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A discrete maximum principle

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CONTENTS

Introduction

In the numerical analysis of finite difference methods a discrete maximum principle plays a central role. In papers [6], [16], this principle is presented for finite difference schemes in the canonical form with a finite difference operator of "positive type" In [17], R. S. Varga gave a discrete maximum principle for an $N \times N$ matrix A. R. S. Varga's maximum principle includes a class of coherent matrices with operators of positive type and furthermore can be applied to operators which are not of positive type. A description of these maximum principles is presented in § 1. In § 2 (cf. [12]) we propose an extension of R. S. Varga's maximum principle to nonlinear mappings and to an *a priori* estimation. An application of these extensions of the maximum principle is given in § 3. Namely, in § 3 (cf. [12]) we prove a difference analogue of the maximum principle for nonlinear elliptic equations with boundary value conditions of the first, second and third kinds. In particular, we prove that a solution v of the difference schema (3.3) attains its maximum on that part of the boundary $\partial\Omega_h$ of a net Ω_h on which the Dirichlet's condition is given.

As we mentioned above, there exist finite difference schemes which satisfy R. S. Varga's maximum principle and which are not in a canonical form with an operator of positive type. Such a finite difference scheme for nonlinear elliptic equations is considered in § 4. Also, in § 4 we solve a nonlinear elliptic equation by a finite difference method of higher order accuracy.

In § 5 we propose a new maximum principle for ordinary differential operators and present its satisfactory conditions. We use this maximum principle in the proof of convergence of the method of lines for nonlinear parabolic equations

$$k(t, x)\, u_t = G(t, x, u, u_{x_1}, \ldots, u_{x_p}, u_{x_1x_1}, \ldots, u_{x_px_p}),$$

which can be degenerated to elliptic equations (cf. § 6 and [9]).

We face another situation when approximating systems of differential equations by the finite difference methods. Then we approximate a vector-function $\vec{u} = (u_1, u_2, \ldots, u_n)$. This fact must be taken into consideration in a definition of a maximum principle. In § 7 and 8 (cf. [10], [11]) we present certain sufficient conditions of satisfying a maximum principle which concerns direction and length of a solution $\vec{u} = (u_1, u_2, \dots, u_n)$ of a system of finite difference equations approximating an elliptic system of differential equations.

§ 1. A maximum principle for linear mappings

A discrete maximum principle is a finite difference analogue of Hopf's theorem [2]. Below we present this principle for an operator of "positive type" (cf. [6], [16]).

A net. Let $\mathscr{X}$ be a set of isolated points $x = (x_1, x_2, \dots, x_N)$ of the real space R^N A finite subset $\mathscr{N}(x) \subset \mathscr{X}$ such that $x \in \mathscr{N}(x)$ is called the *neighbourhood of* x. Let $\bar{\Omega}$ be a non-empty subset of $\mathscr{X}$. The set

$$\partial\Omega = \{x \in \bar{\Omega}\colon \text{there exists a } y \in \mathscr{X} - \bar{\Omega} \text{ such that } x \in \mathscr{N}(y) \text{ or } y \in \mathscr{N}(x)\}.$$

is called the *boundary of* $\bar{\Omega}$. We define the interior Ω of the set $\bar{\Omega}$ as follows: $\Omega = \bar{\Omega} - \partial\Omega$.

The set $\bar{\Omega}$ with the boundary $\partial\Omega$ and the interior Ω is called a *net.*

A connected net. A set $\bar{\Omega}$ is said to be a *connected net* if and only if for every couple of points x, $y \in \bar{\Omega}$, $x \in \Omega$, there exists a finite sequence $\{x^n\}_{n=1}^{m}$ such that

a) $x^n \in \bar{\Omega}$,

b) $x^{n+1} \in \mathscr{N}(x^n)$, $n = 1, 2, \dots, m-1$,

c) $x^1 = x$, $x^m = y$,

A weak connected net. A set $\bar{\Omega}$ is said to be a *weak connected net* if and only if for every $x \in \Omega$ there exist a $y \in \partial\Omega$ and a sequence $\{x^n\}_{n=1}^{m}$ for which conditions a), b), c) hold.

Remark 1. If a connected net $\bar{\Omega}$ has a non-empty boundary $\partial\Omega$, then $\bar{\Omega}$ is also a weak connected net, but the contrary does not hold in general.

A dense family of nets. Finite difference schemes are considered on a family of nets which is dense in the real space R^N, A *dense family of nets* is defined as follows: Assume a parameter $h \in H \subset (0, 1]$ and let $\inf H = 0$. With a parameter h we join a family $\{X_h\}$, where X_h is a set of isolated points $x \in R^N$ such that

$$\sup_{x \in X_h} \inf_{\substack{y \in X_h \\ y \neq x}} \varrho(x, y) \leqslant Ch, \tag{1.1}$$

$C = \text{const} > 0$, $\varrho(x, y)$ – a distance from a point x to y.

A family $\{\bar{\Omega}_h\}$ of nets $\bar{\Omega}_h \subset X_h$, $h \in H$, is called *dense* if a set $\mathscr{X}_h$ satisfies condition (1.1) for every $h \in H$.

An operator of positive type. A linear operator α is said to be of *positive type* if the following conditions are satisfied:

a) $\alpha[u](x) \equiv \sum_{y \in V(x)} a(x, y) u(y)$ for $x \in \Omega$,

b) $a(x, y)$ — real functions which are defined for $x \in \Omega$, $y \in \bar{\Omega}$,

c) $a(x, y) < 0$ for $x \neq y$, $x \in \Omega$, $y \in V(x)$,

d) $\sum_{y \in V(x)} a(x, y) \geqslant 0$, $x \in V(x)$,

where a function $u(x)$ is defined on $\bar{\Omega}$.

A strong maximum principle. Let $u(x)$ be a real function defined on a connected net $\bar{\Omega}$. If the function $u(x)$ attains its positive maximum (or negative minimum) at a point $\mathring{x} \in \Omega$ and if

$$\alpha[u](x) \leqslant 0 \quad \text{for} \quad x \in \Omega \quad (\text{or } \alpha[u](x) \geqslant 0 \text{ for } x \in \Omega),$$

then $u(x) \equiv \text{const}$.

A maximum principle. Let $u(x)$ be a real function which is defined on a weak connected net $\bar{\Omega}$. If the inequality

$$\alpha[u](x) \leqslant 0 \quad \text{for} \quad x \in \Omega \quad (\text{or } \alpha[u](x) \geqslant 0 \text{ for } x \in \Omega),$$

is satisfied, then

$$u(x) \leqslant \max \{0, \max_{y \in \partial\Omega} u(y)\} \quad (u(x) \geqslant \min \{0, \min_{y \in \partial\Omega} u(y)\}).$$

The above maximum principles imply convergence of the following finite difference schemes:

$$\alpha_h[u](x) = \varphi(x) \quad \text{for} \quad x \in \Omega_h, \tag{1.2}$$

$$u(x) = \psi(x) \quad \text{for} \quad x \in \partial\Omega_h, \tag{1.3}$$

where α_h — an operator of the positive type.

A finite difference schemes (1.2) and (1.3) is called canonical (cf. [6]).

A maximum principle for an $N \times N$ matrix A. In [17] R. S. Varga considers a maximum for an $N \times N$ matrix A. R. S. Varga's maximum principle also implies converqence of finite difference schemes of the form (1.2), (1.3). Moreover, this principle can be applied in the analysis of finite difference schemes of different form from (1.2), (1.3) (cf. § 4).

