrals on the right-hand side are equal, and it may be easily proved that they tend to zero with e and ε, i.e. they are $o(1)$. Hence

$$\sum_{p=1}^{N}\int_{\Gamma_p^M}\frac{\partial^3 u_0}{\partial x_3^3}n_3\frac{\log|x-x_p|}{\log e}d\Gamma = \sum_{p=1}^{N}\frac{\partial^3 u_0}{\partial x_3^3}\frac{o(1)}{\log e}.$$

The measure of Γ_p^M is $\varepsilon^2|Y|$, so that

$$\sum_{p=1}^{N} \int_{\Gamma_p^M} \frac{\partial^3 u_0}{\partial x_3^3} n_3 \frac{\log|x-x_p|}{\log e} d\Gamma = \frac{o(1)}{\varepsilon^2 \log e} \int_{\Gamma} \frac{\partial^3 u_0}{\partial x_3^3} n_3 \, d\Gamma.$$

As $\varepsilon^2|\log e|$ is bounded below, this term is $o(1)$, and (5.8) yields

$$\int_{\Gamma^M} \frac{\partial u^{e\varepsilon}}{\partial x_3} n_3 \, d\Gamma = \int_{\Gamma^M} \frac{\partial u}{\partial x_3} n_3 \, d\Gamma + o(1).$$

The justification is now ended as in the case $\varepsilon^2|\log e| \ll 1$.

6. Study of the two temperatures conduction problem

In this section, we study the conduction problem constituted by equations (5.5). These equations have to be completed by boundary conditions.

Different boundary conditions are studied: Dirichlet and Neumann conditions, homogeneous and nonhomogeneous. These conditions are not derived from an asymptotic study of boundary conditions for $u^{e\varepsilon}$, they are a priori settled; but we may expect that homogeneous Dirichlet or Neumann conditions for $u^{e\varepsilon}$ yield similar conditions for U and u.

Therefore we study the following problem:

Find U and u such that

$$\mu \frac{\partial^2 U}{\partial x_i \, \partial x_i} - (U-u) + f = 0 \quad \text{on } \Omega,$$

(6.1)
$$\nu \frac{\partial^2 u}{\partial x_3^2} + (U-u) = 0 \quad \text{on } \Omega,$$

μ, ν two positive numbers,

$$\frac{\partial U}{\partial x_i} n_i = 0 \quad \text{on } \partial\Omega,$$

(6.2)
$$\frac{\partial u}{\partial x_3} n_3 = 0 \quad \text{on } \partial\Omega,$$

$n = (n_1, n_2, n_3)$ the exterior normal to $\partial\Omega$.

Let $W(\Omega)$ be a suitable function space for this problem. The variational formulation is written in a very straightforward way:

Find (U, u) belonging to $W(\Omega)$ such that

(6.3)
$$\forall (V, v) \in W(\Omega) \quad \int_\Omega \mu \frac{\partial U}{\partial x_i} \frac{\partial V}{\partial x_i} dx + \int_\Omega \nu \frac{\partial u}{\partial x_3} \frac{\partial v}{\partial x_3} dx + \int_\Omega (U-u)(V-v)\, dx = \int_\Omega f V \, dx.$$

From this variational formulation, it is obvious that the space $W(\Omega)$ must be $H^1(\Omega) \times P_3^1(\Omega)$ where

$$H^1(\Omega) = \{V \in L^2(\Omega) \quad \text{s.t. } \partial V/\partial x_i \in L^2(\Omega),\ i = 1, 2, 3\},$$

$$P_3^1(\Omega) = \{v \in L^2(\Omega) \quad \text{s.t. } \partial v/\partial x_3 \in L^2(\Omega)\}.$$

$H^1(\Omega)$ is a Hilbert space for the norm

$$\|v\|_{H^1} = \left[\int_\Omega V^2\, dx + \int_\Omega \frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_i}\, dx\right]^{1/2}.$$

