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FIXED POINT SETS OF SMOOTH GROUP ACTIONS ON DISKS AND EUCLIDEAN SPACES. A SURVEY*

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Among the fundamental questions about a smooth action of a compact Lie group G on a smooth manifold M (with or without boundary ∂M) are the description of the set F of points in M left fixed by G (it is known that F is a smooth submanifold of M with $\partial M \cap F = \partial F$) and the deeper question concerning the description of the equivariant normal bundle $N(F)$ of F in M.

Since we may assume that G acts by isometries on M in some riemannian metric on M, the tangent bundle $T(M)$ (considered as a G-vector bundle via the differential of the action of G on M) restricted to F decomposes into the direct sum of $T(F)$ (the trivial summand) and its orthogonal complement (the nontrivial summand) which is canonically equivalent to $N(F)$. In particular, for any $x \in F$, the tangent space $T_x(M)$ (considered as the representation of G) decomposes into the direct sum of the tangent space $T_x(F)$ (the trivial summand) and the normal space $N_x(F)$ (the nontrivial summand).

The Slice Theorem asserts that, for any x in $F - \partial F$ (resp., ∂F), there is an open invariant neighborhood S_x of x in M, such that the action of G on S_x is smoothly equivalent to the action of G on $T_x(M)$ (resp., the half-space of $T_x(M)$). Thus, the description of $N_x(F)$ provides information about the action of G on M near x.

The Equivariant Tubular Neighborhood Theorem asserts, in turn, that for some open invariant neighborhood U of F in M, there is an equivariant diffeomorphism $N(F) \to U$ whose restriction to F (where F is embedded in $N(F)$ via the 0-section) is the inclusion of F in U. Thus, the description of $N(F)$ provides information about the action of G on M near all of F.

Many survey articles on developments in transformation groups have

* This paper is in final form and no version of it will be submitted for publication elsewhere.

been published during the past two decades including a number of recent surveys that have been submitted to the proceedings of the 1983 conference on group actions at Boulder, Colorado (see, e.g., Assadi [3], Bredon [7], Browder [9], Cappell and Shaneson [11], Davis [12], Dovermann, Petrie, and Schultz [18], Edmonds [22], Hsiang [23], Masuda and Petrie [32], Milgram [34], Petrie [46, 47], Schultz [57–60], and Weinberger [66]; see, in particular, Schultz [60] for an excellent survey of obtained results, as well as for an extensive bibliography, and see also Schultz [61] for a large collection of problems in transformation groups).

It turns out that the global invariants of smooth manifolds upon which a compact Lie group G acts smoothly can impose some restrictions on the fixed point sets F and their equivariant normal bundles $N(F)$. A lot of the effort goes toward trying to describe F and $N(F)$ for smooth actions of G on homotopy spheres, disks, and euclidean spaces. Systematic surveys of related results include the articles by Cappell and Shaneson [11], Dovermann, Petrie, and Schultz [18], Masuda and Petrie [32], and Schultz [57–59]. The aim of this paper is to survey further developments in the description of F and $N(F)$ for smooth actions of G on disks and euclidean spaces.

Here is a basic question that we would like to answer:

Which smooth manifolds occur as the fixed point sets of smooth actions of G on disks (resp., euclidean spaces)?

Once we know that a smooth manifold F does occur, the following question arises immediately:

For smooth actions of G on disks (resp., euclidean spaces) with fixed point set F, which smooth G-vector bundles over F occur as the equivariant normal bundles $N(F)$?

If F above consists of finitely many points $x_1, \ldots, x_k$, we simply ask:

Which lists of representations $V_1, \ldots, V_k$ of G occur as the representations of G on the tangent spaces at $x_1, \ldots, x_k$?

We shall discuss the developments related to these questions in seven sections ordered as follows:

§ 1. Preliminary remarks.

§ 2. Fixed point sets of torus actions and p-group actions.

§ 3. Fixed point sets of nilpotent group actions.

§ 4. Group actions with finitely many fixed points.

§ 5. Equivalence of representations at fixed points.

§ 6. Product equivariant normal bundles.

§ 7. Representations at isolated fixed points.

Almost all of the results discussed in this paper were discovered during the past decade and most of the author's work was done during the past five years.

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§ 1. Preliminary remarks

We refer the reader to Bredon's book [8] for background information on transformation groups that we use in this paper.

