DIFFERENT ASPECTS OF DIFFERENTIABILITY
DISSERTATIONES MATHEMATICAE 340 (1995)

CONVOLUTION ALGEBRAS OF SEQUENCES

VICTOR V. ZHARINOV
Steklov Mathematical Institute
Vavilov St. 42, GSP-1, Moscow, Russia
E-mail: victor@zharinov.mian.su

Abstract. We discuss convolution algebras of sequences from the point of view of differential geometry. In particular, we study their differentiations, i.e., linear mappings satisfying the Leibniz rule, and de Rham cohomologies of differential forms over them. Several model examples are considered. We hope that such geometrical approach will throw light on new additional properties of sequences, known as traditional model objects of functional analysis.

Let $\mathbb{C}^{\mathbb{Z}}$ be the linear space of all complex sequences $x = (x^i)$, $x^i \in \mathbb{C}$, $i \in \mathbb{Z}$, with the natural topology of the projective limit of the Euclidean spaces. We shall always assume that a linear subspace $\mathcal{L} \subset \mathbb{C}^{\mathbb{Z}}$ under study is equipped with some locally convex topology (for example, induced from $\mathbb{C}^{\mathbb{Z}}$). Denote by $e_m = (e^i_m) \in \mathbb{C}^{\mathbb{Z}}$ the sequence with the Kronecker components $e^i_m = \delta^i_m$, $m, i \in \mathbb{Z}$. For a subset $\mathcal{L} \subset \mathbb{C}^{\mathbb{Z}}$ we denote by $\mathbb{S}(\mathcal{L})$ the set of all integers $m \in \mathbb{Z}$ such that $e_m \in \mathcal{L}$. We shall call a linear subspace $\mathcal{L} \subset \mathbb{C}^{\mathbb{Z}}$ *standard* if the set $\{e_m : m \in \mathbb{S}(\mathcal{L})\}$ is a basis in $\mathcal{L}$. By $\operatorname{supp} x \subset \mathbb{Z}$ we denote the *support* of a sequence $x \in \mathbb{C}^{\mathbb{Z}}$, i.e., the set of all $i \in \mathbb{Z}$ such that $x^i \neq 0$.

EXAMPLE 1. Let $\mathbb{Z}_{\geq k} = \{k, k+1, k+2, \ldots\}$, $k \in \mathbb{Z}$ (in particular, $\mathbb{N} = \mathbb{Z}_{\geq 0}$). The linear subspaces

$$\mathcal{L}_{\geq k} = \{x \in \mathbb{C}^{\mathbb{Z}} : \operatorname{supp} x \subset \mathbb{Z}_{\geq k}\}, \qquad k \in \mathbb{Z},$$

equipped with the induced topology, are standard.

EXAMPLE 2. The linear subspace

$$\mathcal{S} = \Big\{x = (x^i) : \|x\|_q = \sum_{i \in \mathbb{Z}} |i^q x^i| < \infty \text{ for all } q \in \mathbb{N}\Big\},$$

with the topology given by the norms $\|\cdot\|_q$ is standard.

1991 *Mathematics Subject Classification*: 44A35, 47B37.
The paper is in final form and no version of it will be published elsewhere.

EXAMPLE 3. For the linear subspaces

$$\mathcal{L}(p) = \{x = (x^i) : x^{i+p} = x^i \text{ for all } i \in \mathbb{Z}\}, \quad p = 1, 2, \dots,$$

the sets $\mathbb{S}(\mathcal{L}(p))$ are empty. In particular, these subspaces are not standard.

Notice that if subspaces $\mathcal{L}$ and $\mathcal{M}$ are standard, and $f : \mathcal{L} \to \mathcal{M}$ is a continuous linear mapping, then

$$fx = \sum_{m \in \mathbb{S}(\mathcal{L})} x^m f_m \quad \text{for all } x = (x^i) \in \mathcal{L},$$

where $f_m = fe_m \in \mathcal{M}$, $m \in \mathbb{S}(\mathcal{L})$.