Let $V_N(C)$ be an $N \times N$ vector-space with elements $z = (z_1, z_2, \ldots, z_N)^T$ and complex components $z_i \in C$, $i = 1, 2, \ldots, N$, and let S^n be a subspace of $V_N(C)$ spanned by vectors $\delta_j \in \{e_1, e_2, \ldots, e_N\}$ $1 \leqslant j \leqslant N$, $e_i = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{Ni})$ $(i = 1, 2, \ldots, N)$. The projection of a vector z on the subspace S^n we denote by $P_S z$, where $P_S = \text{diagonal}\,(d_1, d_2, \ldots, d_N)$, $d_i = 0$ if $e_i \notin S^n$ and $d_i = 1$ if $e_i \in S^n$ $(i = 1, 2, \ldots, N)$.

DEFINITION 1.1. An $N \times N$ matrix A satisfies a maximum principle with respect to a subspace S^n of $V_N(C)$ (in symbols $A \in \mathfrak{M}_S$) if and only if every solution z of $Az = P_S b$ satisfies the inequality

$$|z|_\infty \leqslant |P_S b|_\infty \quad \text{for any } b \in V_N(C),$$

where $|z|_\infty = \max_i |z_i|$.

In [17] R. S. Varga presents the following sufficient and necessary condition for this maximum principle:

LEMMA 1.1. *An $N \times N$ matrix A belongs to $\mathfrak{M}_S$ if and only if there exists an A^{-1} and $|A^{-1}P_S|_\infty \leqslant 1$, where $|A|_\infty = \max_i \sum_j |A_{ij}|$, $A = (A_{ij})$.*

The relation $A \in \mathfrak{M}_S$ is equivalent to the conditions: there exists an A^{-1}, $A^{-1}P_S \geqslant 0$ ($A = (A_{ij}) \geqslant 0$ if $A_{ij} \geqslant 0$, $i, j = 1, 2, \ldots, N$) and a matrix A is normalized with respect to a subspace S^n of $V_N(C)$ in the following sense:

DEFINITION 1.2. An $N \times N$ matrix A is normalized with respect to a subspace S^n of $V_N(C)$ (in symbols $A \in \mathfrak{N}_S$) if and only if $A\xi = P_S \xi$ for $\xi = (1, 1, \ldots, 1)^T$

To verify the relation $A \in \mathfrak{M}_S$ we can also use the following corollaries resulting from Lemma 1.1.

COROLLARY 1.1. *If a monotone matrix A belongs to $\mathfrak{N}_S$, then $A \in \mathfrak{M}_S$.*

COROLLARY 1.2. *If a matrix B belongs to $\mathfrak{N}_S \cap \mathfrak{M}_S$ and if a nonsingular matrix A satisfies the inequality*

$$|A^{-1}P_S| \leqslant B^{-1}P_S,$$

then $A \in \mathfrak{M}_S$, where $|A| = (|A_{ij}|)$.

The classes $\mathfrak{M}_S$ and K_S. Let us consider the class K_S of coherent matrices of operators of positive type.

$$\begin{aligned} K_S = \{A = (A_{ij})_{i,j=1}^N:\ & A_{ij} = a(x^i, y^j) \text{ for } x^i \in \Omega \text{ and } y^j \in N(x^i),\\ & A_{ij} = 0 \text{ for } x^i \in \Omega \text{ and } y^i \notin N(x^i) \text{ or } x^i \in \partial\Omega \text{ and } i \neq j,\\ & A_{ii} = 1 \text{ for } x^i \in \partial\Omega\},\end{aligned}$$

where $a(x, y)$ — coefficients of an operator α of positive type which is defined on a connected net $\bar{\Omega}$, $x^i = (x_1^i, x_2^i, \ldots, x_p^i) \in \bar{\Omega}$, an index i follows from the lexicographical ordering of points of the net $\bar{\Omega}$, N is a number of points of $\bar{\Omega}$.

Let S be a subspace of R^N spanned by vectors e_i for $i \in I_S = \{j: x^j = (x_1^j, x_2^j, \ldots, x_p^j) \in \partial\Omega\}$.

Now, we shall prove that if an $N \times N$ matrix A belongs to K_S, then $A \in \mathfrak{M}_S$. Namely, let us observe that every matrix A belonging to the class K_S is of positive type in the sense of the following definition:

DEFINITION 1.3 (cf. [1]). A matrix $B = (B_{ji})$ is said to be of *positive type* if the following conditions are satisfied:

a′) $B_{ji} \leqslant 0$ for $j \neq i$,

b′) $\sum_k B_{jk} \geqslant 0$ for all j, and further there exists a non-empty subset $J(B)$ of the integers $1, 2, \dots, N$ such that $\sum_k B_{jk} > 0$ for all $j \in J(B)$,

c′) for $i \notin J(B)$ there exists a $j \in J(B)$ and a sequence of non-zero elements of B of the form $B_{jk_1}, B_{k_1 k_2}, \dots, B_{k_r j}$.

From the definition of an operator α of positive type and from the definition of K_S we have

$$A_{ji} \leqslant 0 \quad \text{for} \quad j \neq i$$

and

$$\sum_{i=1}^{N} A_{ji} \geqslant 0 \quad \text{for} \quad i = 1, 2, \dots, N.$$

Furthermore, from the assumption that the net $\bar{\Omega}$ is connected and from the condition c′) of the definition of an operator of positive type α it follows that for every $i \notin I_S = J(A)$ there exists a sequence of non-zero elements of A of the form

$$a(x^i, y^{k_1}), a(x^{k_1}, y^{k_2}), \dots, a(x^{k_r}, y^j),$$

where $y^{k_s} \in \mathcal{N}(x^{k_{s-1}})$, $s = 1, 2, \dots, r$. We also have

$$\sum_{i=1}^{N} A_{ji} = 1 \quad \text{for} \quad j \in I_S. \tag{1.4}$$

Thus, all the conditions of Definition 1.3 are satisfied and therefore the matrix A is of positive type. On the other hand, every matrix A of positive type is monotone (cf. [1]). Let $B = A - \bar{A}$, where $\bar{A} = \text{diagonal}(A_1, A_2, \dots, A_N)$, $A_j = \sum_{i=1}^{N} A_{ji}$ if $j \notin I_S$ and $A_i = 0$ if $j \in I_S$. Since the matrix $B \in \mathfrak{N}_S \cap \mathfrak{M}_S$ (cf. (1.4) and Corrollary 1.1), then from Corollary 1.2 it follows that $A \in \mathfrak{M}_S$.

Remark 2. In the class $\mathfrak{M}_S$ there exist matrices which do not belong to K_S. We meet such matrices when approximating differential equations with a higher order of accuracy than two, (cf. [1], § 4).

§ 2. A maximum principle for nonlinear mappings

Below, we present an extension of R. S. Varga's maximum principle to nonlinear mappings (cf. [12]) and to *a priori* estimations. Next, we use this extension to prove stability finite difference schemes for nonlinear elliptic equations with boundary value conditions of the first second and third kinds (cf. § 3).

Let us consider the following system of nonlinear equations:

$$f(\vec{v}) = \vec{b}, \quad \vec{b} = (b^1, b^2, \ldots, b^N)^T \in R^N, \tag{2.1}$$

where a vector-function $f(v) = (f^1(v), f^2(v), \ldots, f^N(v))^T$ is defined for $\vec{v} = (v_1, v_2, \ldots, v_N)^T \in D^N$, D^N – a domain of R^N

Definition 2.1. A mapping $f(\vec{v})$ satisfies a maximum principle with respect to a subspace S of the space R^N (in symbols $f \in \mathfrak{M}_S$) if and only if for any vector $\vec{b} \in R^N$ every solution $\vec{z}$ of $f(\vec{v}) = P_S \vec{b}$ satisfies the inequality

$$|\vec{z}|_\infty \leqslant |P_S \vec{b}|_\infty.$$

From Lemma 1.1 it follows that a vector-function $f(\vec{v})$ satisfies the maximum principle if for every $\vec{v} \in D^N$ there exists a matrix $A(\vec{v})$ such that

$$f(\vec{v}) = A(\vec{v})\vec{v}$$

and $A(v) \in \mathfrak{M}_S$.

