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SOME EXTREMAL RESULTS CONCERNING THE NUMBER OF GRAPH AND HYPERGRAPH COLORINGS

IOAN TOMESCU

Faculty of Mathematics, University of Bucharest, Bucharest, Romania

1. Preliminary definitions and results

Given $h \geqslant 2$, an *h-uniform hypergraph* H is a pair $H = (X, \mathscr{E})$, where X is a finite set of vertices and $\mathscr{E}$ is a set of h-subsets of X, i.e. $|E| = h$ for all $E \in \mathscr{E}$. For $h = 2$ such a pair is called a *graph*. The members of $\mathscr{E}$ are called the *edges* of H. The *order* of the hypergraph H is equal to $|X|$.

An *independent set* of H is a subset S of vertices, $S \subset X$, which does not contain any edge of H. An *r-coloring* of H is a partition of the vertex-set X into r independent subsets. The minimum number of classes in a partition of the vertex-set X into independent sets is called the *chromatic number* of H, and it will be denoted by $\chi(H)$. The number of r-colorings of H will be denoted by $\mathrm{Col}_r(H)$.

A *path* P of length $k \geqslant 1$ in H is a sequence of vertices and edges of H of the form

$$P: \quad x_1, E_1, x_2, E_2, \ldots, x_k, E_k, x_{k+1}$$

such that:

1) The vertices $x_1, \ldots, x_{k+1}$ are pairwise distinct;
2) The edges $E_1, \ldots, E_k$ are pairwise distinct;
3) $x_1 \in E_1$; $x_{k+1} \in E_k$ and $x_p \in E_{p-1} \cap E_p$ for every $p = 2, \ldots, k$.

A hypergraph H is *connected* if, for any two distinct vertices x, y of H, there is a path connecting them.

In [2] it is shown that the maximal number of k-colorings of a graph G of order n with $\chi(G) = k$ is equal to k^{n-k} and the corresponding extremal graph is unique and is the union of a k-clique and $n-k$ isolated vertices. This graph also realizes the minimal number of edges and the maximal number of

$(k+r)$-colorings in the class of k-chromatic graphs of order n for every $0 \leqslant r \leqslant n-k-1$ and it is unique having these properties.

When $h \geqslant 3$ the minimal number of edges that a k-chromatic h-hypergraph H of order n can contain is not known; a similar situation occurs for the maximal number of $(k+r)$-colorings of H. Note that $\mathrm{Col}_{n-h+1}(H) = S(n, n-h+1) - m(H)$, where $m(H)$ is the number of edges of H and $S(n, n-h+1)$ is the Stirling number of the second kind. Thus the two problems are connected. However, the following asymptotic result was obtained in [6]:

Let $C(n, h, k)$ (and $C^*(n, h, k)$, respectively) be the maximum number of k-colorings in the class of h-hypergraphs H of order n (which are connected, respectively), having chromatic number $\chi(H) = k$. Also let $C(n, h) = \max_k C(n, h, k)$ and $C^*(n, h) = \max_k C^*(n, h, k)$.

THEOREM 1. *For every* $h \geqslant 2$ *the following relations hold*:

$$C(n,h)^{1/n} \sim C^*(n, h)^{1/n} \sim \frac{n}{e \ln n}$$

as $n \to \infty$.

2. Minimum number of hypergraph colorings

The problem of determining the minimum number of s-colorings of a hypergraph was completely solved in [5]. In order to present the solution here we need some definitions.

Let $H(n, k, h)$ be the class of h-hypergraphs H having the vertex-set X with $|X| = n$ and such that, for each H, there exists a k-equipartition of X:

$$X = A_1 \cup \ldots \cup A_k \quad (-1 \leqslant |A_i| - |A_j| \leqslant 1 \text{ for any } 1 \leqslant i, j \leqslant k),$$

such that the edge-set of H is composed of all h-subsets of X that are not contained in any class A_i.

The hypergraphs from $H(n, k, h)$ are generated by various k-equipartitions of X and are all pairwise isomorphic.

It is easy to show that for any $n \geqslant (k-1)(h-1)+1$ and $k \geqslant h \geqslant 2$ the maximum number of edges of an h-hypergraph H of order n and chromatic number $\chi(H) = k$ is attained if and only if $H \in H(n, k, h)$.

Let $c(n, k, h, s)$ be the minimum number of s-colorings of an h-hypergraph H having $n \geqslant (k-1)(h-1)+1$ vertices and chromatic number $\chi(H) = k \geqslant 2$.