We denote by $x * y \in \mathbb{C}^{\mathbb{Z}}$ the *convolution* of sequences $x, y \in \mathbb{C}^{\mathbb{Z}}$, where

$$(x * y)^i = \sum_{j \in \mathbb{Z}} x^{i-j} y^j, \quad i \in \mathbb{Z}.$$

For a given pair x, y the convolution $x * y$ is defined iff the series on the right side converges for all $i \in \mathbb{Z}$.

EXAMPLE 4. Clearly,

$$e_m * e_n = e_{m+n} \quad \text{for all } m, n \in \mathbb{Z},$$

and

$$(x * e_m)^i = x^{i-m} \quad \text{for all } x \in \mathbb{C}^{\mathbb{Z}} \text{ and } i, m \in \mathbb{Z}.$$

In particular, $x * e_0 = x$ for all $x \in \mathbb{C}^{\mathbb{Z}}$.

DEFINITION. A linear subspace $\mathcal{A} \subset \mathbb{C}^{\mathbb{Z}}$ is called a *convolution algebra of sequences* (CAS) if $x * y$ exists in $\mathcal{A}$ for all $x, y \in \mathcal{A}$, and the operation $*$ defines a continuous commutative multiplication in $\mathcal{A}$.

EXAMPLE 1(a). The linear subspaces $\mathcal{L}_{\geq k}$ are CAS for all $k \geq 0$. Indeed, the convolution $x * y \in \mathcal{L}_{\geq k+l}$ for all $x \in \mathcal{L}_{\geq k}$ and $y \in \mathcal{L}_{\geq l}$, because

$$(x * y)^i = \sum_{l \leq j \leq i-k} x^{i-j} y^j = 0 \quad \text{for all } i < k + l.$$

EXAMPLE 2(a). The linear subspace $\mathcal{S}$ is a CAS. Indeed, using the inequality

$$(i + j)^q \leq 2^{q-1}(i^q + j^q), \quad i, j, q = 1, 2, \dots,$$

one can easily check that $\|x * y\|_q \leq 2^{q-1}\|x\|_q \cdot \|y\|_q$, hence, $x * y \in \mathcal{S}$ for all $x, y \in \mathcal{S}$.

LEMMA 1. *Let $\mathcal{A}$ be a CAS. Then the set $\mathbb{S}(\mathcal{A})$ is an additive semigroup.*

Proof. In this case $e_m * e_n = e_{m+n} \in \mathcal{A}$ for all $e_m, e_n \in \mathcal{A}$.

DEFINITION. A continuous linear mapping $\zeta : \mathcal{A} \to \mathcal{A}$ is called a *differentiation* of a CAS $\mathcal{A}$ if it satisfies the Leibniz rule

$$\zeta(x * y) = x * (\zeta y) + y * (\zeta x) \quad \text{for all } x, y \in \mathcal{A}.$$

The set of all differentiations of a CAS $\mathcal{A}$ is denoted by $\mathcal{DA}$. Clearly, $\mathcal{DA}$ is an $\mathcal{A}$-module and a Lie algebra.

LEMMA 2. *Let $\mathcal{A}$ be a standard CAS. Then $\zeta \in \mathcal{DA}$ iff*

$$\zeta_{m+n} = e_m * \zeta_n + e_n * \zeta_m \quad \text{for all } m, n \in \mathbb{S}(\mathcal{A}).$$

Proof. Indeed, here

$$\zeta(x*y) - x*(\zeta y) - y*(\zeta x) = \sum_{m,n\in\mathbb{S}(\mathcal{A})} x^m y^n \big(\zeta_{m+n} - e_m * \zeta_n - e_n * \zeta_m\big).$$

LEMMA 3. *Let $\mathcal{A}$ be a standard CAS, and $\mathbb{S}(\mathcal{A}) = \mathbb{Z}$. Then $\zeta \in \mathcal{DA}$ iff*

$$\zeta_m = m \cdot e_m * z \quad \text{for all } m \in \mathbb{S}(\mathcal{A}),$$

where $z = e_{-1} * \zeta_1 \in \mathcal{A}$.