If a vector-function $f(v)$ is continuously differentiable in D^N and if $f(0) = 0$, (here we assume that $O = (0, 0, \ldots, 0) \in D^N$), then for every $\vec{v} \in D^N$ there exists a $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, $0 < \theta_i < 1$ $(i = 1, 2, \ldots, N)$, such that the matrix

$$M(\vec{v}, \theta) = \frac{\partial f^i(\vec{v}\theta_i)}{\partial v_k} \quad (i, k = 1, 2, \ldots, N), \tag{2.2}$$

satisfies the condition $f(\vec{v}) = M(\vec{v}, \theta)\vec{v}$.

Of course, then $f \in \mathfrak{M}_S$ if $M(\vec{v}, \theta) \in \mathfrak{M}_S$ for $\vec{v} \in D^N$

Now, we present an extension of this maximum principle to an *a priori* estimation.

Definition 2.2. A mapping $f(\vec{v})$ satisfies a maximum principle as an *a priori* estimation with respect to a subspace S of the space R^N and with respect to a constant K (in symbols $f \in \mathfrak{M}_S(K)$) if and only if for any $\vec{b} \in R^N$ every solution z of (2.1) satifties the inequality

$$|\vec{z}|_\infty \leqslant |P_S \vec{b}|_\infty + K\,|(E - P_S)\vec{b}|_\infty,$$

where E is the unit matrix.

A sufficient condition for satisfying the relation $f \in \mathfrak{M}_S(K)$ can be written as follows:

Theorem 2.1. *If there exist a vector $\vec{\alpha} \in R^N$ and a monotone matrix $A(v)$ such that for $v \in D^N$*

$$f(\vec{v}) = A(\vec{v})\vec{v},$$

$$A(\vec{v})\xi \geqslant P_S \xi, \quad \xi = (1, 1, \ldots, 1)^T,$$

$$A(\vec{v})\vec{\alpha} \geqslant (E - P_S)\xi,$$

then $f \in \mathfrak{M}_S(|\alpha|_\infty)$.

Proof. Let $\vec{g} = |P_S\vec{b}|_\infty \xi + |(E-P_S)\vec{b}|\vec{\alpha}$ and let $\vec{z}$ be a solution of $f(\vec{v}) = \vec{b}$. Then, we have

$$A(\vec{z})(\vec{g}\mp\vec{z}) = |P_S\vec{b}|_\infty A(\vec{z})\xi + |(E-P_S)\vec{b}|_\infty A(\vec{z})\vec{\alpha}\mp\vec{b}$$
$$\geqslant |P_S\vec{b}|_\infty P_S\xi + |(E-P_S)\vec{b}|_\infty (E-P_S)\xi\mp\vec{b} \geqslant 0.$$

Since the matrix $A(\vec{z})$ is monotone,

$$\vec{g}\mp\vec{z} \geqslant 0.$$

Hence, we have $|\vec{z}|_\infty \leqslant |\vec{g}|_\infty \leqslant |P_S\vec{b}|_\infty + |\vec{\alpha}|_\infty |(E-P_S)\vec{b}|_\infty$. Thus the theorem is proved.

§ 3. A finite difference analogue of a maximum principle for nonlinear elliptic equations

A boundary value problem. Let us consider the following boundary value problem:

(3.1) $$G(x, u, u_x, u_{xx}) = \varphi(x) \quad \text{for} \quad x\in D^p,$$

(3.2) $$B(x)\frac{du}{dn} + C(x)u = \psi(x) \quad \text{for} \quad x\in \partial D^p,$$

where $x = (x_1, x_2, \ldots, x_p)$, $u = u(x)$, $u_x = (u_{x_1}, u_{x_2}, \ldots, u_{x_p})$, $u_{xx} = (u_{x_1x_1}, u_{x_2x_2}, \ldots, u_{x_px_p})$, $\frac{du}{dn}$ — a normal vector, $D^p = \{x\in R^p\colon 0 < x_i < 1,\ i = 1, 2, \ldots, p\}$, ∂D^p — the boundary of D^p

The assumptions. 1° The function $G(x, q, r, s)$, $r = (r_1, r_2, \ldots, r_p)$, $s = (s_1, s_2, \ldots, s_p)$ is defined in $D^p\times R^{2p+1}$ and continuosly differentiable with respect to variables $q, r_1, r_2, \ldots, r_p, s_1, \ldots, s_p$, $G(x, 0, 0, 0) = 0$ for $x\in D^p$, $G_q(x, q, r, s) \leqslant 0$ for $(x, q, r, s)\in D^p\times R^{2p+1}$, the given functions $B(x) \geqslant 0$, $C(x) \geqslant 0$, $\psi(x)$ and $\varphi(x)$ are bounded on ∂D^p and D^p, respectively, $B(x) + C(x) > 0$ for $x\in\partial D^p$, there exists a point $\bar{x} \subseteq D^p$ such that $G_q(\bar{x}, q, r, s) < 0$ for $(q, r, s)\in R^{2p+1}$ or there exists a point $\bar{x}\in\partial D^p$ such that $C(\bar{x}) > 0$.

2° There exist functions $\mu_j(q, r, s) > 0$, $L_j(q, r, s)$ and constants $K_j > 0$, $(j = 1, 2. \ldots, p)$ such that

$$\frac{\partial G}{\partial s_j}(x, q, r, s) \geqslant \mu(q, r, s),$$

$$\left|\frac{\partial G}{\partial r_j}(x, q, r, s)\right| \leqslant L_j(q, r, s),$$

$$L_j(q, r, s) \leqslant K_j\mu_j(q, r, s),$$

for $(x, q, r, s)\in D^p\times R^{2p+1}$.

A finite difference scheme. Let $\bar{\Omega}_h$ be a net defined as follows:

$$\bar{\Omega}_h = \{ih = i_1 h_1, i_2 h_2, \ldots, i_p h_p): i = (i_1, i_2, \ldots, i_p) - \text{integers}, \quad 0 \leqslant i_s \leqslant N_s,\ N_s h_s = 1,\ s = 1, 2, \ldots, p\},$$

$$\partial\Omega_h = \partial^0 \Omega_h \cup \partial^N \Omega_h, \quad \partial^0 \Omega_h = \bigcup_{l=1}^{p} \partial_l^0 \Omega_h, \quad \partial^N \Omega_h = \bigcup_{l=1}^{p} \partial_l^N \Omega_h,$$

$$\partial_l^0 \Omega_h = \{ih = (i_1 h_1, i_2 h_2, \ldots, i_p h_p) \in \bar{\Omega}_h:\ i_l = 0\},$$

$$\partial_l^N \Omega_h = \{ih = (i_1 h_1, i_2 h_2, \ldots, i_p h_p) \in \bar{\Omega}_h:\ i_l = N_l\}, \qquad \Omega_h = \bar{\Omega}_h - \partial\Omega_i.$$