Let G be a compact Lie group and let H be a closed normal subgroup of G. If the quotient group G/H acts on a space X with fixed point set F, then the quotient homomorphism $G \to G/H$ allows us to consider X as a G-space with $X^G = F$. Clearly, the action G on X may not be effective, but it is easy to build up X to a space with an effective action G and with the same fixed point set. For example, consider orthogonal actions of G on D^n and $\boldsymbol{R}^n$ given via a faithful representation $G \to 0(n)$, such that the fixed point set is just the origin. Then, $X \times D^n$ and $X \times \boldsymbol{R}^n$ (with the diagonal actions of G) both have the required properties. Hence, the following proposition holds.

PROPOSITION 1.1. *Let G be a compact Lie group and let H be a closed normal subgroup of G. If F is the fixed point set of a smooth action of G/H on a disk (resp., euclidean space), then there is a smooth effective action of G on a disk (resp., euclidean space) with fixed point set F.*

Let G be a finite group and let H be a subgroup of G with index n (here H need not be normal). Assume H acts on a space X with fixed point set F. Under these hypotheses, Oliver ([35], the proof of Lemma 6) defines an action of G on X^n, the n-fold cartesian product of X, with fixed point set equal to the image of F under the diagonal map $X \to X^n$. It follows easily from the construction of the action of G on X^n that if X is a smooth manifold and H acts smoothly on X, then the action of G on X^n is also smooth. Thus, the following proposition holds.

PROPOSITION 1.2. *Let G be a finite group and let H be a subgroup of G. If F is the fixed point set of a smooth action of H on a disk (resp., euclidean space), then there is a smooth action of G on a disk (resp., euclidean space) with fixed point set F.*

The following corollary follows immediately from Propositions 1.1 and 1.2.

COROLLARY 1.3. *Let G be a compact Lie group and let G_0 be the identity connected component of G. If F is the fixed point set of a smooth action of a subquotient of G/G_0 on a disk (resp., euclidean space), then there is a smooth* *effective action of G on a disk (resp., euclidean space) with fixed point set F.*

Remark 1.4. Unlike for group actions on disks and euclidean spaces, the arguments used to prove Propositions 1.1 and 1.2 do not work for group actions on spheres. Clearly, we can get group actions on spheres from group actions on disks, e.g., as follows: for a compact Lie group G acting smoothly on D^n, restrict the action to the boundary $\partial D^n = S^{n-1}$ or take the equivariant double of D^n; i.e., first consider $D^n \times D^1$ with the diagonal action of G, where G acts trivially on D^1, and then restrict the action to the boundary $\partial(D^n \times D^1) = S^n$. However, there exist smooth group actions on spheres which cannot be obtained in this way including smooth one fixed point actions (cf. Remark 4.6).

Remark 1.5. In turn, we can get group actions on disks and euclidean spaces from group actions on spheres as follows: for a compact Lie group G acting smoothly on S^n with at least one fixed point x, take (using the Slice Theorem) a sufficiently small invariant ε-disk D^n_ε in S^n around x, so that G acts orthogonally on D^n_ε, and now consider the action of G on $S^n - \operatorname{int} D^n_\varepsilon \cong D^n$ and the action of G on $\operatorname{int} D^n \cong \boldsymbol{R}^n$. The latter action we can also obtain by adding an open equivariant collar to D^n along ∂D^n or just by identifying $S^n - \{x\} = \boldsymbol{R}^n$. Note that the action of G on ∂D^n is ortogonal. On the other hand, there exist smooth group actions on D^n which are not orthogonal on ∂D^n (cf. Theorem 6.1).

Remark 1.6. Useful tools for constructing smooth G-manifolds include equivariant thickening procedures (see, e.g., Assadi [1], Edmonds and Lee [20, 21], and Pawałowski [40–45]) and equivariant surgery (see, e.g., Dovermann and Petrie [17], Dovermann and Rothenberg [19], Petrie [46–51], and Petrie and Randall [52]); compare, e.g., Assadi and Browder [4, 5], Assadi and Vogel [6], Bredon [7], Browder [9], Cappell and Shaneson [10], Davis, Hsiang, and Morgan [13], tom Dieck [14], tom Dieck and Petrie [15], Jones [27], Löffler [29], Löffler and Raussen [30], Madsen and Rothenberg [31], Oliver [35, 36, 39], Schultz [54–56], Stein [63], and Weinberger [65]. Equivariant thickenings procedure produce smooth G-manifolds with boundary (and thus boundaries of such manifolds) or open smooth G-manifolds while equivariant surgery can also produce closed smooth G-manifolds which are not equivariant boundaries (cf. Remarks 1.4 and 1.5); see Assadi [1], Chapt. III, Sect. 4 for a brief description of some relations between equivariant thickening and equivariant surgery.