Proof. To check the formula for $m = 0, 1, 2, \dots$ it is enough to use Lemma 2. To treat $m = -1, -2, \dots$ one should use Lemma 2 and the equality

$$0 = \zeta e_0 = \zeta(e_{-1} * e_1) = e_1 * \zeta_{-1} + e_{-1} * \zeta_1.$$

LEMMA 4. *Let $\mathcal{A}$ be a standard CAS, and $\mathbb{S}(\mathcal{A}) = \mathbb{Z}_{\ge k}$, $k \ge 0$. Then $\zeta \in \mathcal{DA}$ iff*

$$\zeta_m = m \cdot e_m * z \quad \text{for all } m \in \mathbb{S}(\mathcal{A}),$$

where $z = \frac{1}{p} e_{-p} * \zeta_p \in \mathbb{C}^{\mathbb{Z}}$, $p = \max\{k, 1\}$.

Proof. The proof is done by direct induction when $k = 0, 1$. For $k \ge 2$ let $z = \frac{1}{k} e_{-k} * \zeta_k$. Denote $z_i = \frac{1}{k+i} e_{-k-i} * \zeta_{k+i}$ for $i = 1, \dots, k-1$. Using Lemma 2, one can derive the system

$$\begin{cases} kz + (k+2)z_2 = (k+1)z_1 + (k+1)z_1, \\ kz + (k+3)z_3 = (k+1)z_1 + (k+2)z_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ kz + (2k-1)z_{k-1} = (k+1)z_1 + (2k-2)z_{k-2}, \\ kz + 2kz = (k+1)z_1 + (2k-1)z_{k-1}. \end{cases}$$

It easy to check that the only solution is $z_1 = z_2 = \dots = z_{k-1} = z$. The proof is completed by induction.

Define the linear mapping $\nabla : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ by the rule $\nabla e_m = m e_m, m \in \mathbb{Z}$. Then $\nabla(x * y) = (\nabla x) * y + x * (\nabla y)$ for all $x, y \in \mathbb{C}^{\mathbb{Z}}$ such that the left and right sides of the equality are defined. In particular, if $\mathcal{A}$ is a ∇-invariant CAS then $\nabla \in \mathcal{DA}$. (We say that a CAS $\mathcal{A}$ is ∇-*invariant* if $\nabla(\mathcal{A}) \subset \mathcal{A}$.)

LEMMA 5. *Let $\mathcal{A}$ be a ∇-invariant CAS. Then its subalgebra of constants is*

$$\mathbb{K} = \begin{cases} \mathbb{C} & \text{if } 0 \in \mathbb{S}(\mathcal{A}), \\ 0 & \text{if } 0 \notin \mathbb{S}(\mathcal{A}). \end{cases}$$

Proof. Indeed, let $x \in \mathbb{K}$. Then $\nabla x = 0$, i.e., $i x^i = 0$ for all $i \in \mathbb{Z}$, hence, $x^i = 0$ for $i \ne 0$, and x^0 is arbitrary. Thus, $x = \lambda e_0$, $\lambda \in \mathbb{C}$.

EXAMPLE 1(b). Every CAS $\mathcal{L}_{\ge k}$, $k \ge 0$, is ∇-invariant. Moreover, $\zeta \in \mathcal{DL}_{\ge k}$ iff $\zeta = z * \nabla$, where $z = e_{-1} * \zeta_1 \in \mathcal{L}_{\ge -1}$ if $k = 0$, and $z = \frac{1}{k} e_{-k} * \zeta_k \in \mathcal{L}_{\ge 0}$ if $k \ge 1$.