Next, let v^i be a value of a function v at the nodal point $(i_1 h_1, i_2 h_2, \ldots, i_p h_p)$ and let

$$\Delta_s v^i = \frac{1}{h_s}\big(v(i_1 h_1, i_2 h_2, \ldots, (i_s+1) h_s, \ldots, i_p h_p) - v^i\big),$$

$$\nabla_s v^i = \frac{1}{h_s}\big(v^i - v(i_1 h_1, i_2 h_2, \ldots, (i_s-1) h_s, \ldots, i_p h_p)\big),$$

$$\bar{\Delta}_s v^i = \tfrac{1}{2}(\Delta_s + \nabla_s) v^i,$$

$$\Delta v^i = (\Delta_1 v^i, \Delta_2 v^i, \ldots, \Delta_p v^i),$$

$$\nabla v^i = (\nabla_1 v^i, \nabla_2 v^i, \ldots, \nabla_p v^i).$$

Now, we can write a difference scheme for Problems (2.1) and (2.2).

$$G(ih, v^i, \bar{\Delta} v^i, \Delta\nabla v^i) = \varphi^i, \quad ih \in \Omega_h, \tag{3.3}$$

$$B^i \Delta_s v^i + C^i v^i = \psi^i, \quad ih \in \partial_l^0 \Omega_h, \tag{3.4}$$

$$B^i \nabla_s v^i + C^i v^i = \psi^i, \quad ih \in \partial_l^N \Omega_h, \tag{3.5}$$

Below, we shall write the difference scheme (2.3), (2.4), (2.5) in the form

$$f(\vec{v}) = \vec{b} \tag{3.6}$$

where $\vec{v} = (v^1, v^2, \ldots, v^N)^T$, $N = (N_1+1)(N_2+1)\ldots(N_p+1)$, $v^{m(i)} = v(ih)$, an index $m(i)$ follows from the lexicographical ordering of points of the net $\bar{\Omega}_h$, $\vec{b} = (b^1, b^2, \ldots, b^N)^T$,

$$b^{m(i)} = \begin{cases} -\varphi^i, & ih \in \Omega_h, \\ \varphi^i, & ih \in \partial\Omega_h, \end{cases} \tag{3.7}$$

$$f(\vec{v}) = (f^1(\vec{v}), f^2(\vec{v}), \ldots, f^N(\vec{v}))^T$$

$$f^{m(i)}(\vec{v}) = \begin{cases} -G(ih, v^i, \bar{\Delta} v^i, \Delta\nabla v^i), & ih \in \Omega_h, \\ B^i \Delta_s v^i + C^i v^i, & ih \in \partial_s^0 \Omega_h,\ s = 1, 2, \ldots, p, \\ B^i \nabla_s v^i + C^i v^i, & ih \in \partial_s^{N_s} \Omega_h,\ s = 1, 2, \ldots, p. \end{cases} \tag{3.8}$$

A priori estimations. To estimate a solution v of the system of equations (3.6) we shall first prove the following theorem:

THEOREM 3.1. *If* $h_j \leqslant h_0 = \min_l (1/K_l)$ $(j = 1, 2, \ldots, p)$, *then the matrix*

$$M(\vec{v}, \theta) = \left(\frac{\partial f^j(\vec{v}\theta_j)}{\partial v_k}\right), \quad (j, k = 1, 2, \ldots, N),$$

is of positive type for every $\vec{v} \in R^N$, $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, $0 < \theta_j < 1$, $(j = 1, 2, \ldots, N)$.

Proof. From (3.8) if follows that

$$(3.9) \quad \frac{\partial f^{m(i)}}{\partial v^k} = \begin{cases} 2\sum_{l=1}^{p} \frac{1}{h_l^2}\frac{\partial G^i}{\partial s_l} - \frac{\partial G^i}{\partial q}, & k = m(i),\ ih \in \Omega_h, \\ -\frac{1}{h_l^2}\frac{\partial G^i}{\partial s_l} \mp \frac{1}{2h_l}\frac{\partial G^i}{\partial r_l}, & \\ \qquad k = m(i) \mp (N_1+1)(N_2+1)\ldots(N_l+1), & \\ \qquad l = 1, 2, \ldots, p,\ ih \in \Omega_h, & \\ \frac{1}{h_l}B^i + C^i, \quad k = m(i),\ ih \in \partial_l^0 \Omega_h \cup \partial_l^N \Omega_h,\ l = 1, 2, \ldots, p, & \\ -\frac{1}{h_i}B^i, \text{ this element appears in the row } m(i)\ ih \in \partial\Omega_h \text{ only once}, & \\ \text{the remaining elements of } M \text{ are equal to zero.} & \end{cases}$$

Let us observe that the matrix $M(\vec{v}, \theta)$ satisfies the conditions a'), b'), c') of Definition 1.3. Namely, from (3.9) for $h_s \leqslant h_0$ $(s = 1, 2, \ldots, p)$, we have

$$M_{jk}(\vec{v}, \theta) \leqslant 0, \quad j \neq k,$$

$$\sum_{k=1}^{N} M_{jk}(\vec{v}, \theta) \geqslant 0, \quad j = 1, 2, \ldots, N.$$

Furthermore

$$\sum_{k=1}^{N} M_{jk}(\vec{v}, \theta) > 0, \quad j \in J(M),$$

and the non-empty set

$$J(M) = \{m(i) \in (1, 2, \ldots, N):\ C(ih) > 0 \text{ for } ih \in \partial\Omega_h \text{ or } \frac{\partial G}{\partial q}(ih, q, r, s) < 0 \text{ for } (ih, q, r, s) \in \Omega_h \times R^{2p+1}\}.$$

A sequence of the form

$$M_{k,k_1}, M_{k_1,k_1\mp 1}, M_{k_1\mp k_1\mp 2}, \ldots, M_{j-1,j}$$

satisfies condition c') of Definition 1.3. Thus, the matrix $M(\vec{v}, \theta)$ is of positive type.

From Theorem 2.2 in [1] it follows that the matrix $M(v, \theta)$ is monotone for $h_s \leqslant h_0$ $(s = 1, 2, \ldots, N)$,

$$\vec{v} \in R^N, \quad \theta = (\theta_1, \theta_2, \ldots, \theta_N) \quad (0 < \theta_j < 1,\ j = 1, 2, \ldots, N).$$

Now, we can formulate a difference analogue of the theorem given in [3] p. 165.

THEOREM 3.2. *If* $B(x) = 0$, $C(x) = 1$, $\psi(x) \geqslant 0$ *(or* $\psi(x) \leqslant 0$*) for* $x \in \partial\Omega_h$ *and if* $\varphi(x) \geqslant 0$ *(or* $\varphi(x) \leqslant 0$*) for* $x \in \Omega_h$, *then a solution* v *of the finite difference scheme* (3.3), (3.4), (3.5) *is non-positive (or non-negative) in* $\bar{\Omega}_n$.

Proof. If v is a solution of (3.3), (3.4), (3.5), then from (3.6) we have

$$M(\vec{v}, \theta)\vec{v} = \vec{b} \quad \text{for certain } \theta = (\theta_1, \theta_2, \ldots, \theta_N).$$

Since the matrix $M(\vec{v}, \theta)$ is monotone, we have $\vec{v} \geqslant 0$ if $\vec{b} \geqslant 0$. From the assumption and (3.7) if follows that $\vec{b} \geqslant 0$ (or $\vec{b} \leqslant 0$) and this ends the proof.

Let $T(M)$ be a non-empty subset of the set $J(M)$ and let S be a subspace of R^N spanned by vectors $e_j = (\delta_{1j}, \delta_{2j}, \ldots, \delta_{Nj})$, $j \in T(M)$.

