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Continuous mappings on continua

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1. Introduction

In view of the fact that a great many recent papers concern new kinds of mappings in connection with particular properties of continua, it has become necessary to systematize these new classes of mappings together with some classical classes of mappings as: homeomorphisms, open mappings and monotone mappings. The main purpose of this paper is to collect results concerning mappings on metric continua, to elaborate these results and to give a synthetic monograph about them.

Thus this paper is a study of various classes of continuous mappings, of relations between them, of methods of producing the new classes, and a study of general and invariance properties of those classes.

Beginning with nine fundamental classes of mappings, we obtain new classes by general operations, such as a composition, inheritance, localization and some other. This line of approach allows me to put in order these variform classes of mappings.

In addition to the results scattered in the literature the paper contains my own original results which concern the above-mentioned problems and also an analysis of the essentiality of the assumptions of various theorems shown by appropriate examples. All the results are collected in tables at the end of sections. These tables contain also unsolved problems.

The results obtained by the author, recapitulated and proved here, were also reported at the topological seminar of PAN, conducted by Professor B. Knaster. The paper has been worked out on the basis of a dissertation which was prepared by the author at the Institue of Mathematics of the Wrocław University under the guidance of Professor J. J. Charatonik.

The author wishes to express his gratitude to Professor J. J. Charatonik, whose confidence and continued interest has made this paper possible. The author is very much indebted to the participants of Professor Knaster's seminar for many helpful remarks and he is grateful to Professor K. Sieklucki for persuading him to prepare this paper.

2. Preliminaries. Special kinds of continua

The topological spaces under consideration will be assumed to be metric and compact and all mappings will be assumed to be continuous and surjective. A continuum means a compact connected space.

A continuum X is said to be *unicoherent* provided that for each two continua C_1 and C_2 such that $C_1 \cup C_2 = X$, the common part $C_1 \cap C_2$ is a continuum (see [36], § 46, X, p. 162). We say that X is *hereditarily unicoherent* if each continuum contained in X is unicoherent.

(2.1) *Each hereditarily unicoherent continuum is unicoherent.*

The inverse implication fails to hold. For example, the unit square is unicoherent and it is not hereditarily unicoherent. A continuum X is called *discoherent* if for each two closed sets C_1 and C_2 such that $X = C_1 \cup C_2$ and $C_1 \neq X \neq C_2$, the set $C_1 \cap C_2$ is not connected (see [36], § 46, X, p. 162). A continuum X is said to be *decomposable* if there exists a decomposition of X into two proper subcontinua C_1 and C_2 such that $X = C_1 \cup C_2$; it is said *indecomposable* if there exists no such decomposition (see [36], § 48, V, p. 204). A continuum is said to be *hereditarily decomposable* (hereditarily indecomposable) if any subcontinuum of it is decomposable (indecomposable).

(2.2) *Every indecomposable continuum is unicoherent and discoherent.*

(2.3) *Every hereditarily indecomposable continuum is hereditarily unicoherent.*

The following proposition characterizes hereditarily unicoherent continua (see [57], Corollary (5.5)).

(2.4) *A continuum X is not hereditarily unicoherent if and only if it contains a decomposable discoherent continuum.*

A space X is said to be *acyclic* if all its homology groups are trivial (see [6], p. 35). We consider only one-dimensional acyclic continua (a one-dimensional continuum is called a curve), and then X is acyclic if and only if every mapping of X onto a circle is inessential (i.e., it is homotopic to a constant mapping) (see [27], p. 150). It is known (see [49], 2.3) that

(2.5) *Every acyclic curve is hereditarily unicoherent.*

The inverse implication fails to hold, because there are hereditarily unicoherent continua which are not one-dimensional (see [4], [29]). Moreover, the so-called standard solenoid, i.e., the inverse limit of circles $S_k = S$ with bonding maps $f_k\colon S_{k+1} \to S_k$ defined by $f_k(e^{2\pi ix}) = e^{4\pi ix}$ for $x \in [0, 1]$ and $k = 1, 2, \ldots$, makes an example of a hereditarily unicoherent curve that is not acyclic (cf. van Dantzig's solenoids described in [16], pp. 73–76).

A continuum X is called *tree-like* (*arc-like*) if for any number $\varepsilon > 0$ ther eexists a mapping $f\colon X \to Y$ such that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each point $y \in Y$ and Y is a one-dimensional polyhedron containing no simple closed curve (Y is an arc, respectively) (see [3], p. 653; cf. [63], where one can find a characterization of tree-like continua in terms of inverse systems). Sometimes arc-like continua are also called "chainable" or "snake-like". From above definitions we infer that

(2.6) *Each arc-like continuum is tree-like.*

Further, from Case's and Chamberlin's characterization of tree-like continua (see [7], p. 74) we infer that

(2.7) *Every tree-like continuum is an acyclic curve.*

It is known that there exists an acyclic curve which is not tree-like (see [7], p. 80). Each two arc-like hereditarily indecomposable (nondegenerate) continua are homeomorphic (see [70], p. 583) and such a continuum is called a *pseudo-arc* (the first example was described in [30]).

A hereditarily decomposable and hereditarily unicoherent continuum is called a *λ-dendroid* (see [11], p. 16). It follows from [13], Corollary, p. 21, that

(2.8) *Every λ-dendroid is a tree-like continuum.*

An arcwise connected and hereditarily unicoherent continuum is called a *dendroid* (see [8], p. 239) and we infer that (see [8], (49), p. 239)

(2.9) *Every dendroid is a λ-dendroid.*

We have the following characterization of a dendroid (see [59], Theorem 2):

(2.10) *An arcwise connected continuum X is a dendroid if and only if the intersection of any two subcontinua of X with nonempty interiors is connected.*

A locally connected and hereditarily unicoherent continuum is called a *dendrite* (see [36], § 51, VI, p. 300). For example, every one-dimensional polyhedron containing no simple closed curve (cf. the definition of tree-like continua) is a dendrite. Since every locally connected continuum is arcwise connected, we conclude that

(2.11) *Every dendrite is a dendroid.*

Recall that a continuum T is a *triod* provided that there are three subcontinua A, B and C of T such that $T = A \cup B \cup C$, $A \cap B \cap C = A \cap B = A \cap C = B \cap C$ and this common part is a proper subcontinuum of each of them. A continuum X is said to be *atriodic* if it does not contain a triod. For example an arc and a circle are atriodic. They are the only atriodic locally connected continua (see [36], § 51, pp. 274–303). We have (see [3] and [69], p. 55; cf. also [78], Theorem 13, p. 50)

(2.12) *A hereditarily decomposable continuum is arc-like if and only if it is hereditarily unicoherent and atriodic.*

Moreover, (see [3]),

(2.13) *Every arc-like continuum is atriodic.*

As an immediate consequence of the definitions we infer that

(2.14) *Every hereditarily indecomposable continuum is atriodic.*

Atriodic continua need not be hereditarily unicoherent (for example a circle is atriodic and it is not even unicoherent), but they are hereditarily bicoherent (see [83], p. 153, the definition of the function $r(x)$). Namely (see [58], Theorem 5.13)

(2.15) *The intersection of each two subcontinua of an atriodic continuum is the union of two continua.*

Recall that a continuum is called *hereditarily divisible by points* if any of its subcontinua contains a point which separates that subcontinuum. Continua of this type were considered in [81] and [67]. Observe that (see [67], Theorem 3.2, p. 350)

(2.16) *Any continuum which is hereditarily divisible by points is a λ-dendroid.*

One can ask whether an atriodic λ-dendroid is hereditarily divisible by points. The answer is negative. We have the following

(2.17) Example. If in the continuum X described in Example 5 of [36], § 48, I, p. 191, we subsitute a copy of such a continuum X instead of any straight line interval, then one can obtain an atriodic λ-dendroid which is not divisible by any point.

We say that a set A is a *set of irreducibility of a continuum X* provided there is a point a of X such that the continuum X is irreducible about the set $A \cup \{a\}$. The following theorem, proved in [55], Theorem 3, is a generalization of Théorème XIX in [34], p. 270.

(2.18) *A set A is a set of irreducibility of a continuum X if and only if there exist no two proper subcontinua P and R of X such that $X = P \cup R$ and $A \subset P \cap R$.*

If a continuum X contains a degenerate set of irreducibility, then X is shortly called *irreducible*. We have (see [77], Theorem 3.2, p. 456)

(2.19) *Every unicoherent atriodic continuum is irreducible.*

A continuum X is said to be a (linear) *graph* if X is the union of a finite number of arcs which are pairwise disjoint except their endpoints (see [83], p. 182). We say that a continuum X is an *n-star* if X is the union of n arcs which are pairwise disjoint except for one given point p, which is the common endpoint of these arcs and p is called the *top* of X. We say that the space X is *of order* $\leqslant m$ at the point p provided for any $\varepsilon > 0$ there is an open set G such that

$$p \in G, \qquad \operatorname{diam} G < \varepsilon \qquad \text{and} \qquad \operatorname{card} \operatorname{Fr}(G) \leqslant m,$$

where $\operatorname{Fr}(G)$ denotes the boundary of G in X (see [36], § 51, I, p. 274). The minimal cardinal number which satisfies this condition is called the *order of X at p* and it is denoted by $\operatorname{ord}_p X$. Menger's theorem, the so-called "n-Beinsatz" (e.g. see [36], § 51, I, p. 277) says that

(2.20) *if X is locally connected and n is a natural number, then* $\operatorname{ord}_p X = n$ *if and only if there exists an n-star in X with the top p and X does not contain an $(n+1)$-star with the top p.*

The following condition characterizes graphs (see [83], p. 182).

(2.21) *A continuum X is a graph if and only if all points of X save a finite number of them are of order 2, and all points are of finite order.*

Note that every graph is homeomorphic to some one-dimensional polyhedron.

If a point p is the top of some 3-star contained in a continuum X, then p is called a *ramification point of X* (in the classical sense) (see [8]). A dendroid having exactly one ramification point is called a *fan* (see [10], p. 6) and this ramifiaction point is called a *top*. Obviously

(2.22) *Every fan is a dendroid.*

A continuum is called *regular* (or *rational*) if it possesses a basis of open sets whose boundaries are finite (or countable, respectively), i.e., if this continuum is of order ω (or $\aleph_0$, respectively) at each point. We note that (see [42], p. 132)

(2.23) *A continuum X is regular if and only if for any number $\varepsilon > 0$, there exists a positiee integer n such that each collection of mutually disjoint subcontinua of X having diameters greater than ε consists of at most n elements.*

We say that a continuum X is *finitely Suslinian* if for any number $\varepsilon > 0$ each collection of mutually disjoint subcontinua of X having diameters greater than ε is finite (see [42]). A continuum is said to be *Suslinian* if each collection of its mutually disjoint nondegenerate subcontinua is countable (cf. [23]). It follows from (2.23) that

(2.24) *Every regular continuum is finitely Suslinian.*

Moreover (see [42], p. 132, and [83], p. 94)

(2.25) *Every finitely Suslinian continuum is hereditarily locally connected.*

(2.26) *Every hereditarily locally connected continuum is rational.*

(2.27) *Every rational continuum is Suslinian.*

Further, since every indecomposable continuum contains an uncountable number of composants (for the definition and for the properties see [36], § 48, VI, pp. 208–215), we infer that

(2.28) *Every Suslinian continum is hereditarily decomposable.*

There is an arc-like Suslinian continuum which is not rational (see [14], p. 178 and [41], p. 135). The *harmonic fan* (see [8], E1, p. 240) is a rational continuum which fails to be locally connected; the *Cantor fan* (see [8], E2, p. 240) is a hereditarily decomposable continuum which is not a Suslinian continuum. Also, hereditarily locally connected continua need not be finitely Suslinian (see [36], p. 270) (every planable hereditarily locally connected continuum is finitely Suslinian) and finitely Suslinian continua need not be regular (see [36], p. 284).

A continuum is called a *local dendrite* if each of its points has a closed neighbourhood which is a dendrite (see [36], § 51, VII, p. 303). We note that (see [36], § 51, VII, Theorems 1 and 4, p. 303)

(2.29) *Every local dendrite is a regular continuum.*

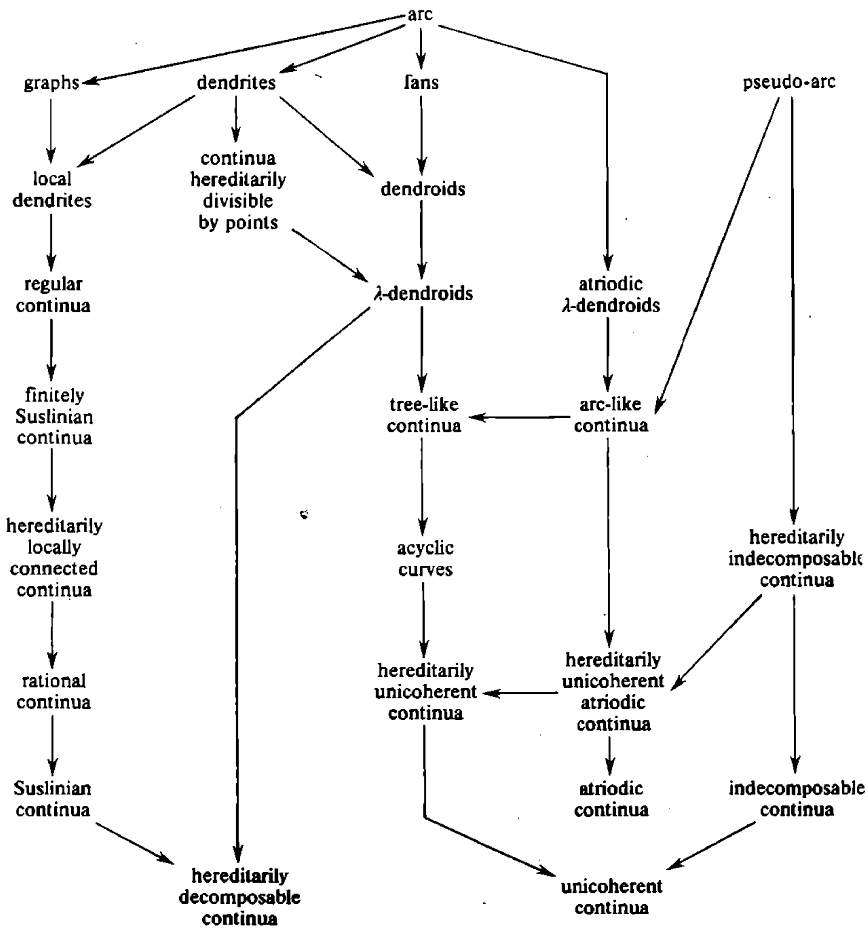
(2.30) *A locally connected oontinuum X is a local dendrite if and only if it contains a finite number of simple closed curves.*

By the definition of a graph we conclude that

(2.31) *Every graph is a local dendrite.*

Table I comprises all inclusions between the classes of continua which are recalled in this section. We use the sign of implication instead of the sign of inclusion.

TABLE I



3. Classes of mappings

To begin with, we recall the definitions of special classes of mappings which are used in the next section to define other classes. Namely, a mapping f from a topological space X onto a topological space Y is said to be

(i) a *homeomorphism* if f is one-to-one and the inverse mapping f^{-1} is continuous;

(ii) *open* if f maps every open set in X onto an open set in Y:

(iii) *atomic* if for each subcontinuum K of X such that the set $f(K)$ is nondegenerate we have $K = f^{-1}(f(K))$ (see [1], [12] and [20]).

The next classes of mappings to be considered are defined by the property which is satisfied by the inverse image of an arbitrary subcontinuum of the image. We say that such a class of mappings is defined by the subcontinua of the image.

The mapping f from a topological space X onto a topological space Y is said to be

(iv) *monotone* if for any $y \in Y$, the set $f^{-1}(y)$ is connected (see [83], p. 25); or, which is equivalent, if the inverse image of any subcontinuum of Y is a continuum (see [36], p. 123);

(v) *confluent* if for each subcontinuum Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q (see [9], p. 213);

(vi) *semi-confluent* if for each subcontinuum Q in Y and for each two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$ (see [52], p. 252);

(vii) *weakly confluent* if for each subcontinuum Q in Y there exists a component C of $f^{-1}(Q)$ such that $f(C) = Q$ (see [50]);

(viii) *joining* if for each subcontinuum Q of Y and for each two components C_1 and C_2 of the set $f^{-1}(Q)$ we have $f(C_1) \cap f(C_2) \neq \emptyset$ (see [56]);

(ix) *atriodic* if for each subcontinuum Q of Y there are two components C_1 and C_2 of the set $f^{-1}(Q)$ such that $f(C_1) \cup f(C_2) = Q$ and for each component C of the set $f^{-1}(Q)$ we have either $f(C) = Q$ or $f(C) \subset f(C_1)$ or $f(C) \subset f(C_2)$ (see [57]).

As an easy consequences of the definitions we infer that

(3.1) *Any homeomorphism is an atomic open mapping.*

(3.2) *Any monotone mapping is confluent.*

(3.3) *Any confluent mapping is semi-confluent.*

(3.4) *Any semi-confluent mappings is joining.*

(3.5) *Any weakly confluent mapping is atriodic.*

It is known (see [9], VI, p. 214; cf. [83], (7.5), p. 148) that

(3.6) *Any open mapping (of a compact space) is confluent.*

Moreover, (see [20], Theorem 1, p. 49)

(3.7) *Any atomic mapping of a continuum is monotone.*

The next theorem is proved in [52], Corollary 3.2, p. 254. We give a new proof of this theorem.

(3.8) THEOREM. *Any semi-confluent mapping is weakly confluent.*

Proof. Let a semi-confluent mapping f map a continuum X onto Y and let Q be an arbitrary subcontinuum of Y. Put $\mathscr{H} = \{H\colon H$ is a subcontinuum of Q and there is a component C of $f^{-1}(Q)$ such that $H \subset f(C)\}$. The family $\mathscr{H}$ is nonempty, because if $y \in Q$, then $\{y\} \in \mathscr{H}$. If $\{H_i\}$ is a sequence of elements of $\mathscr{H}$ such that $H_i \subset H_{i+1}$ for $i = 1, 2, \ldots$, then $\overline{\bigcup_{i=1}^{\infty} H_i} = H_0 \in \mathscr{H}$. In fact, let C_i be components of $f^{-1}(Q)$ such that $H_i \subset f(C_i)$. Since X is compact, the set $f^{-1}(Q)$ is compact and we can choose a convergent subsequence $\{C_{i_n}\}$ of the sequence $\{C_i\}$ (compare [36], § 42, I, Theorem 1, p. 45 and § 42, II). Define $K = \operatorname{Lim}_{n\to\infty} C_{i_n}$. Since $f(K) \subset Q$ and since K is a continuum (compare [36], § 47, II, Theorem 4, p. 170), there is a component C_0 of $f^{-1}(Q)$ such that $K \subset C_0$. By the continuity of f we have $H_0 \subset f(K)$, and thus $H_0 \subset f(C_0)$. This means that $H_0 \in \mathscr{H}$.

Therefore, it follows from Zorn's well-known lemma that there is a maximal element in $\mathscr{H}$. Denote it by R. By the definition of $\mathscr{H}$, we conclude that there is a component C of $f^{-1}(Q)$ such that $R \subset f(C)$. Suppose that there is a point $q \in Q \setminus f(C)$. Take a component C' of $f^{-1}(Q)$ such that $q \in f(C')$. Since f is semi-confluent, we infer that $f(C) \subset f(C') \setminus \{q\}$, contrary to the maximality of R. Thus $f(C) = Q$.

All the inclusions proved and recalled here are essential. We now describe examples which show this.

(3.9) EXAMPLE. Define $f(t) = |t|$ for $t \in [-1, 1]$. The mapping $f\colon [-1, 1] \to [0, 1]$ is open, but it is not monotone.

(3.10) EXAMPLE. Define $f(x, y) = (x, 0)$, where (x, y) denotes a point in the Euclidean plane having x and y as its rectangular coordinates.

Put $A = \{(x, \sin 2\pi/x)\colon\ 0 < x \leqslant 1\}$ and $I_1 = \{(0, y)\colon\ -1 \leqslant y \leqslant 1\}$ and $I_2 = \{(0, y)\colon\ -1 \leqslant y \leqslant 2\}$. The partial mapping $f|A \cup L_1$ is atomic, but it is not open. The partial mapping $f|A \cup I_2$ is monotone, but it is not atomic.

(3.11) EXAMPLE. Define

$$f(x, y) = \begin{cases} (x, y) & \text{if} \quad y \geqslant -1, \\ (x, 2|y+1|-1) & \text{if} \quad y \leqslant -1. \end{cases}$$

The mapping $g = f|A \cup I_2$ is confluent, but it is neither open nor monotone, where A and I_2 are such as in Example (3.10).

(3.12) EXAMPLE. Define $f(t) = |t|$ for $t \in [-1, 2]$. The mapping $f\colon\ [-1, 2] \to [0, 2]$ is semi-confluent, but it is not confluent.

(3.13) EXAMPLE. Define

$$f(t) = \begin{cases} -\big||t+1|-1\big| & \text{for} \quad t \in [-2, 0], \\ \big||-t+1|-1\big| & \text{for} \quad t \in [0, 2]. \end{cases}$$

The mapping $f\colon\ [-2, 2] \to [-1, 1]$ is weakly confluent, but it is not semi-confluent.

(3.14) EXAMPLE. Define

$$f(t) = \begin{cases} (\cos 2\pi t, \sin 2\pi t) & \text{if} \quad t \in [0, 1], \\ (t, 0) & \text{if} \quad t \in [1, 2], \\ (4-t, 0) & \text{if} \quad t \in [2, 4]. \end{cases}$$

The mapping f is joining, but it is not atriodic.

(3.15) EXAMPLE. Define $f(t) = (\cos 2\pi t, \sin 2\pi t)$ for $t \in [0, 3/2]$. The mapping f is atriodic, but it is neither weakly confluent nor joining.

The following equivalence characterizes open mappings (see [36], pp. 67–68; cf. also [50], (1.2), p. 101).

(3.16) *A mapping $f\colon X \to Y$ is open if and only if $\lim\limits_{n\to\infty} y_n = y$ implies $\operatorname*{Ls}\limits_{n\to\infty} f^{-1}(y_n) = f^{-1}(y)$.*

4. Generated classes of mappings

In this section we describe methods which are used to generate new classes of mappings from classes recalled in § 3.

A. The composition of classes of mappings. Let A and B be two classes of mappings each of which contains the class of homeomorphisms. We define

$$AB = \{gf\colon\ g \in A,\ f \in B\}.$$

We have the following dependences:

(4.1) $A \cup B \subset AB$.

(4.2) *If* $A \subset D$ *and* $B \subset E$, *then* $AB \subset DE$.

(4.3) *If* $A(BA) \subset BA$, *then* $AB \subset BA$.

(4.4) *If* $AA = A$, $BB = B$ *and* $AB \subset BA$, *then* $(BA)(BA) = BA$.

(4.5) *The following conditions are equivalent*:

(i) $(BA)(BA) = BA$,

(ii) $A(BA) = BA = (BA)B$,

(iii) $(AB)(AB) \subset BA$,

(iv) $(AB)(AB) = BA$.

This means that if classes A and B are closed with respect to composition and $AB \subset BA$ and $AB \neq BA$, then the class AB is not closed with respect to composition and the class BA is closed with respect to composition.

If we take the class M of monotone mappings and the class O of open mappings, we obtain two new classes of mappings MO and OM in the same way as above.

The class OM coincides (see [50], (3.1)) with the class of quasi-open mappings, which were introduced in [82], p. 9, and which are considered also in [85]. It follows from (3.16) that (see [50], (2.2)).

(4.6) *The mapping* $f\colon X \to Y$ *is an* OM-*mapping if and only if the condition* $\lim\limits_{n\to\infty} y_n = y$ *implies that the set* $\operatorname{Ls}\limits_{n\to\infty} f^{-1}(y_n)$ *intersects each component of* $f^{-1}(y)$.

This characterization of OM-mappings easily implies (see [50], Theorem 2.8) that

(4.7) *The composition of* OM-*mappings is an* OM-*mapping.*

Therefore, by (4.1) and (4.5) we infer that

(4.8) *Any* MO-*mapping is an* OM-*mapping.*

Moreover, (cf. [9], III, p. 214) by (4.2)

(4.9) *Any* OM-*mapping is confluent.*

The class of MO-mappings containing open mappings and monotone mappings (cf. (4.1)) is essentially larger than the class of open mappings and than the class of monotone mappings. Namely, we have the following

(4.10) EXAMPLE. Define

$$f(t) = \begin{cases} |t|-1 & \text{if} \quad 1 \leqslant |t| \leqslant 2, \\ 0 & \text{if} \quad |t| \leqslant 1. \end{cases}$$

The mapping f is an MO-mapping, but it is neither open nor monotone.

The class of OM-mappings is essentially larger than the class of MO-mappings. Namely, we have the following

(4.11) EXAMPLE. We define a mapping $f\colon [0,1] \to [0,1]$ as follows:

$$f(t) = \begin{cases} 3t & \text{if} \quad 0 \leqslant t \leqslant 1/3, \\ 2-3t & \text{if} \quad 1/3 \leqslant t \leqslant 2/3, \\ 0 & \text{if} \quad 2/3 \leqslant t \leqslant 1. \end{cases}$$

The mapping f is an OM-mapping, but it is not an MO-mapping (see [50], Example 3.4).

The mapping g described in Example (3.11) is confluent, but it is not an OM-mapping (to see this cf. (2.6)). Therefore also the class of confluent mappings is essentially larger than the class of OM-mappings.