§ 2. Fixed point sets of torus actions and p-group actions

We say that a smooth manifold B is *stably complex* if there is a smooth embedding of B into some euclidean space, such that the normal bundle admits a complex structure (in particular, all connected components of B are either even or odd dimensional). Equivalently, B is stably complex if the tangent bundle $T(B)$ admits a complex structure perhaps after adding a product bundle $B \times \mathbf{R}^n$ over B (cf. Edmonds and Lee [20], Sect. 2).

Edmonds and Lee ([20], Lemma (3.1)) have indicated that if a finite odd order group G acts smoothly on a smooth manifod M, then the normal bundle of the fixed point set M^G in M admits a complex structure. Hence, if M is stably complex, so is M^G. Moreover, Edmonds and Lee ([20], Proposition (3.2)) have shown that if G is a finite even order group with a normal 2-Sylow subgroup and G acts smoothly on a $\mathbf{Z}_2$-acyclic (and thus stably complex) smooth manifold M, then M^G is a stably complex manifold.

Now, let G be a compact Lie group with $G_0 = T^n$, the n-dimensional torus, where $n \geqslant 1$. If G acts on a $\mathbf{Z}$-acyclic manifold M, then M^{G_0} is also $\mathbf{Z}$-acyclic by the Smith Theory. Recall that the quotient group G/G_0 always acts on M^{G_0} with $(M^{G_0})^{G/G_0} = M^G$. Therefore, the above discussion yields the following proposition.

PROPOSITION 2.1. *Let G be a compact Lie group such that G_0 is abelian (i.e., G_0 is either a trivial group or a torus) and G/G_0 has a normal (possibly trivial) 2-Sylow subgroup. If G acts smoothly on a $\mathbf{Z}$-acyclic smooth manifold M, then M^G is a stably complex manifold.*

Let $\mathscr{P}$ be a class of compact Lie groups G such that G_0 is abelian and G/G_0 has prime power order. According to Proposition 2.1, for each $G \in \mathscr{P}$ acting smoothly on a disk or some euclidean space, the fixed point set F is a stably complex manifold. Beside this restriction on F, there is the obvious restriction which follows from the Smith Theory: F is $\mathbf{Z}$-acyclic when $G = G_0$, and F is $\mathbf{Z}_p$-acyclic when G/G_0 is a nontrivial p-group (p any prime); recall that each $\mathbf{Z}$-acyclic, as well as $\mathbf{Z}_2$-acyclic, smooth manifold is stably complex. The following two theorems assert that these restrictions on F are both necessary and sufficient.

THEOREM 2.2. *Let G be a torus T^n, $n \geqslant 1$. Then, a compact (resp., open) smooth manifold F is the fixed point set of a smooth action of G on a disk (resp., euclidean space) if and only if F is $\mathbf{Z}$-acyclic.*

THEOREM 2.3. *Let G be a compact Lie group such that G_0 is abelian and G/G_0 is a nontrivial p-group (p any prime). Then a compact (resp., open) smooth manifold F is the fixed point set of a smooth action of G on a disk (resp., euclidean space) if and only if F is stably complex and $\mathbf{Z}_p$-acyclic.*

Note that if F above were a closed manifold, then Poincaré's Duality would imply that F is just one point. Thus, Theorems 2.2 and 2.3 answer the question of which smooth manifolds occur as the fixed point sets of smooth actions of a given group $G \in \mathscr{P}$ on disks (resp., euclidean spaces).

In Theorems 2.2 and 2.3 the necessity has been discussed above and the sufficiency follows via equivariant thickening (see Pawałowski [42]). For G

$= \boldsymbol{Z}_p$, the result in Theorem 2.3 goes back to Jones [27], Theorem 2.1 and Section 3. Generalized versions of Jones' Theorem have been obtained (using different methods) by other authors (see, e.g., Assadi [2], Assadi and Browder [4], Dovermann [16], Rothenberg and Sondow [53], and Weinberger [65]).

§ 3. Fixed point sets of nilpotent group actions

Oliver [35] has shown that, for a finite group G not of prime power order, there exists an integer n_G such that a finite CW-complex F is the fixed point set of a finite contractible G-CW-complex if and only if the Euler characteristic $\chi(F)$ is congruent to 1 mod n_G (see, e.g., Oliver [38] for the calculations of the integer n_G). For a compact Lie group G such that G_0 is abelian and G/G_0 is not of prime power order, the same result follows with $n_G = n_{G/G_0}$. Oliver [36] has also shown that for a compact Lie group G such that G_0 is nonabelian, the above result holds with $n_G = 1$.

Recall that for a compact Lie group G, any smooth G-manifold (with or without boundary) has the structure of a G-CW-complex, and the manifold is compact if and only if the complex is finite (see, e.g., Illman [26] and Matumoto and Shiota [33]). Therefore, the following proposition follows immediately from the Smith Theory and the above discussion.