THEOREM 2. *For any* $k+1 \leqslant s \leqslant n-h+1$, $\mathrm{Col}_s(H) = c(n, k, h, s)$ *in the class of h-hypergraphs H of order n and chromatic number* $k \geqslant 2$ *if and only if* $H \in H(n, k, h)$.

For $s = k$ *we distinguish two cases*:

(a) $(k-1)(h-1)+1 \leqslant n \leqslant k(h-1)+1$. *In this case the class of extremal hypergraphs is* $H(n, k, h)$.

(b) $n \geqslant k(h-1)+2$. *In this case the class of extremal hypergraphs strictly contains* $H(n, k, h)$.

Let $X = B_1 \cup \ldots \cup B_k$ be a partition of X, where $|X| = n-1$, $|B_q| = n_q$ for $q = 1, \ldots, k$, and suppose that there exist two indices $i, j \in \{1, \ldots, k\}$ such that $n_i > n_j$ and $n_i \geqslant h-1$. For $x \notin X$ consider the following partitions of $X \cup \{x\}$:

$$B_1 \cup \ldots \cup (B_i \cup \{x\}) \cup \ldots \cup B_k,$$

which generates, as above, an h-hypergraph denoted by H_i, and

$$B_1 \cup \ldots \cup (B_j \cup \{x\}) \cup \ldots \cup B_k,$$

which produces H_j.

The key of the proof of Theorem 2 is the following result:

PROPOSITION 3. *For any* $k+1 \leqslant s \leqslant n-h+1$, $\mathrm{Col}_s(H_i) > \mathrm{Col}_s(H_j)$.

Proof. Let H be the h-hypergraph of order $n-1$ generated by $X = B_1 \cup \ldots \cup B_k$. For an s-coloring π of H, denote by $\varphi_{h-1}(\pi, B_i)$ and $\varphi_{h-1}(\pi, B_j)$ the number of classes of π having at least $h-1$ elements that are contained in B_i and B_j, respectively, and by $\psi_{h-2}(\pi)$ the number of classes of π containing at most $h-2$ elements. One obtains

$$\mathrm{Col}_s(H_t) = \mathrm{Col}_{s-1}(H) + \sum_{\pi} \varphi_{h-1}(\pi, B_t) + \sum_{\pi} \psi_{h-2}(\pi)$$

for $t = i, j$. Since $s \leqslant n-h+1$ and $n_i \geqslant h-1$ one obtains $\sum_{\pi} \varphi_{h-1}(\pi, B_i) > 0$. We must show that

$$(1) \qquad \sum_{\pi} \varphi_{h-1}(\pi, B_i) > \sum_{\pi} \varphi_{h-1}(\pi, B_j).$$

If $n_j \leqslant h-2$ then $\sum_{\pi} \varphi_{h-1}(\pi, B_j) = 0$ and (1) is proved. Otherwise $n_i > n_j \geqslant h-1$. Let $C \subset B_i$ be such that $|C| = n_j$ and let $f\colon C \to B_j$ be a bijection. Let π be an s-coloring of H satisfying $a = \varphi_{h-1}(\pi, B_i) < \varphi_{h-1}(\pi, B_j) = b$. If the classes of π which are included in B_i and in B_j are $C_1, \ldots, C_a$ and $D_1, \ldots, D_b$, respectively, suppose that $\bigcup_{i=1}^{a} C_i \subset C$. In this case if we replace each vertex $x \in C$ by $f(x) \in B_j$ and each vertex $y \in B_j$ by $f^{-1}(y) \in C \subset B_i$ we find another coloring π' of H such that $\varphi_{h-1}(\pi', B_i) = b$ and $\varphi_{h-1}(\pi', B_j) = a$ and this correspondence between colorings is injective.

Otherwise $\bigcup_{i=1}^{a} C_i \not\subset C$ and let $E_1, \ldots, E_d$ $(d \leqslant a)$ be the nonempty sets of the family $\{C_p \backslash C\colon 1 \leqslant p \leqslant a,\ C_p \cap C \neq \emptyset\}$. Let also $C'_1, \ldots, C'_{a'}$ $(a' \leqslant a)$ be the nonempty classes obtained from $C_1, \ldots, C_a$ by deleting elements of $B_i \backslash C$.

In this case we define a coloring π' as follows: The classes among $C_1, \ldots, C_a$ having a nonempty intersection with C will be replaced by

$f(C'_1), \ldots, f(C'_{a'})$ and for the other classes of π every vertex $x \in C$ will be replaced by $f(x) \in B_j$ and every vertex $y \in B_j$ by $f^{-1}(y) \in C \subset B_i$.