B. The hereditary classes of mappings. Let A be an arbitrary class of mappings which contains the class of homeomorphisms. We shall call a mapping $f\colon X \to Y$ *hereditarily* A if for any continuum $K \subset X$ the partial mapping $f|K$ is in A. Taking the class of monotone, confluent, weakly confluent or atriodic mappings for A, we get in this way the classes of hereditarily monotone, hereditarily confluent, hereditarily weakly confluent and hereditarily atriodic mappings, respectively (see [57]). Since only the above hereditary classes of mappings were considered in the literature, we will not consider other hereditary classes of mappings. Note that the class of hereditarily weakly confluent mappings is investigated in [79], where such mappings are called pseudo-monotone. As an easy consequence of the definitions we infer that

(4.12) *Any hereditarily monotone (hereditarily confluent, hereditarily weakly confluent, hereditarily atriodic) mapping defined on a continuum is monotone (confluent, weakly confluent, atriodic, respectively).*

(4.13) *Any hereditarily monotone mapping is hereditarily confluent, any hereditarily confluent mapping is hereditarily weakly confluent, and any hereditarily weakly confluent mapping is hereditarily atriodic.*

Moreover, it follows from (3.7) that

(4.14) *Any atomic mapping of a continuum is hereditarily monotone.*

The class of monotone mappings is not contained even in the class of hereditarily atriodic mapping. We have the following

(4.15) EXAMPLE. *Put* $X = \{(x, y)\colon x = 0, 1 \text{ or } 2 \text{ and } 0 \leqslant y \leqslant 1\} \cup \{(x, y)\colon 0 \leqslant x \leqslant 2 \text{ dan } y = 0 \text{ or } 1\}$ and let a mapping $f\colon X \to f(X)$ identify points which have the second coordinate equal to 1. It is easy to ascertain that the mapping f is monotone and that it is not hereditarily atriodic.

Similarly, the class of open mappings is not contained in the class of hereditarily atriodic mappings. We have the following

(4.16) EXAMPLE. Put $X = \{(x, y)\colon x = 0, 1, 2, \text{ or } 3 \text{ and } -1 \leqslant y \leqslant 1\} \cup$ $\cup \{(x, y)\colon 0 \leqslant x \leqslant 3 \text{ and } y = -1 \text{ or } 1\}$ and let $f(x, y) = (x, |y|)$. The mapping $f|X$ is open, but it is not hereditarily atriodic.

The following examples show that the inverse implications in (4.13) are not true.

(4.17) EXAMPLE. Let X denote the pseudo-arc (cf. § 2) and let $p \in X$. Put $M = (X \times \{0\}) \cup (X \times \{1\}) \cup (\{p\} \times [0, 1])$ and $f(x, t) = x$ for $(x, t) \in M \times I$. It is clear that the mapping $f|M$ is not monotone (thus it is not hereditarily monotone; cf. (4.12)), but it follows from Corollary (3.4) in [57] (cf. [12], p. 243) that $f|M$ is hereditarily confluent.

(4.18) EXAMPLE. The mapping f described in Example (3.9) is hereditarily weakly confluent (cf. [57]. Corollary (3.13) and [71], Theorem 4) and it is not hereditarily confluent.

(4.19) EXAMPLE. Let S denote the unit circle, i.e., $S = \{(x, y)\colon x^2 + y^2 = 1\}$ and let the mapping $f\colon S \to f(S)$ identify all points of some semicircle which is contained in S. It is easy to observe that f is hereditarily atriodic and that it is not hereditarily weakly confluent.

C. The local classes of mappings. Let A be an arbitrary class of mappings and let f map X onto Y. We define: $f \in \mathrm{Loc}(A)$ provided for each point $x \in X$ there is a closed neighbourhood V of the point x such that $f(V)$ is a closed neighbourhood of $f(x)$ and the partial mapping $f|V$ belongs to A.

We have the following easy consequences of the definition:

(4.20) $A \subset \mathrm{Loc}(A)$,

(4.21) *If* $A \subset B$, *then* $\mathrm{Loc}(A) \subset \mathrm{Loc}(B)$,

(4.22) $\mathrm{Loc}(A) = \mathrm{Loc}(\mathrm{Loc}(A))$.

Recall that a mapping f from a topological space X onto a topological space Y is said to be

(i) a *local homeomorphism* if for each point $x \in X$ there exists an open neighbourhood U of x such that $f(U)$ is an open neighbourhood of $f(x)$ and that f restricted to U is a homeomorphism between U and $f(U)$ (see [83], p. 199);

(ii) *locally confluent* (*locally weakly confluent*) provided for each point y of Y there exists a closed neighbourhood F of y in Y such that the partial mapping $f|f^{-1}(F)$ is a confluent mapping (a weakly confluent mapping) of $f^{-1}(F)$ onto F (see [21], p. 239 and [54], p. 60).

We have the following

(4.23) THEOREM. *A mapping* $f\colon X \to Y$ *is a local homeomorphism if and only if* $f \in \mathrm{Loc}(H)$, *where* H *denotes the class of homeomorphisms.*

Proof. Suppose that f is a local homeomorphism and that x is an arbitrary point of X. Let U be an open neighbourhood of x such that $f(U)$ is an open neighbourhood of $f(x)$ and $f|U$ is a homeomorphism and let V be an arbitrary closed neighbourhood of x which is contained in U. Obviously f restricted to V is a homeomorphism. Since the interior of V is an open subset of U and $f|U$ is a homeomorphism, we infer that the image under f of the interior of V is an open subset of $f(U)$. Therefore the image under f of the interior of V is an open subset of Y, because $f(U)$ is an open subset of Y. Thus $f(V)$ is a neighbourhood of $f(x)$. It is closed as an image of a compact set. This means that $f \in \mathrm{Loc}(H)$.

Conversely, suppose that $f \in \mathrm{Loc}(H)$. Let x be an arbitrary point of X and let V be a closed neighbourhood of x such that $f(V)$ is a closed neighbourhood of $f(x)$ and $f|V$ is a homeomorphism. Let U' be an open set in X such that $x \in U' \subset V$. Then $f(U')$ is an open subset of $f(V)$ which contains $f(x)$. Since $f(V)$ is a neighbourhood of $f(x)$, we conclude that there is an open subset G of Y such that $f(x) \in G \subset f(U')$. Put $U = U' \cap f^{-1}(G)$. Sets G and U' are open, and thus U is open. Since $f(U) = f(U' \cap f^{-1}(G)) = f(U') \cap G = G$, we infer that $f(U)$ is open. Moreover, $f|U$

is a homeomorphism, because $(f|V)^{-1}|G = (f|U)^{-1}$ and $(f|V)^{-1}$ is continuous. Consequently we find that f is a local homeomorphism.

Similarly, we have the following

(4.24) THEOREM. *A mapping $f\colon X \to Y$ is locally confluent (locally weakly confluent) if and only if $f \in \mathrm{Loc}\,(K)$, where K denotes the class of confluent mappings (weakly confluent mappings, respectively).*

Proof. Suppose that f is locally confluent (locally weakly confluent) and suppose that x is an arbitrary point of X. There is a closed neighbourhood F of $f(x)$ such that $f|f^{-1}(F)$ is confluent (weakly confluent). But then $V = f^{-1}(F)$ is a closed neighbourhood of x and $f|V$ is confluent (weakly confluent). This means that $f \in \mathrm{Loc}\,(K)$, where K denotes the class of confluent mappings (of weakly confluent mappings, respectively).

Conversely, suppose that $f \in \mathrm{Loc}\,(K)$. Let y be an arbitrary point of Y and let V_x denote a closed neighbourhood of x of $f^{-1}(y)$ such that $f(V_x)$ is a closed neighbourhood of y and $f|V_x$ is confluent (weakly confluent). Since $f^{-1}(y)$ is compact, there is a finite collection $V_{x_1}, V_{x_2}, \ldots, V_{x_n}$ such that the set $f^{-1}(y)$ is contained in the union of the interiors of V_{x_i}. Put $F' = \bigcap_{i=1}^{n} f(V_{x_i})$. The set F' is a closed neighbourhood of y. Let F be a closed neighbourhood of y such that F is contained in F' and $f^{-1}(F)$ is contained in the union of the interiors of V_{x_i} (such F exists, becuase $f^{-1}(y)$ is contained in the union of the interiors of V_{x_i}). We will prove that $f|f^{-1}(F)$ is confluent (weakly confluent). Let Q be an arbitrary subcontinuum of F. Suppose now that K denotes the class of confluent mappings. Let C be an arbitrary component of $f^{-1}(Q)$. There is a point z such that $z \in C \cap V_{x_i}$ for some i. Since $f|V_{x_i}$ is confluent, we infer that the component C' of $(f|V_{x_i})^{-1}(Q)$ containing the point z is such that $f(C') = (f|V_{x_i})(C') = Q$. But $z \in C \cap C'$ and C is a component of $f^{-1}(Q)$, and thus $C' \subset C$. Therefore we obtain $f(C) = Q$. This means that f is locally confluent. Suppose now that K denotes a class of weakly confluent mappings. Since $f|V_{x_i}$ is weakly confluent for some arbitrary i and since $Q \subset f(V_{x_i})$, we infer that there is a component C' of $(f|V_{x_i})^{-1}(Q)$ such that $f(C') = (f|V_{x_i})(C') = Q$. Then the component C of $f^{-1}(Q)$ such that $C' \subset C$ satisfies the equality $f(C) = Q$. This means that f is locally weakly confluent. The proof of Theorem (4.24) is complete.

Since the investigated classes of local mappings in the literature coincide with the corresponding classes Loc by the above theorems, we call the mappings belonging to Loc (A) *locally A*.

We will now study classes of locally monotone mappings and locally MO-mappings. Firstly, we have

(4.25) $\mathrm{O} = \mathrm{Loc}\,(\mathrm{O})$.

In fact, the inclusion $\mathrm{O} \subset \mathrm{Loc}\,(\mathrm{O})$ holds by (4.20). Let $f: X \to Y$ and let $f \in \mathrm{Loc}\,(\mathrm{O})$. Then for each $x \in X$ there is a closed neighbourhood V of x such that $f|V$ is open by the definition of Loc (O). Therefore f maps elements of some open basis of X onto open sets in Y. Hence the image under f of an arbitrary open set in X is an open set in Y. This means that f is open, i.e., the inclusion $\mathrm{Loc}\,(\mathrm{O}) \subset \mathrm{O}$ holds.

It follows from (4.21) and (4.25) that (cf. [83], p. 199)

(4.26) *Every local homeomorphism is an open mapping.*

The inverse is not true, because for example the mapping f described in Example (3.9) is open, but it is not a local homeomorphism.

The following theorem characterizes local homeomorphisms (see [53], Theorem 4, p. 856 and [83], (6.21), p. 200):

(4.27) *Let a mapping f map a space X onto a continuum Y. The mapping f is a local homeomorphism if and only if f is open and there is a positive integer n such that* $\mathrm{card}\, f^{-1}(y) = n$ *for each $y \in Y$.*

(4.28) Example. Let (r, φ) denote a point of the Euclidean plane having r and φ as its polar coordinates. Take the unit circle $S = \{(r, \varphi)\colon r = 1, 0 \leqslant \varphi \leqslant 2\pi\}$. We define a mapping f as follows:

$$f(r, \varphi) = \begin{cases} (r, 2\varphi) & \text{if} \quad 0 \leqslant \varphi \leqslant \pi, \\ (r, -2\varphi) & \text{if} \quad \pi \leqslant \varphi \leqslant 2\pi. \end{cases}$$

The mapping f is continuous on $S = f(S)$ and for each $y \in S$ we have $\mathrm{card}\, f^{-1}(y) = 2$. The mapping f is not open. Therefore the assumptions in (4.27) are essential. However, if we take a mapping h of S onto itself defined by the formula $h(r, \varphi) = (r, 2\varphi)$, then h is a local homeomorphism, but it is not a homeomorphism. Thus the class of local homeomorphisms is essentially larger than the class of homeomorphisms.

Just as for open mappings, the class of locally OM-mappings coincides with the class of OM-mappings; namely we have

(4.29) $\mathrm{OM} = \mathrm{Loc}\,(\mathrm{OM})$.

Proof. The inclusion $\mathrm{OM} \subset \mathrm{Loc}\,(\mathrm{OM})$ holds by (4.20). We will prove that $\mathrm{Loc}\,(\mathrm{OM}) \subset \mathrm{OM}$. Let a mapping f map X onto Y and let $f \in \mathrm{Loc}\,(\mathrm{OM})$. It follows from (4.6) that it suffices to show that for each sequence $\{y_n\}$ of points of Y with $\lim\limits_{n\to\infty} y_n = y_0$ the set $\mathop{\mathrm{Ls}}\limits_{n\to\infty} f^{-1}(y)$ intersects any component of the set $f^{-1}(y_0)$. Let C be an arbitrary component of the set $f^{-1}(y_0)$. Since $f \in \mathrm{Loc}\,(\mathrm{OM})$, we infer that there is a closed neighbourhood V of an arbitrary point c of C such that $f(V)$ is a closed neighbourhood of y_0 and the partial mapping $f|V$ is an OM-mapping. We conclude from (4.6) that $\operatorname*{Ls}_{n\to\infty}(f|V)^{-1}(y_n)\cap C\neq\emptyset$. But $\operatorname*{Ls}_{n\to\infty}(f|V)^{-1}(y_n)\subset \operatorname*{Ls}_{n\to\infty}f^{-1}(y_n)$; thus $\operatorname*{Ls}_{n\to\infty}f^{-1}(y_n)\cap C\neq\emptyset$.

Equality (4.29) implies the following inclusions:

(4.30) $\quad \mathrm{MO}\subset \mathrm{Loc}\,(\mathrm{MO})\subset \mathrm{OM}.$

Both the above inclusions are essential. The mapping f described in Example (4.11) is a locally MO-mapping, but it is not an MO-mapping. Similarly, there are OM-mappings which are not locally MO-mappings. This can be seen by the following

(4.31) EXAMPLE. Let (x, y) be a point of the Euclidean plane having x and y as its rectangular coordinates. Put (Fig. 1)

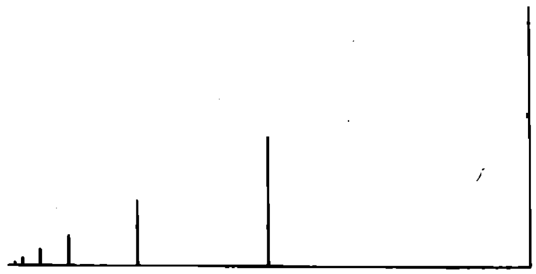


Fig. 1

$$I = \{x, y): 0\leqslant x\leqslant 1,\ y = 0\},$$
$$A_n = \{(x, y):\ x = 1/2^n,\ 0\leqslant y\leqslant 1/2^{n-1}\}\quad \text{for}\quad n = 0, 1, 2, \ldots,$$
$$T = I\cup\bigcup_{n=0}^{\infty} A_n$$

and define mappings $f: T\to f(T)$ and $g: f(T)\to I$ as follows:

$$f(x, y) = \begin{cases}(x, y) & \text{if}\quad y\leqslant x,\\ (x, x) & \text{if}\quad x\leqslant y,\end{cases}$$
$$g(x, y) = (x-y, 0)\quad \text{for}\quad (x, y)\in f(T).$$

The mapping f is monotone and the mapping g is open, and thus their composition $h = gf$ is an OM-mapping. We will show that the mapping h is not a locally MO-mapping.

Suppose, on the contrary, that the mapping h is a locally MO-mapping. Then there is a closed neighbourhood V of the point $p = (0, 0)$ such that $h(V)$ is a closed neighbourhood of $h(p)$ and the partial mapping $h|V$ is an MO-mapping. Therefore there are a monotone mapping f_1 and an open mapping g_1 such that $h|V = f_1g_1$. Since open mappings do not increase the orders of points (see [83], (7.3), p. 147, compare [36], § 51, I, p. 277), we infer that for $A_n\subset V$ either $g_1(A_n)$ is an arc such that $g_1(A_n)\cap g_1(I\cap V) = g_1(1/2^n, 0)$ or $g_1(A_n)\subset g_1(I\cap V)$ and $g_1(1/2^n, 1/2^{n-1}) =$

$= g_1(p)$. In the first case, since $f_1 g_1(1/2^n, 0) = f_1 g_1(p)$, we obtain a contradiction of the monotoneity of f_1, because in this case the set $f_1^{-1}(p)$ has at least two components, one contained in $g_1(A_n)$ and one contained in $g_1(I \cap V)$. In the second case, since $f_1 g_1(1/2^n, 1/2^n) = g_1(p)$ and $g_1(1/2^n, 1/2^n) \neq g_1(p)$ we infer that there is a point $q \neq p$ in the arc I such that $h(q) = h(p)$. This contradicts the fact that the mapping $h|I$ is a homeomorphism.

Locally monotone mappings have the following property:

(4.32) THEOREM. *Any locally monotone mapping $f: X \to Y$ is a locally MO-mapping and there is a positive integer n such that the set $f^{-1}(y)$ has at most n components for each $y \in Y$.*

Proof. A locally monotone mapping $f: X \to Y$ is a locally MO-mapping by (4.1) and (4.21). Since $f \in \mathrm{Loc}(M)$, we infer that for each point $y \in Y$ there are sets $V_1, V_2, \ldots, V_{n_y}$ such that the set $f^{-1}(y)$ is contained in the union of the interiors of V_i, $f(V_i)$ are closed neighbourhoods of y and $f|V_i$ are monotone for $i = 1, 2, \ldots, n_y$. Therefore there is a closed neighbourhood V_y of y such that the set $f^{-1}(V_y)$ is contained in the union of the interiors of V_i. Thus for each $y' \in V_y$ the set $f^{-1}(y')$ has at most n_y components. Since Y is compact, we conclude that there is a positive integer n such that for each $y \in Y$ the set $f^{-1}(y)$ has at most n components.

The class of locally MO-mappings is essentially larger than the class of locally monotone mappings. This can be seen from Example (3.9), where the described mapping f is open, and thus locally MO, but it is not locally monotone. Similarly, there are locally monotone mappings which are not monotone and which are not local homeomorphisms. For example, f described in Example (4.10) is such a mapping.

We will now study the dependences between the classes of locally confluent, of locally semi-confluent and of locally weakly confluent mappings. It is clear by (4.20) that

(4.33) *Any confluent mapping is locally confluent, any semi-confluent mapping is locally semi-confluent and any weakly confluent mapping is locally weakly confluent.*

All the inclusions mentioned in (4.33) are essential. We describe below an example which shows this.

(4.34) EXAMPLE. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates. Put

$$X_1 = \{(x, 2+\sin 2\pi/x):\ 0 < x \leqslant 1\} \cup \{(0, y):\ 0 \leqslant y \leqslant 12\} \cup$$
$$\cup \{1, y):\ 0 \leqslant y \leqslant 2\} \cup \{(x, 6+\sin 2\pi/x):\ -1 \leqslant x \leqslant 0\},$$
$$X_2 = \{(x, y):\ (x, -y) \in X_1\}$$

and let $X = X_1 \cup X_2$. We define a mapping $f\colon X \to f(X)$ as follows:

$$f(x, y) = (x, |8 - |y + 4||) \quad \text{for} \quad (x, y) \in X.$$

It is easy to ascertain that the mapping f is locally confluent and that it is neither atriodic nor joining.

It follows from (3.3), (3.8) and (4.21) that

(4.35) *Any locally confluent mapping is locally semi-confluent and any locally semi-confluent mapping is locally weakly confluent.*

The mapping f described in Example (3.12) is locally semi-confluent but it is not locally confluent. Similarly, there are locally weakly confluent mappings which are not locally semi-confluent. This can be seen from the following

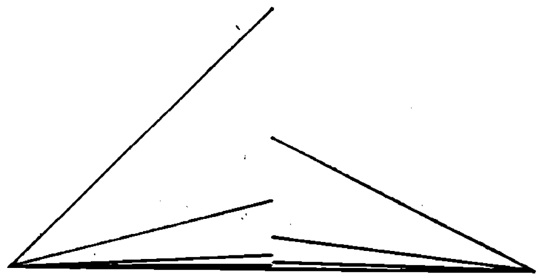


Fig. 2

(4.36) EXAMPLE. Let a continuum X consist of a straight segment joining the point $(-1, 0)$ with the point $(1, 0)$, of straight segments joining the point $(-1, 0)$ with points $(0, 1/2^{2n})$ and of straight segments joining the point $(1, 0)$ with points $(0, 1/2^{2n+1})$ for $n = 0, 1, 2, \ldots$ (Fig. 2). The mapping $f\colon X \to [-1, 1]$ defined by $f(x, y) = (x, 0)$ for $(x, y) \in X$ is locally weakly confluent (even weakly confluent), but it is not locally semi-confluent.

The following theorem characterizes locally weakly confluent mappings (see [54], Theorem (2.11), p. 62):

(4.37) *A mapping f of X onto Y is locally weakly confluent if and only if there is a positive number ε such that for each continuum Q of diameter less than ε in Y there exists a component C of $f^{-1}(Q)$ such that $f(C) = Q$.*

A similar theorem holds also for locally confluent mappings; namely

(4.38) THEOREM. *A mapping f of X onto Y is locally confluent if and only if there is a number $\varepsilon > 0$ such that for each continuum Q of diameter less than ε in Y each component of $f^{-1}(Q)$ is mapped by f onto Q.*

Proof. Suppose that f is locally confluent. Then there are sets $F_1, F_2, \ldots, F_n$ such that $Y = \operatorname{Int} F_1 \cup \operatorname{Int} F_2 \cup \ldots \cup \operatorname{Int} F_n$ and $f|f^{-1}(F_i)$

is a confluent mapping for $i = 1, 2, \ldots, n$. The family $\{\mathrm{Int}\, F_1, \mathrm{Int}\, F_2, \ldots$ $\ldots, \mathrm{Int}\, F_n\}$ is an open covering of Y. Therefore, by Corollary 4d of [36], § 41, VI, p. 24, there is a number $\varepsilon > 0$ such that each continuum Q of diameter less than ε in Y is contained in some F_i. Since $f|f^{-1}(F_i)$ is confluent, each component C of $f^{-1}(Q) = \big(f|f^{-1}(F_i)\big)^{-1}(Q)$ is such that $f(C) = \big(f|f^{-1}(F_i)\big)(C) = Q$.

Conversely, suppose that there is a number $\varepsilon > 0$ such that for each continuum Q of diameter less than ε in Y each component C of $f^{-1}(Q)$ is such that $f(C) = Q$. Let y be an arbitrary point of Y and let V be the closed ball with diameter equal to $\varepsilon/2$ and with centre at y. Obviously $f|f^{-1}(V)$ is a confluent mapping, because each continuum Q contained in V has diameter less than ε. Therefore f is a locally confluent mapping.

An analogous condition for locally semi-confluent mappings is only sufficient, but it need not be necessary. This can be seen from the following

(4.39) Example. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates. We put

$$T = \{(0, y)\colon\ 0 \leqslant y \leqslant 1\} \cup \{(x, 0)\colon\ -1 \leqslant x \leqslant 1\}$$

and we define a mapping $f\colon T \to [0, 1]$ as follows:

$$f(x, y) = \begin{cases} (x, y) & \text{if} \quad 0 < x \leqslant 1, \\ (y/2, 0) & \text{if} \quad x = 0, \\ (|3x/2|, y) & \text{if} \quad -2/3 \leqslant x < 0, \\ (3/2 \cdot x + 2, y) & \text{if} \quad -1 \leqslant x \leqslant -2/3. \end{cases}$$

Observe that the mapping f is locally semi-confluent, but there are "small" continua contained in $[0, 1]$ for which the condition of semi-confluence is not satisfied.

We do not consider the classes of locally atomic mappings, of locally atriodic mappings and of locally joining mappings, because we do not have any interesting theorem of these classes except general dependences.

D. Other methods of the generation of classes of mappings. If we have some class of mappings defined by subcontinua of the image (cf. § 3), then we obtain new classes by taking the class of subcontinua of the image with nonempty interiors or the class of irreducible subcontinua of the image instead of the class of all subcontinua of the image.

In the literature three such classes of mappings were considered. Namely, a mapping of a space X onto a space Y is said to be

(i) *quasi-monotone* if for any subcontinuum Q in Y with a nonempty interior the set $f^{-1}(Q)$ has a finite number of components and f maps each of them onto Q (see [80], p. 136);

(ii) *weakly monotone* if for any continuum Q in Y with a nonempty interior each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q

(see [80], p. 136, where these mappings are called quasi-monotone and the spaces considered are locally connected continua, see also [75], p. 418);

(iii) *pseudo-confluent* if for each irreducible continuum Q in Y there is a component C of $f^{-1}(Q)$ such that $Q = f(C)$ (see [51], 5.3).

As an easy consequences of the definitions we infer that

(4.40) *Any quasi-monotone mapping is weakly monotone.*

(4.41) *Any confluent mapping is weakly monotone.*

(4.42) *Any weakly confluent mapping is pseudo-monotone.*

Moreover, it follows from Theorem (4.32) that

(4.43) *Any locally monotone mapping is quasi-monotone.*

Hereditarily confluent mappings have the following property:

(4.44) THEOREM. *If a hereditarily confluent mapping f maps a continuum X onto Y, then the inverse image of any subcontinuum of Y with a nonempty interior is connected. Moreover, if Q is a subcontinuum of Y such that $Q = A \cap B$ and sets $f^{-1}(A)$ and $f^{-1}(B)$ are connected, then the set $f^{-1}(Q)$ is connected.*

Proof. We claim that

(4.44.1) *if A and B are proper subcontinua of Y such that $Y = A \cup B$, then sets $f^{-1}(A)$ and $f^{-1}(B)$ are connected.*

In fact, let A' be an arbitrary component of $f^{-1}(A)$. Since f is confluent, we have $f(A') = A$. Thus there is a component B' of $f^{-1}(B)$ such that $A' \cap B' \neq \emptyset$. Observe that

(4.44.2) $f^{-1}(A) \cap B' = A' \cap B' = f^{-1}(B) \cap A'$.

Indeed, if $(f^{-1}(A) \cap B') \setminus A' \neq \emptyset$, then there is a component A'' of the set $(f|A' \cup B')^{-1}(A)$ which is contained in B'. Since $A \setminus B \neq \emptyset$, $f(A' \cup B') = Y$ and $f(B') = B$, we obtain a contradiction of the confluence of $f|A' \cup B'$, because $f(A'') = (f|A' \cup B')(A'') \neq A$. Similarly one can prove the second equality. Thus (4.44.2) holds.

