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TRANSFORMS OF BOEHMIANS

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Abstract. Boehmians are defined by an algebraic construction which is similar to the construction of a field of quotients. If the construction is applied to a function space and the multiplication is interpreted as convolution, the construction yields a space of generalized functions. Those spaces provide a natural setting for extensions of transforms like the Fourier, Laplace, Radon, or Zak transforms. Since the abstract algebraic definition of Boehmians allows different interpretations, not necessarily based on the convolution product, those transforms are actually isomorphisms between spaces of Boehmians.

Introduction. The construction of Boehmians given in [3] was motivated by the idea of regular operators introduced by T. K. Boehme in [1]. The algebraic construction of Boehmians is similar to the construction of the field of quotients. The main difference is that Boehmians can be constructed even if the ring has divisors of zero. This feature enables us to define a space of "convolution quotients", similar to the Mikusiński operators [2], but without the restriction on the support. Basic properties of Boehmians and connections with other theories of generalized functions are discussed in [4], [6], and [7].

The purpose of this note is to show that some transforms have natural extensions onto appropriately defined spaces of Boehmians. Then they become mappings between different spaces of Boehmians. Such an extension is possible if the transform satisfies the following two conditions. First, it has to be a homomorphism between function spaces. Second, it has to map delta sequences in one space to delta sequences in the other space. The meaning of these two conditions becomes clear after one looks at the examples given below.

We discuss three transforms. The first one is the Fourier transform. We define two extensions of the transform. The first one is based on the fact that the Fourier transform changes convolution into pointwise multiplication. The other one on the fact that it transforms pointwise multiplication into convolution. These two

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extensions of the Fourier transform act on different spaces of Boehmians. The second transform discussed here is the Radon transform. This transform changes convolution on $\mathbb{R}^q$ into a partial convolution on a cylinder. Both the Fourier transform and the Radon transform are examples of integral transforms. The third transform considered here is the Zak transform, which is not an integral transform. This transform changes convolution on $\mathbb{R}$ into a partial convolution on $\mathbb{R}^2$. In all these examples, the delta sequences have to be defined in such a manner that the transform carries delta sequences to delta sequences.

In Section 2 of this note we give a general definition of Boehmians. The definition presented here is more general than those previously considered. The main difference is that we allow two different "convolutions" in the basic space. This generality is necessary to include some new applications of Boehmians [13].

Functions are assumed here to be complex-valued unless otherwise stated. The norm in $\mathbb{R}^q$ is the Euclidean norm

$$\|x\| = \|(x_1, \ldots, x_q)\| = \sqrt{x_1^2 + \ldots + x_q^2}.$$

By B_ε we mean the ε-ball at the origin: $B_\varepsilon = \{x \in \mathbb{R}^q : \|x\| < \varepsilon\}$.

2. Boehmians. In order to construct a space of Boehmians we need the following:

(1) a complex vector space $\mathcal{G}$,

(2) a commutative semigroup $(\mathcal{T}, \odot)$,

(3) an operation $\otimes : \mathcal{G} \times \mathcal{T} \to \mathcal{G}$ such that for all $f, g \in \mathcal{G}$, $\varphi, \psi \in \mathcal{T}$, and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} (f \otimes \varphi) \otimes \psi &= f \otimes (\varphi \odot \psi), \\ (f + g) \otimes \varphi &= f \otimes \varphi + g \otimes \varphi, \\ \lambda(f \otimes \varphi) &= (\lambda f \otimes \varphi), \end{aligned}$$

(4) a family Δ of sequences of elements of $\mathcal{T}$ such that if $f \in \mathcal{G}$ and $f \neq 0$, then for every $\{\varphi_n\} \in \Delta$ there is $n_0 \in \mathbb{N}$ such that $f \otimes \varphi_{n_0} \neq 0$, and if $\{\varphi_n\}, \{\psi_n\} \in \Delta$, then $\{\varphi_n \odot \psi_n\} \in \Delta$.

Elements of Δ are called *delta sequences*. A pair of sequences $(\{f_n\}, \{\varphi_n\})$ is called a *quotient of sequences* if $f_n \in \mathcal{G}$ for all $n \in \mathbb{N}$, $\{\varphi_n\} \in \Delta$, and $f_n \otimes \varphi_m = f_m \otimes \varphi_n$ for all $m, n \in \mathbb{N}$. A quotient of sequences $(\{f_n\}, \{\varphi_n\})$ will be denoted simply by f_n/φ_n. Two quotients of sequences f_n/φ_n and g_n/ψ_n are called equivalent if $f_n \otimes \psi_n = g_n \otimes \varphi_n$ for all $n \in \mathbb{N}$. It is easy to check that this is an equivalence relation. The equivalence class of f_n/φ_n is denoted by $[f_n/\varphi_n]$. Finally, the space of all such equivalence classes is denoted by $\mathfrak{B}(\mathcal{G}, \mathcal{T}, \odot, \otimes, \Delta)$. Elements of $\mathfrak{B}(\mathcal{G}, \mathcal{T}, \odot, \otimes, \Delta)$ are called *Boehmians*.