By this procedure the b classes of π included in B_j and containing each at least $h-1$ elements have been transformed into b classes included in B_i and containing at least $h-1$ elements, namely $f^{-1}(D_1), \ldots, f^{-1}(D_b)$. The sets $E_1, \ldots, E_d \subset B_i \backslash C$ will be adjoined to these subsets of B_i in the following manner:

Let g: $\{E_1, \ldots, E_d\} \to \{f^{-1}(D_1), \ldots, f^{-1}(D_b)\}$ be an injection (there exists one since $d \leqslant a < b$). In this case every class $g(E_p)$ will be transformed into $g(E_p) \cup E_p$ for $p = 1, \ldots, d$. In this way we have obtained an s-coloring π' of H such that

$$\varphi_{h-1}(\pi', B_i) \geqslant b, \quad \varphi_{h-1}(\pi', B_j) \leqslant a.$$

We shall prove that for every s-coloring π of H such that $\varphi_{h-1}(\pi, B_i) < \varphi_{h-1}(\pi, B_j)$ we can choose an injection g such that the correspondence between the colorings π and π' be injective.

Indeed, g may be defined in $(b)_d = b(b-1)\ldots(b-d+1)$ ways. On the other hand, if all classes of π having an empty intersection with C are fixed, there are $(a')_d = a'(a'-1)\ldots(a'-d+1)$ partitions generating the same splitting $C'_1, \ldots, C'_{a'}$; $E_1, \ldots, E_d$ of the family of those classes having a nonempty intersection with C.

Hence the inequalities $(a')_d \leqslant (a)_d < (b)_d$ imply that we can define for every s-coloring π of H such that $a = \varphi_{h-1}(\pi, B_i) < \varphi_{h-1}(\pi, B_j) = b$ another s--coloring π' of H such that $\varphi_{h-1}(\pi', B_i) \geqslant b$ and $\varphi_{h-1}(\pi', B_j) \leqslant a$ and this correspondence between partitions is injective but not surjective, and (1) is proved.

We have

$$\lim_{n \to \infty} c(n, k, h, s)^{1/n} = \max_{s_1 + \ldots + s_k = s} (s_1 \ldots s_k)^{1/k}$$

and, as in the case of the maximum number of s-colorings, this quantity does not depend on $h \geqslant 2$ [5].

3. Maximum number of colorings of connected graphs

Let G be a graph of order n and $P(G, \lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda$ be its chromatic polynomial. It is well known that $P(G, \lambda)$ can be expressed in terms of the number of k-colorings as follows:

$$P(G, \lambda) = \sum_{k=1}^{n} (\lambda)_k \operatorname{Col}_k(G),$$

hence

$$\operatorname{Col}_k(G) = \frac{1}{k!} P(G, k) \quad \text{if } k = \chi(G).$$

The n-vertex cycle will be denoted by C_n and the graph consisting of C_n and another vertex which is joined by an edge to a single vertex of C_n by C_n^1. A *numbering* of the vertices of a graph G of order n is a bijection f: $V(G) \to \{1, \ldots, n\}$.

It is natural to ask what the maximum number of colorings is for some restricted classes of graphs which do not contain the graph composed of a k-clique and $n-k$ isolated vertices, e.g. for connected graphs. The following theorem was proved in [4]:

THEOREM 4. *The maximum number of 3-colorings of a connected graph G having n vertices and chromatic number $\chi(G) = 3$ is $\frac{1}{3}(2^{n-1}-1)$ for odd n and $\frac{2}{3}(2^{n-2}-1)$ for even n.*

If n is odd, the unique connected graph for which this maximum is reached is C_n, while if n is even, the unique graph is C_{n-1}^1.

Let $K(n, k)$ denote the class of connected graphs G of order n containing a unique k-clique such that the graph obtained from G by the contraction of the k-clique to a unique vertex is a tree.

The number of labeled graphs in $K(n, k)$ is equal to $\binom{n-1}{k-1}n^{n-k}$ for $4 \leqslant k \leqslant n$ [3].

In [3] it is also proved that the minimum number of edges of a connected graph G of order n with $\chi(G) = k$ is equal to $\binom{k}{2}+n-k$.

Extremal graphs relative to this property are precisely the graphs from $K(n, k)$ for $\chi(G) = k \geqslant 4$. The following conjecture was proposed in [3]:

CONJECTURE 1. The maximum number of k-colorings of a connected graph G of order n having $\chi(G) = k \geqslant 4$ is equal to $(k-1)^{n-k}$ and the extremal graphs all belong to $K(n, k)$.