PROPOSITION 3.1. *Let G be a compact Lie group and let M be a compact contractible smooth G-manifold. Then the following two statements hold.*

(1) $G \in \mathscr{P}$ *implies* $\chi(M^G) = 1$.

(2) $G \notin \mathscr{P}$ *implies* $\chi(M^G) \equiv 1 \pmod{n_G}$.

Oliver ([37], Theorem 1) has described finite CW-complexes which occur (up to homotopy equivalence) as the fixed point sets of smooth actions of a given compact Lie group G on disks (cf. Theorems 2.2 and 2.3). In turn, the author has described compact smooth manifolds which occur (up to diffeomorphism) as the fixed point sets of smooth actions of a given (up to cyclic factor) finite nilpotent group G on disks. Recall that a finite group G is nilpotent if and only if G is the direct product of its Sylow subgroups, and also: if and only if all Sylow subgroups of G are normal. The announced result is contained in the following two theorems (cf. Theorem 3.6).

THEOREM 3.2. *Let p be an integer greater than 1. Then, a compact smooth manifold F is the fixed point set of a smooth action of $\boldsymbol{Z}_{pq}$ on a disk for some integer q relatively prime to p if and only if F is stably complex and $\chi(F) = 1$.*

THEOREM 3.3. *Let G be a finite nilpotent group with exactly n noncyclic Sylow subgroups, say, of p_1-power, ..., p_n-power orders for distinct primes $p_1, \ldots, p_n$. Then, a compact smooth manifod F is the fixed point set of a smooth action of $G \times \boldsymbol{Z}_q$ on a disk for some integer q relatively prime to the order of G, if and only if:*

$n=1$: *F is stably complex and* $\chi(F)=1$.
$n=2$: *F is stably complex and* $\chi(F)\equiv 1 \pmod{p_1 p_2}$.
$n\geqslant 3$: *F is stably complex.*

In Theorems 3.2 and 3.3, the sufficiency is shown via equivariant thickening (see Pawałowski [42, 45]) and the necessity follows from Proposition 2.1 and 3.1 and the calculations of the Oliver integer n_G (see Oliver [38], Sect. 4). Those calculations show that, for a finite nilpotent group G not of prime power order, n_G depends only on the noncyclic Sylow subgroups of G; more precisely:

(1) If G has at most one noncyclic Sylow subgroup, then $n_G=0$.

(2) If G has exactly two noncyclic Sylow subgroups, say, of p_1-power and p_2-power orders for two distinct primes p_1 and p_2, then $n_G=p_1 p_2$.

(3) If G has at least three noncyclis Sylow subgroups, then $n_G=1$.

Now, recall that Edmonds and Lee ([20], Proposition (5.3) and Remark (5.4)) have shown that for a given closed stably complex smooth manifold F such that each connected component of F has the same dimension, there is a smooth action of $\boldsymbol{Z}_{pq}$ on some euclidean space with fixed point set F provided p and q are two relatively prime integers both sufficiently large with respect to the dimension of F. Arguing similarly, we can construct a smooth action of $\boldsymbol{Z}_{pqr}\times\boldsymbol{Z}_{pqr}$ on a disk with fixed point set F (the same as above) provided p, q, and r are three relatively prime integers, again, all sufficiently large with respect to the dimension of F (note that $n_G=1$ for $G=\boldsymbol{Z}_{pqr}\times\boldsymbol{Z}_{pqr}$).

The author has removed both the dependence of p, q, and r on the dimension of F and the restriction of the dimension of the connected components of F (recall, however, that a stably complex manifold has connected components either even or odd dimensional; see § 2). The result is contained in the following proposition (see Pawałowski [45] for the proof).

PROPOSITION 3.4. *Let p, q, and r be three relatively prime integers all greater than* 1. *Let F be a compact (resp., closed) stably complex smooth manifold. Then there is a smooth action of $\boldsymbol{Z}_{pqr}\times\boldsymbol{Z}_{pqr}$ (resp., $\boldsymbol{Z}_{pq}$) on a disk (resp., euclidean space) with fixed point set F.*

The following corollary follows from Proposition 3.4 and Corollary 1.3.

COROLLARY 3.5. *Let G be a compact Lie group such that G/G_0 has a nilpotent subquotient with three or more noncyclic Sylow subgroups (resp., G/G_0 has a nilpotent subquotient not of prime power order) and let F be a compact (resp., closed) stably complex smooth manifold. Then there is a smooth action of G on a disk (resp., euclidean space) with fixed point set F.*

Finally, Corollary 3.5 and Proposition 2.1 give the following theorem.