Moreover, the linear mapping $z \mapsto \zeta = z * \nabla$ defines the isomorphisms

$$*\nabla : \mathcal{L}_{\geq -1} \rightsquigarrow \mathcal{DL}_{\geq 0}, \quad \text{and} \quad *\nabla : \mathcal{L}_{\geq 0} \rightsquigarrow \mathcal{DL}_{\geq k}, \quad \text{for all } k \geq 1.$$

In particular, $\mathcal{DL}_{\geq 0}$ is a free one-dimensional $\mathcal{L}_{\geq 0}$-module.

EXAMPLE 2(b). The CAS $\mathcal{S}$ is ∇-invariant. Moreover, $\zeta \in \mathcal{DS}$ iff $\zeta = z * \nabla$ with $z = e_{-1} * \zeta_1 \in \mathcal{S}$. In particular, $\mathcal{DS}$ is a free one-dimensional $\mathcal{S}$-module, and the linear mapping $z \mapsto \zeta = z * \nabla$ defines the isomorphism

$$*\nabla : \mathcal{S} \rightsquigarrow \mathcal{DS}.$$

For a pair $x, y \in \mathbb{C}^{\mathbb{Z}}$ define its *Poisson bracket* $\{x, y\} \in \mathbb{C}^{\mathbb{Z}}$ by the formula

$$\{x, y\} = x * \nabla y - y * \nabla x,$$

if the convolutions on the right side exist. The Poisson bracket $\{\cdot,\cdot\}$ is skew-symmetric, and satisfies the Jacobi identity (when all entries exist).

EXAMPLE 1(c). Clearly, $\{x, y\} \in \mathcal{L}_{\geq -2}$ for all $x, y \in \mathcal{L}_{\geq -1}$. Moreover, in this case,

$$\{x, y\}^{-2} = \sum_{-1 \leq j \leq -1} j(x^{-2-j}y^j - y^{-2-j}x^j) = (-1)(x^{-1}y^{-1} - y^{-1}x^{-1}) = 0.$$

Hence, the Poisson bracket $\{\cdot,\cdot\}$ defines a Lie algebra structure on $\mathcal{L}_{\geq -1}$. Moreover, the isomorphism $*\nabla : \mathcal{L}_{\geq -1} \rightsquigarrow \mathcal{DL}_{\geq 0}$ is an isomorphism of Lie algebras.

Also, the Poisson bracket $\{\cdot,\cdot\}$ defines a Lie algebra structure in $\mathcal{L}_{\geq 0}$. Moreover, the isomorphism $*\nabla : \mathcal{L}_{\geq 0} \rightsquigarrow \mathcal{DL}_{\geq k}$ is an isomorphism of Lie algebras for any $k \geq 1$.

R e m a r k. The isomorphism of linear spaces

$$e_1 * : \mathcal{L}_{\geq -1} \rightsquigarrow \mathcal{L}_{\geq 0},$$

acting by the rule $x \mapsto e_1 * x$, defines another Poisson structure on $\mathcal{L}_{\geq 0}$ given by the Poisson bracket

$$\{x, y\}' = e_1 * \{e_{-1} * x, e_{-1} * y\} = e_{-1} * \{x, y\} \quad \text{for all } x, y \in \mathcal{L}_{\geq 0}$$

(notice that here $\{x, y\} \in \mathcal{L}_{\geq 1}$). It is easy to check that the Poisson structures $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}'$ are different (i.e., the corresponding Lie algebras are nonisomorphic).

EXAMPLE 2(c). The Poisson bracket $\{\cdot,\cdot\}$ defines a Lie algebra structure on $\mathcal{S}$. Moreover, the isomorphism $*\nabla : \mathcal{S} \rightsquigarrow \mathcal{DS}$ is an isomorphism of Lie algebras.

Let $\mathcal{L}$, $\mathcal{M}$ be $\mathcal{A}$-modules, where $\mathcal{A}$ is some CAS. A linear mapping $\Omega : \mathcal{L} \to \mathcal{M}$ is called $\mathcal{A}$-*linear* if in addition $\Omega(x * f) = x * \Omega(f)$ for all $x \in \mathcal{A}$ and $f \in \mathcal{L}$. We denote by $\mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{M})$ the set of all $\mathcal{A}$-linear mappings $\Omega : \mathcal{L} \to \mathcal{M}$.