From (4.44.2) we conclude that

(4.44.3) *Any component of $f^{-1}(A)$ intersects exactly one component of $f^{-1}(B)$ and any component of $f^{-1}(B)$ intersects exactly one component of $f^{-1}(A)$.*

Now, suppose on the contrary that the set $f^{-1}(A)$ is not connected (if $f^{-1}(B)$ is not connected the proof is quite similar). Thus $f^{-1}(A) = P \cup R$, where P and R are closed, nonempty and disjoint. Put

$$P' = \bigcup \{K\colon K \text{ is a component of } f^{-1}(B) \text{ and } K \cap P \neq \emptyset\}$$

and

$$R' = \bigcup\{K\colon K \text{ is a component of } f^{-1}(B) \text{ and } K \cap R \neq \emptyset\}.$$

Since P' and R are closed, we infer that sets P' and R' are closed. Moreover, it follows from (4.44.3) that sets $P' \cap R'$, $P \cap R'$ and $P' \cap R$ are empty. Therefore, we have a decomposition of X into two closed disjoint sets $P \cup P'$ and $R \cup R'$, contrary to the connectedness of X. This means that (4.44.1) holds.

Let Q be an arbitrary subcontinuum of Y with a nonempty interior. Consider two cases.

Case 1. The set $Y \setminus Q$ is connected. Then $Y = Y/Q \cup Q$, and by (4.44.1) we conclude that the set $f^{-1}(Q)$ is connected.

Case 2. The set $Y \setminus Q$ is not connected. It follows from Theorem 3 in [36], § 47, I, p. 168, that there are proper subcontinua A and B of Y such that $Y = A \cup B$ and $Q = A \cap B$. By (4.44.1) we infer that the sets $f^{-1}(A)$ and $f^{-1}(B)$ are connected. Suppose on the contrary that $f^{-1}(Q) = P \cup R$, where P and R are closed, nonempty and disjoint. Take an open subset U of X such that $P \subset U \subset \overline{U} \subset X \setminus R$. Let K be a component of $f^{-1}(B) \cap \overline{U}$ intersecting P. The component K contains the points which do not belong to $f^{-1}(Q)$ (see [36], § 47, III, p. 172). Thus

(4.44.4) $f(K) \setminus A \neq \emptyset$.

Consider the continuum $K \cup f^{-1}(A)$ and the mapping $f|K \cup f^{-1}(A)$. The set $C = \big(f|K \cup f^{-1}(A)\big)^{-1}\big(f(K)\big)$ is contained in the set $f^{-1}(B) \cap \big(K \cup \cup f^{-1}(A)\big) = K \cup P \cup R$. Since $(K \cup P) \cap R \neq \emptyset$ and $f(P) = f(R) = Q$ by the confluence of f, we infer that the set C has some component W contained in R. But $f(R) \subset A$, and thus, by (4.44.4), $f(W) = \big(f|K \cup f^{-1}(A)\big)(W) \neq f(K)$. Hence $f|K \cup f^{-1}(A)$ is not confluent, contrary to the hereditary confluence of f. One can observe that the proof of case 2 is also the proof of the additional proposition of Theorem (4.44).

Theorem (4.44) implies the following

(4.45) COROLLARY. *Any hereditarily confluent mapping is quasi-monotone.*

The following examples show that other inclusions fail to hold.

(4.46) EXAMPLE. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates. Put

$$X = \{(x, \sin 2\pi/x)\colon 0 < x \leqslant 1\} \cup \{(0, y)\colon -3/2 \leqslant y \leqslant 1\}.$$

The mapping f is defined by $f(x, y) = (x, |y+1|-1)$ for $(x, y) \in X$ and the mapping g maps $f(X)$ onto $gf(X)$ and it identifies points $(0, 1)$, $(0, 0)$ and $(0, -1)$. The composition gf is a quasi-monotone mapping and it is neither pseudo-confluent nor locally weakly confluent. Moreover, it is also neither atriodic nor joining.

(4.47) EXAMPLE. Let a continuum X consist of a straight segment joining the point (1, 0) with the origin and of straight segments joining the same point with points $(0, 1/n)$ for $n = 1, 2, \ldots$ The mapping $f\colon X \to [0,1]$ defined by $f(x, y) = x$ is open, but it is not quasi-monotone.

(4.48) EXAMPLE. The mapping f described in Example (3.13) is hereditarily weakly confluent, but it is not weakly monotone.

(4.49) EXAMPLE. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates. We put

$$T = \{(0, y)\colon\ 0 \leqslant y \leqslant 1\} \cup \{(x, 0)\colon\ -1 \leqslant x \leqslant 1\}$$

and we define a mapping $f\colon [0, 1] \to T$ as follows :

$$f(t) = \begin{cases} (-3t-1/2, 0) & \text{if} \quad 0 \leqslant t \leqslant 1/6, \\ (6t-2, 0) & \text{if} \quad 1/6 \leqslant t \leqslant 2/6, \\ (0, 6t-2) & \text{if} \quad 2/6 \leqslant t \leqslant 3/6, \\ (0, -6t+4) & \text{if} \quad 3/6 \leqslant t \leqslant 4/6, \\ (6t-4, 0) & \text{if} \quad 4/6 \leqslant t \leqslant 5/6, \\ (-12t+11, 0) & \text{if} \quad 5/6 \leqslant t \leqslant 1. \end{cases}$$

The mapping f is pseudo-confluent and it is neither weakly monotone nor locally weakly confluent. This mapping is also neither atriodic nor joining.

(4.50) EXAMPLE. Let (x, y) be such as in Example (4.49). Put

$$X = \{(x, \sin 2\pi/x)\colon\ 0 < x \leqslant 1\} \cup \{(0, y)\colon\ -5 \leqslant y \leqslant 1\} \cup \\ \cup \{(x, -4 + \sin 2\pi/x)\colon\ -1 \leqslant x < 0\}$$

and define

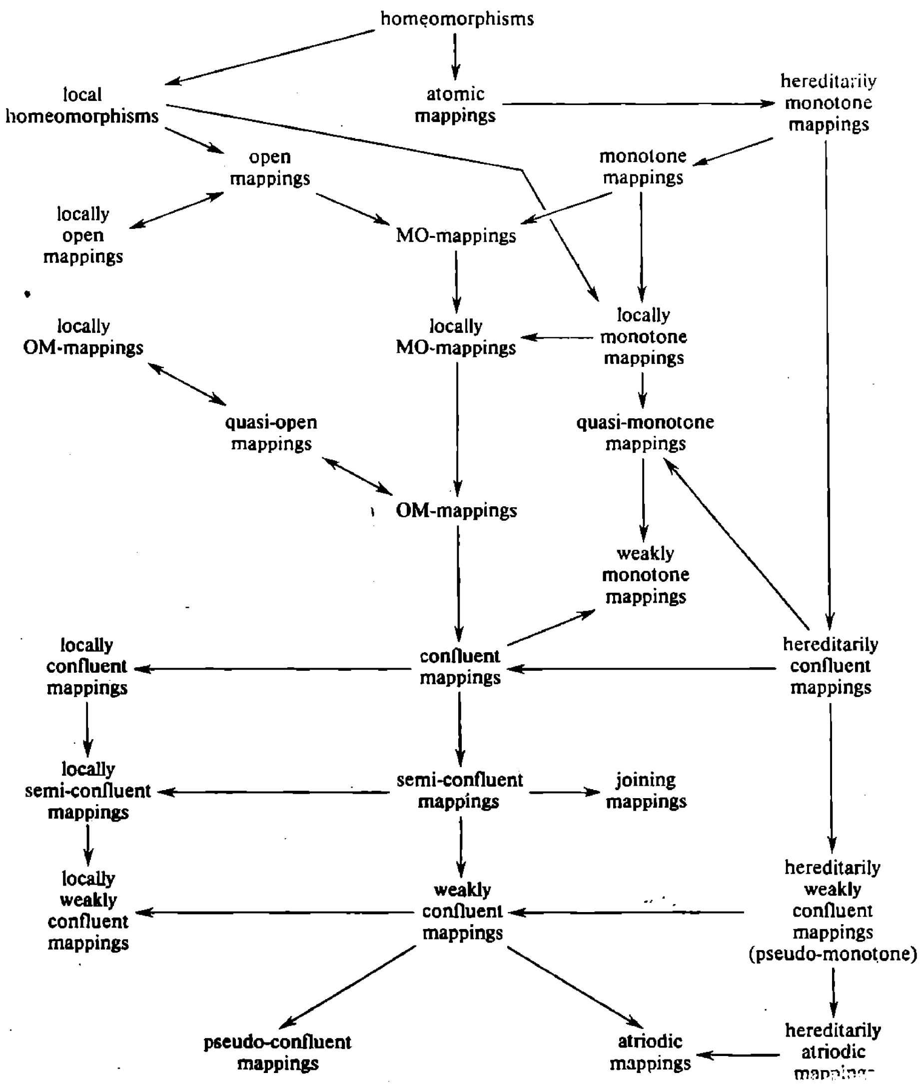
$$f(x, y) = (x, |y+2|) \quad \text{for} \quad (x, y) \in X.$$

The mapping f is locally confluent, joining and atriodic, but it is not pseudo-confluent.

E. Remark. In [22] B. B. Epps considered strongly confluent mappings. They are omitted in this paper, because they differ in type from the mappings investigated here.

Table II comprises all possible inclusion between the classes of mappings on continua which are considered here. These inclusions are essential, which is shown in the examples described in Sections 3 and 4. We use the sign of implication instead of the sign of inclusion.

TABLE II


homeomorphisms
local homeomorphisms
atomic mappings
hereditarily monotone mappings
open mappings
monotone mappings
locally open mappings
MO-mappings
locally OM-mappings
locally MO-mappings
locally monotone mappings
quasi-open mappings
quasi-monotone mappings
OM-mappings
weakly monotone mappings
locally confluent mappings
confluent mappings
hereditarily confluent mappings
locally semi-confluent mappings
semi-confluent mappings
joining mappings
locally weakly confluent mappings
weakly confluent mappings
hereditarily weakly confluent mappings (pseudo-monotone)
pseudo-confluent mappings
atriodic mappings
hereditarily atriodic

5. General properties of mappings

In this section we investigate some general properties of classes of mappings which are defined in the previous two sections.

A. The composition property. Let A be an arbitrary class of mappings. We say that the class A has the *composition property* if for each two mappings $f\colon X \to Y$ and $g\colon Y \to Z$ belonging to A their composition gf belongs to A. As an immediate consequence of the definitions we infer that

(5.1) *The composition property characterizes the following mappings: homeomorphisms, atomic mappings, open mappings, monotone mappings.*

It follows from (4.27) that

(5.2) *The class of local homeomorphims has the composition property.*

Proposition (2.3) in [57] says that

(5.3) *The composition property characterizes the following mappings: hereditarily monotone, hereditarily confluent and hereditarily weakly confluent.*

Moreover (cf. [9], III, p. 214; [50], 4.4 and [51], 1.5)

(5.4) *Classes of confluent mappings, of weakly confluent mappings and of pseudo-confluent mappings have the composition property.*

(5.5) *The class of* OM-*mappings has the composition property, but the class of* MO-*mappings does not have it.*

The above proposition is implied by (4.5) and Example (4.11) (cf. (4.7), see also [50], Theorem 2.8 and Example 3.5).

We also have (cf. [83], Theorem (8.3) (ii), p. 153) the following

(5.6) THEOREM. *The class of quasi-monotone mappings has the composition property.*

Proof. Let mappings $f\colon X \to Y$ and $g\colon Y \to Z$ be quasi-monotone and let Q be an arbitrary subcontinuum of Z with a nonempty interior. Since g is quasi-monotone, we infer that the set $g^{-1}(Q)$ has a finite number of components and g maps each of them onto Q. Therefore each component of $g^{-1}(Q)$ has a nonempty interior in Y, because Q has a nonempty interior. Since f is quasi-monotone, we conclude that for each component C of $g^{-1}(Q)$ the set $f^{-1}(Q)$ has a finite number of components and f maps each of them onto C, respectively. Hence the set $(gf)^{-1}(Q)$ has a finite number of components and gf maps each of them onto Q, because any component of $(fg)^{-1}(Q)$ is a component of $f^{-1}(C)$ for some component C of $g^{-1}(Q)$. This means that gf is quasi-monotone, and thus the class of quasi-monotone mappings has the composition property.

Just, as the class of MO-mappings, the remaining classes of mappings do not have the composition property. This can be seen from the following examples.

(5.7) EXAMPLE. Let (r, φ) denote a point of the Euclidean plane having r and φ as its polar coordinates. Put (Fig. 3)

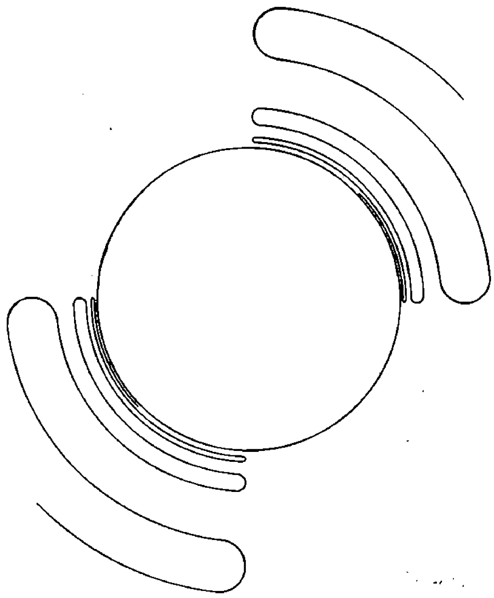


Fig. 3

$$X_1 = \left\{\left(r, \frac{\pi}{2r}\sin\frac{\pi}{r-1}\right) : 1 < r \leqslant 2\right\},$$

$$X_2 = \{(r, \varphi) : (r, \varphi - \pi) \in X_1\},$$

$$S = \{(1, \varphi) : 0 \leqslant \varphi \leqslant 2\pi\}$$

and

$$X = X_1 \cup X_2 \cup S,$$

and define a mapping $f\colon X \to f(X)$ as follows: $f(r, \varphi) = (r, 2\varphi)$. It is clear that f is a local homeomorphism (cf. (4.27)). The monotone mapping $g\colon f(X) \to [0, 1]$ identifies points of $f(S)$. The mapping gf is not locally monotone, because there exists no closed neighbourhood V of $(1, \pi/2)$ in X such that $gf(V)$ is a closed neighbourhood of $gf(1, \pi/2)$ and such that gf/V is monotone.

(5.8) Example. The mappings f and g described in Example (4.31) are both locally MO-mappings, but their composition gf is not.

(5.9) Example. Let a continuum X consist of a straight segment joining the point (1, 0) with the origin, of a straight segment joining the point (1, 0) with (1/2, −1/2) and of straight segments joining the same point with points $(0, 1/n)$ for $n = 1, 2, \ldots$ The mapping $f\colon X \to f(X)$ defined by $f(x, y) = (x, |y|)$ is quasi-monotone. The mapping g of $f(X)$ onto [0, 1] defined by $g(x, y) = x$ is open. The composition gf is not weakly monotone.

(5.10) Example. Let a continuum X consist of a straight segment joining the point (−1, 0) with (1, 0), of a straight segments joining the same point with points $(-1/n, -1/n)$ and of straight segments joining the point (1, 0) with points $(1/n, 1/n)$ for $n = 2, 3, \ldots$ (see Fig. 4). The

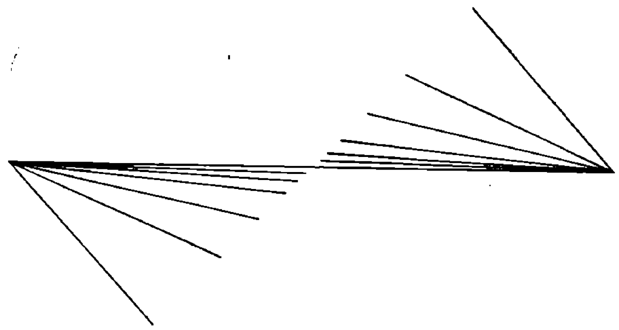


Fig. 4

mappings $f\colon X \to f(X)$ and $g\colon f(X) \to [-1, 1]$ defined by

$$f(x, y) = \begin{cases} (x, y) & \text{if} \quad 0 \leqslant x, \\ (x, 0) & \text{if} \quad x \leqslant 0, \end{cases}$$

and

$$g(x, y) = x$$

are semi-confluent, but their composition is neither locally semi-confluent nor joining.

(5.11) Example. Let a continuum X be such as in Example (4.50). The mappings $f\colon X \to f(X)$ and $g\colon f(X) \to gf(X)$ are defined as follows:

$$f(x, y) = (x, |y+2|-2),$$

$$g(x, y) = \begin{cases} (x, 0) & \text{if} \quad -1 \leqslant y, \\ (x, y+1) & \text{if} \quad y \leqslant -1. \end{cases}$$

The mapping f is locally confluent and the mapping g is monotone, but their composition is not even locally weakly confluent.

(5.12) EXAMPLE. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates. Put

$$X = \{(x, -1):\ -1 \leqslant x \leqslant 0\} \cup \{(0, y):\ -1 \leqslant y \leqslant 1\} \cup \{(x, 1):\ 0 \leqslant x \leqslant 3\}.$$

and define $f\colon X \to f(X)$ and $g\colon f(X) \to gf(X)$ by $f(x, y) = (x, |y|)$ and $g(x, y) = (|x-1|), y)$. The mappings f and g are both hereditarily atriodic, but their composition is not even atriodic.

Recall that a mapping f from X onto Y is called *light* if $\dim f^{-1}(y) = 0$ for each $y \in Y$ (see [83], p. 130). Theorem (2.13) of [54], p. 63 says that the composition of light locally weakly confluent (light locally confluent) mappings is locally weakly confluent (locally confluent, respectively). Such a property for semi-confluent (locally semi-confluent) mappings is not true. This can be seen from Example (5.8).

In [52], Theorem 3.3, p. 254, in [54], Theorem 2.5, p. 61 and in [50], 4.6 some particular properties of the composition of locally confluent, locally weakly confluent and semi-confluent mappings are proved.

B. The composition factor property. Let A be an arbitrary class of mappings. We say that the class A has the *composition factor property* if for each mapping $f\colon X \to Y$ belonging to A the equality $f = gh$ (we assume that g and h are continuous) implies that the mapping g belongs to A.

As a direct consequence of Whyburn's factorization theorem (see [83] (4.1), p. 141) we infer that

(5.13) *If A is a class of mappings having the composition factor property, then for each f belonging to A there exists a unique factorization of f into two mappings g and h, i.e., $f = gh$, where h is monotone and g is light and belongs to A.*

The following propositions say which classes of mappings have the composition factor property. Obviously,

(5.14) *The class of homeomorphisms has the composition factor property.*

It follows from (3.1) and (3.2) of [83], p. 140 that

(5.15) *Classes of open mappings and of monotone mappings have the composition factor property.*

As immediate consequences of definitions we infer that (cf. [9], IV, p. 214 and [52], Theorem 3.5, p. 254)

(5.16) *The composition factor property have the following mappings: confluent, semi-confluent, weakly confluent and pseudo-confluent.*

Moreover (see [54], Theorem (2.7), p. 61)

(5.17) *The class of locally weakly confluent mappings has th; composition factor property.*

Similarly,

(5.18) *The class of locally confluent mappings has the composition factor property.*

In fact, let f map X onto Y and let g map Y onto Z. Suppose that V is a closed neighbourhood of z in Z such that the partial mapping $h|h^{-1}(V)$ is confluent, where $h = gf$. Since $h|h^{-1}(V) = \big(g|g^{-1}(V)\big)\ \big(f|h^{-1}(V)\big)$, we conclude that $g|g^{-1}(V)$ is confluent by (5.16). Therefore g is locally confluent.

In the some easy way as for confluent mappings one can infer that

(5.19) *Classes of quasi-monotone mappings and of weakly monotone mappings have the composition factor property.*

From (4.6) we conclude that

(5.20) *The class of* OM-*mappings has the composition factor property.*

Further, it is clear that

(5.21) *The class of joining mappings has the composition factor property.*

The following problem is open:

(5.22) *Does the class of atomic mappings have the composition factor property* (*on continua*)?

We conjecture that the answer is positive.

The remaining classes of mappings do not have the composition factor property. This can be seen from the following examples.

(5.23) Example. Let (r, φ) denote a point of the Euclidean plane having r and φ as its polar coordinates. Take the unit circle $S = \{(1, \varphi)\colon 0 \leqslant \varphi \leqslant 2\pi\}$. The mapping h of S onto itself defined by $h(r, \varphi) = (r, 2\varphi)$ is a local homeomorphism. Let $f\colon S \to f(S)$ be a mapping which identifies points $(1, 0)$ and $(1, \pi)$ and let a mapping $g\colon f(S) \to S$ be such that $h = gf$ (there is exactly one such g). The mapping g is not a local homeomorphism.

(5.24) Example. The mapping f described in Example (4.10) is locally monotone and it is a composition of two mappings, namely of a monotone mapping g defined by

$$g(t) = \begin{cases} t+1 & \text{if} \quad -2 \leqslant t \leqslant -1, \\ 0 & \text{if} \quad -1 \leqslant t \leqslant 1, \\ t-1 & \text{if} \quad 1 \leqslant t \leqslant 2 \end{cases}$$

and of an open mapping h defined by $h(t) = |t|$. We have $f = hg$ and g is not locally monotone.

(5.25) EXAMPLE. Let f denote the step-function which is extended to the whole interval $[0, 1]$ (see [35], § 16, II, p. 150). The mapping f': $[-1, 1] \to [0, 1]$ defined by $f'(t) = f(|t|)$ is an MO-mapping, because $|\ |$ is open and f is monotone. The mapping f' is a composition of two mappings, namely $f' = hg$, where

$$g(t) = \begin{cases} t & \text{if} \quad -1 \leqslant t \leqslant 0, \\ f(t) & \text{if} \quad 0 \leqslant t \leqslant 1 \end{cases}$$

and

$$h(t) = \begin{cases} f(|t|) & \text{if} \quad -1 \leqslant t \leqslant 0, \\ t & \text{if} \quad 0 \leqslant t \leqslant 1. \end{cases}$$

As in Examples (4.11) and (4.31) one can show that the mapping h is not a locally MO-mapping.

The following Example is a modification of Example (2.8) of [57].

(5.26) EXAMPLE. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates and let f be the step-function extended to the whole interval $[0, 1]$ and let C denote the Cantor standard ternary set lying in $[0, 1]$ (see [35], § 16, II, p. 150). Consider the continuum X defined by

$$X = \{(x, y)\colon x \in C \text{ and } 0 \leqslant y \leqslant 1\} \cup \{(x, 0)\colon 0 \leqslant x \leqslant 1\} \cup \\ \cup \{(1/2, y)\colon -1 \leqslant y \leqslant 0\}$$

and the mapping $f'\colon X \to f'(X)$ defined as follows:

$$f'(x, y) = \begin{cases} (f(x), 0) & \text{if} \quad 0 \leqslant y, \\ (f(x), y) & \text{if} \quad y \leqslant 0. \end{cases}$$

The mapping f' is hereditarily monotone and $f' = hg$, where

$$g(x, y) = (f(x), y)$$

and

$$h(x, y) = \begin{cases} (x, 0) & \text{if} \quad 0 \leqslant y, \\ (x, y) & \text{if} \quad y \leqslant 0. \end{cases}$$

Since the mapping h restricted to the set $\{(x, 1)\colon 0 \leqslant x \leqslant 1\} \cup \{(1, y)\colon 0 \leqslant y \leqslant 1\} \cup \{(x, 0)\colon 1/2 \leqslant x \leqslant 1\} \cup \{(1/2, y)\colon -1 \leqslant y \leqslant 0\}$ is not weakly confluent, we conclude that h is not even hereditarily weakly confluent.

(5.27) EXAMPLE. Let (x, y) be as in Example (5.26). We put

$$T = \{(0, y)\colon 0 \leqslant y \leqslant 1\} \cup \{(x, 0)\colon -1 \leqslant x \leqslant 1\}$$

and we define a mapping $f\colon [0,1] \to T$ as follows:

$$f(t) = \begin{cases} (-6t, 0) & \text{if} \quad 0 \leqslant t \leqslant 1/6, \\ (6t-2, 0) & \text{if} \quad 1/6 \leqslant t \leqslant 2/6, \\ (0, 6t-2) & \text{if} \quad 2/6 \leqslant t \leqslant 3/6, \\ (0, -6t+4 & \text{if} \quad 3/6 \leqslant t \leqslant 4/6, \\ (6t-4, 0) & \text{if} \quad 4/6 \leqslant t \leqslant 5/6, \\ (-6t+6, 0) & \text{if} \quad 5/6 \leqslant t \leqslant 1. \end{cases}$$

The mapping f is hereditarily atriodic and it is a composition of two mappings g and h, i.e., $f = hg$, where $g\colon [0,1] \to g([0,1])$ identifies points 0 and 1; and $h\colon g([0,1]) \to T$ is such that $f = hg$ (there is exactly one such h). The second mapping h is not even atriodic, and thus it is not hereditarily atriodic.

(5.28) Example. Let a continuum X consist of a straight segment joining the point $(0, 2)$ with the point $(0, -2)$, of straight segments joining the point $(0, 1)$ with points $(1/n, 2)$ for $n = 1, 2, \ldots$ and of straight segments joining the point $(0, -1)$ with points $(-1/n, -2)$ for $n = 1, 2, \ldots$ (see Fig. 5). Define a mapping f from X onto $f(X)$ and define a mapping

Fig. 5

from $f(X)$ onto the interval $[-1, 1]$ as follows:

$$f(x, y) = (x, 1-|2-|y+1||)$$

and

$$g(x, y) = (0, y).$$

It is easy to ascertain that the mapping gf is locally semi-confluent and that the mapping g is not locally semi-confluent.

The following theorem comprises also Theorem (2.7) of [57].

(5.29) *Let a weakly confluent mapping f map a continuum X onto Y. If a mapping $g\colon Y \to Z$ is such that gf is atomic (hereditarily monotone, hereditarily confluent, hereditarily weakly confluent, atriodic, hereditarily atriodic), then the mapping g is atomic (hereditarily monotone, hereditarily confluent, hereditarily weakly confluent, atriodic, hereditarily atriodic, respectively).*

In fact, if gf is atomic (in the remaining cases the proof is similar) and Q is an arbitrary subcontinuum of Y such that the set $g(Q)$ is nondegenerate, then there is a continuum K in X such that $f(K) = Q$, because f is weakly confluent. Therefore, since $gf(K) = g(Q)$, we infer that $K = (gf)^{-1}\big((gf)(K)\big)$. Thus $Q = g^{-1}\big(g(Q)\big)$. This means that g is atomic.