If we define

$$\lambda[f_n/\varphi_n] = [\lambda f_n/\varphi_n] \quad \text{and} \quad [f_n/\varphi_n] + [g_n/\psi_n] = [(f_n \otimes \psi_n + g_n \otimes \varphi_n)/\varphi_n \otimes \psi_n]$$

then $\mathfrak{B}(\mathcal{G}, \mathcal{T}, \odot, \otimes, \Delta)$ becomes a vector space. It is important to note that every $f \in \mathcal{G}$ can be identified with the Boehmian $[(f \otimes \varphi_n)/\varphi_n]$ where $\{\varphi_n\} \in \Delta$. This identification turns out to be an isomorphism of $\mathcal{G}$ with a subspace of $\mathfrak{B}(\mathcal{G}, \mathcal{T}, \odot, \otimes, \Delta)$. The operation $\otimes$ can be extended onto $\mathfrak{B}(\mathcal{G}, \mathcal{T}, \odot, \otimes, \Delta) \times \mathcal{T}$ if we define $F \otimes \varphi = [(f_n \otimes \varphi)/\varphi_n]$. Then, for every $k \in \mathbb{N}$, we have $[f_n/\varphi_n] \otimes \varphi_k = f_k$.

3. The Fourier transform. In this section we define the Fourier transform of integrable Boehmians. To obtain the space of *integrable Boehmians* we take:

$\mathcal{G} = \mathcal{L}^1(\mathbb{R}^q) =$ the space of Lebesgue integrable functions on $\mathbb{R}^q$,

$\mathcal{T} = \mathcal{D}(\mathbb{R}^q) =$ the space of infinitely differentiable functions on $\mathbb{R}^q$ with compact support,

$\otimes = \odot = *$ (i.e., the convolution defined by $(f * \varphi)(y) = \int_{\mathbb{R}^q} f(x)\varphi(y - x)\,dx$),

$\Delta = \Delta_0 =$ the family of sequences of real-valued functions $\varphi_1, \varphi_2, \ldots \in \mathcal{D}(\mathbb{R}^q)$ such that

$$\int_{\mathbb{R}^q} \varphi_n(x)\,dx = 1 \quad \text{for all } n \in \mathbb{N},$$

$$\int_{\mathbb{R}^q} |\varphi_n(x)| dx < M \quad \text{for some } M > 0 \text{ and all } n \in \mathbb{N},$$

$$\sigma(\varphi_n) \to 0 \quad \text{as } n \to \infty \quad (\text{where } \sigma(\varphi) = \inf\{\, \varepsilon > 0 : \operatorname{supp}\varphi \subset B_\varepsilon\}).$$

The Fourier transform of $f \in \mathcal{L}^1(\mathbb{R}^q)$ is defined as

$$\mathcal{F}f(s) = \int_{\mathbb{R}^q} f(x) e^{-i\langle x, s\rangle}\,dx$$

and the Fourier transform of an integrable Boehmian $[f_n/\varphi_n]$ is defined as

$$\mathcal{F}[f_n/\varphi_n] = [\mathcal{F}f_n/\mathcal{F}\varphi_n].$$

The Boehmian $[\mathcal{F}f_n/\mathcal{F}\varphi_n]$ is an element of the space $\mathfrak{B}(\mathcal{C}(\mathbb{R}^q), \mathcal{S}(\mathbb{R}^q), \cdot, \cdot, \Delta_p)$ where

$\mathcal{C}(\mathbb{R}^q) =$ the space of continuous functions on $\mathbb{R}^q$,

$\mathcal{S}(\mathbb{R}^q) =$ the space of rapidly decreasing functions on $\mathbb{R}^q$,

$\cdot$ denotes pointwise multiplication,

$\Delta_p =$ the family sequences of real-valued functions $\varphi_1, \varphi_2, \ldots \in \mathcal{S}(\mathbb{R}^q)$ which converge to 1 uniformly on compact sets.

When checking that the Fourier transform can be extended in this way one realizes that the following two properties are crucial:

$\mathcal{F}(f * g) = (\mathcal{F}f) \cdot (\mathcal{F}g)$.

If $\{\varphi_n\} \in \Delta_0$, *then* $\{\mathcal{F}\varphi_n\} \in \Delta_p$.

A more detailed discussion of this extension of the Fourier transform can be found in [5].