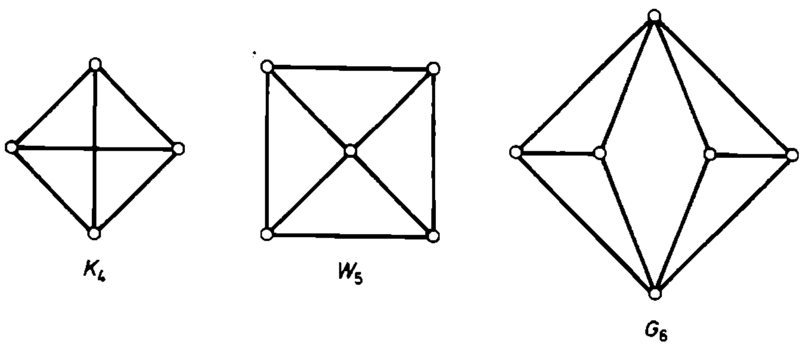

Fig. 1

In [7] this conjecture is proved for connected planar graphs G having $\chi(G) = 4$ as follows.

We denote by $G_4(n)$ the class of connected graphs G of order n containing exactly four triangles such that the graph obtained from G by deleting any edge from $E(G)$ is not connected or contains at most three triangles. It can be easily shown that if $G \in G_4(n)$ then by contracting, in an arbitrary order, all triangles in G, the resulting graph is a tree.

In [7] it is shown that for every connected planar graph G of order n containing at least four triangles there exists a spanning subgraph H of G such that $H \in G_4(n)$.

PROPOSITION 5. *For every $G \in K(n, k)$ the chromatic polynomial satisfies*

$$P(G, \lambda) = (\lambda)_k (\lambda - 1)^{n-k}.$$

Note that all graphs in $K(n, k)$ are planar for $1 \leqslant k \leqslant 4$.

THEOREM 6. *If G is a connected planar 4-chromatic graph of order n then $P(G, \lambda) \leqslant (\lambda)_4(\lambda - 1)^{n-4}$ for every integer λ, $\lambda \geqslant 4$, and for every $\lambda \geqslant 4$ equality holds if and only if $G \in K(n, 4)$.*

Proof. Let G be a connected planar 4-chromatic graph of order n. By Grünbaum's theorem [1], G has at least four triangles. By deleting some edges from $E(G)$, one can obtain a graph H of order n containing exactly four triangles and such that $H \in G_4(n)$. It follows that $\chi(H) = 3$ or $\chi(H) = 4$. Also $P(G, \lambda) \leqslant P(H, \lambda)$ for every $\lambda \geqslant 0$.

In [7] it was proved that if $G \in G_4(n)$ is a graph containing as subgraph none of the following graphs: the complete graph K_4, the wheel W_5 and the graph G_6, represented in Fig. 1, then there exists a numbering f of the vertices of G satisfying:

(i) Every vertex x such that $f(x) > 1$ is adjacent to a vertex y such that $f(y) < f(x)$.

(ii) There exists no vertex x which belongs to two triangles $T_1 = \{x, a, b\}$ and $T_2 = \{x, u, v\}$ (T_1 and T_2 may contain one or two vertices in common) and $f(x) > \max(f(a), f(b), f(u), f(v))$.

We shall show that if our H does not contain any of the graphs K_4, W_5, G_6 then $P(H, \lambda) < (\lambda)_4(\lambda - 1)^{n-4}$ for every integer λ, $\lambda \geqslant 4$.

Let f be a numbering of the vertices of H satisfying (i) and (ii). Let us color the vertices from $V(H) = V(G)$ sequentially with λ colors in the order given by the function f. Vertex z having $f(z) = 1$ can be colored with λ colors. By (i) any other vertex $x \neq z$ can be colored with at most $\lambda - 1$ colors. If $T = \{u, v, w\}$ is a triangle of H and $f(w) > \max(f(u), f(v))$ it follows that w can be colored with at most $\lambda - 2$ colors since adjacent vertices u and v have been previously colored. By (ii) there exist at least four vertices which can be colored with at most $\lambda - 2$ colors.

It follows that $P(H, \lambda) \leqslant \lambda(\lambda - 2)^4(\lambda - 1)^{n-5} < (\lambda)_4(\lambda - 1)^{n-4}$ for $\lambda \geqslant 4$ since the last inequality is equivalent to $\lambda^2 - 5\lambda + 5 > 0$, which holds for $\lambda \geqslant 4$.