LEMMA 6. *Let $\mathcal{L}, \mathcal{M} \subset \mathbb{C}^{\mathbb{Z}}$ be standard $\mathcal{A}$-modules over some standard CAS $\mathcal{A}$. Then a linear mapping $\Omega \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{M})$ iff*

$$\Omega_{m+n} = e_m * \Omega_n \quad \textit{for all } m \in \mathbb{S}(\mathcal{A}) \textit{ and } n \in \mathbb{S}(\mathcal{L}).$$

Proof. It is enough to note that

$$\Omega(x*y) - x*\Omega y = \sum_{m\in\mathbb{S}(\mathcal{A})}\ \sum_{n\in\mathbb{S}(\mathcal{L})} x^m y^n \big(\Omega_{m+n} - e_m * \Omega_n\big)$$

for all $x = \sum_{m\in\mathbb{S}(\mathcal{A})} x^m e_m \in \mathcal{A}$ and $y = \sum_{n\in\mathbb{S}(\mathcal{L})} y^n e_n \in \mathcal{L}$.

Lemma 7. *Let $\mathcal{L},\mathcal{M}\subset\mathbb{C}^{\mathbb{Z}}$ be standard $\mathcal{A}$-modules over some standard CAS $\mathcal{A}$. Suppose that there exists $p\in\mathbb{S}(\mathcal{L})$ with the following property: for every $n\in\mathbb{S}(\mathcal{L})$ one can find $m\in\mathbb{S}(\mathcal{A})$ such that $m+n-p\in\mathbb{S}(\mathcal{A})$. Then a linear mapping $\Omega\in\mathrm{Hom}_{\mathcal{A}}(\mathcal{L},\mathcal{M})$ iff*

$$\Omega_n = e_n*\omega, \qquad \omega = e_{-p}*\Omega_p \in e_{-p}*\mathcal{M}.$$

Proof. Indeed, for a given $n\in\mathbb{S}(\mathcal{L})$ choose some $m\in\mathbb{S}(\mathcal{A})$ with $m+n-p\in\mathbb{S}(\mathcal{A})$. Then

$$e_m*\Omega_n = \Omega_{m+n} = \Omega_{(m+n-p)+p} = e_{m+n-p}*\Omega_p,$$

hence, $\Omega_n = e_{-m}*e_{m+n-p}*\Omega_p = e_n*\omega$.

Remark. If $\mathbb{S}(\mathcal{A}) = \mathbb{Z}$ or $\mathbb{Z}_{\ge k},\ k\ge 0$, then every $p\in\mathbb{S}(\mathcal{L})$ has the property of Lemma 7.

Definition. Let $\mathcal{A}$ be a CAS. A *q-form* Ω over $\mathcal{A}$ is a mapping

$$(\mathcal{D}\mathcal{A})^q \ni (\zeta_1,\dots,\zeta_q)\mapsto \Omega(\zeta_1,\dots,\zeta_q)\in\mathcal{A}$$

which is $\mathcal{A}$-linear in every component ζ_μ, $\mu = 1,\dots,q$, and skew-symmetric in $\zeta_1,\dots,\zeta_q$. We denote by $\Lambda^q\mathcal{A}$ the linear space of all q-forms over $\mathcal{A}$, $q = 0,1,\dots$ In particular, $\Lambda^0\mathcal{A} = \mathcal{A}$, $\Lambda^1\mathcal{A} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{D}\mathcal{A},\mathcal{A})$.

Clearly, $\Lambda^q\mathcal{A} = 0$ for all $q>p$ if $\mathcal{D}\mathcal{A}$ is a free p-dimensional $\mathcal{A}$-module.