C. The product property. A product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of two mappings f_i: $X_i \to Y_i$ $(i = 1, 2)$ is defined by

$$(f_1 \times f_2)(x_1, x_2) = \big(f_1(x_1), f_2(x_2)\big) \quad \text{for} \quad x_1 \in X_1 \text{ and } x_2 \in X_2.$$

It is easy to observe that the following two formulas hold:

(5.30) $(f_1 \times f_2)(A \times B) = f_1(A) \times f_2(B)$,

(5.31) $(f_1 \times f_2)^{-1}(C \times D) = f_1^{-1}(C) \times f_2^{-1}(D)$,

where $A \subset X_1$, $B \subset X_2$, $C \subset Y_1$ and $D \subset Y_2$.

Let A be an arbitrary class of mappings. We say that the class A has the *product property* if for each two mappings f_1 and f_2 belonging to A their product $f_1 \times f_2$ belongs to A.

As an immediate consequence of the definitions from (5.30) and (5.31) we infer that (cf. [49], (1.9))

(5.32) *Classes of homeomorphisms, of monotone mappings and of open mappings have the product property.*

Since $(h_1 g_1) \times (h_2 g_2) = (h_1 \times h_2)(g_1 \times g_2)$, we conclude by (5.32) that (cf. [49], (1.10))

(5.33) *The product property characterizes the following mappings:* MO-*mappings and* OM-*mappings.*

We have the following

(5.34) THEOREM. *If a class A of mappings has the product property, then a class of locally A mappings also has the product property.*

Proof. Let mappings f_1: $X_1 \to Y_1$ and f_2: $X_2 \to Y_2$ belong to a class of locally A mappings. Therefore for arbitrary points $x_1 \in X_1$ and $x_2 \in X_2$ there are closed neighbourhoods V_1 in X_1 and V_2 in X_2 of x_1 and x_2, respectively, such that $f_1(V_1)$ and $f_2(V_2)$ are closed neighbourhoods of $f_1(x_1)$ and $f_2(x_2)$ in Y_1 and in Y_2, respectively, and the partial mappings $f_1|V_1$ and $f_2|V_2$ belong to A. Then the set $V_1 \times V_2$ is a closed neighbourhood of (x_1, x_2) in $X_1 \times X_2$ and then the set $(f_1 \times f_2)(V_1 \times V_2)$ is a closed neighbourhood of $(f_1 \times f_2)(x_1, x_2)$ in $Y_1 \times Y_2$ by (5.30). Moreover, since the class A has the product property and since $(f_1 \times f_2)|V_1 \times V_2 = (f_1|V_1) \times (f_2|V_2)$, we conclude that the mapping $(f_1 \times f_2)|V_1 \times V_2$ belongs to A, because $f_1|V_1$ and $f_2|V_2$ belong to A. This means that the mapping $f_1 \times f_2$

is a locally A mapping. Hence the class of locally A mappings has the product property.

By (5.32), (5.33) and Theorem (5.34) we obtain the following

(5.35) COROLLARY. *The product property characterizes the following mappings: local homeomorphisms, locally monotone mappings and locally* MO-*mappings.*

The remaining classes of mappings do not have the product property. This can be seen from the following two examples.

(5.36) EXAMPLE. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates. Put

$$X = \{(x, \sin 2\pi/x): 0 < x \leqslant 1\} \cup \{(0, y): -1 \leqslant y \leqslant 1\}$$

and define a mapping $f: X \to I = [0, 1]$ by $f(x, y) = x$. The mapping f is atomic. Let $h: I \to I$ be the identity mapping onto I, i.e., $h(t) = t$. The product mapping $f \times h: X \times I \to I \times I$ is not even hereditarily atriodic because $f \times h$ restricted to the set $\big(X \times \{0, 1/2, 1\} \cup \{(1, 0)\} \times I\big) \cup \big(\{(0, 1)\} \times \times \{t: 0 \leqslant t \leqslant 1/3 \text{ or } 2/3 \leqslant t \leqslant 1\}\big) \cup \big(\{(0, -1)\} \times \{t: 1/3 \leqslant t \leqslant 2/3\}\big)$ is not atriodic.

The following example is a modification of Example of [60].

(5.37) EXAMPLE. Let (r, t) denote a point of the Euclidean plane having r and t as its polar coordinates. If $0 < a < b$, then we put

$$K(a, b) = \left\{\left(r, \frac{\pi r}{3(b-a)} |2r - (a+b)| \sin \frac{1}{(r-a)(b-r)}\right): a < r < b\right\}.$$

Define

$$K_n = K\left(\frac{1}{n+1}, \frac{1}{n}\right),$$

$$S_n = \{(1/n, t): 0 \leqslant t \leqslant 2\pi)\},$$

$$X = \{(0, 0)\} \cup \bigcup_{n=1}^{\infty} (K_n \cup S_n) \text{ (see Fig. 6)}$$

$$f(r, t) = (r, 3t) \quad \text{for each } (r, t) \in X$$

and

$$Y = f(X).$$

It is easy to observe that the mapping f is confluent and quasi-monotone (cf. [60], Example).

Now we consider the products $X \times I$ and $Y \times I$ with $I = [0, 1]$ and, to simplify the notation we shall use (r, t, s) instead of $\big((r, t), s\big)$ where $(r, t) \in X$ or $(r, t) \in Y$ and $s \in I$. Fix a natural n and consider the following sets for $i = 0, 1, 2$:

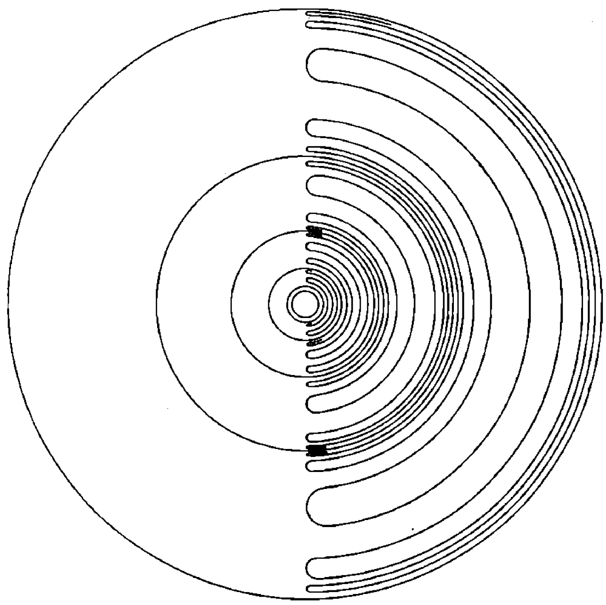


Fig. 6

$$M_i = \left\{\left(r, t, \frac{\pi+t}{2\pi n}+\frac{i}{n}\right): (r, t) \in f(S_{n+1}\cup K_n \cup S_n), \ -\pi \leqslant t \leqslant \pi, \ r \leqslant \frac{2n+1}{2n(n+1)}\right\},$$

$$A_i = \left\{\left(r, t, \frac{\pi+3t}{2\pi n}\right): (r, t) \in S_{n+1}\cup K_n \cup S_n, \ -\frac{\pi}{3} \leqslant t \leqslant \frac{\pi}{3}, \ r \leqslant \frac{2n+1}{2n(n+1)}\right\},$$

$$B_i = \left\{\left(r, t+\frac{2\pi}{3}, \frac{\pi+3t}{2\pi n}\right): r = \frac{1}{n+1} \text{ and } -\frac{\pi}{3} \leqslant t \leqslant \frac{\pi}{3}\right\},$$

and

$$C_i = \left\{\left(r, t+\frac{4\pi}{3}, \frac{\pi+3t}{2\pi n}\right): r = \frac{1}{n+1} \text{ and } -\frac{\pi}{3} \leqslant t \leqslant \frac{\pi}{3}\right)\right\}.$$

It is easy to observe that

(5.37.1) *Sets A_0, A_1 and A_2 are irreducible continua.*

Let $h: I \to I$ be the identity mapping onto I, i.e., $h(s) = s$ for each $s \in I$. It is easy to check that the mapping $f\times h: X\times I \to Y\times I$ has the following properties

(5.37.2) $(f\times h)^{-1}(M_i) = A_i \cup B_i \cup C_i$ *for* $i = 0, 1, 2$.

(5.37.3) $(f\times h)(A_i) = M_i$ *for* $i = 0, 1, 2$.

Since $M_0 \cap M_1 \neq \emptyset$, we conclude that $M_0 \cup M_1$ is an irreducible continuum by (5.37.1) and (5.37.3). Sets $A_0 \cup B_1$, $A_1 \cup C_0$ and $B_0 \cup C_1$ are components of the set $(f \times h)^{-1}(M_0 \cup M_1)$ and none of them is mapped onto the whole of $M_0 \cup M_1$. Therefore

(5.37.4) *$f \times h$ is neither pseudo-confluent nor locally weakly confluent because the diameter of $M_0 \cup M_1$ is less than* $3/n$ (cf. (4.37)).

Since $M_0 \cap M_1 \neq \emptyset$ nad $M_1 \cap M_2 \neq \emptyset$, we infer that the set $M_0 \cup \cup M_1 \cup M_2$ is a continuum by (5.37.1) and (5.37.3). Sets $A_0 \cup B_1 \cup C_2$, $A_1 \cup \cup B_2 \cup C_0$ and $A_2 \cup B_0 \cup C_1$ are components of the set $(f \times h)^{-1}(M_0 \cup M_1 \cup \cup M_2)$ and the image under $f \times h$ of the union of each two of them is not equal to $M_0 \cup M_1 \cup M_2$. Therefore

(5.37.5) *$f \times h$ is not atriodic.*

Let $D = \left\{(r, t, s) \in Y: \dfrac{2n+1}{2n(n+1)} \leqslant r\right\}$. The set D is a subcontinuum of Y with a nonempty interior. Since M_0 and M_2 are continua (by (5.37.1) and (5.37.3)) and since $D \cap M_0 \neq \emptyset \neq D \cap M_2$, we infer that the set $D \cup M_0 \cup M_2$ is a continuum with a nonempty interior. Sets $A_0 \cup A_2 \cup (f \times h)^{-1}(D)$, B_0, B_2, C_0 and C_2 are components of the set $(f \times h)^{-1}(D \cup M_0 \cup M_2)$ (cf. (5.37.2)). Since $(f \times h)(B_0) \cap (f \times h)(B_2) = \emptyset$, we infer that

(5.37.6) *$f \times h$ is neither weakly monotone nor joining.*

Example (5.37) is a solution of the problem asked by A. Lelek (see [49], Problem I); it is also an answer to the problem asked in [74] in the Table. Moreover, it is an answer to the following problem, asked by A. Lelek in his letter to the author: Is the product of two confluent mappings always weakly confluent? The answer is negative by (5.37.4).

D. The product factor property. Let A be an arbitrary class of mappings. We say that the class A has the *product factor property* if $f \times g \in A$ implies that $f \in A$ and $g \in A$. This property is investigated in [74], where it is proved that

(5.38) *Let a class A of mappings satisfy the following conditions:*
(i) *$f \in A$, then $f|f^{-1}(B) \in A$ for each closed set $B \subset Y$ and*
(ii) *if $gf \in A$ and f is open, then $g \in A$.*
Then the class A has the product factor property.

Thus, by proposition of (5.B) we infer that (cf. [74], Corollaries)

(5.39) *The following mappings have the product factor property: homeomorphisms, atomic mappings, monotone mappings, open mappings,* OM-*mappings, quasi-monotone mappings, weakly monotone mappings, confluent mappings, semi-confluent mappings and weakly confluent mappings.*

From (5.16), (5.21) and (5.38) it is easy to find that

(5.40) *Classes of joining mappings and of pseudo-confluent mappings have the product factor property.*

We have the following general theorem:

(5.41) THEOREM. *If is a class of mappings, then the class of hereditarily A mappings has the product factor property.*

Indeed, let a mapping $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ belong to the class of hereditarily A mappings and let K be an arbitrary subcontinuum of X_1 and let $x_2 \in X_2$. Then $K \times \{x_2\}$ is a subcontinuum of $X_1 \times X_2$, and thus the mapping $(f_1 \times f_2)|K \times \{x_2\}$ belongs to the class A. But $(f_1 \times f_2)|K \times \times \{x_2\} = f_1|K$ (the equality is given with respect to homeomorphisms). This means that f_1 belongs to the class of hereditarily A mappings. Similarly, f_2 belongs to the class of hereditarily A mappings. Consequently the class of hereditarily A mappings has the product factor property.

Theorem (5.41) implies that

(5.42) *The following mappings have the product factor property: hereditarily monotone mappings, hereditarily confluent mappings, hereditarily weakly confluent mappings and hereditarily atriodic mappings.*

Further, it is clear that property (i) of (5.38) is satisfied for a class of atriodic mappings. Thus, by (5.29) and (5.38) we conclude that

(5.43) *The class of atriodic mappings has the product factor property.*

Similarly (cf. [54], Theorem 2.6, p. 61) by (5.17), (5.18) and (5.38) we conclude that

(5.44) *Classes of locally confluent mappings and of locally weakly confluent mappings have the product factor property.*

We also have the following

(5.45) THEOREM. *The following classes of mappings have the product factor property: local homeomorphisms, locally monotone mappings, locally semi-confluent mappings.*

Proof. Let a mapping $f\colon X \to Y$ be a local homeomorphism (a locally monotone mapping, a locally semi-confluent mapping), let B be a closed subset of Y and let $x \in f^{-1}(B)$. There is a closed neighbourhood V of x such that $f(V)$ is a closed neighbourhood of $f(x)$ and $f|V$ is a homeomorphism (a monotone mapping, a semi-confluent mapping, respectively). Then the set $V \cap f^{-1}(B)$ is a closed neighbourhood of x in $f^{-1}(B)$ and the set $(f|V)\big(V \cap f^{-1}(B)\big) = f(V) \cap B$ is a closed neighbourhood of $f(x)$ in B.

Since the class of homeomorphisms (of monotone mappings, of semi-confluent mappings) has property (i) of (5.38) (for semi-confluent mappings it is proved in [52], Corollary 3.8, p. 255; for homeomorphisms and for monotone mappings it is obvious), we conclude that the partial mapping $(f|V)|(f|V)^{-1}(B)$ is a homeomorphism (a monotone mapping, a semi-confluent mapping, respectively). This means that $f|f^{-1}(B)$ is a local homeomorphism (a locally monotone mapping, a locally semi-confluent mapping, respectively). Therefore the class of local homeomorphisms (of locally monotone mappings, of locally semi-confluent mappings) has property (i) of (5.38).

Consequently, it suffices to show that the class of local homeomorphisms (of locally monotone mappings, of locally semi-confluent mappings) has property (ii) of (5.38). We will now prove this.

Indeed, let a mapping $f\colon X \to Y$ be open and let a mapping $g\colon Y \to Z$ be such that the composition gf is a local homeomorphism (a locally monotone mapping, a locally semi-confluent mapping). Assume $y \in Y$ and take $x \in X$ such that $f(x) = y$. Then there is a closed neighbourhood V of x such that $gf(V)$ is a closed neighbourhood of $gf(x)$ and the partial mapping $gf|V$ is a homeomorphism (a monotone mapping, a semi-confluent mapping, respectively). Since the mapping f is open, we conclude that the set $f(V)$ is a closed neighbourhood of y. The image of $f(V)$ under g is a closed neighbourhood of $g(y)$, because $g(f(V)) = gf(V)$. Since $gf|V = (g|f(V))(f|V)$, we infer that the partial mapping $g|f(V)$ is a homeomorphism (a monotone mapping, a semi-confluent mapping) by (5.14) (by (5.15) and by (5.16), respectively). Hence the mapping g is a local homeomorphism (a locally monotone mapping, a locally semi-confluent mapping, respectively). Thus also (ii) of (5.38) holds for these classes of mappings. Therefore the proof of Theorem (5.45) is completed by (5.38).

The following problems remain open (cf. [74], Table):

(5.46) PROBLEM. *Does it follow that the class of* MO-*mappings* (*of locally* MO-*mappings*) *has the product factor property*?

E. The limit property. Let A be an arbitrary class of mappings. We say that A has the (weak) limit property if for each two spaces X and Y (if for each space X and for each locally connected space Y) the set of all onto mappings $f\colon X \to Y$ belonging to A is closed in the space Y^X, where Y^X denotes the space of all continuous mappings $f\colon X \to Y$ with the compact-open topology. It is known that (see [37], p. 797)

(5.47) *The class of monotone mappings has the weak limit property.*

It follows from Theorems 1 and 2 of [59] that

(5.48) *Classes of confluent mappings and of semi-confluent mappings have the weak limit property.*

Moreover, by Theorem 3 of [59] we infer that

(5.49) *The class of pseudo-monotone mappings has the limit property.*

Since the class of OM-mappings (of quasi-monotone mappings, of weakly monotone mappings, and of locally confluent mappings) coincides with the class of confluent mappings onto locally connected spaces (see [9], p. 214 and p. 215; [50], Corollary 5.2 and [80] Theorem (2.1), p. 137; and Theorem (2.3), p. 138; cf. § 6 here) from (5.48) we conclude that

(5.50) *The following mappings have the weak limit property:* OM-*mappings, quasi-monotone mappings, weakly monotone mappings and locally confluent mappings.*

We will now prove that the class of joining mappings has the weak limit property. Firstly, recall that (see [59], Lemmas 1 and 2)

(5.51) *If G is an open subset of X and P is a closed subset of X, and if H is an open subset of Y and R is a closed subset of Y, then the sets $\{g: g^{-1}(R) \subset G\}$ and $\{g: g(P) \subset H\}$ are open in Y^X.*

Moreover (cf. [59], Lemma 3),

(5.52) *If f maps X onto Y and G is an open subset of X, then the set $\{y: f^{-1}(y) \subset G\}$ is open in Y.*

We have the following

(5.53) THEOREM. *The class of joining mappings has the weak limit property.*

Proof. Assume that Y is locally connected and let Φ denote the set of all joining mappings from X onto Y. Suppose that $f \in \overline{\Phi}$. We should prove that f is joining. Suppose on the contrary that f is not joining and let Q be subcontinuum of Y and let C_1 and C_2 be components of $f^{-1}(Q)$ such that their images $Q_1 = f(C_1)$ and $Q_2 = f(C_2)$ are disjoint. Then the set $f^{-1}(Q)$ is not connected between C_1 and $f^{-1}(Q_2)$. Thus (see [36], § 47, II, Theorem 3, p. 170) there are closed sets A_1 and A_1' such that $f^{-1}(Q) = A_1 \cup A_1'$, $A_1 \cap A_1' = \emptyset$, $C_1 \subset A_1$ and $f^{-1}(Q_2) \subset A_1'$. Then $f(A_1) \cap Q_2 = \emptyset$. Thus the set A_1' is not connected between $f^{-1}(f(A_1))$ and C_2 (cf. ibid.). Therefore there are closed sets A_2 and A_3 such that $A_1' = A_2 \cup A_3$, $A_2 \cap A_3 = \emptyset$, $C_2 \subset A_2$ and $f^{-1}(f(A_1)) \cap A_2 = \emptyset$. Hence we have

(5.53.1) $f^{-1}(Q) = A_1 \cup A_2 \cup A_3$, $A_i \cap A_j = \emptyset$ *for* $i \neq j$ *and* $i, j = 1, 2, 3$ *and* $f(A_1) \cap f(A_2) = \emptyset$.

Since X is normal and f is continuous, there are open sets G_1, G_2 and G_3 such that

(5.53.2) $A_i \subset G_i$, $\overline{G}_i \cap \overline{G}_j = \emptyset$ *for* $i \neq j$ *and* $i, j = 1, 2, 3$ *and* $f(\overline{G}_1) \cap f(\overline{G}_2) = \emptyset$.

Since Y is normal, there are open sets H_1 and H_2 such that

(5.53.3) $f(\bar{G}_i) \subset H_i$, $H_i \cap H_j = \emptyset$ *for* $i \neq j$ *and* $i, j = 1, 2$.

Consider subsets Ψ_1 and Ψ_2 of Y^X defined as follows:

(5.53.4) $\Psi_i = \{g\colon\ g(\bar{G}_i) \subset H_i\}$ *for* $i = 1, 2$.

Since the sets $\bar{G}_1$ and $\bar{G}_2$ are closed and the sets H_1 and H_2 are open, we conclude that the sets Ψ_1 and Ψ_2 are open in Y^X, by (5.51). Moreover, $f \in \Psi_1 \cap \Psi_2$ (cf. (5.53.3)), and thus

(5.53.5) *the set* $\Psi_1 \cap \Psi_2$ *is an open neighbourhood of* f.

Put $G = G_1 \cup G_2 \cup G_3$. Since f is continuous and G is open, from (5.52) we infer that

(5.53.6) *the set* $U = \{y\colon f^{-1}(y) \subset G\}$ *is open in* Y.

Moreover, by (5.53.1) and (5.53.2) we have

(5.53.7) $Q \subset U$.

The local connectedness of Y implies that there is a connected open set V such that

(5.53.8) $Q \subset V \subset \bar{V} \subset U$.

We will show that

(5.53.9) $f^{-1}(\bar{V}) \subset G$.

Indeed, if $x \in f^{-1}(\bar{V})$, then $f(x) \in \bar{V}$. Hence $f(x) \in U$ by (5.53.8). Since $x \in f^{-1}f(x)$, we conclude that $x \in f^{-1}(f(x)) \subset G$ by (5.53.6). Thus (5.53.9) holds.

Consider the subset Δ of Y^X defined as follows:

(5.53.10) $\Delta = \{g\colon\ g^{-1}(\bar{V}) \subset G\}$.

Since the set G is open and since the set $\bar{V}$ is closed, we infer by (5.51) that the set Δ is open in Y^X. Moreover, by (5.53.9) we have $f \in \Delta$. Thus

(5.53.11) Δ *is an open neighbourhood of* f *in* Y^X.

Put

(5.53.12) $\Gamma_i = \{g\colon\ g(A_i) \subset V\}$ *for* $i = 1, 2$.

Since the sets A_1 and A_2 are closed and the set V is open, we conclude that the sets Γ_1 and Γ_2 are open in Y^X, by (5.51). Moreover, $f \in \Gamma_1 \cap \Gamma_2$, because $f(A_i) \subset Q \subset V$ by (5.53.1) and (5.53.8). Thus the set $\Delta \cap \Psi_1 \cap \Psi_2 \cap \cap \Gamma_1 \cap \Gamma_2$ is an open neighbourhood of f by (5.53.5) and (5.53.11). Since $f \in \Phi$, we conclude that there is a g such that

(5.53.13) $g \in \Phi \cap \Delta \cap \Psi_1 \cap \Psi_2 \cap \Gamma_1 \cap \Gamma_2$.

Since $g \in \Delta$, we have, by (5.53.10), the decomposition

$$g^{-1}(\bar{V}) = \big(g^{-1}(\bar{V}) \cap G_1\big) \cup \big(g^{-1}(\bar{V}) \cap G_2\big) \cup \big(g^{-1}(\bar{V}) \cap G_3\big)$$

of $g^{-1}(V)$ into three separated sets (cf. (5.53.2)).

Let $i = 1, 2$. Since the set $A_i \subset f^{-1}(Q)$ and since $g \in \Gamma_i$, we infer that $g(A_i) \subset V$ by (5.53.12). Thus $A_i \subset g^{-1}(\overline{V})$. This implies that $A_i \subset g^{-1}(\overline{V}) \cap G_i$, because $A_i \subset G_i$ by (5.53.2). We conclude that there is a component K_i of the set $g^{-1}(\overline{V})$ which is contained in the set $g^{-1}(\overline{V}) \cap G_i$. Thus we infer that $g(K_i) \subset g(G_i) \subset H_i$, because $g \in \Psi_i$ and $K_i \subset g^{-1}(\overline{V}) \cap \cap G_i \subset \overline{G}_i$.

Consequently K_1 and K_2 are components of the set $g^{-1}(\overline{V})$ such that $g(K_1) \cap g(K_2) = \emptyset$ by (5.53.3). This means that the mapping g is not joining, because $\overline{V}$ is a subcontinuum of Y. This contradicts the fact that $g \in \Phi$ (cf. (5.53.13)). The proof of Theorem (5.53) is complete.

We have (cf. [59], Theorem 4) the following

(5.54) THEOREM. *The class of weakly confluent mappings has the limit property.*

Proof. Let Φ denote the set of all weakly confluent mappings from X onto Y. Suppose that $f \in \overline{\Phi}$. We should prove that f is weakly confluent. Suppose, on the contrary, that f is not weakly confluent, and let Q be a subcontinuum of Y such that no component of $f^{-1}(Q)$ is mapped onto Q under f. Let C be an arbitrary component of $f^{-1}(Q)$. Then there is a point y_C such that $y_C \in Q \setminus f(C)$. Therefore the set $f^{-1}(Q)$ is not connected between C and $f^{-1}(y_C)$ (see [36], § 47, II, Theorem 3, p. 170). Thus there are closed sets A_C and A_C' such that $f^{-1}(Q) = A_C \cup A_C'$, $A_C \cap \cap A_C' = \emptyset$, $C \subset A_C$ and $f^{-1}(y_C) \subset A_C'$. We infer that the set A_C is open in $f^{-1}(Q)$. Since the set $f^{-1}(Q)$ is compact and the collection $\{A_C: C \text{ is a component of } f^{-1}(Q)\}$ covers $f^{-1}(Q)$, we infer that there are closed-open sets $A_1, A_2, \ldots, A_n$ in $f^{-1}(Q)$ and points $y_1, y_2, \ldots, y_n$ such that $f^{-1}(Q) = A_1 \cup \cup A_2 \cup \ldots \cup A_n$ and $y_i \in Q \setminus f(A_i)$ for each $i = 1, 2, \ldots, n$. Put $B_i = A_i \setminus (A_{i+1} \cup A_{i+2} \cup \ldots \cup A_n)$. Since A_i are closed-open in $f^{-1}(Q)$, we infer that B_i are closed-open in $f^{-1}(Q)$. Therefore, B_i are closed in X. Moreover,

(5.54.1) $f^{-1}(Q) = B_1 \cup B_2 \cup \ldots \cup B_n$, $B_i \cap B_j = \emptyset$ *and* $y_i \in Q \setminus f(B_i)$ *for each* $i = 1, 2, \ldots, n$.