By Tutte's formula for $P(W_n, \lambda)$ in [8], we have

$$P(W_n, \lambda) = \lambda[(\lambda-2)^{n-1}+(-1)^{n-1}(\lambda-2)]$$

whence $P(W_5, \lambda) = (\lambda)_3(\lambda^2-5\lambda+7)$.

One deduces that if $H \in G_4(n)$ contains W_5 as an induced subgraph then

$$P(H, \lambda) = (\lambda)_3(\lambda-1)^{n-5}(\lambda^2-5\lambda+7).$$

Hence $(\lambda)_4(\lambda-1)^{n-4}-P(H, \lambda) = \lambda(\lambda-1)^{n-4}(\lambda-2)(\lambda-4) > 0$ for every $\lambda \geqslant 5$. Equality holds for $\lambda = 4$ but in this case we have $\chi(H) = 3$. Hence G is obtained from H by joining by an edge at least two nonadjacent vertices x and y.

Let H_1 be the graph obtained from H by joining x and y, and let H_2 be the graph obtained from H by identifying x and y. It follows that $P(H, \lambda) = P(H_1, \lambda)+P(H_2, \lambda)$ and $P(G, \lambda) \leqslant P(H_1, \lambda)$; $\chi(H_2) = 3$ or 4. Hence

$$P(H, 4) = P(H_1, 4)+P(H_2, 4) \geqslant P(G, 4)+P(H_2, 4) > P(G, 4)$$

since $P(H_2, 4) > 0$. It follows that

$$P(G, \lambda) < (\lambda)_4(\lambda-1)^{n-4}$$

holds for every $\lambda \geqslant 4$.

By a straightforward calculation one deduces that

$$P(G_6, \lambda) = (\lambda)_3(\lambda^3-7\lambda^2+18\lambda-16).$$

Hence if $H \in G_4(n)$ contains G_6 as an induced subgraph one can write

$$P(H, \lambda) = (\lambda)_3(\lambda^3-7\lambda^2+18\lambda-16)(\lambda-1)^{n-6} < (\lambda)_4(\lambda-1)^{n-4}$$

for $\lambda \geqslant 4$ since the last inequality is equivalent to $2\lambda^2-11\lambda+13 > 0$, which is true for $\lambda \geqslant 4$.

It remains to consider the case that $H \in G_4(n)$ contains an induced subgraph isomorphic to K_4. But in this case $H \in K(n, 4)$ and $P(H, \lambda) = (\lambda)_4(\lambda-1)^{n-4}$. Since we have only $P(G, \lambda) \leqslant P(H, \lambda)$ we must show that if G is a connected planar 4-chromatic graph of order n containing K_4 then for every integer $\lambda \geqslant 4$, $P(G, \lambda) = (\lambda)_4(\lambda-1)^{n-4}$ holds if and only if $G \in K(n, 4)$.

If $G \notin K(n, 4)$ then there exists a graph $H \in K(n, 4)$ which is obtained from G by deleting some edges from $E(G)$. Let x and y be two vertices which are adjacent in G but nonadjacent in H. It follows that $P(H, \lambda) = P(H_1, \lambda)+P(H_2, \lambda)$ where H_1 and H_2 are defined as above. Since $H \in K(n, 4)$ it follows that $\chi(H_2) = 4$, hence $P(H_2, \lambda) > 0$ for every $\lambda \geqslant 4$. Because $P(H_1, \lambda) \geqslant P(G, \lambda)$ we conclude that $P(H, \lambda) > P(G, \lambda)$ for every $\lambda \geqslant 4$ and the proof is complete.

COROLLARY. *For every connected planar 4-chromatic graph of order n the following inequality holds*:

$$\mathrm{Col}_4(G) \leqslant 3^{n-4}.$$

We have equality if and only if $G \in K(n, 4)$.

This result is a support for the following conjecture:

CONJECTURE 2. The maximum number of p-colorings in the class of connected planar 4-chromatic graphs of order n is reached for a graph G if and only if $G \in K(n, 4)$, for every $4 \leqslant p \leqslant n-1$.

Another conjecture which extends both Conjectures 1 and 2 is the following:

CONJECTURE 3. The maximum number of p-colorings of a connected graph G of order n with $\chi(G) = k \geqslant 4$ is reached only for graphs in $K(n, k)$, for every $4 \leqslant p \leqslant n-1$.

Note that Conjectures 2 and 3 are true for $p = n-1$ since $\mathrm{Col}_{n-1}(G) = \binom{n}{2} - |E(G)|$ and $|E(G)|$ is minimum only for $G \in K(n, k)$.

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