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RIGID SETS IN MANIFOLDS*

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1. Introduction

Much of geometric topology has been devoted to the topic of taming compacta in manifolds. A compactum X in a manifold M is said to be *tame* if there is a homeomorphism of M that takes X onto some particularly "nice" embedding of X in M. The "nice" embedding may be a polyhedron, a standard embedding, an embedding that lies in some hyperplane, or, in the case of Hilbert cube manifolds, a Z-set.

The study of taming compacta necessitates examples of nontame or *wild* compacta. In 3-space there are numerous interesting examples of wild arcs, spheres, and Cantor sets. Many examples in n-space are modifications of the examples in 3-space. There are surprisingly few interesting examples of wild embeddings in the Hilbert cube. The intuitive reason for this fact is that taming theory (the study of Z-sets) is simpler in the Hilbert cube; hence, the construction of wild sets is more difficult.

In 1966 J. Martin constructed a 2-sphere in 3-space that was wild at each of its points [4]; furthermore, each point was wild in a different way so that the sphere was rigidly embedded. A set X in a space M is said to be *rigidly embedded* if the only homeomorphism of X onto itself that extends to a homeomorphism of M is the identity. Shortly after Martin's example, H. G. Bothe [2] constructed a rigid simple closed curve in 3-space. Much later Shilepsky [6] constructed a rigid Cantor set in 3-space, but Shilepsky used work of Sher [5] that was published shortly after the examples of Martin and Bothe.

The constructions of Martin, Bothe, and Shilepsky are quite different. However, the method of Shilepsky has recently been generalized to higher dimensions and the Hilbert cube [3], [7] where it has been used to construct rigid embeddings of spheres and other compacta as well as Cantor sets.

Shilepsky's results depended on Sher's work which has no easy

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generalization to higher dimensions. In what follows we outline a self-contained approach to constructing a rigid Cantor set in 3-space which does generalize to higher dimensions [3], [7].

2. Antoine Cantor sets

Consider the embedding of k solid tori ($k \geqslant 4$) in a solid torus T as shown in Figure 1 (where $k = 4$). We call such an embedding an *Antoine embedding*. An

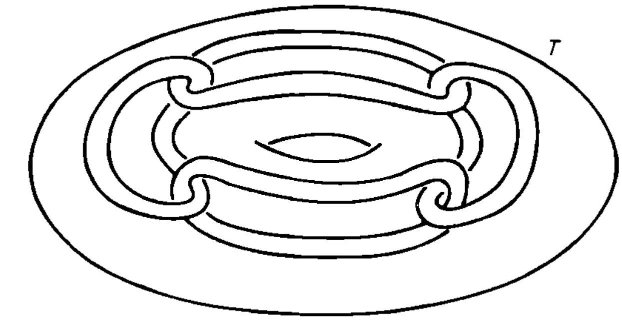

Fig. 1

Antoine Cantor set X is a Cantor set in E^3 that is the intersection of nested compact 3-manifolds M_0, M_1, $M_2, \ldots$ so that M_0 is an unknotted solid torus, and, for each component R of M_i, $R \cap M_{i+1}$ is an Antoine embedding in R. We call M_i an *Antoine defining sequence* for X.

Given an Antoine defining sequence M_0, M_1, $M_2, \ldots$ we define a *necklace* of M_0, M_1, $M_2, \ldots$ to be any set of the form $N \cap M_{j+1}$ where N is a component of M_j. In Antoine's original construction [1] each necklace had four components. A simple way to construct a rigid Cantor set is to construct an Antoine Cantor set with a given Antoine defining sequence so that different necklaces have a different number of components.

The following lemma seems to be well known.

LEMMA 2.1. *Let $X = \bigcap M_i$ and $Y = \bigcap N_i$ be Antoine Cantor sets with Antoine defining sequences M_i, N_i, resp. If M_0 and N_0 are linked solid tori as shown in Figure 2, then it is impossible*

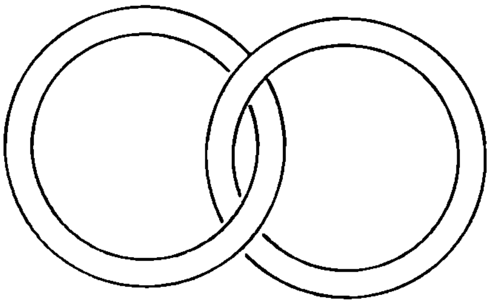

Fig. 2

to separate X from Y by a 2-sphere in E^3. Furthermore, if $X' \subseteq X$ and $Y' \subseteq Y$ are compacta and either $X' \neq X$ or $Y' \neq Y$, then it is possible to separate X' from Y' by a 2-sphere in E^3.

3. Rigid sets

The following lemma is the heart of our argument.

LEMMA 3.1. *Let $X = \bigcap M_i$ and $Y = \bigcap N_i$ be Antoine Cantor sets with Antoine defining sequences M_i and N_i, respectively. If h: $E^3 \to E^3$ is a homeomorphism such that $h(X) \subseteq Y$, then some necklace of M_i has the same number of components as some necklace of N_i.*

Proof. Let n be the largest integer so that $h(X)$ is contained in a component W' of N_n. Let m be the smallest integer so that for each component R of M_{m+1}, $h(R \cap X)$ is contained in a component of $W' \cap N_{n+1}$. Let W be a component of M_m so that there are linked components R, S of $W \cap M_{m+1}$ so that $h(R \cap X)$ and $h(S \cap X)$ are contained in different components R' and S' of $W' \cap N_{n+1}$. By Lemma 2.1 $h(R \cap X) = R' \cap Y$, $h(S \cap X) = S' \cap Y$, and R', S' are linked. An inductive argument shows that $W \cap M_{m+1}$ and $W' \cap N_{n+1}$ have the same number of components.

THEOREM 3.2. *Let Z be an Antoine Cantor set with an Antoine defining sequence M_i so that different necklaces of M_i have a different number of components. Then Z is rigid.*

Proof. Suppose h: $E^3 \to E^3$ is a homeomorphism such that $h(Z) = Z$ and for some $x \in Z$, $h(x) \neq x$. Choose components R, S of some M_i, M_j, respectively, so that $x \in R$, $h(x) \in S$, $R \cap S = \emptyset$, and $h(R \cap Z) \subset S$. Consider the Antoine Cantor sets $R \cap Z$ and $S \cap Z$ along with the defining sequences $M_i \cap R$ and $M_i \cap S$. Lemma 3.1 now gives a contradiction and our proof is complete.

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