Since X is normal and f is continuous, there are open sets $G_1, G_2, \ldots, G_n$ such that

(5.54.2) $B_i \subset G_i$, $\overline{G}_i \cap \overline{G}_j = \emptyset$ *and* $y_i \in Q \setminus f(\overline{G}_i)$ *for each* $i \neq j$ *and* $i, j = 1, 2, \ldots, n$.

Since Y is normal, there are open sets $H_1, H_2, \ldots, H_n$ such that

(5.54.3) $f(\overline{G}_i) \subset H_i \subset Y \setminus \{y_i\}$ *for each* $i = 1, 2, \ldots, n$.

Consider subsets $\Psi_1, \Psi_2, \ldots, \Psi_n$ of Y^X defined as follows

(5.54.4) $\Psi_i = \{g: g(\overline{G}_i) \subset H_i\}$ *for* $i = 1, 2, \ldots, n$.

Since the sets $\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_n$ are closed and the sets $H_1, H_2, \ldots, H_n$ are open, we conclude that sets $\Psi_1, \Psi_2, \ldots, \Psi_n$ are open in Y^X by (5.51). Moreover, by (5.54.3), $f \in \Psi_1 \cap \Psi_2 \cap \ldots \cap \Psi_n$. Thus

(5.54.5) *the set* $\Psi_1 \cap \Psi_2 \cap \ldots \cap \Psi_n$ *is an open neighbourhood of* f.

Put $G = G_1 \cup G_2 \cup \ldots \cup G_n$. Since G is open and Q is closed, from (5.51) we conclude that

(5.54.6) *the set* $\Delta = \{g\colon\ g^{-1}(Q) \subset G\}$ *is open in* Y^X.

Moreover, it follows from (5.54.1) and (5.53.2) that $f \in \Delta$. Thus, (5.54.5) and (5.54.6) imply that the set $\Delta \cap \Psi_1 \cap \Psi_2 \cap \ldots \cap \Psi_n$ is an open neighbourhood of f in Y^X. Since $f \in \Phi$, there is g such that

(5.54.7) $g \in \Phi \cap \Delta \cap \Psi_1 \cap \Psi_2 \cap \ldots \cap \Psi_n$.

Since $g \in \Delta$, we have, by (5.54.6), the decomposition

$$g^{-1}(Q) = \big(g^{-1}(Q) \cap G_1\big) \cup \big(g^{-1}(Q) \cap G_2\big) \cup \ldots \cup \big(g^{-1}(Q) \cap G_n\big)$$

of $g^{-1}(Q)$ into separated sets (cf. (5.54.2)). Let K be an arbitrary component of $g^{-1}(Q)$. Then K is contained in the set $g^{-1}(Q) \cap G_i$ for some $i = 1, 2, \ldots, n$. Since $g \in \Psi_i$ (cf. (5.54.7)), we infer that $g(K) \subset g(\bar{G}_i) \subset H_i$ by (5.54.4). Thus $y_i \in Q \setminus g(K)$ by (5.54.3). Consequently $g(K) \neq Q$ for each component K of $g^{-1}(Q)$. This means that the mapping g is not weakly confluent, because Q is a subcontinuum of Y. This contradicts the fact that $g \in \Phi$ (cf. (5.54.7)). The proof of Theorem (5.54) is complete.

One can observe that in the proof of Theorem (5.53), and also in the proof of Theorem (5.54), we do not use the assumption of the metrizability of spaces. Thus these theorems are true in general for compact Hausdorff's spaces. In particular, from Theorem (5.54) we obtain an answer to the question asked in [59].

Further, one can generalize Theorem (5.54) as follows:

(5.55) *Let n be a positive integer and let A denote a class of all mappings $f\colon X \to Y$ such that for each subcontinuum Q of Y the image of the union of some n components of $f^{-1}(Q)$ is equal to Q. Then the class A has the limit property.*

This proposition is in connection with the following

(5.56) PROBLEM. *Has a class of atriodic mappings the (weak) general limit property (on continua)?*

Atriodic mappings do not have the limit property on compact spaces. This can be seen from the following

(5.57) EXAMPLE. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates. We put

$$X = \{(x, y)\colon\ x = 0 \text{ or } 1/n \text{ for } n = 1, 2, \ldots \text{ and } 1 \leqslant |y| \leqslant 2\} \cup$$
$$\cup \{(x, y)\colon\ -1 \leqslant x \leqslant 0 \text{ and } y = -1 \text{ or } 1\} \cup \{(-2, y)\colon\ -1 \leqslant y \leqslant 1\}$$

and for $n = 1, 2, \ldots$ we define $f_n\colon X \to f_n(X) = Y$ as follows:

$$f_n(x, y) = \begin{cases} (x, y-1) & \text{if} \quad -1 \leqslant x \text{ and } 1 \leqslant y, \\ (x, y+1) & \text{if} \quad -1 \leqslant x \text{ and } y \leqslant -1, \\ (x, 0) & \text{if} \quad x = -1 \text{ and } |y| \leqslant 1, \\ (1/n, y) & \text{if} \quad x = -2. \end{cases}$$

It is easy to ascertain that mappings f_n are atriodic for $n = 1, 2, \ldots$ and that the sequence $\{f_n\}$ converges uniformly to a mapping f_0 which is not atriodic (recall that the topology of the uniformly convergence of Y^X coincides with the compact-open topology provided Y is metric).

Similarly, the remaining classes of mappings do not have the (weak) limit property either. This can be seen from the following examples.

(5.58) EXAMPLE. Let (r, φ) denote a point of the Euclidean plane having r and φ as its polar coordinates. Put

$$X = \{(1, \varphi)\colon 0 \leqslant \varphi \leqslant 2\pi\} \cup \{(r, \varphi)\colon 0 \leqslant r \leqslant 1 \text{ and } \varphi = 0 \text{ or } \pi\}$$

and for each $n = 1, 2, \ldots$ define $f_n\colon X \to X$ by

$$f_n(r, \varphi) = \begin{cases} \left(r, \dfrac{2(\pi n - 1)}{\pi n}\varphi\right) & \text{if} \quad -\pi/2 \leqslant \varphi \leqslant \pi/2, \\ \left(r, \dfrac{2}{\pi n}\varphi + \dfrac{\pi n - 2}{n}\right) & \text{if} \quad \pi/2 \leqslant \varphi \leqslant \pi, \\ \left(r, \dfrac{2}{\pi n}\varphi + \dfrac{2 - \pi n}{n}\right) & \text{if} \quad -\pi \leqslant \varphi \leqslant -\pi/2. \end{cases}$$

It is clear that the mapping f_n is a homeomorphism for each $n = 1, 2, \ldots$ and that the sequence $\{f_n\}$ is uniformly convergent to a mapping f_0 which is neither hereditarily atriodic nor open.

(5.59) EXAMPLE. Let (x, y) be such as in Example (5.57). For each $n = 1, 2, \ldots$ define a mapping f_n from $[0, 1]$ onto the unit circle as follows:

$$f_n(t) = \big(\cos 2\pi(1+1/n)t,\ \sin 2\pi(1+1/n)t\big)$$

for each $t \in [0, 1]$. Then f_n is locally semi-confluent and the limit function of the sequence $\{f_n\}$ is not locally weakly confluent.

(5.60) EXAMPLE. For each $n = 1, 2, \ldots$ we define a mapping f_n from $[0, 1]$ onto itself by

$$f_n(t) = \frac{6n}{3n-2}\, t \quad \text{for} \quad t \in \left[0, \frac{3n-2}{6n}\right],$$

and

$$f_n(t) = \begin{cases} 1 & \text{for} \quad t \in \left[\frac{3n-2}{6n}, \frac{1}{2}\right], \\ -2t+2 & \text{for} \quad t \in [1/2, 1]. \end{cases}$$

The sequence $\{f_n\}$ is a uniformly convergent sequence of locally monotone mappings, and its limit is not locally monotone.

(5.61) EXAMPLE. It is easy to construct a sequence $\{f_n\}$ of homeomorphisms from $[0, 1]$ onto itself which is uniformly convergent to the step-function f extended to the whole interval $[0, 1]$ (see [35], § 16, II, p. 150) and such that $f_n(0) = 0$ and $f_n(1) = 1$ for any $n = 1, 2, \ldots$ For any $n = 1, 2, \ldots$ we define a mapping g_n from the interval $[-1, 1]$ onto the interval $[0, 1]$ as follows:

$$g_n(t) = \begin{cases} f_n(t) & \text{if} \quad 0 \leqslant t \leqslant 1, \\ |t| & \text{if} \quad -1 \leqslant t \leqslant 0. \end{cases}$$

Mappings g_n are open, and their limit is not locally MO-mapping (cf. Example (5.25)).

Now we will use the construction from Example 2 of [59] in the following:

(5.62) EXAMPLE. Let C denote the Cantor ternary set lying in the unit interval $I = [0, 1]$. There is a sequence $\{f_n\}$ of homeomorphisms from I onto itself such that $f_n(C) = C$ for any $n = 1, 2, \ldots$ and such that it converges uniformly to the mapping f_0, where

$$f_0(t) = \begin{cases} 0 & \text{if} \quad t \in [0, 2/3], \\ 3t-2 & \text{if} \quad t \in [2/3, 1]. \end{cases}$$

Put $N = (I \times \{0\}) \cup (C \times I)$ and $g_n(x, y) = (f_n(x), y)$ for each $(x, y) \in N$ and $n = 0, 1, 2, \ldots$ The mappings g_n are homeomorphisms for $n = 1, 2, \ldots$ and they converge uniformly to g_0. Define a mapping ψ of I onto itself as follows:

$$\psi(t) = \begin{cases} -t+1/4 & \text{if} \quad 0 \leqslant t \leqslant 1/4, \\ 3t-3/4 & \text{if} \quad 1/4 \leqslant t \leqslant 1/2, \\ -t+5/4 & \text{if} \quad 1/2 \leqslant t \leqslant 3/4, \\ 2t-1 & \text{if} \quad 3/4 \leqslant t \leqslant 1. \end{cases}$$

Consider an equivalence relation ϱ defined on N as follows:

$(x, y)\, \varrho\, (x', y')$ if and only if

either $(x, y) = (x', y')$ or $x = x' = 0$ and $\psi(y) = \psi(y')$.

TABLE III

	composition property	composition factor property	product property	product factor property	limit property	
					weak	general
1	2	3	4	5	6	7
homeomorphisms	+ (5.1)	+ (5.14)	+ (5.32)	+ (5.39)	− (5.58)	− (5.58)
local homeomorphisms	+ (5.2)	− (5.23)	+ (5.35)	+ (5.45)	− (5.58)	− (5.58)
hereditarily monotone mappings	+ (5.3)	− (5.26)	− (5.36)	+ (5.42)	− (5.58)	− (5.58)
atomic mappings	+ (5.1)	? (5.22)	− (5.36)	+ (5.39)	− (5.58)	− (5.58)
monotone mappings	+ (5.1)	+ (5.15)	+ (5.32)	+ (5.39)	+ (5.47)	− (5.62)
open mappings	+ (5.1)	+ (5.15)	+ (5.32)	+ (5.39)	− (5.58)	− (5.58)
MO-mappings	− (5.5)	− (5.25)	+ (5.33)	? (5.46)	− (5.61)	− (5.62)
locally monotone mappings	− (5.7)	− (5.24)	+ (5.35)	+ (5.45)	− (5.60)	− (5.60)
locally MO-mappings	− (5.8)	− (5.25)	+ (5.35)	? (5.46)	− (5.61)	− (5.62)
OM-mappings	+ (5.5)	+ (5.20)	+ (5.33)	+ (5.39)	+ (5.50)	− (5.62)
hereditarily confluent mappings	+ (5.3)	− (5.26)	− (5.36)	+ (5.42)	− (5.58)	− (5.58)
quasi-monotone mappings	+ (5.6)	+ (5.19)	− (5.37)	+ (5.39)	+ (5.50)	− (5.62)
weakly monotone mappings	− (5.9)	+ (5.19)	− (5.37)	+ (5.39)	+ (5.50)	− (5.62)
confluent mappings	+ (5.4)	+ (5.16)	− (5.37)	+ (5.39)	+ (5.48)	− (5.62)
locally confluent mappings	− (5.11)	+ (5.18)	− (5.37)	+ (5.44)	+ (5.50)	− (5.62)
semi-confluent mappings	− (5.10)	+ (5.16)	− (5.37)	+ (5.39)	+ (5.48)	− (5.62)

Table III cont.

1	2	3	4	5	6	7
locally semi-confluent mappings	− (5.10)	− (5.28)	− (5.37)	+ (5.45)	− (5.59)	− (5.59)
joining mappings	− (5.10)	+ (5.21)	− (5.37)	+ (5.40)	+ (5.53)	− (5.62)
hereditarily weakly confluent mappings	+ (5.3)	− (5.26)	− (5.36)	+ (5.42)	− (5.58)	− (5.58)
weakly confluent mappings	+ (5.4)	+ (5.16)	− (5.37)	+ (5.39)	+ (5.54)	+ (5.54)
locally weakly confluent mappings	− (5.11)	+ (5.17)	− (5.37)	+ (5.44)	− (5.59)	− (5.59)
pseudo-confluent mappings	+ (5.4)	+ (5.16)	− (5.37)	+ (5.40)	+ (5.49)	+ (5.49)
hereditarily atriodic mappings	− (5.12)	− (5.27)	− (5.36)	+ (5.42)	− (5.58)	− (5.58)
atriodic mappings	− (5.12)	− (5.27)	− (5.37)	+ (5.43)	? (5.56)	− (5.57)

Denote by φ the canonical mapping from N onto N/ϱ. Put $M = N/\varrho$ and $h_n(q) = \varphi\big(g_n(\varphi^{-1}(q))\big)$ for each $q \in M$ and $n = 0, 1, 2, \ldots$

The sequence $\{h_n\}$ is a sequence of homeomorphisms which converges uniformly to a mapping h_0 which is not joining not locally confluent and not weakly monotone (all this is easy to observe).

For some other general results concerning the limit property see [58].

F. Remark. Papers [44], [51], [71] and [72] deal also with some other general properties of the mappings which are considered and investigated here.

Table III sums up the properties of mappings studied in § 5. The sign "+" (the sign "−") denotes that the corresponding class of mappings has (or does not have, respectively) the property corresponding to given sign. The number under the sign is the number of the proposition which justifies the use of the sign in question.

6. Mappings on some special spaces

In this section we will recall some known relations and show some other relations between mappings, investigated here on some special kinds of spaces.

A. Local homeomorphisms onto tree-like continua. It has been proved by Whyburn in [83], Corollary, p. 199 that every local homeomorphism of a continuum onto a dendrite is a homeomorphism. A more general result is contained in [18] (cf. [39], p. 56). In [52] I have proved that every local homeomorphism of a continuum onto a λ-dendroid is a homeomorphism. Similarly, every local homeomorphism of an arc-like continuum is a homeomorphism (see [73], Theorem 2.0, p. 261). The most general result of this kind that has been proved is contained in my paper. Namely, in [61] it is proved that

(6.1) *Every local homeomorphism of a continuum onto a tree-like continuum is a homeomorphism.*

The proof of this proposition contains a certain simple method which, I hope, can be used in the proofs of theorems about local homeomorphisms. This method reduces the investigations of local homeomorphisms on continua to the investigations of local homeomorphisms on polyhedra.

In [56], Theorem 8, p. 287 it is proved that if f is a mapping from a continuum X onto a continuum Y which is hereditarily divisible by points, and f is such that $\operatorname{card} f^{-1}(y) = n$ for each y belonging to Y and for some positive integer n, then f is a homeomorphisms. Similarly (see [56], Theorem 10, p. 289), if f is a joining mapping from a continuum X onto a λ-dendroid Y such that $\operatorname{card} f^{-1}(y) = n$ for each $y \in Y$ and for some n, then f is a homeomorphisms. It follows from (4.27) that these implications also generalize Whyburn's theorem.

There are several problems which concern the above relations between local homeomorphisms and homeomorphisms (see [53], Problem 12, p. 858; [56], Problem 14, p. 290, [72], Questions 2, 3 and 4, p. 261 and 262, compare [56], Examples 15 and 16, p. 290).

B. Weakly monotone and locally confluent mappings onto locally connected spaces. It follows from [9], p. 214-215, [50], Corrollary 5.2

and [80], Theorems (2.1) and (2.3), p. 137–138 that

(6.2) *All locally confluent mappings and all weakly monotone mappings onto locally connected spaces are* OM-*mappings and quasi-monotone mappings.*

C. Atomic mappings, locally confluent mappings and weakly monotone mappings on arcwise connected continua. It is known that (see [20], Corollary 9, p. 53; cf. also ibid. Theorems 5 and 6, p. 52)

(6.3) *If f is an atomic mapping of an arcwise connected continuum onto a nondegenerate continuum, then f is a homeomorphism.*

It is easy to infer from (6.2) (see [50], 5.3) that

(6.4) *All locally confluent mappings onto hereditarily arcwise connected spaces are confluent.*

We also have the following

(6.5) THEOREM. *Any weakly monotone mapping onto arcwise connected space is joining.*

Proof. Suppose on the contrary that a weakly monotone mapping f maps a space X onto an arcwise connected continuum Y and Q is a subcontinuum of Y such that there are components C_1 and C_2 of $f^{-1}(Q)$ with $f(C_1) \cap f(C_2) = \emptyset$. From Theorem 1 of [58] we conclude that there is a subcontinuum R of Y with a nonempty interior, such that $R \cap f(C_1) \neq \emptyset$ and $R \cap f(C_2) = \emptyset$. Then the set $Q \cup R$ is a subcontinuum of Y with a nonempty interior, and C_2 is a component of $f^{-1}(Q \cup R)$. Since $f(C_2) \neq Q \cup R$, we obtain a contradiction of the weak monotoneity of f.

D. Hereditarily confluent mappings onto arcwise connected spaces and onto hereditarily decomposable spaces. We will now prove two generalizations of Theorems (4.4) and (4.5) of [57], which are also a solution of Problem (4.8) of [57] and which partially answer the following question asked in [57], Problem (4.7).

(6.6) PROBLEM. *For which class of continua any hereditarily confluent mapping is monotone?*

We have the following

(6.7) THEOREM. *Any hereditarily confluent mapping of a continuum onto an arcwise connected space is monotone.*

Proof. Let an hereditarily confluent mapping f map a continuum X onto an arcwise connected continuum Y and let y be an arbitrary point of Y such that there is a subcontinuum Q of Y with a nonempty interior such that $y \in Y \setminus Q$. Then

(6.7.1) *the set $f^{-1}(y)$ is connected.*

In fact, since Y is arcwise connecteed, there is an arc yy' with end-points y and y' such that $yy' \cap Q = \{y'\}$. The set $f^{-1}(Q \cup yy')$ is connected, because f is confluent and the set $f^{-1}(Q)$ is connected (cf. Theorem (4.44)). The hereditary confluence of f implies that the partial mapping $f|f^{-1}(Q \cup \cup yy')$ is hereditarily confluent. Since any nondegenerate subarc A of yy' has a nonempty interior in $Q \cup yy'$, we infer by Theorem (4.44) that the set

$$\big(f|f^{-1}(Q \cup yy')\big)^{-1}(A) = f^{-1}(A)$$

is connected. This easily implies that the inverse image of any point of yy' is connected. Thus (6.7.1) holds.

Since for each two different points of Y there is a subcontinuum Q of Y with a nonempty interior such that Q contains one of these points and does not contain the other (see [58], Theorem 1), we infer that the inverse image of any point of Y, except at most one point of Y, is connected by (6.7.1). Thus an inverse image of any nondegenerate subcontinuum of Y is connected by the confluence of f.

Let b be a given point of Y. Then there is a decreasing sequence $\{B_n\}$ of nondegenerate subcontinua of Y such that $\{b\} = \bigcap_{n=1}^{\infty} B_n$. Since $f^{-1}(b) = \bigcap_{n=1}^{\infty} f^{-1}(B_n)$ and the sequence $\{f^{-1}(B_n)\}$ is a decreasing sequence of continua, we infer that the set $f^{-1}(b)$ is a continuum (see [36], § 47, II, Theorem 5, p. 170). This means that f is monotone. The proof of Theorem (6.7) is complete.

(6.8) THEOREM. *Any hereditarily confluent mapping of a continuum onto a hereditarily decomposable continuum is hereditarily monotone.*

Proof. Let a hereditarily confluent mapping f map a continuum X onto a hereditarily decomposable continuum Y and let y be an arbitrary point of Y. Take a subcontinuum Q of Y which is irreducible with respect to the property that $y \in Q$ and the set $f^{-1}(Q)$ is connected (cf. [53], Lemma 7, p. 856). Suppose that Q is nondegenerate. Then $Q = A \cup B$, where A and B are proper subcontinua of Q, because Q is decomposable. Since the set $f^{-1}(Q)$ is a continuum and since the mapping $f|f^{-1}(Q)$ is hereditarily confluent, we infer that the sets $f^{-1}(A)$ and $f^{-1}(B)$ are connected by Theorem (4.44). But also either $y \in A$ or $y \in B$, contrary to the choice of Q. This means that Q is degenerate, i.e., $Q = \{y\}$. Therefore the set $f^{-1}(y)$ is connected.

Consequently the mapping f is monotone. Since any subcontinuum of a hereditarily decomposable continuum is also hereditarily decomposable, we conclude that the mapping f is hereditarily monotone. The proof of Theorem (6.8) is complete.

E. Monotone mappings on hereditarily indecomposable continua and on hereditarily unicoherent continua. It has been proved in [20], Theorem 4, p. 51 that

(6.9) *A continuum X is hereditarily indecomposable if and only if each monotone mapping from X is atomic.*

Similarly, we have the following characterization of hereditarily unicoherent continua (see [57], Corollary (3.2)) by monotone mappings.

(6.10) *A continuum X is hereditarily unicoherent if and only if each monotone mapping from X is hereditarily monotone.*

F. Mappings onto hereditarily indecomposable continua. It follows from [12], p. 243; [50], 5.7 and from [57], Corollary 3.4 that

(6.11) *The following conditions are equivalent provided X is a continuum:*

(i) *X is hereditarily indecomposable,*

(ii) *any mappings of a continuum onto X is confluent,*

(iii) *any confluent mapping of a continuum onto X is hereditarily confluent,*

(iv) *the projection mapping $p\colon X \times I \to X$ is hereditarily confluent, where $I = [0, 1]$ and $p(x, t) = x$ for each $(x, t) \in X \times I$.*

G. Mappings onto atriodic continua. Using the same methods as in the proof of Theorem (3.6) of [57], we will now generalize this theorem. Namely, we have the following

(6.12) THEOREM. *Any mapping of a continuum onto an atriodic continuum is hereditarily atriodic.*

Proof. Let a mapping f map a continuum X onto an atriodic continuum Y and let Q be an arbitrary subcontinuum of Y. We shall prove three properties of $f^{-1}(Q)$, which are needed in the sequel.

(6.12.1) *For each point $q \in Q$ there is a component C_q of $f^{-1}(Q)$ such that for each component C of the set $f^{-1}(Q)$ it is not true that $f(C_q) \subset f(C) \neq f(C_q)$.*

Let C_1 be an arbitrary component of $f^{-1}(Q)$ such that $q \in f(C_1)$ and let $\mathscr{C} = \{H\colon f(C_1) \subset H$ and there is a component K of $f^{-1}(Q)$ such that $f(K) = H\}$. Denote by $\mathscr{D}$ a maximal totally ordered subcollection of $\mathscr{C}$ and put $D = \overline{\bigcup \{H\colon H \in \mathscr{D}\}}$. It is clear that $D \in \mathscr{C}$, and thus there is a component C_q of $f^{-1}(Q)$ such that $f(C_q) = D$. Then C_q satisfies the required conditions.

(6.12.2) *There are no three components C_1, C_2 and C_3 of $f^{-1}(Q)$ such that the sets $f(C_1)$, $f(C_2)$ and $f(C_3)$ are pairwise disjoint.*

Indeed, suppose on the contrary that C_1, C_2 and C_3 are components of the set $f^{-1}(Q)$ such that the sets $f(C_1)$, $f(C_2)$ and $f(C_3)$ are pairwise disjoint. Then there are continua K_1, K_2 and K_3 such that $C_i \subset K_i \neq C_i$ and $f(K_i) \cap f(K_j) = \emptyset$ for any $i \neq j$ and $i, j = 1, 2, 3$. Since

$$Q = \bigcap_{n=1}^{3} \big(Q \cup f(K_n)\big) = \big(Q \cup f(K_j)\big) \cap \big(Q \cup f(K_i)\big) \quad \text{and} \quad f(K_i) \setminus Q \neq \emptyset$$

for any $i \neq j$ and $i, j = 1, 2, 3$, we infer that the set $Q \cup \bigcup_{i=1}^{3} f(K_i)$ is a triod. But Y is atriodic, a contradiction.

(6.12.3) *There are no components C_1, C_2 and C_3 of $f^{-1}(Q)$ such that $f(C_1) \cap f(C_i) \neq \emptyset \neq f(C_i) \setminus \big(f(C_1)\big) \cup f(C_j)$, for $i \neq j$ and $i, j = 2, 3$.*

Indeed, suppose on the contrary that there are such components C_1, C_2 and C_3 of the set $f^{-1}(Q)$. Then there is a continuum K_1 in X such that $C_1 \subset K_1 \neq C_1$ and such that $f(C_i) \setminus \big(f(K_1) \cup f(C_j)\big) \neq \emptyset$ for $i \neq j$ and $i, j = 2, 3$. The set $f(K_1) \cap \big(f(C_1) \cup f(C_2) \cup f(C_3)\big)$ is the union of two components (cf. (2.15)). Since also the set $f(C_2) \cap f(C_3)$ is the union of two components (cf. (2.15)), we can choose an open set U in Y such that $\bar{U}$ contains only that component M of the set

$$R = \big(f(C_2) \cap f(C_3)\big) \cup \Big(f(K_1) \cap \big(f(C_1) \cup f(C_2) \cup f(C_3)\big)\Big)$$

which contains the set $f(C_1)$. Consider the components P_1, P_2 and P_3 of $\big(f(K_1) \cup M\big) \cap \bar{U}$, $\big(f(C_2)\big) \cup M \cap \bar{U}$ and $\big(f(C_3) \cup M\big) \cap \bar{U}$, respectively, such that $M \subset P_1 \cap P_2 \cap P_3$. Since $M \subset P_i \cap P_j \subset R \cap \bar{U} = M$ for $i \neq j$ and $i, j = 1, 2, 3$, we infer that $M = P_1 \cap P_2 \cap P_3 = P_1 \cap P_2 = P_1 \cap P_3 = P_2 \cap P_3$. Moreover, it follows from Theorem 1 of [36], § 47, III, p. 172 that M is a proper subcontinuum of each of the continua P_1, P_2 and P_3. This means that the set $P_1 \cup P_2 \cup P_3$ is a triod. But Y is atriodic, a contradiction.