It may be proved easily that $P_3^1(\Omega)$ is a Hilbert space for the norm

$$\|v\|_{P_3^1} = \left[\int_\Omega v^2\, dx + \int_\Omega (\partial v/\partial x_3)^2\, dx\right]^{1/2}.$$

Then $W(\Omega)$ is a Hilbert space for the norm

$$\|V, v\|_W = [\|V\|_{H^1}^2 + \|v\|_{P_3^1}^2]^{1/2}.$$

Obviously, the bilinear form

$$A[(U, u), (V, v)] = \int_\Omega \mu \frac{\partial U}{\partial x_i}\frac{\partial V}{\partial x_i}\, dx$$

$$+ \int_\Omega v \frac{\partial u}{\partial x_3}\frac{\partial v}{\partial x_3}\, dx + \int_\Omega (U-u)(V-v)\, dx$$

is bicontinuous and symmetric on $W(\Omega)$, and the linear form

$$L(V, v) = \int_\Omega f V\, dx$$

is continuous on the same space.

Then, to use the Lax–Milgram theorem, we just have to prove that the form A is coercive on $W(\Omega)$, that is to say,

$$\exists \alpha > 0 \text{ s.t. } \forall (V, v) \in W(\Omega) \quad \alpha \|V, v\|_W^2 \leqslant A[(V, v), (V, v)].$$

In fact, this is untrue: indeed, for $v = V = \varrho =$ a constant, we have $A[(V, v), (V, v)] = 0$.

This feature is classical in Neumann boundary problems and the suitable set is the quotient space $\mathscr{W} = W/\mathscr{R}$ where $\mathscr{R}$ is the following equivalence relation:

$$(V_1, v_1)\mathscr{R}(V_2, v_2) \quad \Leftrightarrow \quad \exists \varrho \in \boldsymbol{R} \text{ s.t. } V_1 - V_2 = v_1 - v_2 = \varrho.$$

The norm on $\mathscr{W}$ is

$$\|\hat{V}, \hat{v}\|_{\mathscr{W}} = \inf_{\varrho \in \boldsymbol{R}} \|V + \varrho, v + \varrho\|_{W(\Omega)}$$

where $(\hat{V}, \hat{v})$ is the equivalence class of (V, v).

The variational formulation (6.3) makes sense in $\mathscr{W}$ if and only if

$$\int_\Omega f\,dx = 0$$

which is the classical compatibility condition in Neumann boundary problems.

The continuity of A and L on $\mathscr{W}$ may be easily proved. Then the Lax–Milgram theorem may be applied under the condition that A is coercive on $\mathscr{W}$, which comes down to the following lemma:

LEMMA 6.1. *There exists a real constant C such that*

$$\forall(\hat{V},\hat{v})\in\mathscr{W}\quad \|\hat{V},\hat{v}\|_* \leqslant C\left[\|V-v\|^2_{L^2}+\sum_{i=1}^{3}\left\|\frac{\partial V}{\partial x_i}\right\|^2_{L^2}+\left\|\frac{\partial v}{\partial x_3}\right\|^2_{L^2}\right]^{1/2}$$

where (V, v) is any element of $(\hat{V},\hat{v})$.

Proof. Let (V, v) belong to $W(\Omega)$. We may write $v=(v-V)+V$ and

$$\int_\Omega v^2\,dx \leqslant 2\int_\Omega (V-v)^2\,dx + 2\int_\Omega V^2\,dx.$$

V belonging to $H^1(\Omega)$, the Poincaré inequality holds true for it, thus

$$\int_\Omega v^2\,dx \leqslant C\left[\int_\Omega (V-v)^2\,dx+\int_\Omega \frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_i}\,dx+\left|\int_\Omega V\,dx\right|^2\right].$$

This inequality holds true for any (V, v) in $W(\Omega)$, therefore

$$\int_\Omega (V+\varrho)^2\,dx+\int_\Omega (v+\varrho)^2\,dx \leqslant C\left[\int_\Omega (V-v)^2\,dx+\int_\Omega \frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_i}\,dx+\left|\int_\Omega (V+\varrho)\,dx\right|^2\right].$$