THEOREM 3.6. *Let G be a compact Lie group such that G_0 is abelian and the following two conditions hold:*

(1) *G/G_0 has a nilpotent subquotient with three or more noncyclic Sylow subgroups (resp., G/G_0 has a nilpotent subquotient not of prime power order).*

(2) *G/G_0 has a normal, possibly trivial, 2-Sylow subgroup. Then, a compact (resp., closed) smooth manifold F is the fixed point set of a smooth action of G on a disk (resp., euclidean space) if and only if F is a stably complex manifold.*

§ 4. Group actions with finitely many fixed points

The question of which compact Lie groups have smooth fixed point free actions on disks (resp., euclidean spaces) has been answered completely. The result is contained in the following two theorems.

THEOREM 4.1. *A compact Lie group G has a smooth fixed point free action on a disk if and only if $G \notin \mathcal{P}$ and $n_G = 1$.*

THEOREM 4.2. *A compact Lie group G has a smooth fixed point free action on some euclidean space if and only if $G \notin \mathcal{P}$.*

Theorem 4.1 is due to Oliver [35, 36] and Theorem 4.2 has been shown independently by Assadi [1] and Edmonds and Lee [21].

The next two theorems answer the question of which compact Lie groups have smooth actions on disks (resp., euclidean spaces) with two or more isolated fixed points (cf. Theorems 4.1 and 4.2).

THEOREM 4.3. *For an integer $k > 1$, a compact Lie group G has a smooth action on a disk with exactly k fixed points if and only if $G \notin \mathcal{P}$ and $k \equiv 1$ (mod n_G).*

THEOREM 4.4. *For an integer $k > 1$, a compact Lie group G has a smooth action on some euclidean space with exactly k fixed points if and only if $G \notin \mathcal{P}$.*

In Theorems 4.3 and 4.4, the necessity follows from the Smith Theory and Proposition 3.1 (similarly as the necessity in Theorems 4.1 and 4.2) and the sufficiency can be obtained via equivariant thickening (see Pawałowski [42, 43]; cf. Assadi [1] and Edmonds and Lee [20]).

Remark 4.5. Recall that the class of finite groups $G \notin \mathcal{P}$ with $n_G = 1$ include the alternating group A_5 as the smallest group, $S_4 \times \boldsymbol{Z}_3$ and $A_4 \times S_3$ as the smallest solvable groups, and $\boldsymbol{Z}_{30} \times \boldsymbol{Z}_{30}$ as the smallest nilpotent group (see, e.g., Oliver [35], p. 175). Moreover, $n_G = 1$ for each compact Lie group G with nonabelian G_0 (cf. § 3). Recall also that for a compact Lie group $G \notin \mathcal{P}$, $n_G = 0$ if and only if G_0 is abelian and G/G_0 is "cyclic mod p" for some prime p; i.e., G/G_0 has a normal subgroup $P \in \mathcal{P}$ such that $(G/G_0)/P$ is cyclic (see, e.g., Oliver [38], p. 259).

Remark 4.6. Unlike for group actions on disks and euclidean spaces, the question of which compact Lie groups G have smooth actions on homotopy spheres with exactly one fixed point seems to be of a rather deep nature and, according to the author's knowledge, it is still unsolved (see Stein [63] for a construction of a smooth action of the binary icosahedral group on S^7 with exactly one fixed point, and see Petrie [48, 49] for further examples of smooth one fixed point actions on homotopy spheres). Here, one may conjecture that the answer to the above question is affirmative if and only if $G \notin \mathscr{P}$ and $n_G = 1$. One may also conjecture that for an integer $k > 1$, a compact Lie group G has a smooth action on a homotopy sphere with exactly $k+1$ fixed points if and only if $G \notin \mathscr{P}$ and $k \equiv 1 \pmod{n_G}$; cf. Theorems 4.1 and 4.3 and Remark 1.5.

§ 5. Equivalence of representations at fixed points

Let G be a compact Lie group and let M be a smooth G-manifold. It follows from the Slice Theorem that each point $x \in M^G$ has an open invariant neighborhood S_x in M such that the tangent bundle $T(M)$ restricted to S_x and the product bundle $S_x \times T_x(M)$ over S_x are equivalent as G-vector bundles. Therefore, if x varies within a fixed connected component of M^G, then as representations of G:

$$T_x(M) \cong \boldsymbol{R}^n \oplus V,$$

where n is the dimension of the connected component containing x, $\boldsymbol{R}^n$ has the trivial action of G, and V is a fixed representation of G with $V^G = 0$ (cf. the introduction of this paper).