Now, let p be an arbitrary point of Q and let C_p be a component of $f^{-1}(Q)$ which is determined by (6.12.1). If $f(C_p) = Q$, then the condition of the atriodicity of f for the continuum Q is satisfied. Suppose that $f(C_p) \neq Q$. Then there is a point q such that $q \in Q \setminus f(C_p)$. Let C_q be a component of $f^{-1}(Q)$ which is defined by (6.12.1). Then

(6.12.4) $f(C_p) \cup f(C_q) = Q$.

Indeed, suppose on the contrary that $f(C_p) \cup f(C_q) \neq Q$. Then there is a point r belonging to $Q \setminus \big(f(C_p) \cup f(C_q)\big)$; take the component C_r of $f^{-1}(Q)$ which is defined by (6.12.1) .We can assume that $f(C_p) \cap$

$\cap f(C_q) \neq \emptyset$ (in any other case the proof is the same) by (6.12.2). Then $f(C_r) \cap (f(C_p) \cup f(C_q)) = \emptyset$, because in the opposite case we obtain a contradiction of (6.12.3) simply by substituting C_p, C_q and C_r for C_1, C_2 and C_3, respectively. Therefore there is a point s of Q which does not belong to $f(C_p) \cup f(C_q) \cup f(C_r)$. Let C_s be as above. It follows from (6.12.3) that $f(C_s) \cap (f(C_p) \cup f(C_q)) = \emptyset$, and by (6.12.2) we conclude that $f(C_s) \cap f(C_r) \neq \emptyset$. Thus $(f(C_p) \cup f(C_q)) \cap (f(C_r) \cup f(C_s)) = \emptyset$. Therefore there is a point x such that $x \in Q \setminus (f(C_p) \cup f(C_q) \cup f(C_r) \cup f(C_s))$; take the component C_x of $f^{-1}(Q)$ determined by (6.12.1). It follows from (6.12.2) that either $f(C_x) \cap \cap (f(C_p) \cup f(C_q)) \neq \emptyset$ or $f(C_x) \cap (f(C_r) \cup f(C_s)) = \emptyset$. In both cases we obtain a contradiction by (6.12.3). Hence (6.12.4) holds.

Now, let C be an arbitrary component of $f^{-1}(Q)$ and suppose that $f(C)$ is not contained either in $f(C_q)$ or in $f(C_p)$. Then $f(C) \cap f(C_p) \neq \emptyset \neq f(C) \cap f(C_q)$ and if the set $f(C)$ contains some component of the set $f(C_p) \cap f(C_q)$ (this set has at most two components by (2.15)), we have $f(C_p) \setminus (f(C) \cup f(C_q)) \neq \emptyset \neq f(C_q) \setminus (f(C) \cup f(C_p))$ contrary to (6.12.3). Thus suppose that the set $f(C)$ does not contain a component W of $f(C_p) \cap f(C_q)$ and $W \cap f(C) \neq \emptyset$. The set $W \cap f(C)$ has at most two components by (2.15). Denote one of them by M. Since the set $R = f(C_p) \cap f(C_q) \cap f(C)$ has a finite number of components (by (2.15)), there is an open set U in Y such that $\overline{U}$ contains only that component of the set R which contains M. Consider the components P_1, P_2 and P_3 of $f(C_p) \cap f(C) \cap \overline{U}$, $f(C_q) \cap f(C) \cap \overline{U}$ and $W \cap f(C)$, respectively, such that $M \subset P_1 \cap P_2 \cap P_3$. Since $M \subset P_i \cap P_j \subset R \cap \overline{U} = M$ for $i \neq j$ and $i, j = 1, 2, 3$, we infer that $M = P_1 \cap P_2 \cap P_3 = P_1 \cap P_2 = P_1 \cap P_3 = P_2 \cap P_3$. Moreover, M is a proper subcontinuum of each of the continua P_1, P_2 and P_3 by Theorem 1 of [36], § 47, III, p. 172. This means that the set $P_1 \cup P_2 \cup P_3$ is a triod, which contradicts the fact that Y is atriodic.

Therefore we conclude that any component C of $f^{-1}(Q)$ is contained either in $f(C_p)$ or in $f(C_q)$. Hence the condition of the atriodicity of f for the continuum Q is satisfied. Consequently the mapping f is atriodic. Moreover, since the property of being an atriodic continuum is hereditary on continua, we infer that the mapping f is hereditarily atriodic. The proof of Theorem (6.12) is complete.

The above theorem and Theorem (3.7) of [57] imply the following

(6.13) COROLLARY. *Let X be a continuum. Then the following conditions are equivalent:*

(i) *X is atriodic,*

(ii) *any mapping of a continuum onto X is hereditarily atriodic,*

(iii) *any atriodic mapping of a continuum onto X is hereditarily atriodic,*

(iv) *the projection mapping* $p: X \times I \to X$ *is hereditarily atriodic, where* $I = [0, 1]$ *and* $p(x, t) = x$ *for each* $(x, t) \in X \times I$.

There is an atriodic hereditarily unicoherent continuum such that there is a mapping of a continuum onto it, which is not weakly confluent. This is an answer to Question (3.11) of [57]. This can be seen from the following

(6.14) EXAMPLE. Let X denote the indecomposable arc-like continuum described in Example 3 of [36], § 48, V, p. 205. The mapping $f: X \to f(X)$ identifies the points (0, 0) and (1, 0). Then the continuum $f(X)$ is a hereditarily unicoherent atriodic continuum which separates the plane. Moreover, the mapping f is not weakly confluent.

We will now give a new proof of a theorem which shows that the indecomposability of $f(X)$ of the above example is essential (this theorem was first proved in [71], Theorem 4, with the use of the method of inverse systems). Recall that (see [71], Lemma, cf. also [82], Théoréme 1, p. 182)

(6.15) *Any mapping of a continuum onto an arc is weakly confluent.*

We have (cf. (2.13)) the following

(6.16) THEOREM. *Any mapping of a continuum onto an arc-like continuum is hereditarily weakly confluent.*

Proof. Let a mapping f map a continuum X onto an arc-like continuum Y and let Q be an arbitrary subcontinuum of Y. Since Y is arc-like, for each $n = 1, 2, \ldots$ there is a mapping g_n from Y onto $[0, 1]$ such that diameter of $g_n^{-1}(t)$ is less than $1/n$ for any $t \in [0, 1]$. It follows from (6.15) that the mapping $g_n f$ is weakly confluent. Thus there is a continuum C_n of X such that $g_n f(C_n) = g_n(Q)$, we can assume that the sequence $\{C_n\}$ is convergent (see [36], § 42, I, p. 45 and ibid. II, Theorem, p. 47) and let C be its limit. Then C is a continuum (see [36], § 46, II, Theorem 14, p. 139). We will prove that $f(C) = Q$. Indeed, if $x \in C$, then there is a sequence $\{x_n\}$ of points of X with $x_n \in C_n$ for any $n = 1, 2, \ldots$ such that its limit is equal to x. Since the diameter of $g_n^{-1}(g_n f(x_n))$ is less than $1/n$ and since $g_n f(x_n) \in g_n(Q)$, we infer that there is a point $y_n \in Q$ such that the distance between $f(x_n)$ and y_n is less than $1/n$. Thus $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} y_n \in Q$. Therefore $f(C) \subset Q$. Now, if $y \in Q$, then there is a point $x_n \in C_n$ such that $g_n f(x_n) = g_n(y)$. Thus the distance between $f(x_n)$ and y is less than $1/n$. We can assume that the sequence $\{x_n\}$ is convergent (because X is compact) and let x be its limit. Then $x \in C$, and the distance between $f(x)$ and y is equal to zero. Hence $f(x) = y \in f(C)$. This implies that $Q \subset f(C)$.

Consequently the mapping f is weakly confluent. Moreover, since the arc-likeness is hereditarily on continua, we conclude that f is hereditarily weakly confluent. The proof of Theorem (6.16) is complete.

Theorems (2.12) and (6.16) imply (cf. also [57], Theorem (3.9) and Corollary (3.12)) that

(6.17) COROLLARY. *Let a continuum X be hereditarily decomposable. The following conditions are equivalent:*

(i) *X is arc-like,*

(ii) *any mapping of a continuum onto X is hereditarily weakly confluent,*

(iii) *any weakly confluent mapping of a continuum onto X is hereditarily weakly confluent.*

(iv) *the projection mapping $p\colon X\times I\to X$ is hereditarily weakly confluent, where $I=[0,1]$ and $p(x,t)=x$ for each $(x,t)\in X\times I$.*

The above considerations are associated with the following problem: (see [45], Problem 1): characterize continua which have the property that any mapping of a continuum onto X is weakly confluent. It is known (see [57], Examples (3.14) and (13.5)) that there are hereditarily decomposable triodic continua having this property.

Note also (see [15]) that

(6.18) *A continuum X is atriodic and Suslinian if and only if there exists no weakly confluent mapping of X onto a simple triod.*

We will now study invariance properties of mappings. Namely, we will recall and show some results which say which classes of continua (recalled in § 2) are preserved under the mappings investigated here.

7. Images of unicoherent continua

In this section we will investigate unicoherent invariances of mappings; namely, unicoherent continua and classes of continua which are contained in the class of unicoherent continua (cf. Table I).

A. Images of unicoherent continua in general. It has been proved (see [83], (8.61), p. 154) that

(7.1) *Quasi-monotone images of unicoherent continua are unicoherent.*

All the remaining classes of mappings except the classes which are contained in the class of quasi-monotone mappings (cf. Table II) do not preserve the unicoherence of continua. This can be seen from the following examples:

(7.2) EXAMPLE. The natural projection of the standard solenoid onto the first circle $S_1 = S$ is an open mapping. The solenoid is a hereditarily unicoherent, atriodic, indecomposable continuum and S is not unicoherent, It is not irreducible either (cf. remarks below (2.5) here, see also [9], p. 218).

(7.3) EXAMPLE. Let a mapping f from the unit interval $I = [0, 1]$ identify points $1/n$ with the point 0 for $n = 1, 2, \ldots$ The mapping f is joining and $f(I)$ is not unicoherent, not a local dendrite and not an irreducible continuum (see Fig. 7).

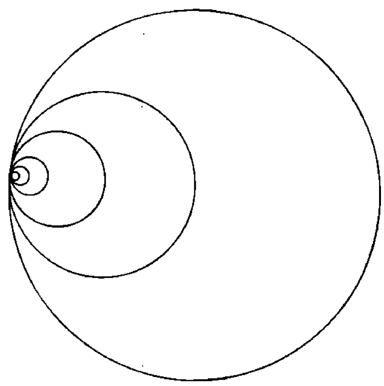


Fig. 7

(7.4) EXAMPLE. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates and put

$$A = \{(x, \sin \pi/x)\colon 0 < x \leqslant 1\} \cup \{(0, y)\colon -1 \leqslant y \leqslant 1\}.$$

The continuum A is an atriodic λ-dendroid. Let a mapping f identify the points $(0, 1)$ and $(0, -1)$ (remark that f is quasi-monotone), and denote the image of X under f by Y (see Fig. 8). The continuum Y

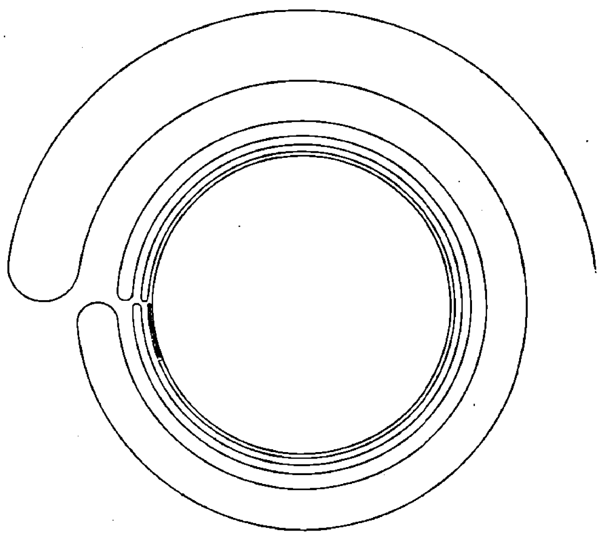


Fig. 8

si unicoherent (it is also irreducible), but it is not hereditarily unicoherent, because it contains a circle $S = f(J)$, where J is a closed interval joining the points $(0, 1)$ and $(0, -1)$. Let a mapping g map Y onto S such that. $g^{-1}(g(f(0, 1)))$ is a one-point set (it is clear that there is such a mapping). One can show that the mapping g is hereditarily weakly confluent.

B. Images of hereditarily unicoherent continua. It follows from Theorem (4.44) (cf. [57], Theorem (5.6)) that

(7.5) *Hereditarily confluent images of hereditarily unicoherent continua are hereditarily unicoherent.*

Moreover,

(7.6) *Locally monotone images of hereditarily unicoherent continua are hereditarily unicoherent.*

Indeed, let a locally monotone mapping f map a hereditarily unicoherent continuum X onto Y and let Q be an arbitrary subcontinuum of Y. It suffices to show that Q is unicoherent. Since the mapping f is confluent (cf. Table II), there is a component C of $f^{-1}(Q)$ such that $f(C) = Q$. It is clear that the mapping $f|C$ is locally monotone. Thus $f|C$ is quasi-monotone (cf. Table II). The continuum C is unicoherent, and thus $f(C)$ is unicoherent by (7.1), i.e., Q is unicoherent.

The following problem remains open (cf. [57], Question (5.8)):

(7.7) PROBLEM. *Is a hereditarily weakly confluent image of a hereditarily unicoherent continuum also hereditarily unicoherent?*

It follows from Examples (7.2), (7.3) and (7.4) and from the example described below that the remaining classes of mappings except the classes which are contained in the class of hereditarily confluent mappings or in the class of locally monotone mappings do not preserve the hereditary unicoherence of continua.

(7.8) EXAMPLE. Let a mapping f from the unit interval $[0, 1]$ onto the unit circle S identify the points 0 and 1. The mapping f is hereditarily atriodic (cf. Theorem (6.12)).

C. Images of acyclic curves. The one-dimensionality of continua is not preserved by mappings in general. In fact each locally connected continuum can be represented as the image of the Menger universal curve under a mapping that is monotone and open (see [2], p. 348). One can observe also that the locally homeomorphic image of a curve is a curve. We will now investigate images of acyclic curves. It is known that (see [64], p. 328)

(7.9) *If* $\dim X \geqslant 2$, *then there exists a weakly confluent mapping of X onto a 2-cell.*

Let K be a continuum which consists of all points in the plane having polar coordinates (r, φ) for which $r = 1$, $r = 2$ or $r = (2+e^{\varphi})/(1+e^{\varphi})$ (see Fig. 9). By an easy modification of the proof of Theorem 1 of [24]. pp. 541 and 542 one can find that

(7.10) *If a mapping f maps an acyclic continuum X into K, then $f(K)$ is contained in a single arc component of K.*

We have the following

(7.11) THEOREM. *The atriodic (joining, pseudo-confluent) image of an acyclic curve is at most one-dimensional.*

Proof. The proof for pseudo-confluent mappings is given in [51], Theorem 5.5; we will now only prove (7.11) for atriodic mappings and joining mappings by an easy modification of that proof. Suppose on the contrary that an atriodic (joining) mapping f maps an acyclic curve X onto a continuum Y of dimension $\dim Y \geqslant 2$. By (7.9), there exists a

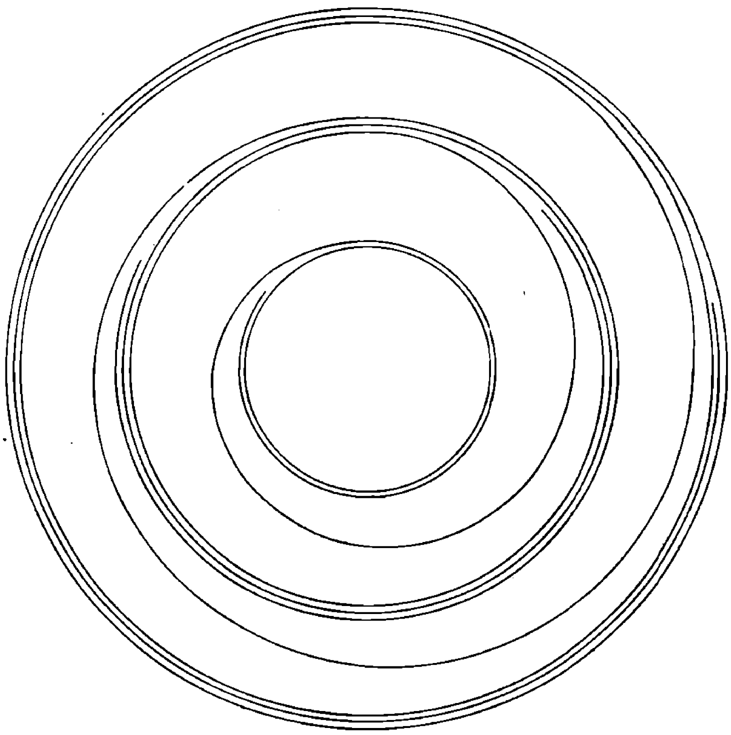


Fig. 9

weakly confluent mapping g of Y onto the 2-cell I^2. Let L be a homeomorphic copy of K (described above) which is contained in I^2. Since g is weakly confluent, we infer that there is a subcontinuum Q in Y such that $g(Q) = L$. Now, let C be an arbitrary component of $f^{-1}(Q)$. Since C is acyclic (see [36], pp. 332 and 354), we conclude that $gf(C)$ is contained in some single arc component of L by (7.10).

If f is atriodic, then there are two components C_1 and C_2 of $f^{-1}(Q)$ such that $Q = f(C_1) \cup f(C_2)$. Since the set $gf(C_1) \cap gf(C_2)$ is nonempty, we conclude that $gf(C_1 \cup C_2)$ is contained in some single arc component of L. But L has three arc components, and thus $gf(C_1 \cup C_2) = g(Q) \neq L$, a contradiction of the choice of Q.

If f is joining, then for each two components C_1 and C_2 of $f^{-1}(Q)$ we have $f(C_1) \cap f(C_2) \neq \emptyset$. Thus $gf(C_1) \cap gf(C_2) \neq \emptyset$. Therefore the set $gf(C_1)$ and the set $gf(C_2)$ are contained in the same arc component of L. Consequently $gf(f^{-1}(Q))$ is contained in some single arc component of L. But L has three arc components, and thus $gf(f^{-1}(Q)) = g(Q_0) \neq L$, a contradiction of the choice of Q. The proof of Theorem (7.11) is complete.

One can observe that

(7.12) LEMMA. *If a space X is of dimension greater than* 1, *then for* *every positive number ε there is a subcontinuum R of X of diameter less than ε and of dimension greater than* 1.

Indeed, since X is of dimension greater than 1, there is a closed subset V of diameter less than ε and of dimension greater than 1. By Theorem 3 of [36], § 46, V, p. 148, there is a continuous mapping of V into the Cantor discontinuum C such that the quasi-components of V (hence its components, see [36], § 47, II, Theorem 2, p. 169) coincide with the point inverses of f. If the condition $\dim f^{-1}(t) \leqslant 1$ is satisfied for all t of $f(V)$, then $\dim f(V) \geqslant \dim V - 1 \geqslant 1$ by Theorem 1 of [36], § 45, VI, p. 114. But $\dim f(V) \leqslant \dim C = 0$, a contradiction. Therefore $\dim f^{-1}(t) \geqslant 2$ for some $t \in f(V)$. This means that some component R of V is of dimension greater than 1.

Now, we will generalize Theorem 1.2 of [15]. Namely

(7.13) THEOREM. *The locally weakly confluent image of an acyclic curve is a curve.*

Proof. Suppose on the contrary that a locally weakly confluent mapping f maps an acyclic curve X onto a continuum Y of dimension $\dim Y \geqslant 2$. By (4.37) there is a positive number ε such that for each continuum Q of diameter less than ε in Y there exists a component C of $f^{-1}(Q)$ such that $f(C) = Q$. By Lemma (7.12) we conclude that there is a subcontinuum R of Y of diameter less than ε and of dimension $\dim R \geqslant 2$. By (7.9) there exists a weakly confluent mapping g of R onto the 2-cell I^2. Let L be a homeomorphic copy of K (described above) which is contained in I^2. Since g is weakly confluent, we infer that there is a continuum Q in R such that $g(Q) = L$. Since Q is of diameter less than ε, there is a component C of $f^{-1}(Q)$ such that $f(C) = Q$. Thus $gf(C) = L$. Since C is acyclic (see [36], pp. 332 and 354), we conclude that $gf(C)$ is contained in some single arc component of L by (7.10). This contradicts the fact that L has three arc components and $gf(C) = L$. The proof of Theorem (7.13) is complete.

The quasi-monotone image of an acyclic curve need not be a curve. This results from the following

(7.14) EXAMPLE. Let a continuum A be such as in Example (7.4). There is a mapping f from A which maps the segment J of the convergence of A onto the unit square I^2 and which maps the line approximating J homeomorphically onto the line approximating I^2. Such a mapping is quasi-monotone. But the continuum $f(A)$ is of dimension 2 (thus it contains also indecomposable continua, cf. [64], p. 328).

It has been proved in [41], p. 230 that

(7.15) *The confluent image of an acyclic continuum is acyclic.*

Thus, by Theorem (7.11), we conclude that (cf. [49], 2.7)

(7.16) *The confluent image of an acyclic curve is an acyclic curve.*

It follows from the example described below that the locally confluent image of an acyclic curve need not be an acyclic curve.

(7.17) EXAMPLE. Let B_0 be the simplest example of an indecomposable continuum described in Example 1 of [36], § 48, V, p. 204, and let (x, y, z) denote a point of the Euclidean 3-space having x, y and z as its rectangular coordinates. Put

$$B_1 = \{(x, y, 0)\colon (x-3/2, y) \in B_0 \text{ and } x-3/2 \geqslant 1/2\},$$

$$B_2 = \{(x, y, 0)\colon (x+3/2, y) \in B_0 \text{ and } x+3/2 \leqslant 1/2\},$$

$$I_0 = \{(-3/2, y, 0)\colon -1 \leqslant y \leqslant 0\},$$

$$I_1 = \{(7/3, y, 0)\colon -1 \leqslant y \leqslant 0\},$$

$$I_n = \left\{\left(\frac{5}{2\cdot 3^n} - \frac{3}{2}, y, 0\right)\colon -1 \leqslant y \leqslant 0\right\} \quad \text{for} \quad n = 2, 3, \ldots,$$

$$A = \{(x, -1, 0)\colon -3/2 \leqslant x \leqslant 5/2\},$$

$$C_y = \{(x, y, \sin 2\pi/(x-1))\colon 1 \leqslant x \leqslant 2\} \cup \{(1, y, z)\colon -1 \leqslant z \leqslant 1\},$$

$$C'_y = \{(x, y, \sin 2\pi/x)\colon -1 \leqslant x \leqslant 0\} \cup \{(0, y, z)\colon -1 \leqslant z \leqslant 1\}$$

and define

$$X = A \cup B_1 \cup B_2 \cup \bigcup_{n=0}^{\infty} I_n \cup \bigcup \{C_y \cup C'_y : y \geqslant 1/2 \text{ and } y \text{ belongs to the Cantor ternary set lying in the interval } [0, 1]\}.$$

It is easy to observe that the set X is a continuum which is hereditarily divisible by points (cf. Table I). We define a locally confluent mapping f of X as follows:

$$f(x, y, z) = \begin{cases} (x-1/2, y, z) & \text{if } 1 \leqslant x, \\ (1/2, y, z) & \text{if } 0 \leqslant x \leqslant 1, \\ (x+1/2, y, z) & \text{if } x \leqslant 0. \end{cases}$$

The continuum $f(X)$ is not hereditarily unicoherent and it contains an indecomposable continuum which can be mapped onto B_0 by a monotone mapping.

Similarly the weakly confluent image of an acyclic curve need not be acyclic. This can be seen from the following

(7.18) EXAMPLE. Let (r, φ) denote a point of the Euclidean plane having r and φ as its polar coordinates. The weakly confluent mapping f defined by $f(t) = (1, 4\pi t)$ for each $t \in [0, 1]$ maps the unit interval $[0, 1]$ onto the unit circle S. It is also locally semi-confluent.

The following problem remains open:

(7.19) PROBLEM. *Is a semi-confluent (hereditarily weakly confluent) image of an acyclic (one-dimensional) continuum also an acyclic (one-dimensional, respectively) continuum?*

D. Images of tree-like continua. We have the following two general theorems about images of tree-like continua, which have been proved by J. Krasinkiewicz (see [31], Theorem 3.1 and [32], Main Theorem).

(7.20) *A continuous imgae of a tree-like continuum is tree-like if and only if it is an acyclic curve.*

(7.21) *If f is a mapping of a tree-like continuum X onto a curve Y such that $f(C)$ is tree-like for every irreducible continuum C contained in X, then Y is tree-like.*

From (7.16) and (7.20) we obtain the following theorem which was first proved in [63], p. 472.

(7.22) *The confluent image of a tree-like continuum is tree-like.*

Similarly, a positive solution of Problem (7.19) would imply a positive answer to the following open question (see [52], Question 5, p. 263 and [57], Question 5.25, see also [26], Theorem 2).

(7.23) PROBLEM. *Is the semi-confluent (hereditarily weakly confluent), image of a tree-like continuum also tree-like?*

The remaining classes of mappings except the classes contained in the class of confluent mappings do not preserve the tree-likeness of continua. This can be seen from Examples (7.3), (7.4), (7.8), (7.17) and (7.18).