It is possible to interchange the roles of convolution and pointwise multiplication in extending the Fourier transform. For this purpose consider the space of Boehmians defined by:

$\mathcal{G} = \mathcal{C}_0(\mathbb{R}^q)$ = the space of continuous functions with compact support,
$\mathcal{T} = \mathcal{D}(\mathbb{R}^q)$,
$\otimes = \odot = \cdot$,
$\Delta = \Delta_\times$ = the family of sequences of real-valued functions $\varphi_1, \varphi_2, \ldots \in \mathcal{D}(\mathbb{R}^q)$ such that $\varphi_n(x) = \varphi(x/n)$ for some $\varphi \in \mathcal{D}(\mathbb{R}^q)$ such that $\varphi(0) = 1$.

Note that the space $\mathfrak{B}(\mathcal{C}_0(\mathbb{R}^q), \mathcal{D}(\mathbb{R}^q), \cdot, \cdot, \Delta_\times)$ can be identified with $\mathcal{C}(\mathbb{R}^q)$. Indeed, if $f \in \mathcal{C}(\mathbb{R}^q)$ and $\{\varphi_n\} \in \Delta_\times$, then $[f\varphi_n/\varphi_n] \in \mathfrak{B}(\mathcal{C}_0(\mathbb{R}^q), \mathcal{D}(\mathbb{R}^q), \cdot, \cdot, \Delta_\times)$. Conversely, if $[f_n/\varphi_n] \in \mathfrak{B}(\mathcal{C}_0(\mathbb{R}^q), \mathcal{D}(\mathbb{R}^q), \cdot, \cdot, \Delta_\times)$, then the sequence $f_1, f_2, \ldots$ converges to an element of $\mathcal{C}(\mathbb{R}^q)$.

The Fourier transform of a Boehmian $[f_n/\varphi_n] \in \mathfrak{B}(\mathcal{C}_0(\mathbb{R}^q), \mathcal{D}(\mathbb{R}^q), \cdot, \cdot, \Delta_\times)$ can be defined as before:

$$\mathcal{F}[f_n/\varphi_n] = [\mathcal{F}f_n/\mathcal{F}\varphi_n].$$

Now the Boehmian $[\mathcal{F}f_n/\mathcal{F}\varphi_n]$ is an element of $\mathfrak{B}(\mathcal{W}(\mathbb{R}^q), \mathcal{S}(\mathbb{R}^q), *, *, \Delta_\mathcal{S})$ where

$\mathcal{W}(\mathbb{R}^q)$ = the space of all continuous functions which are bounded by a polynomial,

$\Delta_\mathcal{S}$ = the family of sequences of real-valued functions $\delta_1, \delta_2, \ldots \in \mathcal{S}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^q} \delta_n(x)\,dx = 1 \quad \text{for all } n \in \mathbb{N},$$

$$\int_{\mathbb{R}^q} |\delta_n(x)|\,dx < M \quad \text{for some } M > 0 \text{ and all } n \in \mathbb{N},$$

$$\lim_{n\to\infty} \int_{\|x\|\ge\varepsilon} |\delta_n| = 0 \quad \text{for every } \varepsilon > 0.$$

We thus have a simple definition of the Fourier transform which can be applied to any continuous function. As before, the extension is possible because:

$\mathcal{F}(fg) = (\mathcal{F}f) * (\mathcal{F}g)$.
If $\{\varphi_n\} \in \Delta_\times$, *then* $\{\mathcal{F}\varphi_n\} \in \Delta_\mathcal{S}$.

More general definitions of the Fourier transform can be found in [8] and [9].

4. The Radon transform. In this section we give a brief description of the Radon transform of Boehmians introduced in [11] and [12]. The domain of this extension is the space of integrable Boehmians $\mathfrak{B}(\mathcal{L}^1(\mathbb{R}^q), \mathcal{D}(\mathbb{R}^q), *, *, \Delta_0)$. The range is a space of Boehmians defined on the cylinder:

$$\Sigma = \{(p, \xi) \in \mathbb{R} \times \mathbb{R}^q : \|\xi\| = 1\} = \mathbb{R} \times S^{q-1}.$$

For this space of Boehmians we take:

$\mathcal{G} = \mathcal{L}(\Sigma) =$ the space of all functions $f(p,\xi)$ on Σ such that $f(\cdot,\xi)$ is Lebesgue integrable for almost all $\xi \in S^{q-1}$,

$\mathcal{T} = \mathcal{D}(\Sigma) =$ the space of all infinitely differentiable functions on Σ with compact support,