The differential $d\Omega\in\Lambda^{q+1}\mathcal{A}$ of a q-form $\Omega\in\Lambda^q\mathcal{A}$ is defined by the rule

$$\begin{aligned}(d\Omega)(\zeta_0,\dots,\zeta_q) = {}& \frac{1}{q+1}\sum_{\mu=0}^{q}(-1)^\mu\zeta(\Omega(\zeta_0,\dots,\widehat{\zeta_\mu},\dots,\zeta_q))\\ &+\frac{1}{q+1}\sum_{\mu<\nu}(-1)^{\mu+\nu}\Omega([\zeta_\mu,\zeta_\nu],\zeta_0,\dots,\widehat{\zeta_\mu},\dots,\widehat{\zeta_\nu},\dots,\zeta_q),\end{aligned}$$

for all differentiations $\zeta_0,\dots,\zeta_q\in\mathcal{D}\mathcal{A}$. In particular, $(dx)(\zeta) = \zeta x$ for all $x\in\mathcal{A}$ and $\zeta\in\mathcal{D}\mathcal{A}$. Clearly, $d\circ d = 0$, hence, the de Rham type complex

$$0\to\Lambda^0\mathcal{A}\xrightarrow{d}\Lambda^1\mathcal{A}\xrightarrow{d}\Lambda^2\mathcal{A}\xrightarrow{d}\dots$$

is defined. We denote by $H^q(\mathcal{A})$ the cohomology spaces of this complex, $q = 0,1,\dots$

Example 1(d). Case $k=0$. By definition, $\Lambda^0\mathcal{L}_{\ge0} = \mathcal{L}_{\ge0}$. Further,

$$\Lambda^1\mathcal{L}_{\ge0} = \mathrm{Hom}_{\mathcal{L}_{\ge0}}(\mathcal{D}\mathcal{L}_{\ge0},\mathcal{L}_{\ge0}) \xrightarrow{\sim} \mathcal{L}_{\ge0},$$

the isomorphism is given by the rule $\Omega\mapsto\omega = \Omega(e_{-1}*\nabla)$, so $\Omega(\zeta) = x*\omega$, for all $\zeta = z*\nabla\in\mathcal{D}\mathcal{L}_{\ge0}$ with $z\in\mathcal{L}_{\ge-1}$, and $x = e_1*z\in\mathcal{L}_{\ge0}$. Moreover, following the proof of Lemma 7, one can easily check that $\Omega(e_{-1} * \nabla) = e_{-1} * \Omega(\nabla)$ for any $\Omega \in \Lambda^1\mathcal{L}_{\geq 0}$, hence, also, $\Lambda^1\mathcal{L}_{\geq 0} \rightsquigarrow \mathcal{L}_{\geq 1}$, with $\Omega \mapsto \Omega(\nabla)$. Clearly, $\Lambda^q\mathcal{L}_{\geq 0} = 0$ for all $q \geq 2$, because $\mathcal{D}\mathcal{L}_{\geq 0}$ is a free 1-dimensional $\mathcal{L}_{\geq 0}$-module.

Let $x \in \mathcal{L}_{\geq 0}$. Then $dx = 0$ iff $x = \lambda e_0$, $\lambda \in \mathbb{C}$ (see Lemma 5). Hence, $H^0(\mathcal{L}_{\geq 0}) = \mathbb{C}$. Further, let $\Omega \in \Lambda^1\mathcal{L}_{\geq 0}$. Then $\Omega = dx$ for some $x \in \Lambda^0\mathcal{L}_{\geq 0}$ iff $\Omega(\zeta) = (dx)(\zeta) = \zeta(x)$, i.e., iff $\nabla x = \omega$, where $\omega = \Omega(\nabla) \in \mathcal{L}_{\geq 1}$ (see above). The last equation reads $i\,x^i = 0,\ i \leq 0,\ i\,x^i = \omega^i,\ i \geq 1$, and has the obvious solution $x^i = 0,\ i \leq 0,\ x^i = \frac{1}{i}\omega^i,\ i \geq 1$. Hence, $H^1(\mathcal{L}_{\geq 0}) = 0$.