A MAXIMUM PRINCIPLE. *Every solution* $\vec{v}$ *of the system equations* $f(\vec{v}) = P_s\vec{b}$ *satisfies the inequality*

$$|\vec{v}|_\infty \leqslant |D^1 P_S \vec{b}|_\infty \quad \textit{for any } \vec{b} \in R^N,$$

where $D^1 = $ *diagonal* $(D_1^1, D_2^1, \ldots, D_N^1)$,

$$D^1_{m(i)} = \begin{cases} \left(\dfrac{\partial G^i}{\partial q}\right)^{-1} & \textit{for} \quad m(i) \in T(M),\ ih \in \Omega_h, \\ (C(ih))^{-1} & \textit{for} \quad m(i) \in T(M),\ ih \in \partial\Omega_h, \\ 1 & \textit{for} \quad m(i) \notin T(M). \end{cases}$$

Proof. The vector $\vec{v}$ satisfies the following system of equations:

$$M(\vec{v}, \theta)\vec{v} = P_S\vec{b} \quad \text{and} \quad D^1 M(\vec{v}, \theta)\vec{v} = D^1 P_S \vec{b}.$$

From Theorem 3.1 it follows that

$$D^1 M(\vec{v}, \theta) - D^2 \in \mathfrak{N}_s \cap \mathfrak{M}_s,$$

where $D^2 = $ diagonal $(D_1^2, D_1^2, \ldots, D_N^2)$,

$$D^2_{m(i)} = \begin{cases} 0 & \text{for} \quad m(i) \notin J(M),\ ih \in \bar{\Omega}_h, \\ -\dfrac{\partial G^i}{\partial q} & \text{for} \quad m(i) \in J(M) - T(M),\ ih \in \Omega_h, \\ C(ih) & \text{for} \quad m(i) \in J(M) - T(M),\ ih \in \partial\Omega_h. \end{cases}$$

Since

$$D^1 M(\vec{v}, \theta) - D^2 \leqslant D^1 M(\vec{v}, \theta)$$

and

$$(D^1 M(\vec{v}, \theta))^{-1} \leqslant (D^1 M(\vec{v}, \theta) - D^2)^{-1},$$

from Corollary 1.2, we have

$$|\vec{v}|_x \leqslant |D^1 P_s \vec{b}|_x .$$

Thus the theorem is proved.

Now, let us assume that

$$B(x) = \begin{cases} 1 & \text{for} \quad x \in \partial\Omega - \Sigma, \\ 0 & \text{for} \quad x \in \Sigma, \end{cases}$$

$$C(x) = \begin{cases} 0 & \text{for} \quad x \in \partial\Omega - \Sigma, \\ 1 & \text{for} \quad x \in \Sigma, \end{cases}$$

where $\Sigma \subset \partial\Omega$ and $\varphi(x) = 0$, $x \in \Omega$; $\psi(x) = 0$, $x \in \partial\Omega - \Sigma$; $T(M) = \{m(i): ih \in \Sigma\}$. This maximum principle implies the following corollary.

COROLLARY 3.1. *Let the above assumptions hold. Every solution v of the difference scheme* (3.3), (3.4), (3.5) *attains its extremum on that part of the boundary* $\partial\Omega_h$ *on which Dirichlet's condition is given. Then v satisfies the inequality*

$$\max_{ih \in \bar{\Omega}_h} |v(ih)| \leqslant \max_{ih \in \Sigma_h} |\psi(ih)|,$$

where $\Sigma_h \subset \Sigma$.

AN EXTENSION OF THE MAXIMUM PRINCIPLE. *Let* $B(x) = 0$, $C(x) = 1$ *for* $x \in \partial\Omega$. *Then every solution v of the finite difference scheme* (3.3), (3.4), (3.5) *satisfies the following inequality*:

$$\max_{\bar{\Omega}_h} |v(ih)| \leqslant \max_{\partial\Omega_h} |\psi(ih)| + \exp(\gamma) \max_{\Omega_h} |\varphi(ih)| \tag{3.10}$$

for $\gamma = \min_{1 \leqslant i \leqslant p} (L_i + \sqrt{L_i^2 + 2\mu_i})/\mu_i$, $h_s \leqslant h_0$ $(s = 1, 2, \ldots, p)$.

Proof. The vector $\vec{v}$ satisfies the system of equations

$$f(\vec{v}) = \vec{b},$$

where the vector $\vec{b}$ and $f(\vec{v})$ are given by (3.7) and (3.8). Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)^T$, $\alpha_j = \exp(\gamma) - (1 + \gamma h_1)^j$, $j = 1, 2, \ldots, N_1$; $\alpha_j = \alpha_{j-N_1}$, $j = (N_1 + 1), (N_1 + 2), \ldots, N$, and let S be a subspace of R^N spanned by the vector $e_j = (\delta_{j1}, \delta_{j2}, \ldots, \delta_{jN})$, $j \in J(M) = \{m(i) \in \{1, 2, \ldots, N\}: ih \in \partial\Omega_h\}$. Now, we shall show that the mapping $f(\vec{v})$ satisfies the assumptions of Theorem 2.1. Namely, from assumption 2° it follows that

$$e_j M(\vec{v}, \theta)\vec{\alpha} = M_{j,j-1}\alpha_{j-1} + M_{jj}\alpha_j + M_{j,j+1}\alpha_{j+1}$$
$$\geqslant (\mu_1 - \tfrac{1}{2}h_1 L_1)\gamma^2 - L_1\gamma, \quad j \notin S, \; h_1 \leqslant h_0,$$

where the matrix $M(\vec{v}, \theta)$ is given by (3.9). Hence, we have

$$M(\vec{v}, \theta)\xi \geqslant P_s\xi, \quad \xi = (1, 1, \ldots, 1)^T$$

and

$$M(\vec{v}, \theta)\vec{\alpha} \geqslant (E - P_s)\xi \quad \text{if} \quad (\mu_1 - \tfrac{1}{2}h_1 L_1)\gamma^2 - L_1\gamma \geqslant 1.$$

But

$$(\mu_1 - \tfrac{1}{2}h_1 L_1)\gamma^2 - L_1\gamma \geqslant 1 \quad \text{for} \quad \gamma = (L_1 + \sqrt{L_1^2 + 2\mu_1})/\mu_1.$$

Thus, inequality (3.10) is true. Changing the coordinates of R^N, we can obtain inequality (3.10) for $\gamma = \min_i L_i + \sqrt{L_i^2 + 2\mu_i}/\mu_i$, and this ends the proof.

§ 4. A finite difference scheme of higher order accuracy

In this section we consider an elliptic equation ($p = 1$) of form (3.1). Namely, we deal with the following boundary value problem:

$$G(x, u, u'') = \varphi(x), \quad 0 < x < 1, \tag{4.1}$$

$$u(0) = a_0, \quad u(1) = a_N. \tag{4.2}$$

Also, we assume that the functions $G(x, q, s)$ and $\varphi(x)$ satisfy assumptions 1°, 2° in § 3.

We approximate problem (4.1), (4.2) by the following finite difference scheme:

$$\begin{aligned} v^i &= a_i, \quad i = 0, N, \\ G(ih, v^i, l_h v^i) &= \varphi^i, \quad i = 1, 2, \ldots, N-1, \end{aligned} \tag{4.3}$$

where $Nh = 1$, $v^i = v(ih)$,

$$l_h v^i = \begin{cases} \dfrac{1}{h^2}(v^{i-1} - 2v^i + v^{i+1}), & i = 1, N-1, \\ \dfrac{1}{12h^2}(-v^{i-2} + 16v^{i-1} - 30v^i + 16v^{i+1} - v^{i-2}), & i = 2, 3, \ldots, N-2. \end{cases}$$

It is easy to verify that if $u \in C^6(0, 1)$ then the finite difference scheme (4.3) approximates the boundary value problem (4.1), (4.2) with accuracy $O(h^2)$

at the points h, $1-h$ and with accuracy $O(h^4)$ at the points $2h, 3h, \ldots, (N-2)h$.