$\otimes = \odot = \bullet$, where $f \bullet g$ is the partial convolution defined by

$$(f \bullet g)(p,\xi) = \int_{\mathbb{R}} f(t,\xi) g(p-t,\xi)\, dt,$$

$\Delta = \Delta_\Sigma =$ the family of all sequences of real-valued functions $\varphi_1, \varphi_2, \ldots \in \mathcal{D}(\Sigma)$ such that

$$\int_{\mathbb{R}} \varphi_n(p,\xi)\, dp = 1 \quad \text{for every } \xi \in S^{q-1} \text{ and every } n \in \mathbb{N},$$

$$\sup_{\xi\in S^{q-1}} \int_{\mathbb{R}} |\varphi_n(p,\xi)| dp < M \quad \text{for some } M > 0 \text{ and all } n \in \mathbb{N},$$

$$\sup_{\xi\in S^{q-1}} \sigma(\varphi_n(\cdot,\xi)) \to 0 \quad \text{as } n \to \infty.$$

The Radon transform of $f \in \mathcal{L}^1(\mathbb{R}^q)$ can be defined by

$$(\mathcal{R}f)(p,\xi) = \int_{\mathbb{R}^q} f(x)\delta(p - \langle x,\xi\rangle)\, dx, \quad (p,\xi) \in \Sigma.$$

The Radon transform of a Boehmian $[f_n/\varphi_n] \in \mathfrak{B}(\mathcal{L}^1(\mathbb{R}^q), \mathcal{D}(\mathbb{R}^q), *, *, \Delta_0)$ is defined by

$$\mathcal{R}[f_n/\varphi_n] = [\mathcal{R}f_n/\mathcal{R}\varphi_n].$$

The Boehmian $[\mathcal{R}f_n/\mathcal{R}\varphi_n]$ is an element of $\mathfrak{B}(\mathcal{L}(\Sigma), \mathcal{D}(\Sigma), \bullet, \bullet, \Delta_\Sigma)$. Again, this simple extension of the Radon transform is possible because:

$\mathcal{R}(f * g) = (\mathcal{R}f) \bullet (\mathcal{R}g)$.

If $\{\varphi_n(x)\} \in \Delta_0$, *then* $\{(\mathcal{R}\varphi_n)(p,\xi)\} \in \Delta_\Sigma$.

5. The Zak transform. Our final example is the Zak transform. To define the domain of the extended Zak transform we take:

$\mathcal{G} = \mathcal{L}^2(\mathbb{R}) =$ the space of square integrable functions on $\mathbb{R}$,

$\mathcal{T} = \mathcal{S}(\mathbb{R}) =$ the space of rapidly decreasing functions on $\mathbb{R}$,

$\otimes = \odot = *$,

$\Delta = \Delta_{\mathcal{S}}$.

The Zak transform of $f \in \mathcal{L}^2(\mathbb{R})$ is defined by

$$(\mathcal{Z}f)(t,\omega) = \sum_{k=-\infty}^{\infty} f(t+k)\, e^{-2\pi i k\omega}.$$

The extension to $\mathfrak{B}(\mathcal{L}^2(\mathbb{R}), \mathcal{S}(\mathbb{R}), *, *, \Delta_{\mathcal{S}})$ is defined as

$$\mathcal{Z}[f_n/\varphi_n] = [\mathcal{Z}f_n/\mathcal{Z}\varphi_n]$$

where $[\mathcal{Z}f_n/\mathcal{Z}\varphi_n] \in \mathfrak{B}(\mathcal{P}, \widetilde{\mathcal{P}}, ©, ©, \Delta_{\mathcal{Z}})$. Here

$\mathcal{P}$ = the space of all functions on $\mathbb{R}^2$ such that f is square integrable on $[0,1]^2$, $f(t, \omega+1) = f(t,\omega)$ for all $t,\omega \in \mathbb{R}$, $f(t+1,\omega) = e^{2\pi i\omega} f(t,\omega)$ for all $t,\omega \in \mathbb{R}$,

$\mathcal{T} = \widetilde{\mathcal{P}} = \{\mathcal{Z}f : f \in \mathcal{S}(\mathbb{R})\}$,

$\odot = \otimes =$©, where f©g is the partial convolution defined by

$$(f©g)(t,\omega) = \int_0^1 f(s,\omega) g(t-s,\omega)\, ds,$$

$\Delta_{\mathcal{Z}} = \{\{\mathcal{Z}\varphi_n\} : \{\varphi_n\} \in \Delta_{\mathcal{S}}\}$.

Although the Zak transform is not an integral transform, this extension is possible because, as before, we have the following:

$\mathcal{Z}(f * g) = (\mathcal{Z}f)©(\mathcal{Z}g)$.

If $f \in \mathcal{L}^2(\mathbb{R})$ *and* $g \in \mathcal{S}(\mathbb{R})$, *then* $\mathcal{Z}(f * g) = (\mathcal{Z}f)©(\mathcal{Z}g)$.

The presented extension of the Zak transform is discussed in more detail in [14].

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