E. Images of λ-dendroids. It follows from Theorem XIV of [9], p. 217, from Theorem 5.2 of [52], p. 262 and from Corollary (5.17) of [57] that

(7.24) *The semi-confluent (hereditarily weakly confluent) image of a λ-dendroid is a λ-dendroid.*

The remaining classes of mappings except the classes contained in one of the two classes of mappings mentioned in (7.24) do not preserve

λ-dendroids. This can be seen from Examples (7.3), (7.4), (7.8), (7.17) and (7.18).

F. Images of continua which are hereditarily divisible by points. The invariance of continua which are hereditarily divisible by points under mappings has not yet been investigated. Firstly, remark that (7.24) and (6.1) imply that

(7.25) *The locally homeomorphic image of a continuum which is hereditarily divisible by points is also such a continuum.*

Moreover, we have the following

(7.26) THEOREM. *The monotone image of a continuum which is hereditarily divisible by points is also such a continuum.*

Proof. Let a monotone mapping f map a continuum X which is hereditarily divisible by points onto a continuum Y and let Q be an arbitrary subcontinuum of Y. Take a subcontinuum C of X which is minimal with respect to the property $f(C) = Q$ (such a continuum exists by the monotoneity of f). Since the continuum X is hereditarily unicoherent (cf. Table I), we infer that the mapping $f|C$ is monotone by (6.10). According to the assumptions there is a point c and there are proper subcontinua A and B of C such that $A \cap B = \{c\}$ and $A \cup B = C$. Suppose that the point $f(c)$ does not separate the continuum Q. Then the set $Q \setminus \{f(c)\}$ is connected. Therefore $(f|C)^{-1}(Q \setminus \{f(c)\})$ is connected by Theorem 9 of [36], § 46, I, p. 131. Thus

$$\text{either} \quad (f|C)^{-1}(Q \setminus \{f(c)\}) \subset A \setminus \{c\} \quad \text{or} \quad (f|C)^{-1}(Q \setminus \{f(c)\}) \subset B \setminus \{c\}.$$

But then we have either $f(A) = Q$ or $f(B) = Q$ contrary to the choice of C. This means that the point $f(c)$ separates the continuum Q. Consequently the continuum Y is hereditarily divisible by points. The proof of (7.26) is complete.

It follows from (2.16), (6.8) and (7.26) that

(7.27) COROLLARY. *The hereditarily confluent image of a continuum which is hereditarily divisible by points is also such a continuum.*

The following problem remains open:

(7.28) PROBLEM. *Is an open (locally monotone) image of a continuum which is hereditarily divisible by points also such a continuum?*

It follows from Examples (7.3), (7.8), (7.17) and (7.18) and from the example described below that the remaining classes of mappings do not preserve the continua considered here.

(7.29) EXAMPLE. Consider the Euclidean plane endowed with the rectangular coordinate system $0xy$. Let C be the Cantor ternary set situated in the interval $[0, 1]$. Put (see Fig. 10)

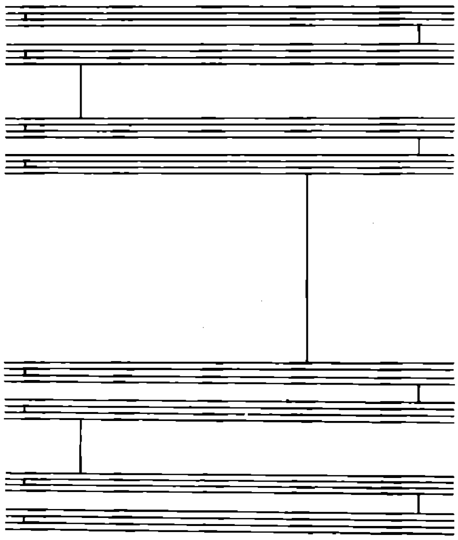


Fig. 10

$$X = \{(x, y)\colon x \in C \text{ and } 0 \leqslant y \leqslant 1\} \cup \bigcup_{n=1}^{\infty} \{(x, 1/3^{2n-1})\colon x \text{ belongs to the intervals contiguous to } C \text{ with lengths } 1/3^{2n-1}\} \cup \bigcup_{n=1}^{\infty} \{(x, 1-(1/3^{2n}))\colon x \text{ belongs to the contiguous intervals to } C \text{ with lengths } 1/3^{2n}\}.$$

The continuum X is irreducible between each point with abscissa 0 and each point with abscissa 1. Moreover, X is hereditarily divisible by points. The mapping $f\colon X \to f(X)$ identifies the point (x, y) with the point (x', y') if and only if $y = y'$ and either x and x' belong to the same closed interval contiguous to C with length $1/3^{2n-1}$ and $y \leqslant 1/3^{2n-1}$ or x and x' belong to the same closed interval contiguous to C with length $1/3^{2n}$ and $y \geqslant 1-(1/3^{2n})$ for some $n = 1, 2, \ldots$ It is easy to observe that the mapping f is confluent, quasi-monotone and hereditarily weakly confluent. Moreover, $f(X)$ is an irreducible continuum, which has no separating point.

F. Images of dendroids. Since the arcwise connectedness is an invariant under an arbitrary continuous mapping (see [83], p. 39) by (7.24) we infer that

(7.30) *The semi-confluent (hereditarily weakly confluent) image of a dendroid is a dendroid.*

Moreover (see [58], Theorem 3),

(7.31) *The weakly monotone image of a dendroid is a dendroid.*

Further, we will prove that

(7.32) THEOREM. *The locally confluent image of a dendroid is a dendroid.*

Proof. Let a locally confluent mapping f map a dendroid X onto a continuum Y. Then the continuum X is tree-like (cf. Table I). Thus also X is an acyclic curve. By Theorem (7.13) (cf. Table II) we conclude that Y is one-dimensional. From (7.21) we infer that it suffices to show that the image under f of an arbitrary arc in X is tree-like. Let A be an arbitrary arc in X and let C be a component of $f^{-1}f(A)$ such that $A \subset C$. The partial mapping $f|C$ is locally confluent.

Indeed, let ε be a positive number such that if a subcontinuum Q of Y is of diameter less than ε, then each component of $f^{-1}(Q)$ is mapped by f onto Q (cf. Theorem (4.38)). If Q is a subcontinuum of $f(A)$ of diameter less than ε and if C' is a component of the set $C \cap f^{-1}(Q)$, then C' is also a component of $f^{-1}(Q)$, because C is a component of $f^{-1}f(A)$ and $f^{-1}(Q) \subset f^{-1}f(A)$. Thus $f(C') = Q$. This means that the mapping $f|C$ is locally confluent by Theorem (4.38).

Since $A \subset C \subset f^{-1}f(A)$, we infer that the equality $f(C) = f(A)$ holds. The continuum $f(A)$ is locally connected as a continuous image of the arc A. Therefore the mapping $f|C$ is confluent by (6.2). The continuum C is a dendroid as a subcontinuum of a dendroid X. Thus the equality $f(C) = f(A)$ implies that $f(A)$ is a dendroid by (7.30). Hence (cf. Table II) $f(A)$ is a tree-like continuum (cf. Table I). The proof of Theorem (7.32) is complete.

Each class of mappings considered here which is not contained in the class of semi-confluent mappings, in the class of hereditarily weakly confluent mapping or in the class of locally confluent mappings does not necessarily preserve dendroids. This can be seen from Examples (732), (7.8) and (7.18).

G. Images of fans. It follows from [10], Theorem 12, p. 32, from [52], Theorem 5.6, p. 263 and from [57], Corollary 5.23 that

(7.33) *The semi-confluent (hereditarily weakly confluent) image of a fan is a fan (or an arc).*

We also have the following

(7.34) COROLLARY. *The locally confluent image of a fan is a fan (or an arc).*

Indeed, let a locally confluent mapping f map a fan X onto Y. Then X is a dendroid (cf. Table I). Therefore Y is a dendroid by Theorem

(7.32). Since a dendroid is hereditarily arcwise connected, we infer that f is confluent by (6.4). Hence Y is a fan by (7.33) (cf. Table II).

Examples (7.3), (7.8) and (7.18) and also example described below (cf. [58], Example 3) imply that any class of mappings investigated here which is not contained in any of the three classes mentioned in (7.33) and (7.34) does not preserve fans.

(7.35) EXAMPLE. Let A denote a harmonic fan lying in the plane $0xy$ and consisting of a straight segment J joining the point $(0,1)$ with the point $(1,0)$ and of straight segments joining the same point with points $(1+(1/n), 0)$ for $n = 1, 2, \ldots$ Denote by B the image of A under the reflection r with respect to the line $x = 0$. Put $X = A \cup B$ (see Fig. 11).

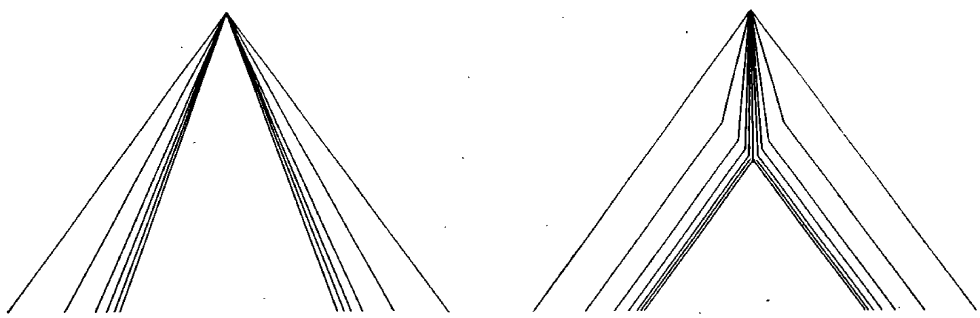


Fig. 11

The mapping $f\colon X \to f(X)$ identifies two different points (x, y) and (x', y') if and only if $y = y' = 1/2$ and either $(x, y) \in J$ and $(x', y') = r(x, y)$ or $(x', y') \in J$ and $(x, y) = r(x', y')$. Since the inverse image under f of each subcontinuum of $f(X)$ with a nonempty interior is connected, we conclude that f is quasi-monotone. The continuum $f(X)$ is a dendroid having two ramification points.

H. Images of dendrites. Since local connectedness is an invariant under arbitrary continuous mappings, we conclude from (7.30), (7.31) and (7.32) that

(7.36) *The semi-confluent (locally confluent, weakly monotone, hereditarily weakly confluent) image of a dendrite is a dendrite.*

The remaining classes of mappings considered here, except the classes which are contained in the classes mentioned above in (7.36), do not preserve dendrites. This can be seen from Examples (7.3), (7.8) and (7.18).

E. D. Tymchatyn in [79], Corollary 2, characterizes continua which are weakly confluent images of dendrites.

Note also the following interesting theorem (see [83], Theorem 2.4, p. 188):

TABLE IV

	dendrites	fans	dendroids	λ-dendroids	continua hereditarily divisible by points	tree-like continua	acyclic curves	hereditarily unicoherent continua	unicoherent continua
1	2	3	4	5	6	7	8	9	10
local homeomorphisms	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	+ (7.25)	+ (7.22)	+ (7.16)	+ (7.6)	+ (7.1)
hereditarily monotone mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	+ (7.26)	+ (7.22)	+ (7.16)	+ (7.6)	+ (7.1)
atomic mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	+ (7.26)	+ (7.22)	+ (7.16)	+ (7.6)	+ (7.1)
monotone mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	+ (7.26)	+ (7.22)	+ (7.16)	+ (7.6)	+ (7.1)
open mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	? (7.28)	+ (7.22)	+ (7.16)	− (7.2)	− (7.2)
MO-mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	? (7.28)	+ (7.22)	+ (7.16)	− (7.2)	− (7.2)
locally monotone mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	? (7.28)	+ (7.22)	+ (7.16)	+ (7.6)	+ (7.1)
locally MO-mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	? (7.28)	+ (7.22)	+ (7.16)	− (7.2)	− (7.2)
OM-mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	? (7.28)	+ (7.22)	+ (7.16)	− (7.2)	− (7.2)
hereditarily confluent mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	+ (7.27)	+ (7.22)	+ (7.16)	+ (7.5)	+ (7.1)
quasi-monotone mappings	+ (7.36)	− (7.35)	+ (7.31)	− (7.4)	− (7.29)	− (7.4)	− (7.4)	− (7.4)	+ (7.1)
weakly monotone mappings	+ (7.36)	− (7.35)	+ (7.31)	− (7.4)	− (7.29)	− (7.4)	− (7.4)	− (7.2)	− (7.2)
confluent mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	− (7.29)	+ (7.22)	+ (7.16)	− (7.2)	− (7.2)

Table IV cont.

1	2	3	4	5	6	7	8	9	10
locally confluent mappings	+ (7.36)	+ (7.34)	+ (7.32)	− (7.17)	− (7.17)	− (7.17)	− (7.17)	− (7.2)	− (7.2)
semi-confluent mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	− (7.29)	? (7.23)	? (7.19)	− (7.2)	− (7.2)
locally semi-confluent mappings	− (7.18)	− (7.18)	− (7.18)	− (7.17)	− (7.17)	− (7.17)	− (7.17)	− (7.2)	− (7.2)
joining mappings	− (7.3)	− (7.3)	− (7.3)	− (7.3)	− (7.3)	− (7.3)	− (7.3)	− (7.3)	− (7.3)
hereditarily weakly confluent mappings	+ (7.36)	+ (7.33)	+ (7.30)	+ (7.24)	− (7.29)	? (7.23)	? (7.19)	? (7.7)	− (7.4)
weakly confluent mappings	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.2)	− (7.2)
locally weakly confluent mappings	− (7.18)	− (7.18)	− (7.18)	− (7.17)	− (7.17)	− (7.17)	− (7.17)	− (7.2)	− (7.2)
pseudo-confluent mappings	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.2)	− (7.2)
hereditarily atriodic mappings	− (7.8)	− (7.8)	− (7.8)	− (7.8)	− (7.8)	− (7.8)	− (7.8)	− (7.8)	− (7.4)
atriodic mappings	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.18)	− (7.2)	− (7.2)

(7.37) *If $f\colon X \to Y$ is light and open, then for each dendrite D in Y and any point x of $f^{-1}(D)$ there exists a dendrite E in X containing x which is mapped topologically onto D under f.*

Such an implication is not true for dendroids (see [62], Example 2).

Table IV sums up the invariance properties studied in § 7. The sign "+" and the sign "−" denote that the corresponding class of mappings preserves or does not preserve, respectively the corresponding class of continua. The number under the sign is the number of the proposition which justifies the use the sign in question (cf. Table II).

8. Images of irreducible continua and of atriodic continua

In this section we consider problems concerning the invariance of irreducible continua and of atriodic continua, and also the invariance of unicoherent subclasses of atriodic continua (any continuum belonging to such a class is irreducible by (2.19)) under an arbitrary class of mappings considered in this paper.

A. Images of irreducible continua in general and of indecomposable continua. Theorem 3 in [27] is an answer to the question which was posed in [38] and which was first partially solved in [6]. In Theorem 4 of [55] we generalize this theorem, and using (2.18) we infer that

(8.1) *If X is a continuum, A is a set of irreducibility of X and f is a quasi-monotone mapping from X onto Y, then $f(A)$ is a set of the irreducibility of Y.*

Therefore (cf. [36], § 48, VI, Theorem 7′, p. 213),

(8.2) *The quasi-monotone image of an irreducible (indecomposable) continuum is irreducible (indecomposable, respectively).*

Each class of mappings considered here which is not contained in the class of quasi-monotone mappings does not preserve either irreducible continua or indecomposable continua. This can be seen from Examples (7.2), (7.3) and (7.4) and from the following

(8.3) EXAMPLE. Let X denote a pseudo-arc. Obviously there is a continuous mapping from X onto the interval $[0, 1]$. Each such mapping is hereditarily weakly confluent by Theorem (6.16) (cf. Table I).

B. Images of atriodic continua. It has been proved in [57], Proposition (5.19) that

(8.4) *The confluent image of an atriodic continuum is atriodic.*

The following problem remains open (cf. [57], Question (5.18)):

(8.5) PROBLEM. *Is the semi-confluent (hereditarily weakly confluent) image of an atriodic continua always atriodic?*

Examples (7.3) and (7.14) and two examples described below show that classes of mappings which are not contained either in the class of semi-confluent mappings or in the class of hereditarily weakly confluent mappings do not preserve the atriodicity of continua.

(8.6) EXAMPLE. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates. Put

$$X = \{(x, \sin 2\pi/x):\ 0 < x \leqslant 1\} \cup \{(0, y):\ -5 \leqslant y \leqslant 1\} \cup \cup \{(x,\ -4+\sin 2\pi/x):\ -1 \leqslant x < 0\}$$

and define a mapping $f\colon X \to f(X)$ as follows:

$$f(x, y) = (x, |y+2|) \quad \text{for each } (x, y) \in X.$$

It is easy to ascertain that f is a locally confluent mapping which maps the atriodic λ-dendroid X onto the atriodic continuum $f(X)$.

(8.7) EXAMPLE. The continuum X and the mapping f described in Example (5.20) of [57] have the following properties: f is weakly confluent and hereditarily atriodic, X is an atriodic λ-dendroid which is also irreducible, and $f(X)$ is a simple triod (cf. (6.18)).

C. Images of hereditarily unicoherent atriodic continua. It follows from (7.5), (7.6) and (8.4) (cf. Table II) that

(8.8) *The hereditarily confluent (locally monotone) image of a hereditarily unicoherent atriodic continuum is also such a continuum.*

The following problem remains open (cf. Problems (7.7) and (8.5)).

(8.9) *Is the hereditarily weakly confluent image of a hereditarily unicoherent atriodic continuum also such a continuum?*

Examples (7.2), (7.3), (7.4) and (7.8) show that classes of mappings which are not contained either in the class of locally monotone mappings or in the class of hereditarily weakly confluent mappings do not preserve hereditarily unicoherent atriodic continua.

D. Images of hereditarily indecomposable continua. It is an immediate consequence of the definition of the confluence that

(8.10) *The confluent image of a hereditarily indecomposable continuum is hereditarily indecomposable.*

The following problem is open:

(8.11) *Is the semi-confluent (locally confluent, locally semi-confluent) image of a hereditarily indecomposable continuum also such a continuum?*

It is my conjecture that there is a locally confluent mapping of a pseudo-arc (which is hereditarily indecomposable; compare Table I) onto a continuum which is not hereditarily indecomposable, but I do not know any such example.

Example (8.3) and the example described below show that no class of mappings considered here which is not contained in the class of locally semi-confluent mappings preserves hereditarily indecomposable continua.

(8.12) EXAMPLE. Let A denote a pseudo-arc and let a and b be two arbitrary different points of A. The mapping $f\colon A \to f(A)$ identifies two different points x and y of A if and only if either $x = a$ and $y = b$ or $x = b$ and $y = a$. It is easy to observe by Theorem 2″ of [36], § 48, VI, p. 210 that the continuum $f(A)$ is indecomposable. Therefore any proper subcontinuum of $f(A)$ has an empty interior (see [36], § 48, V, Theorem 2, p. 207). Thus f is quasi-monotone. Moreover, it is clear that f is joining and that the continuum $f(A)$ is not hereditarily indecomposable.

E. Images of arc-like continua. It follows from [5], p. 47 and from [74], Theorem 1.0, p. 259 that

(8.13) *The image of an arc-like continuum under an* OM-*mapping is arc-like.*

We have (see [43], p. 94 and [52], p. 263; cf. [49], Problem II) the following.

(8.14) PROBLEM. *Is the confluent (semi-confluent) image of an arc-like continuum always arc-like?*

Also the following problem remains open (cf. [57], Question (5.25)):

(8.15) PROBLEM. *Is the hereditarily confluent (hereditarily weakly confluent) image of an arc-like continuum always arc-like?*

Examples (7.3), (7.4), (7.8), (7.16), (7.18) and (8.6) show that the remaining classes of mappings which are not contained in the class of semi-confluent mappings as well as the class of hereditarily atriodic mappings do not preserve the arc-likeness of continua.

Continuous images of arc-like continua are characterized in [40].

F. Images of an pseudo-arc. Known theorems which speak about the invariance of the pseudo-arc under mappings follow from theorems which speak about the invariance of the arc-likeness of continua and from theorems which say which mappings preserve the hereditarily indecomposability of continua because each two arc-like hereditarily indecomposable (nondegenerate) continua are homeomorphic (see [69], p. 583). Namely, we conclude by (8.10) and (8.13) (cf. Table II) that

(8.16) *The image of a pseudo-arc under an* OM-*mappings is a pseudo-arc.*

Examples (8.3) and (8.12) show that mappings which belong to the one of the classes of mappings investigated here and which are not locally semi-confluent do not preserve a pseudo-arc. The answers to the following question are unknown (cf. Problems (8.11), (8.4) and (8.15)):

(8.17) PROBLEM. *Is the confluent (hereditarily confluent, locally confluent, semi-confluent, locally semi-confluent) image of a pseudo-arc always a pseudo-arc?*

G. Images of atriodic λ-dendroids. It follows from (7.24) and (8.4) (cf. (2.12)) that

(8.18) *The confluent image of an atriodic λ-dendroid is an atriodic λ-dendroid.*

Moreover, it has been proved in [57], Corollary 5.17 that

(8.19) *The hereditarily weakly confluent image of an atriodic λ-dendroid is an atriodic λ-dendroid.*

The following theorem results from Theorem 1 of [26]:

(8.20) *The semi-confluent image of an atriodic λ-dendroid is an atriodic λ-dendroid.*

All the investigated classes of mappings which are not contained in the class of semi-confluent mappings, as well as hereditarily atriodic mappings, have the property of not preserving atriodic λ-dendroids. This can be seen from Examples (7.3), (7.4), (7.8), (7.16), (7.18) and (8.6).

H. Images of an arc. It follows from Theorem 5.5 of [52], p. 262 and from Corollary (5.21) of [57] that

(8.21) *The semi-confluent (hereditarily weakly confluent) image of an arc is an arc.*

Therefore, we infer by (6.4) (cf. Table II) that

(8.22) *The locally confluent image of an arc is an arc.*

Examples (7.3), (7.8) and (7.18) show that an arc is an invariant only for those classes of mappings among the classes investigated here which are contained in one of the three classes of mappings mentioned in (8.21) and in (8.22).

Note also that (see [54], Corollary 3.3, p. 67)

(8.23) *The locally weakly confluent image of an arc (of a circle) is either an arc or a circle.*

TABLE V

	arc	atriodic λ-dendroids	psedo-arc	arc-like continua	hereditarily indecomposable continua	hereditarily unicoherent atriodic continua	atriodic continua	indecomposable continua	irreducible continua
1	2	3	4	5	6	7	8	9	10
local homeomorphisms	+ (8.21)	+ (8.18)	+ (8.16)	+ (8.13)	+ (8.10)	+ (8.8)	+ (8.4)	+ (8.2)	+ (8.2)
hereditarily monotone mappings	+ (8.21)	+ (8.18)	+ (8.16)	+ (8.13)	+ (8.10)	+ (8.8)	+ (8.4)	+ (8.2)	+ (8.2)
atomic mappings	+ (8.21)	+ (8.18)	+ (8.16)	+ (8.13)	+ (8.10)	+ (8.8)	+ (8.4)	+ (8.2)	+ (8.2)
monotone mappings	+ (8.21)	+ (8.18)	+ (8.16)	+ (8.13)	+ (8.10)	+ (8.8)	+ (8.4)	+ (8.2)	+ (8.2)
open mappings	+ (8.21)	+ (8.18)	+ (8.16)	+ (8.13)	+ (8.10)	− (7.2)	+ (8.4)	− (7.2)	− (7.2)
MO-mappings	+ (8.21)	+ (8.18)	+ (8.16)	+ (8.13)	+ (8.10)	− (7.2)	+ (8.4)	− (7.2)	− (7.2)
locally monotone mappings	+ (8.21)	+ (8.18)	+ (8.16)	+ (8.13)	+ (8.10)	+ (8.8)	+ (8.4)	+ (8.2)	+ (8.2)
locally MO-mappings	+ (8.21)	+ (8.18)	+ (8.16)	+ (8.13)	+ (8.10)	− (7.2)	+ (8.4)	− (7.2)	− (7.2)
OM-mappings	+ (8.21)	+ (8.18)	+ (8.16)	+ (8.13)	+ (8.10)	− (7.2)	+ (8.4)	− (7.2)	− (7.2)
hereditarily confluent mappings	+ (8.21)	+ (8.18)	? (8.17)	? (8.15)	+ (8.10)	+ (8.8)	+ (8.4)	+ (8.2)	+ (8.2)
quasi-monotone mappings	+ (8.21)	− (7.4)	− (8.12)	− (7.4)	− (8.12)	− (7.4)	− (7.14)	+ (8.2)	+ (8.2)
weakly monotone mappings	+ (8.21)	− (7.4)	− (8.12)	− (7.4)	− (8.12)	− (7.2)	− (7.14)	− (7.2)	− (7.2)
confluent mappings	+ (8.21)	+ (8.18)	? (8.17)	? (8.14)	+ (8.10)	− (7.2)	+ (8.4)	− (7.2)	− (7.2)
locally confluent mappings	+ (8.22)	− (8.6)	? (8.17)	− (8.6)	? (8.11)	− (7.2)	− (8.6)	− (7.2)	− (7.2)

TABLE V cont.