For smooth actions of a compact Lie group G on homotopy spheres, disks or euclidean spaces, the question of the equivalence of the representations of G at any two fixed points and the weaker question of the equality of the dimensions of any two fixed points set connected components go back to Smith ([62], the footnote on p. 406), Hsiang and Hsiang ([24], Problem 16), and Bredon ([8], the second remark on p. 58).

If $G \in \mathscr{P}$, the answers to these questions are affirmative for smooth actions of G on disks and euclidean spaces, as well as for smooth actions of G on homotopy spheres with at least three fixed points, because in these cases, the fixed point sets of G are connected by the Smith Theory.

Let $\mathscr{R}$ be the class of compact Lie groups G such that in the quotient group G/G_0, each element has prime power order. Clearly, $\mathscr{R} \supset \mathscr{P}$ and $\mathscr{R} - \mathscr{P}$ includes finite groups such as the symmetric groups S_3 and S_4, the alternating groups A_4, A_5, and A_6, the nonabelian groups of order pq (p and q two primes with $p|(q-1)$), and the dihedral groups of order $2p^a$ (p odd prime, $a \geqslant 1$), as well as the extensions of these finite groups by a compact connected Lie group, and the extensions of finite prime power order groups by a compact connected nonabelian Lie group; e.g., the classical groups $SO(n)$ and $O(n)$ for $n \geqslant 3$.

THEOREM 5.1. *Let G be a compact Lie group. Then the following three conditions are equivalent.*

(1) *For any smooth action of G on a disk (resp., euclidean space) at any two fixed points the representations of G are equivalent.*

(2) *For any smooth action of G on a disk (resp., euclidean space) each fixed point set connected component has the same dimension.*

(3) $G \in \mathscr{R}$.

THEOREM 5.2. *Let G be a compact Lie group. Then the following three conditions are equivalent.*

(1) *For any smooth action of G on a sphere (resp., homotopy sphere) with at least three fixed points at any two fixed points the representations of G are equivalent.*

(2) *For any smooth action of G on a sphere (resp., homotopy sphere) each fixed point set connected component has the same dimension.*

(3) $G \in \mathscr{R}$.

In Theorems 5.1 and 5.2, (3) implies (1) by Proposition 7.1 and 7.2 of Pawałowski [43], (1) implies (2) by the Slice Theorem (cf. the beginning of this section) and in order to show that (2) implies (3), for any compact Lie group G, such that G/G_0 has a cyclic subgroup not of prime power order, the author has constructed smooth actions of G on disks, spheres, and euclidean spaces with fixed point set connected components of different dimensions (see Pawałowski [43], Example 6.1, cf. § 6 of this paper).

Remark 5.3. Assume $G \in \mathscr{R} - \mathscr{P}$ and $n_G \neq 0$ (resp., $G \in \mathscr{R} - \mathscr{P}$); cf. Remark 4.5. Then, it follows from Theorem 4.3 (resp., Theorem 4.4) that G has a smooth action on a disk (resp., euclidean space) with two or more isolated fixed points. However, at different fixed points, the representations of G cannot be different (i.e., inequivalent) by Theorem 5.1.

Remark 5.4. In the case of smooth actions of a compact Lie group G on homotopy spheres with exactly two fixed points, the question of the equivalence of the representations of G at the fixed points in full generality is still unsolved. Here, beside the linear equivalence, one studies extensively also other kinds of equivalences (such as the topological equivalence and Smith equivalence) between the representations of G at the fixed points (see, e.g., Cappell and Shaneson [11], Dovermann, Petrie, and Schultz [18], and Masuda and Petrie [32] for surveys of related results; see also Illman [25] and Laitinen and Traczyk [28]).

§ 6. Product equivariant normal bundles

Let G be a compact Lie group acting smoothly on M, a disk, some euclidean space or a sphere; in the latter case assume further that there are at least three points left fixed by G. Let $F_0, F_1, \ldots, F_k$ be the fixed point set connected components and let $V_0, V_1, \ldots, V_k$ be representations of G such that for some representation W of G, as representations of G:

$$T_x(M) \cong \boldsymbol{R}^{n_i} \oplus V_i \oplus W,$$

where $x \in F_i$, $n_i = \dim F_i$, and $\boldsymbol{R}^{n_i}$ has the trivial action of G for $i = 0, 1, \ldots, k$. Then $V_i^G = 0$ and $W^G = 0$ (cf. the beginning of § 5).