In the linear case, $(G(x, u, u_{xx}) = u''(x)+q(x)u(x))$, J. H. Bramble and B. E. Hubbard (cf. [1]) proved that the finite difference scheme (4.3) is convergent as fast as $O(h^4)$ in spite of $O(h^2)$ approximation at the points h, $1-h$. Below, we solve the finite difference scheme (4.3) using an explicit form of an inverse matrix to a coherent matrix.

Let us write the finite difference scheme (4.3) in the form

$$f(\vec{v}) = \vec{b}, \tag{4.4}$$

where $\vec{v} = (v^0, v^1, \ldots, v^N)^T$, $f(\vec{v}) = (f^0(\vec{v}), f^1(\vec{v}), \ldots, f^N(\vec{v}))^T$,

$$f^i(\vec{v}) = \begin{cases} v^i, & i = 0, N, \\ -G(ih, v^i, l_n v^i), & i = 1, 2, \ldots, N-1, \end{cases}$$

$\vec{b} = (b^0, b^1, \ldots, b^N)^T$,

$$b^i = \begin{cases} a_i, & i = 0, N, \\ -\varphi(ih), & i = 1, 2, \ldots, N-1, \end{cases}$$

The coherent matrix

$$M(\vec{v}, \theta) = \left(\frac{\partial f^i(\vec{v}\theta_i)}{\partial v^k}\right),$$

$\theta = (\theta_0, \theta_1, \ldots, \theta_N)$, $0 < \theta_i < 1$, $(i = 0, 1, \ldots, N)$, can be presented in the following form:

$$M(\vec{v}, \theta) = B^1(\vec{v}, \theta) A(0) + B^2(\vec{v}, \theta),$$

where $B^1 = \text{diagonal}\ (B_0^1, B_1^1, \ldots, B_N^1)$

$$B_i^1 = \begin{cases} 1, & i = 0, N, \\ \dfrac{1}{h^2}\dfrac{\partial G^i}{\partial s}, & i = 1, N-1, \\ \dfrac{1}{12h^2}\dfrac{\partial G^i}{\partial s}, & i = 2, 3, \ldots, N-2, \end{cases}$$

$B^2(\vec{v}, \theta) = \text{diagonal}\ (B_0^2, B_1^2, \ldots, B_N^2)$,

$$B_i^2 = \begin{cases} \theta, & i = 0, N, \\ -\dfrac{\partial G^i}{\partial q}, & i = 1, 2, \ldots, N, \end{cases}$$

$G^i = G(ih, \theta_i v^i, \theta_i l_h v^i)$ $(i = 1, 2, \ldots, N-1)$,

$$(4.5)\qquad A(a) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & & & \\ -1 & 2+a & -1 & 0 & 0 & & & \\ 1 & -16 & 30+a & -16 & 1 & & & \\ 0 & 1 & -16 & 30+a & -16 & & & \\ & & & & & & & \\ & & & & & 30+a & -16 & -1 \\ & & & & & -1 & 2+a & -1 \\ & & & & & 0 & 0 & 0 \end{bmatrix},$$

$0 \leqslant a \leqslant 36$.

The inverse matrix $A^{-1}(a)$. The explicit form of $A^{-1}(a) = (Y_k^{(a)}(n))$, where $Y_k^{(a)}(n)$ is the general solution of the following system of difference equations:

$$(4.6)\qquad y(n+4)-16y(n+3)+(30+a)y(n+2)-16y(n+1)+y(n) = \delta_{nk},\quad k, n = 0, 1, \ldots, N,$$

δ_{nk} – the Kronecker's symbol.

To find $Y_k^{(a)}(n)$ we compute the roots $\lambda_1(a), \lambda_2(a), \lambda_3(a), \lambda_4(a)$ of the characteristic polynomial

$$W(a) = \left(\lambda^2-(8+\sqrt{36-a})\lambda+1\right)\left(\lambda^2-(8-\sqrt{36-a})\lambda+1\right),$$

$$\lambda_1(a) = \tfrac{1}{2}\left(8+\sqrt{36-a}-\sqrt{96+16\sqrt{36-a}-a}\right),$$

$$\lambda_2(a) = \tfrac{1}{2}\left(8+\sqrt{36-a}+\sqrt{96+16\sqrt{36-a}-a}\right),$$

$$\lambda_3(a) = \tfrac{1}{2}\left(8-\sqrt{36-a}-\sqrt{96-16\sqrt{36-a}-a}\right),$$

$$\lambda_4(a) = \tfrac{1}{2}\left(8-\sqrt{36-a}+\sqrt{96-16\sqrt{36-a}-a}\right).$$

Let us note that $\lambda_1(0) = 7-4\sqrt{3}$, $\lambda_2(0) = 7+4\sqrt{3}$, $\lambda_3(0) = \lambda_4(0) = 1$. Thus, the function

$$(4.6)\qquad Y^{(a)}(n) = \begin{cases} C_1\lambda_1^n+C_2\lambda_2^n+C_3 n+C_4 & \text{for}\quad a = 0,\\ C_1\lambda_1^n+C_2\lambda_2^n+C_3\lambda_3^n+C_4\lambda_4^n & \text{for}\quad 0 < a < 36, \end{cases}$$

is the general solution of the homogeneous equation (4.6). We find the general solution $Y_k^{(a)}(n)$ of (4.6) in the following form:

$$(4.7)\qquad Y_k^{(a)}(n) = Y^{(a)}(C_1^k, C_2^k, C_3^k, C_4^k, n)+y_k^{(a)}(n),$$

where $y_k^{(a)}(n)$ is a solution of (4.6) which can be obtained by the method variation of the constants C_1, C_2, C_3, C_4. We note that

$$y_k^{(a)}(n) = \begin{cases} \sum_{t=0}^{n-1} \frac{\delta_{t+2,k}}{12}\left[t-n+2+\frac{1}{8\sqrt{3}}(\lambda_1^{t+2}\lambda_2^n-\lambda_1^n\lambda_2^{t+2})\right], & a=0,\\ \sum_{t=0}^{n-1}\left[\frac{\lambda_2^n\lambda_1^{t+1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}-\frac{\lambda_1^n\lambda_1^{t+1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}+\right. & \\ \left.+\frac{\lambda_4^n\lambda_3^{t+1}}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}-\frac{\lambda_3^n\lambda_4^{t+1}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_3)}\right]\delta_{t+2,k}, & \\ & 0<a<36, \end{cases}$$

for $k, n = 0, 1, 2, \ldots, N$.

Thus, we have

$$(4.8)\qquad Y_k^{(a)}(n) = Y^{(a)}(n, C_1^k, C_2^k, C_3^k, C_4^k)+y_k^{(a)}(n), \quad k, n = 0, 1, \ldots, N.$$

The constants $C_1^k, C_2^k, C_3^k, C_4^k$, $k = 0, 1, \ldots, N$ are found from the conditions:

$$(4.9)\qquad \begin{aligned} &Y_k^{(a)}(0) = \delta_{0k},\\ &-Y_k^{(a)}(0)+(2+a)\,Y_k^{(a)}(1)-Y_k^{(a)}(2) = \delta_{1k},\\ &-Y_k^{(a)}(N-2)+(2+a)\,Y_k^{(a)}(N-1)-Y_k^{(a)}(N) = \delta_{N-1,k},\\ &Y_k^{(a)}(N) = \delta_{Nk}, \end{aligned}$$