1	2	3	4	5	6	7	8	9	10
semi-confluent mappings	+ (8.21)	+ (8.20)	? (8.17)	? (8.14)	? (8.11)	– (7.2)	? (8.5)	– (7.2)	– (7.2)
locally semi-confluent mappings	– (7.18)	– (7.18)	? (8.17)	– (7.18)	? (8.11)	– (7.2)	– (8.6)	– (7.2)	– (7.2)
joining mappings	– (7.3)	– (7.3)	– (8.12)	– (7.3)	– (8.12)	– (7.3)	– (7.3)	– (7.2)	– (7.3)
hereditarily weakly confluent mappings	+ (8.21)	+ (8.19)	– (8.3)	? (8.15)	– (8.3)	? (8.9)	? (8.5)	– (8.3)	– (7.4)
weakly confluent mappings	– (7.18)	– (7.16)	– (8.3)	– (7.16)	– (8.3)	– (7.2)	– (8.7)	– (7.2)	– (7.2)
locally weakly confluent mappings	– (7.18)	– (7.18)	– (8.3)	– (7.18)	– (8.3)	– (7.2)	– (8.6)	– (7.2)	– (7.2)
pseudo-confluent mappings	– (7.18)	– (7.16)	– (8.3)	– (7.16)	– (8.3)	– (7.2)	– (8.7)	– (7.2)	– (7.2)
hereditarily atriodic mappings	– (7.8)	– (7.8)	– (8.3)	– (7.8)	– (8.3)	– (7.8)	– (8.7)	– (8.3)	– (7.4)
atriodic mappings	– (7.18)	– (7.16)	– (8.3)	– (7.16)	– (8.3)	– (7.2)	– (8.7)	– (7.2)	– (7.2)

This proposition is a generalization of Corollary II.4 of [17]. The hereditarily atriodic (joining, pseudo-confluent) image of an arc need not be either an arc or a circle (for example see [51], Example 3.6, cf. Examples (4.49) and (7.3) here). Moreover, every dendrite having only a finite number of end-points is the image of [0, 1] under a pseudo-confluent mapping (see [28], p. 247).

Table V sums up the invariance properties studied in § 8. The numbers under the signs are the numbers of the propositions which justify the use of signs (cf. Table II).

9. Images of hereditarily decomposable continua

In this section we consider problems concerning the invariance of some classes of continua which are subclasses of the class of hereditarily decomposable continua (cf. Table I).

A. Images of hereditarily decomposable continua in general. The next two theorems generalize some earlier results (cf. [9], XII, p. 217, [52], Theorem 5.1, p. 261)

(9.1) THEOREM. *The atriodic (pseudo-confluent) image of a hereditarily decomposable continuum is hereditarily decomposable.*

Proof. Let an atriodic (pseudo-confluent) mapping f map a hereditarily decomposable continuum X onto a continuum Y and suppose on the contrary that Q is an indecomposable subcontinuum of Y (then Q is also an irreducible subcontinuum of Y). In both cases if f is atriodic or if f is pseudo-confluent, there are two components C_1 and C_2 of $f^{-1}(Q)$ such that $f(C_1 \cup C_2) = Q$. Then either $f(C_1)$ or $f(C_2)$ is equal to Q, because if both $f(C_1)$ and $f(C_2)$ are proper subcontinua, then the set $f(C_1) \cup f(C_2)$ is contained in the union of some two composants of Q (for the definition of a composant see [36], § 48, VI, p. 208). Thus $f(C_1) \cup f(C_2)$ is a proper subset of Q by Theorem 7 of [36], § 48, VI, p. 212, a contradiction. Consequently, we can assume that $f(C_1) = Q$. Hence C_1 contains an indecomposable continuum by Theorem 4 of [36], § 48, V, p. 208, a contradiction of the assumptions regarding X.

(9.2) THEOREM. *The joining image of a hereditarily decomposable continuum is hereditarily decomposable.*

Proof. Let a joining mapping f map a hereditarily decomposable continuum X onto a continuum Y and suppose on the contrary that Q is an indecomposable subcontinuum of Y. If, for each component C of $f^{-1}(Q)$, $f(C)$ is a proper subcontinuum of Q, then $f(C)$ is contained in some single composant of Q (for the definition of a composant and for its properties see [36], § 48, VI, p. 208). Since the intersection of any two components of $f^{-1}(Q)$ is nonempty (f is joining), we infer that $ff^{-1}(Q)$ is contained in some single composant of Q, a contradiction.

Consequently there is a component C of $f^{-1}(Q)$ such that $f(C) = Q$. Then C contains an indecomposable continuum (see [36], § 48, V, Theorem 4, p. 208).

A class of mappings which contains either the class of quasi-monotone mappings or the class of locally confluent mappings does not preserve the hereditary decomposability of continua. This can be seen from Examples (7.12) and (7.17).

Note also that (see [57], Theorem (5.9))

(9.3) *The hereditarily unicoherent image of a hereditarily decomposable continuum is hereditarily decomposable.*

B. Images of Suslinian continua. It has been proved in [51], 5.2 that

(9.4) *The pseudo-confluent image of a Suslinian continuum is Suslinian.*

Similarly, we have the following

(9.5) THEOREM. *The locally weakly confluent (atriodic, joining) image of a Suslinian continuum is Suslinian.*

Proof. Let a locally weakly confluent (atriodic, joining) mapping f map a Suslinian continuum X onto Y. Suppose on the contrary that there is an uncountable collection $\mathscr{C}$ of mutually disjoint nondegenerate subcontinua of Y. Then for each positive number ε there is a collection $\mathscr{C}_\varepsilon$ of nondegenerate subcontinuum of Y having diameters less than ε and such that each member of $\mathscr{C}_\varepsilon$ is contained in some member of $\mathscr{C}$ (see [36], § 47, III, Theorem 4, p. 173). We can assume that $\mathscr{C}_\varepsilon$ are collections of mutually disjoint continua. In any case, if f is locally weakly confluent, or if f is atriodic or if f is joining, there is some nondegenerate component of the inverse image of each member of $\mathscr{C}_\varepsilon$ for sufficiently small ε (cf. (4.37)). Taking these nondegenerate components, we obtain an uncountable collection of mutually disjoint subcontinua of X, which contradicts the fact that X is Suslinian.

The remaining two classes which are not contained in any of the classes mentioned above are quasi-monotone mappings and weakly monotone mappings. Exactly these two classes do not preserve Suslinian continua by Example (7.14).

C. Images of rational continua. It follows from (3.8) and (3.9) of [49] that

(9.6) *The image of a rational continuum under an* OM-*mapping is rational.*

Since the confluent image of a rational continuum is hereditarily decomposable (cf. Tables I and II and (9.1)), we infer, by Theorem (6.8) that

(9.7) *The hereditarily confluent image of a rational continuum is rational.*

The following problem remains open (cf. [49], Problem III):

(9.8) PROBLEM. *Is the confluent (locally confluent, semi-confluent, locally semi-confluent, joining) image of a rational continuum always rational?*

Example (7.14) and the example described below show that the other classes of mappings considered in this paper do not preserve the rationality of continua.

(9.9) EXAMPLE. The arc-like nonrational continuum described in [14], p. 178 can be shown to be the continuous image of the rational continuum described in [48], Example 3. Thus it is the hereditarily weakly confluent image of this rational continuum by Theorem (6.16).

The following proposition as well as Corollary (9.1) are partial answers to Problem (9.8) (see [51], 5.1)

(9.10) *The locally connected pseudo-confluent image of a rational continuum is rational.*

We have the following

(9.11) COROLLARY. *The locally connected image of a rational continuum under a locally weakly confluent mapping is rational.*

Proof. Let a locally weakly confluent mapping f map a rational continuum X onto a locally connected continuum Y and let ε be a positive number such that for each subcontinuum Q of Y of diameter less than ε the mapping $f|f^{-1}(Q)$ is weakly confluent (cf. (4.37)). It follows from Theorem 3 of [36], § 50, II, p. 257 that there are locally connected continua $Y_1, Y_2, \ldots, Y_n$ of diameters less than ε and such that $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_n$. It suffices to show that each Y_i is rational for $i = 1, 2, \ldots, n$ by Theorem 6 of [36], § 51, IV, p. 286.

Fix $i = 1, 2, \ldots, n$. It is clear that $X_i = f^{-1}(Y_i)$ is a regular space as a subset of a regular space. Moreover, $g = f|X_i$ is weakly confluent. Given two points $y, y' \in Y_i$, there is an open subset $G \subset X_i$ such that $g^{-1}(y) \subset G$, $\bar{G} \cap g^{-1}(y') = \emptyset$ and the set $\bar{G} \setminus G$ is countable (see [36], § 51, IV, Theorem 9 (ii), p. 287). Then $U = Y_i \setminus g(\bar{G} \setminus G)$ is an open subset of Y_i which contains y and y'. Moreover, we have $g^{-1}(U) \subset G \setminus (X \setminus \bar{G})$. Thus the set $g^{-1}(U)$ is not connected between $g^{-1}(y)$ and $g^{-1}(y')$. Since the points y and y' lie in distinct components of U, we find that U is not connected between y and y', because the components of the open set U coincide with quasi-components in the locally connected space Y_i. Since $Y_i \setminus U = g(\bar{G} \setminus G)$ is countable, Y is rational (cf. [89], p. 97). The end of this proof uses the methods of [51].

D. Images of hereditarily locally connected continua. It has been proved in [54], Theorem 3.1, p. 64 (cf. also [76], Theorem 7) that

(9.12) *The locally weakly confluent image of a hereditarily locally connected continuum is hereditarily locally connected.*

Since the continuous image of a locally connected continuum is locally connected, we infer by (6.2) and (9.12) (cf. Table II) that

(9.13) *The weakly monotone image of a hereditarily locally connected continuum is hereditarily locally connected.*

It is clear that

(9.14) LEMMA. *If a mapping $f\colon X \to f(X)$ is joining and ab is an arc in $f(X)$, then for each two components C_1 and C_2 of $f^{-1}(ab)$ with $a \in f(C_1)$ and $b \in f(C_2)$ we have the equality $f(C_1) \cup f(C_2) = ab$.*

Now we will prove (cf. [51], Theorem 4.7)

(9.15) THEOREM. *The pseudo-confluent (atriodic, joining) image of a hereditarily locally connected continuum is hereditarily locally connected.*

Proof. Let a pseudo-confluent (atriodic, joining) mapping f map a hereditarily locally connected continuum X onto Y. Suppose on the contrary that Y is not hereditarily locally connected. It follows from Theorem 2 of [36], § 50, IV, p. 269 that there is a convergent sequence $\{Q_n\}$ of pairwise disjoint subcontinua of Y such that $\operatorname{Lim}_{n\to\infty} Q_n$ is nondegenerate and disjoint with each Q_n for $n = 1, 2, \ldots$ Let $\{a_n\}$ and $\{b_n\}$ be the sequences of points of Y such that $\{a_n, b_n\} \subset Q_n$ for each $n = 1, 2, \ldots$ and $\lim_{n\to\infty} a_n = a \neq b = \lim_{n\to\infty} b_n$.

Since Y is locally connected, it is an immediate consequence of Theorem 3 of [36], § 50, II, p. 257 that there are locally connected continua Q'_n in Y such that $Q_n \subset Q'_n$, $Q'_n \cap Q'_m = \emptyset = Q'_m \cap \operatorname{Lim}_{n\to\infty} Q'_n$ for each $n \neq m$ and $n, m = 1, 2, \ldots$ Since each Q'_n is arcwise connected (see [36], § 50, I, Theorem 2, p. 253 and ibid., II, Theorem 1, p. 254) and since $\{a_n, b_n\} \subset Q'_n$ for each $n = 1, 2, \ldots,$ we infer that there is a convergent sequence $\{a_n b_n\}$ of arcs such that $a_n b_n \cap a_m b_m = \emptyset = a_m b_m \cap \operatorname{Lim}_{n\to\infty} a_n b_n$.

Since f is pseudo-confluent (atriodic, joining, cf. Lemma (9.14)), we conclude that there are continua A_n and B_n in X such that $f(A_n \cup B_n) = a_n b_n$ for each $n = 1, 2, \ldots$ We can assume that the sequences $\{A_n\}$ and $\{B_n\}$ are convergent (see [36], § 42, I, Theorem 1, p. 45 and ibid. § 42, II) and put $A = \operatorname{Lim}_{n\to\infty} A_n$ and $\operatorname{Lim}_{n\to\infty} B_n = B$. Then the sets A and B are continua (cf. [36], § 47, II, Theorem 4, p. 170) of the convergence in X (for the definition see ibid., p. 245). Since X is hereditarily locally connected, we infer that A and B are degenerate by Theorem 2 of [36], § 50, IV, p. 269. Therefore $f(A \cup B)$ is at most a two-point set, but $f(A \cup B) = \operatorname{Lim}_{n\to\infty} f(A_n) \cup \operatorname{Lim}_{n\to\infty} f(B_n) = \operatorname{Lim}_{n\to\infty} a_n b_n$ is a nondegenerate continuum, a contradiction. The proof of Theorem (9.15) is complete.

E. Images of finitely Suslinian continua. It has been proved in [51], Theorem 4.6, that

(9.16) *The pseudo-confluent image of a finitely Suslinian continuum is finitely Suslinian.*

Since the continuous image of a locally connected continuum is locally connected, we infer by (6.2) and (9.16) (cf. Tables I and II) that

(9.17) *The weakly monotone image of a finitely Suslinian continuum is finitely Suslinian.*

We also have the following

(9.18) THEOREM. *The locally weakly confluent (atriodic, joining) image of a finitely Suslinian continuum is finitely Suslinian.*

Proof. Let a locally weakly confluent (atriodic, joining) mapping f map a finitely Suslinian continuum X onto Y. It follows from (4.37) and from Lemma (9.14) that there is a positive number ε_0 such that

(9.18.1) *if K is an arc in Y of diameter less than ε_0, then there are continua A and B in X such that $f(A \cup B) = K$.*

By the continuity of f we infer that

(9.18.2) *for each $\varepsilon > 0$ there is a $\delta > 0$ such that if A is of diameter less than δ, then $f(A)$ is of diameter less than ε.*

Let $\mathscr{K}$ be a collection of pairwise disjoint subcontinua of Y with diameters greater than ε. Since Y is hereditarily locally connected (cf. Table I, (9.12) and (9.15)), it is hereditarily arcwise connected (see [36], § 50, I, Theorem 2, p. 253 and ibid. Theorem 1, p 254). Thus there is a collection $\mathscr{K}'$ of pairwise disjoint arcs such that $1/2 \min(\varepsilon_0, \varepsilon) \leqslant \operatorname{diam} K \leqslant \varepsilon_0$ for each $K \in \mathscr{K}'$ and $\operatorname{card}\mathscr{K} = \operatorname{card}\mathscr{K}'$.

Thus, by (9.18.1), for each $K \in \mathscr{K}'$ there are continua A_K and B_K such that $f(A_K \cup B_K) = K$. Since $\operatorname{diam} K \geqslant 1/2 \min(\varepsilon_0, \varepsilon)$, there is $\delta > 0$ (by (9.18.2)) such that either $\operatorname{diam} A_K \geqslant \delta$ or $\operatorname{diam} B_K \geqslant \delta$ for each $K \in \mathscr{K}'$. We can assume that $\operatorname{diam} A_K \geqslant \delta$ for each $K \in \mathscr{K}'$. The collection $\mathscr{C} = \{A_K: K \in \mathscr{K}'\}$ is a collection of pairwise disjoint continua with diameters greater that δ. Thus $\operatorname{card}\mathscr{C}$ is finite. Since $\operatorname{card}\mathscr{C} = \operatorname{card}\mathscr{K}' = \operatorname{card}\mathscr{K}$, we infer that the collection $\mathscr{K}$ is finite. This means that Y is finitely Suslinian.

F. Images of regular continua. It has been proved in [51], Theorem 4.5, that

(9.19) *The pseudo-confluent image of a regular continuum is regular.*

Thus, by (6.2) (cf. Table II) we conclude that

(9.20) *The weakly monotone image of a regular continuum is regular.*

By an easy modification of the proof of Theorem (9.18) and by (2.23) one can infer that

(9.21) *The locally weakly confluent (atriodic, joining) image of a regular continuum is regular.*

G. Images of local dendrites. We have

(9.22) *Let a monotone mappings f map a hereditarily locally connected continuum X onto Y. Then for each simple closed curve S in Y there is a simple closed curve S' in X such that $f(S') = S$.*

Indeed, let S be a simple closed curve in Y and let A and B be subarcs of S such that $S = A \cup B$, $A \cap B = \{a, b\}$ and $a \neq b$. Take $a' \in f^{-1}(a)$ and $b' \in f^{-1}(b)$. Sets $f^{-1}(A)$ and $f^{-1}(B)$ are continua by the monotoneity of f. According to the assumption regarding X there are arcs A' and B' with endpoints a' and b' such that $A' \subset f^{-1}(A)$ and $B' \subset f^{-1}(B)$. The union $A' \cup B'$ contains a simple closed curve S' such that $f(S') = S$.

From (2.30) and (9.22) (cf. Table I) we conclude that

(9.23) *The monotone image of a local dendrite is a local dendrite.*

We will now prove the following

(9.24) THEOREM. *The locally semi-confluent (hereditarily weakly confluent) image of a local dendrite is a local dendrite.*

Proof. Let a locally semi-confluent (hereditarily weakly confluent) mapping f map a local dendrite X onto Y. By Whyburn's factorization theorem (see [83], (4.1), p. 141) we infer that there are a monotone mapping $f_1: X \to X'$ and a light mapping $f_2: X' \to Y$ such that $f(x) = f_2 f_1(x)$ for each $x \in X$. The continuum X' is a local dendrite by (9.23). Thus there is an $\varepsilon_1 > 0$ such that (see [36], § 51, VII, Theorem 2, p. 303)

(9.24.1) *if C is a subcontinuum of X' of diameter less than ε_1, then it is a dendrite.*

Since the mapping f_2 is light, we infer (see [83], (4.41), p. 131) that

(9.24.2) *there is a $\delta_1 > 0$ such that if K is a subcontinuum of Y of diameter less than δ_1, then each component of the set $f_2^{-1}(K)$ is of diameter less than ε_1.*

The continuum Y is locally connected (as a continuous image of a locally connected continuum). Suppose on the contrary that Y is not a local dendrite. Then Y contains infinitely many simple closed curves by (2.30). Therefore (see [36], § 51, VII, Theorem 5, p. 304) Y contains some simple closed curves with arbitrarily small diameters. Since f is locally semi-confluent (hereditarily weakly confluent), there is a positive number ε_2 such that if K is a subcontinuum of Y of diameter less than ε_2, then there is a continuum C of X such that $f(C) = K$ and $f'|C$ is semi-confluent (hereditarily weakly confluent) (compare [52], Theorem 3.7, p. 255). We conclude that there is a simple closed curve S in Y of diameter less than δ_1 and less than ε_2. Then there is a continuum C of X such that $f(C) = S$ and $f|C$ is semi-confluent (hereditarily weakly confluent). Since $f|C = (f_2|f_1(C))(f_1|C)$ (and since f_1 is monotone if f is hereditarily weakly confluent), we infer that $f_2|f_1(C)$ is semi-confluent (hereditarily weakly confluent) by (5.16) (by (5.29), respectively). Moreover, since the diameter of $f_1(C)$ is less than ε_1 (by (9.24.2)), we infer that the continuum $f_1(C)$ is a dendrite by (9.24.1). Thus $f(C) = (f_2|f_1(C))(f_1|C)(C)$ is a dendrite by (7.36), a contradiction, because $f(C) = S$.

From (6.2) and (9.24) (cf. Table II), we infer that

(9.25) *The weakly monotone image of a local dendrite is a local dendrite.*

The following problem remains open:

(9.26) PROBLEM. *Is the hereditarily atriodic image of a local dendrite always a local dendrite?*

It is my conjecture that the answer is positive. The class of joining mappings does not preserve local dendrites by Example (7.9) and the class of weakly confluent mappings does not preserve local dendrites either. This can be seen from the following

(9.27) EXAMPLE. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates. Put

$$A_n = \left\{\left(t\frac{2n+1}{2n(n+1)}+(1-t)\frac{1}{n}, \frac{t}{n}\right):\ 0 \leqslant t \leqslant 1\right\},$$

$$A_n' = \left\{\left(t\frac{2n+3}{2n(n+1)}+(1-t)\frac{1}{n}, \frac{t}{n}\right):\ 0 \leqslant t \leqslant 1\right\},$$

$$S_n = \left\{\left(\frac{1}{n}(1+\cos 2\pi t), \frac{1}{n}(1+\sin 2\pi t)\right):\ 0 \leqslant t \leqslant 1\right\}$$

for each $n = 1, 2, \ldots$ and put $I = [0, 1]$.

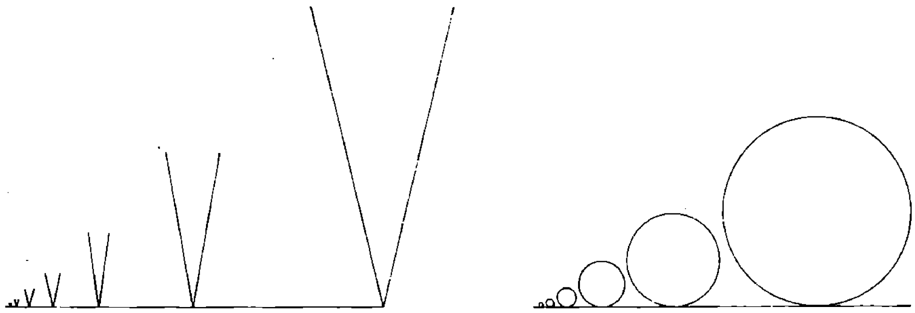


Fig. 12

It is easy to observe that the set

$$X = I \cup \bigcup_{n=1}^{\infty} (A_n \cup A_n')$$

is a dendrite (see Fig. 12) and that the set

$$Y = I \cup \bigcup_{n=1}^{\infty} S_n$$

is not a local dendrite.

The weakly confluent mapping f from X onto Y is defined first by putting

$$a_n(x, y) = 2\cdot n\cdot(n+1)\cdot\left(\frac{\left(x-\frac{1}{n}\right)^2+y^2}{4n^2+8n+5}\right)^{1/2}$$

and then

$$f(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in I, \\ \left(\frac{1}{n}\big(1+\cos(-4\pi a_n(x, y)-\pi)\big), \frac{1}{n}\big(1+\sin(-4\pi a_n(x, y)-\pi)\big)\right) & \text{if } (x, y) \in A_n, \\ \left(\frac{1}{n}\big(1+\cos(4\pi a_n(x, y)-\pi)\big), \frac{1}{n}\big(1+\sin(4\pi a_n(x, y)-\pi)\big)\right) & \text{if } (x, y) \in A_n'. \end{cases}$$

TABLE VI

	graphs	local dendrites	regular continua	finitely Suslinian continua	hereditarily locally connected continua	rational continua	Suslinian continua	hereditarily decomposable continua
1	2	3	4	5	6	7	8	9
local homeomorphisms	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	+ (9.6)	+ (9.4)	+ (9.2)
hereditarily monotone mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	+ (9.6)	+ (9.4)	+ (9.2)
atomic mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	+ (9.6)	+ (9.4)	+ (9.2)
monotone mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	+ (9.6)	+ (9.4)	+ (9.2)
open mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	− (9.6)	− (9.4)	+ (9.2)
MO-mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	+ (9.6)	+ (9.4)	+ (9.2)
locally monotone mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	+ (9.6)	+ (9.4)	+ (9.2)
locally MO-mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	+ (9.6)	+ (9.4)	+ (9.2)
OM-mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	+ (9.6)	+ (9.4)	+ (9.2)
hereditarily confluent mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	+ (9.7)	+ (9.4)	+ (9.2)
quasi-monotone mappings	+ (9.29)	+ (9.25)	+ (9.20)	+ (9.17)	+ (9.13)	− (7.14)	− (7.14)	− (7.12)
weakly monotone mappings	+ (9.29)	+ (9.25)	+ (9.20)	+ (9.17)	+ (9.13)	− (7.14)	− (7.14)	− (7.12)
confluent mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	? (9.8)	+ (9.4)	+ (9.2)

TABLE VI cont.

1	2	3	4	5	6	7	8	9
locally confluent mappings	+ (9.28)	+ (9.24)	+ (9.21)	+ (9.18)	+ (9.12)	? (9.8)	+ (9.5)	– (7.17)
semi-confluent mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	? (9.8)	+ (9.4)	+ (9.2)
locally semi-confluent mappings	+ (9.28)	+ (9.24)	+ (9.21)	+ (9.16)	+ (9.12)	? (9.8)	+ (9.5)	– (7.17)
joining mappings	– (7.3)	– (7.3)	+ (9.21)	+ (9.18)	+ (9.15)	? (9.8)	+ (9.5)	+ (9.2)
hereditarily weakly confluent mappings	+ (9.28)	+ (9.24)	+ (9.19)	+ (9.16)	+ (9.12)	– (9.9)	+ (9.4)	+ (9.1)
weakly confluent mappings	+ (9.28)	– (9.27)	+ (9.19)	+ (9.16)	+ (9.12)	– (9.9)	+ (9.4)	+ (9.1)
locally weakly confluent mappings	+ (9.28)	– (9.27)	+ (9.21)	+ (9.16)	+ (9.12)	– (9.9)	+ (9.5)	– (7.17)
pseudo-confluent mappings	? (9.30)	– (9.27)	+ (9.19)	+ (9.16)	+ (9.15)	– (9.9)	+ (9.4)	+ (9.1)
hereditarily atriodic mappings	? (9.30)	? (9.26)	+ (9.21)	+ (9.18)	+ (9.15)	– (9.9)	+ (9.5)	+ (9.1)
atriodic mappings	? (9.30)	– (9.27)	+ (9.21)	+ (9.18)	+ (9.15)	– (9.9)	+ (9.5)	+ (9.1)

H. Images of graphs. It has been proved in [54], Theorem (4.3), p. 70 that

(9.28) *The locally weakly confluent image of a graph is a graph.*

Note also that in [54], Corollary (4.8), p. 73 we find the characterization of graphs which are locally weakly confluent images of an arbitrary graph.

From (6.2) and (9.28) (cf. Table II) we conclude that

(9.29) *The weakly monotone image of a graph is a graph.*

We have the following

(9.30) PROBLEM. *Is the atriodic (hereditarily atriodic, pseudo-confluent) image of a graph also a graph?*

The joining image of a graph need not be a graph by Example (7.3).

Table VI sums up the invariance properties studied in § 9. The numbers under the signs denote the numbers of the proposition which justify the use of these signs (cf. Table II).

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Added in the proof:

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Problem (9.30) is solved positively for pseudo-confluent mappings in [88]. It is proved in [90] that semi-confluent mappings preserve tree-like continua and one-dimensional acyclic continua (cf. Problems (7.19) and (7.23)). In [91] E. D. Tymchatyn gave an example of a confluent mapping which does not preserve the rationality of a curve what solves Problem (9.8). Papers [86], [87] and [89] contain some other esults in these directions.

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