Example 1(d). Case $k \geq 1$. Notice first that here the linear mapping $\nabla : \mathcal{L}_{\geq k} \to \mathcal{L}_{\geq k}$ has the inverse ∇^{-1}, defined by the rule $(\nabla^{-1}y)^i = \frac{1}{i}y^i$ for $i \geq k$, and $= 0$ in the opposite case, $y \in \mathcal{L}_{\geq k}$. By definition $\Lambda^0\mathcal{L}_{\geq k} = \mathcal{L}_{\geq k}$, but now $dx = 0,\ x \in \mathcal{L}_{\geq k}$, iff $x = 0$ (Lemma 5, again), hence, $H^0(\mathcal{L}_{\geq k}) = 0$. Further,

$$\Lambda^1\mathcal{L}_{\geq k} = \mathrm{Hom}_{\mathcal{L}_{\geq k}}(\mathcal{D}\mathcal{L}_{\geq k}, \mathcal{L}_{\geq k}) = \mathrm{Hom}_{\mathcal{L}_{\geq 0}}(\mathcal{D}\mathcal{L}_{\geq k}, \mathcal{L}_{\geq k}),$$

the last equality is derived as in the proof of Lemma 7. The rule $\Omega \mapsto \Omega(\nabla)$ defines the isomorphism $\Lambda^1\mathcal{L}_{\geq k} \rightsquigarrow \mathcal{L}_{\geq k}$. A form $\Omega \in \Lambda^1\mathcal{L}_{\geq k}$ is exact, i.e., $\Omega = dx$ for some $x \in \mathcal{L}_{\geq k}$, iff $\Omega(\nabla) = \nabla x$, i.e., if $\nabla x = \Omega(\nabla) \in \mathcal{L}_{\geq k}$. This equation has a unique solution $x = \nabla^{-1}(\Omega(\nabla))$, hence $H^1(\mathcal{L}_{\geq k}) = 0$, also. Finally, if $\Omega \in \Lambda^q\mathcal{L}_{\geq k}$, $q \geq 2$, and $\zeta_1, \dots, \zeta_q \in \mathcal{D}\mathcal{L}_{\geq k}$, then, according to the above representation for $\Lambda^1\mathcal{L}_{\geq k}$, we get $\Omega(\zeta_1, \dots, \zeta_q) = z_1 * \ldots * z_q * \Omega(\nabla, \dots, \nabla) = 0$, by skew-symmetry. Hence, $\Lambda^q\mathcal{L}_{\geq k} = 0$ for $q \geq 2$, again.

Example 2(d). By definition, $\Lambda^0\mathcal{S} = \mathcal{S}$, and according to Lemma 5, $dx = 0,\ x \in \mathcal{S}$, iff $x = \lambda e_0$, $\lambda \in \mathbb{C}$, hence, $H^0(\mathcal{S}) = \mathbb{C}$. Further,

$$\Lambda^1\mathcal{S} = \mathrm{Hom}_{\mathcal{S}}(\mathcal{D}\mathcal{S}, \mathcal{S}) \rightsquigarrow \mathcal{S},$$

where the isomorphism is given by the formula $\Omega \mapsto \Omega(\nabla)$. Let $\Omega \in \Lambda^1\mathcal{S}$. Then $\Omega = dx$ for some $x \in \Lambda^0\mathcal{S}$ iff $\nabla x = \omega$, where $\omega = \Omega(\nabla) \in \mathcal{S}$. The last equation reads $i\,x^i = \omega^i,\ i \in \mathbb{Z}$, and has a solution iff $\omega^0 = 0$. The decomposition $\omega = \omega^0 e_0 + \phi$, with $\phi^0 = \omega^0 - \omega^0 = 0$, shows that $H^1(\mathcal{S}) = \mathbb{C}$. For a basis in $H^1(\mathcal{S})$ one can take $\Omega = e_{-1} * de_1$. Clearly, the spaces $\Lambda^q\mathcal{S} = 0$ for all $q \geq 2$, because $\mathcal{D}\mathcal{S}$ is a free 1-dimensional $\mathcal{S}$-module.