Hence, for $a = 0$

$$\begin{aligned} C_1^k &= -\frac{\delta_{1k}}{(\lambda_1-1)^2}+\frac{\delta_{N-1,k}-\delta_{1k}}{\lambda_2^{N-2}-\lambda_1^{N-2}},\\ C_2^k &= \frac{\delta_{1k}-\delta_{N-1,k}}{(\lambda_2^{N-2}-\lambda_1^{N-2})(\lambda_2-1)^2},\\ C_3^k &= \frac{1}{N}(\delta_{Nk}+C_1^k+C_2^k-\delta_{0k}-\lambda_1^N C_1^k-\lambda_2^N C_2^k),\\ C_4^k &= \delta_{0k}-C_1^k-C_2^k, \end{aligned}$$

and for $0 < a < 36$,

$$C_1^k = \frac{D_1^k}{D}, \quad C_2^k = \frac{D_2^k}{D},$$

$$C_3^k = \frac{D_3^k}{D}, \quad C_4^k = \frac{D_4^k}{D},$$

where $\Lambda_i^1 = (2+a)\lambda_i-\lambda_i^2-1$, $\Lambda_i^2 = (2+a)\lambda_i^{N-1}-\lambda_i^N-\lambda_i^{N-2}$,

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \Lambda_1^1 & \Lambda_2^1 & \Lambda_3^1 & \Lambda_4^1 \\ \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 & \Lambda_4^2 \\ \lambda_1^N & \lambda_2^N & \lambda_3^N & \lambda_4^N \end{vmatrix},$$

$$D_1^k = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \delta_{1k} & \Lambda_2^1 & \Lambda_3^1 & \Lambda_4^1 \\ \delta_{N-1,k}+\psi_k^{(a)}(N) & \Lambda_2^2 & \Lambda_3^2 & \Lambda_4^2 \\ \delta_{Nk}+\chi_k^{(a)}(N) & \lambda_2^N & \lambda_3^N & \lambda_4^N \end{vmatrix},$$

$$D_2^k = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \Lambda_1^1 & \delta_{1k} & \Lambda_3^1 & \Lambda_4^1 \\ \Lambda_1^2 & \delta_{N-1,k}+\psi_k^{(a)}(N) & \Lambda_3^2 & \Lambda_4^2 \\ \lambda_1^N & \delta_{Nk}+\chi_k^{(a)}(N) & \lambda_3^N & \lambda_4^N \end{vmatrix},$$

$$D_3^k = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \Lambda_1^1 & \Lambda_2^1 & \delta_{1k} & \Lambda_4^1 \\ \Lambda_1^2 & \Lambda_2^2 & \delta_{N-1,k}+\psi_k^{(a)}(N) & \Lambda_4^2 \\ \lambda_1^N & \lambda_2^N & \delta_{Nk}+\chi_k^{(a)}(N) & \Lambda_4^N \end{vmatrix},$$

$$D_k^4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \Lambda_1^1 & \Lambda_2^1 & \Lambda_3^1 & \delta_{1k} \\ \Lambda_2^1 & \Lambda_2^2 & \Lambda_3^2 & \delta_{N-1,k}+\psi_k^{(a)}(N) \\ \lambda_1^N & \lambda_2^N & \lambda_3^N & \delta_{Nk}+\chi_k^{(a)}(N) \end{vmatrix},$$

$$\psi_k^{(a)}(N) = y_k^{(a)}(N-2)-(2+a)\,y_k^{(a)}(N-1)+y^{(a)}(N),$$

$$\chi_k^{(a)}(N) = -y_k^{(a)}(N), \quad k = 0, 1, \ldots, N.$$

Thus, we have

$$A^{-1}(a) = \big(Y_k^{(a)}(n)\big), \quad k, n = 0, 1, \ldots, N. \tag{4.10}$$

The matrix $A(0)$ is monotone and normalized with respect to the subspace S of R^N spanned by the vectors $e_0 = (1, 0, \ldots, 0)$, $e_{N+1} = (0, 0, \ldots, 1)$. Furthermore (cf. [1])

$$0 \leqslant e_i A^{-1}(0) B^{-1} e_k^T \leqslant 3h^2, \quad k = 1,\ N-1,\ i = 0, 1, \ldots, N,$$

where $B = \text{diagonal}\left(1, \frac{1}{h^2}, \frac{1}{12h^2}, \ldots, \frac{1}{12h^2}, \frac{1}{h^2}, 1\right)$.

Thus $BA(0) \in \mathfrak{N}_S \cap \mathfrak{M}_S$. On the other hand, from the assumptions 1^0, 2^0 if follows that

$$M(\vec{v}, \theta) = B^1(\vec{v}, \theta)\,A(a)+B^2(\vec{v}, \theta) \geqslant \mu(q, s)\,BA(0).$$

Hence, we have

$$M^{-1}(\vec{v}, \theta) \leqslant \frac{1}{\mu} A^{-1}(0)\,B^{-1}$$

and from Corollary 1.2 we have $\mathfrak{M}M \in \mathfrak{M}_S$ and

$$e_i M^{-1}(\vec{v}, \theta) e_k^T \leqslant \frac{3}{\mu} h^2, \quad k = 1, N-1, \ i = 0, 1, \ldots, N.$$

A bound error. Let $u \in C^{(6)}(0,1) \cap C^0[0,1]$ be a solution of (4.1), (4.2) and let v be a solution of the finite difference scheme (4.3), (4.4). Then, we have

$$f(\vec{v}) = \vec{b} \quad \text{and} \quad f(\vec{ru}) = \vec{b} + \vec{\nu} u,$$

where $\vec{ru} = (u(0), u(h), \ldots, u(1))^T$, $\nu = (\nu_0, \nu_1, \ldots, \nu_N)^T$,

$$\nu_i(u) = \begin{cases} 0, \quad i = 0, N, \\ -G(ih, u(ih), u''(ih)) + O_i(h^2), \quad i = 1, N-1, \\ -G(ih, u(ih), u''(ih)) + O_i(h^4), \quad i = 2, 3, \ldots, N-2, \end{cases}$$

$O_i(h^\alpha) \to 0$ as fast as $h^\alpha \to 0$, $\alpha > 0$, $i = 1, 2, \ldots, N-1$. But for certain $\theta = (\theta_0, \theta_1, \ldots, \theta_N)$, $0 < \theta_i < 1$ $(i = 0, 1, \ldots, N)$

$$f(\overrightarrow{r_h u}) - f(\vec{v}) = M(\vec{z}, \theta) \vec{z},$$

where $\vec{z} = \overrightarrow{r_h u} - \vec{v}$, $M(\vec{z}, \theta) = B^1(\vec{z}, \theta) A(0) + B^2(\vec{z}, \theta)$. Hence

$$M(\vec{z}, \theta) \vec{z} = \vec{O}(u), \quad \vec{O} = (O, O_1, O_2, \ldots, O_{N-1}, O)^T$$

and

$$\vec{z} = M^{-1}(\vec{z}, \theta) \vec{O}(u).$$

Since

$$\|M^{-1}(\vec{z}, \theta)\|_\infty \leqslant \frac{1}{\mu} \quad \text{and} \quad 0 \leqslant M_{ik}^{-1}(\vec{z}, \theta) \leqslant \frac{3}{\mu} h^2,$$

$k = 1, N-1$; $i = 0, 1, \ldots, N$, we have

$$z_i = M_{i1}^{-1} O_1(h^2) + M_{iN-1}^{-1} O_{N-1}(h^2) + \sum_{k=2}^{N-2} M_{ik}^{-1} O_i(h^4)$$

and there exists a constant $K_0 > 0$ such that

$$|\vec{z}|_\infty \leqslant \frac{K_0}{\mu} h^4 = Ch^4, \tag{4.11}$$

where C is a constant.