If $G \in \mathscr{M}$ (i.e., each cyclic subgroup of G/G_0 has prime power order), then $n_i = n_j$ and $V_i \cong V_j$ for all $0 \leqslant i, j \leqslant k$ (see Theorems 5.1 and 5.2). If $G \notin \mathscr{M}$, there is a weaker relationship between the n_i and the V_i, namely: for all $0 \leqslant i, j \leqslant k$, $\boldsymbol{R}^{n_i} \oplus V_i$ and $\boldsymbol{R}^{n_j} \oplus V_j$ are equivalent when restricted to any $P \in \mathscr{P}(G)$, the family of all subgroups of G which are in $\mathscr{P}$, because the fixed point set M^P is connected by the Smith Theory.

Note that for $i = 0, 1, \ldots, k$, the product bundle $F_i \times (V_i \oplus W)$ over F_i and the normal bundle $N(F_i)$ are locally equivalent as G-vector bundles. Clearly, if they are equivalent, F_i is a stably parallelizable manifold.

Now, we give examples of (stably) parallelizable manifolds $F_0, F_1, \ldots, F_k$ such that for complex representations $V_0, V_1, \ldots, V_k$ of G with $V_i^G = 0$, the relationship

$$(*) \qquad \operatorname{res}_P(\boldsymbol{C}^{n_i} \oplus V_i) \cong \operatorname{res}_P(\boldsymbol{C}^{n_j} \oplus V_j)$$

for each $P \in \mathscr{P}(G)$, $0 \leqslant i, j \leqslant k$, is both necessary and sufficient for the product bundles $F_i \times (V_i \oplus W)$ over F_i to occur as the equivariant normal bundles $N(F_i)$ for some complex representation W of G. Here, n_i is the greatest integer in $\dim F_i/2$ and $\boldsymbol{C}^{n_i}$ has the trivial action of G.

Let $F_0, F_1, \ldots, F_k$ be compact parallelizable smooth manifolds either even or odd dimensional such that each F_i has the structure of a CW-complex containing as a deformation retract a subcomplex L_i which is either a point or a wedge of circles, and assume $\sum_{i=0}^{k} \chi(L_i) = 1$ (e.g., $F_0 = D^2_{(k)}$, the 2-disk with k holes, $F_1 = \{x_1\}, \ldots, F_k = \{x_k\}$). If $n_i > 0$, F_i has a boundary, and we write DF_i for the double of F_i (e.g., if $F_i = D^2_{(k_i)}$, DF_i is the orientable surface of genus k_i).

Let G be a finite cyclic group not of prime power order and let $V_0, V_1, \ldots, V_k$ be complex representations of G with $V_i^G = 0$ for $i = 0, 1, \ldots, k$.

With the above hypotheses on G, V_i, and F_i, the following three theorems hold.

THEOREM 6.1. *There is a stably complex action of G on a disk containing all F_i as the fixed point set connected components with equivariant normal* *bundles equivalent to $F_i \times (V_i \oplus W)$ for some complex representation W of G if and only if the relationship* $(*)$ *holds.*

THEOREM 6.2. *Assume further each $n_i > 0$. Then there is a stably complex action of G on a sphere containing all DF_i as the fixed point set connected components with equivariant normal bundles equivalent to $DF_i \times (V_i \oplus W)$ for some complex representation W of G if and only if the relationship* $(*)$ *holds.*

THEOREM 6.3. *Assume further $F_0 = \{x_0\}$ and $n_i > 0$ for $i = 1, \ldots, k$. Then there is a stably complex action of G on a disk (resp., euclidean space), containing $F_0, DF_1, \ldots, DF_k$ as the fixed point set connected components, such that the representation of G at x_0 is equivalent to $V_0 \oplus W$ and the equivariant normal bundles of DF_i ($i = 1, \ldots, k$) are equivalent to $DF_i \times (V_i \oplus W)$ for some complex representation W of G if and only if the relationship* $(*)$ *holds.*

Hereafter, by a *stably complex action* of G on $M = D^n$, S^n or $\boldsymbol{R}^n$ we mean a smooth action of G on M such that the tangent bundle $T(M)$ stably admits the structure of a complex G-vector bundle.

In Theorems 6.1, 6.2, and 6.3, the necessity of the relationship $(*)$ follows from the Smith Theory (cf. the beginning of this section). The sufficiency is proved using equivariant thickening (see Pawałowski [44]).

Remark 6.4. For $G = \boldsymbol{Z}_{pq}$ with $(p, q) = 1$ and any sequence of integers $0 \leqslant n_0 \leqslant n_1 \leqslant \ldots \leqslant n_k$, put

$$V_i = n_i t^{p+q} \oplus (n_k - n_i)(t^p \oplus t^q), \qquad i = 0, 1, \ldots, k,$$

where t denotes the 1-dimensional complex representation of G given via multiplication in $\boldsymbol{C}$ by the primitive pqth rooth of unity. Then $\mathrm{res}_P(\boldsymbol{C}^{n_i} \oplus V_i) \cong \mathrm{res}_P(\boldsymbol{C}^{n_j} \oplus V_j)$ for each $P \in \mathscr{P}(G)$, $0 \leqslant i, j \leqslant k$. Therefore, using Theorems 6.1, 6.2, or 6.3, it is easy to construct examples of smooth actions of G on disks, spheres, and euclidean spaces with fixed point set connected components of different dimensions (cf. Pawałowski [43, 44]).