Thus for any real number ϱ

$$\|V+\varrho,\, v+\varrho\|^2_{W(\Omega)} \leqslant C\left[\int_\Omega (V-v)^2\,dx+\int_\Omega \frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_i}\,dx+\int_\Omega \frac{\partial v}{\partial x_3}\frac{\partial v}{\partial x_3}\,dx+\left|\int_\Omega (V+\varrho)\,dx\right|^2\right]$$

and so

$$\|\hat{V},\hat{v}\|_* = \inf_{\varrho\in\mathbf{R}}\|V+\varrho,\, v+\varrho\|_W \leqslant C\inf_{\varrho\in\mathbf{R}}\left[\int_\Omega (V-v)^2\,dx+\int_\Omega \frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_i}\,dx+\int_\Omega \frac{\partial v}{\partial x_3}\frac{\partial v}{\partial x_3}\,dx+\left|\int_\Omega (V+\varrho)\,dx\right|^2\right]^{1/2}.$$

Taking $\varrho = -\int_\Omega V\,dx$, we see that $\inf_{\varrho\in\mathbf{R}}|\int_\Omega (V+\varrho)\,dx|^2 = 0$. Hence

$$\|\hat{V}, \hat{v}\|_W \leqslant C\left[\int_\Omega (V-v)^2\,dx + \int_\Omega \frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_i}\,dx + \int_\Omega \frac{\partial v}{\partial x_3}\frac{\partial v}{\partial x_3}\,dx\right]^{1/2}.$$

The lemma is thus proved and we may state

THEOREM 6.2. *If f is such that $\int_\Omega f\,dx = 0$, then the variational problem* (6.3) *has a solution which is unique up to an additive constant* (ϱ, ϱ).

Remarks. (i) (on the equivalence of classical and variational formulations). Let (U, u) be a solution of the variational problem (6.3). It is possible to prove that (U, u) satisfies (6.1) in a suitable sense; but it is more difficult to give sense to (6.2), for, in the space $P_3^1(\Omega)$, only the derivative with respect to x_3 has a certain regularity and we lack a trace theorem in this space.

(ii) (on a trace theorem in $P_3^1(\Omega)$). If $\partial\Omega$ is supposed to be smooth and such that the set of points where the tangent plane is parallel to $0x_3$ is of null measure in $\partial\Omega$, then, using methods developed in [6], we may (it is not done here) define a trace on $\partial\Omega$ of a function from $P_3^1(\Omega)$ and the trace operator is continuous from $P_3^1(\Omega)$ to $L_h^2(\partial\Omega)$, where $L_h^2(\partial\Omega)$ is defined in the following way:

Let h be equal to $a(x)n_3$ where $a(x)$ is a positive continuous function defined on $\partial\Omega$ and n_3 the third component of the normal n. Then

$$L_h^2(\partial\Omega) = \left\{v \text{ measurable on } \partial\Omega \text{ s.t. } \int_{\partial\Omega} h(x)|v|^2\,d\Gamma < +\infty\right\}.$$

(iii) (on other boundary conditions). With the trace properties stated above the following definition of $W_0(\Omega)$ makes sense:

$$W_0(\Omega) = \{(V, v)\in W(\Omega) \text{ s.t. } V|_{\partial\Omega} = 0 \text{ in } H^{1/2}(\partial\Omega) \text{ and } v|_{\partial\Omega} = 0 \text{ in } L_h^2(\partial\Omega)\}.$$

We may now study the following Dirichlet problem:
Find (U, u) belonging to $W_0(\Omega)$ such that

$$\forall (V, v)\in W_0(\Omega) \qquad A[(U, u), (V, v)] = L(V, v).$$

Likewise, we may study nonhomogeneous Dirichlet and Neumann problems and prove the existence and uniqueness of solutions of variational problems [4].

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