§ 7. Representations at isolated fixed points

In § 6 we have studied (stably) parallelizable fixed point set connected components and their product equivariant normal bundles, and in this section we discuss the special case of smooth group actions with finitely many fixed points and representations thereat.

Let G be a finite group not of prime power order and let $V_1, \ldots, V_k$ be complex representations of G with $V_i^G = 0$ for $i = 1, \ldots, k$. Assume further that each $P \in \mathscr{P}(G)$ is contained in exactly one Sylow subgroup of G. Then the following two theorems hold (cf. Theorems 4.3 and 4.4).

THEOREM 7.1. *There is a stably complex action of G on a disk with exactly k fixed points at which the representations of G are equivalent to*

$V_1 \oplus W, \ldots, V_k \oplus W$ for some complex representation W of G if and only if $k \equiv 1 \pmod{n_G}$ and $\mathrm{res}_P(V_i) \cong \mathrm{res}_P(V_j)$ for each $P \in \mathscr{P}(G)$, $1 \leqslant i, j \leqslant k$.

THEOREM 7.2. *There is a stably complex action of G on some euclidean space with exactly k fixed points at which the representations of G are equivalent to $V_1 \oplus W, \ldots, V_k \oplus W$ for some complex representation W of G if and only if $\mathrm{res}_P(V_i) \cong \mathrm{res}_P(V_j)$ for each $P \in \mathscr{P}(G)$, $1 \leqslant i, j \leqslant k$.*

In Theorems 7.1 and 7.2, the necessity of $k \equiv 1 \pmod{n_G}$ follows from Proposition 3.1 and the necessity of $\mathrm{res}_P(V_i) \cong \mathrm{res}_P(V_j)$ follows from the Smith Theory (cf. the beginning of § 6). The sufficiency of these conditions is proved using equivariant thickening (see Pawałowski [44]).

Remark 7.3. Note that each finite nilpotent group G fulfils the condition that each $P \in \mathscr{P}(G)$ is contained in exactly one Sylow subgroup of G as well as it does each extension G of the form

$$0 \to H \to G \to K \to 0,$$

where H is a finite nilpotent group, K is a finite group whose order is a product of distinct primes, and the orders of H and K are relatively prime (see Pawałowski [43] for the case $H = \boldsymbol{Z}_r$ and $K = \boldsymbol{Z}_s$). Independently, using equivariant surgery, Petrie [51] has obtained a similar result to Theorem 7.1 for abelian G (cf. Petrie and Randall [52]), and Tsai [64] has done it when H above is abelian, $K = \boldsymbol{Z}_s$, s is a prime, and the following additional restriction holds: each representation V_i is a direct sum of $\mathrm{ind}_H^G(S_i)$ and R_i, where S_i is a complex representation of H and R_i is a complex representation of G/H considered also as a complex representation of G via the quotient homomorphism $G \to G/H$.

Remark 7.4. Using Theorems 7.1 and 7.2, we can construct examples of smooth actions of G on disks and euclidean spaces with isolated fixed points and inequivalent representations of G at different fixed points. For example, if $G = \boldsymbol{Z}_{pq}$ for $(p, q) = 1$, we can do it on euclidean spaces; the first such examples are due to Edmonds and Lee [20]. Note that it follows from Theorem 5.1 that in order to obtain such examples of smooth actions of G, it is necessary to assume that G has a cyclic subgroup not of prime power order (otherwise $G \in \mathscr{R}$) and, for smooth actions on disks, it is also necessary to assume that $n_G \neq 0$ (otherwise $k = 1$). This holds, e.g., when G is a finite nilpotent group with at least two noncyclic Sylow subgroup (cf. Remark 4.5) or G is the dihedral group of order $2pq$ for two relatively prime odd integers p and q; in the latter case $n_G = 2$ (cf. Pawałowski [43], Theorem 7.4, Example 7.5, and Remark 7.6).

Added in October 1986. Recently the author has improved the results of Theorems 3.2 and 3.3 by showing which compact smooth manifolds can occur as the fixed point sets of smooth G-actions on disks for a *given* compact Lie group G such that G_0 is abelian and G/G_0 is nilpotent.

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