EXAMPLE. Let us consider the following boundary value problem:

$$\left(\frac{d^2u}{dx^2}\right)^3+p_1(x)\left(\frac{d^2u}{dx^2}\right)^2+p_2(x)\frac{d^2u}{dx^2}-au = \varphi(x), \qquad 0<x<1,\ a\geqslant 0, \tag{4.12}$$

$$u(0) = a_0, \quad u(1) = a_N. \tag{4.13}$$

An approximated solution v of (4.12), (4.13) is found from (4.4;. In this case we have $G(x,q,s) = s^3+p_1(x)s^2+p_2(x)s-aq$. For $a = 0$, system (4.4) is solved by Newton's method

$$\vec{v}^{(m+1)} = \vec{v}^{(m)}-A^{-1}(0)B^{-f}(\vec{v}^{(m)})f(\vec{v}^{(m)}),$$

where $B^f(\vec{v}) = \text{diagonal}\,(B_0^f, B_1^f, \dots, B_N^f)$,

$$B_i^f = \begin{cases} 1, & i = 0, N, \\ \dfrac{1}{h^2}\dfrac{\partial G^i}{\partial s}, & i = 1, N-1, \\ \dfrac{1}{12h^2}\dfrac{\partial G^i}{\partial s}, & i = 2,3,\dots,N-2. \end{cases}$$

Below, we present the results of the computations.

Table 1

$p_1(x) = p_2(x) = 1$, $\varphi(x) = -\sin^3 x+\sin^2 x-\sin x$, $a = 0$, $v^{(0)} = 0$.

x	$u = \sin x$	$\vec{v}^{(m)}$	$\vec{r_h u}-\vec{v}^{(m)}$	m
0.0	0.000000	0.000000	0.000000	0
0.1	0.099835	0.099833	0.000002	5
0.2	0.198672	0.198669	0.000003	5
0.2	0.295523	0.295520	0.000003	5
0.4	0.389422	0.389418	0.000004	5
0.5	0.479430	0.479426	0.000004	5
0.6	0.565647	0.564642	0.000005	5
0.7	0.644223	0.644218	0.000005	5
0.8	0.717362	0.717356	0.000006	5
0.9	0.783333	0.783327	0.000006	5
1.0	0.841471	0.841471	0.000000	0

The error $\vec{z} = \vec{r_h u}-\vec{v} \approx Ch^4$, $h = 0.1$, $C = 0.1$.

Table 2

$p_1(x) = \exp(x)$, $p_2(x) = \exp(2x)$, $\varphi(x) = 3\exp(3x)$, $a = 0$, $\bar{v}^{(0)} = 0$.

x	$u = \exp(x)$	$\bar{v}^{(m)}$	$\vec{r_h u} - \bar{v}^{(m)}$	m
0.0	1.00000	1.00000	0.00000	0
0.1	1.10517	1.10518	0.00001	6
0.2	1.22140	1.22142	0.00002	6
0.3	1.34986	1.34987	0.00001	6
0.4	1.49182	1.49184	0.00002	6
0.5	1.64872	1.64874	0.00002	6
0.6	1.82212	1.82214	0.00002	6
0.7	2.01375	2.01377	0.00002	6
0.8	2.22554	2.22556	0.00002	6
0.9	2.45960	2.45962	0.00002	6
1.0	2.71828	2.71828	0.00000	0

The error $\bar{z} = \vec{r_h u} - \bar{v} \approx Ch^4$, $h = 0.1$, $C = 1$.

Table 3

$p_1(x) = 16\exp(4x)$, $p_2(x) = 256\exp(8x)$, $\varphi(x) = 49152\exp(12x)$, $a = 0$, $\bar{v}^{(0)} = 0$.

x	$u = \exp(4x)$	$\bar{v}^{(m)}$	$\vec{r_h u} - \bar{v}^{(m)}$	m
0.0	1.00000	1.00000	0.00000	0
0.1	1.49182	1.50287	0.01105	10
0.2	2.22554	2.24444	0.01889	10
0.3	3.32012	3.34673	0.02661	10
0.4	4.95303	4.98750	0.03447	10
0.5	7.38906	7.43161	0.04255	10
0.6	11.02320	11.07420	0.05100	10
0.7	16.44460	16.50455	0.05995	10
0.8	24.53250	24.60170	0.07920	10
0.9	36.59820	36.59820	0.07390	10
1.0	54.59820	54.59820	0.00000	0

The error $\bar{z} = \vec{r_h u} - \bar{v} \approx Ch^4$, $h = 0.1$, $C = 500$.

§ 5. A maximum principle for systems of ordinary differential equations

Let us consider the following initial value problem (cf. [8]):

(5.1) $$Pv = D(t, v)\frac{dv}{dt} - A(t, v)v = b(t), \quad 0 < t \leqslant T,$$

(5.2) $$v(0) = c,$$

where $v = (v^1, v^2, \ldots, v^N)^T$, $b = (b^1, b^2, \ldots, b^N)^T$, $c = (c^1, c^2, \ldots, c^N)^T$, D = diagonal $(D^1, D^2, \ldots, D^N)$, $A = (A_{ij})$, $i, j = 1, 2, \ldots, N$.

We assume that the entries of the given matrices $A(t, v)$, $D(t, v)$ and the vector $b(t)$ are real functions in $[0, T] \times R^N$ and $[0, T]$, respectively. Furthermore, we assume that problem (1), (2) has a continuous solution $v(t)$ in $[0, T]$.

DEFINITION 5.1. An operator P of form (5.1) satisfies the maximum principle with respect to a subspace S of the space R^N (in symbols $P \in \mathfrak{M}_S$) if and only if for any b, c every continuous solution $v(t)$ of the problem

$$(5.3) \qquad D(t, v)\frac{dv}{dt} = A(t, v)v + P_S b(t), \quad 0 < t \leqslant T,$$

$$(5.4) \qquad v(0) = c,$$

satisfies the following inequality:

$$(5.5) \qquad \|v\| \leqslant \max\{|c|_\infty, \|P_S b\|, \|P_S v\|\},$$

where $|c|_\infty = \max_i |c^i|$, $\|v\| = \max_i \max_{0 \leqslant t \leqslant T} |v^i(t)|$.

An application of this principle to an approximation of parabolic equations by the method of lines is presented in [9].

THEOREM 5.1. *Let* $v \in C^0[0, T] \cap C^1(0, T]$ *be a solution of* (5.3), (5.4). *If the following conditions are satisfied:*

1° *the matrix* $A(t, v)$ *is of positive type for* $(t, v) \in \Omega = [0, T] \times R^N$,

2° $\sum_k A_{jk}(t, v) = 1$ *for* $k \in I_S = \{i\colon e_i \in S\}$, $(t, v) \in \Omega$,

3° $D^k(t, v) \leqslant 0$ *for* $k \notin I_S$, $(t, v) \in \Omega$,

then

$$\|v\| \leqslant \max\{|c|_\infty, \|P_S b\|; |P_S v(T)|_\infty\}.$$

Proof. Let $t_1, t_2, \ldots, t_N$ be points of the interval $[0, T]$ satisfying the relations

$$v^i(t_i) = \max_{0 \leqslant t \leqslant T} v^i(t)$$

and let

$$v^p(t_p) = \max_{1 \leqslant l \leqslant N} v^i(t_l).$$

Now we show that

$$(5.6) \qquad v^p(t_p) \leqslant \max\{|c|_\infty, \|P_S b\|, |P_S v(T)|_\infty\}.$$

If $t_p = 0$ or $v^p(t_p) \leqslant 0$, then inequality (5.6) is satisfied. Thus, let $0 < t_p \leqslant T$ and $v^p(t_p) > 0$. From assumption 1° and the definition of the point t_p it follows that

$$(5.7)\quad v^p(t_p)\sum_{j=1}^{N} A_{pj}(tp, v(t_p)) \leqslant \sum_{j=1}^{N} A_{pj}(t_p, v(t_p))\, v^j(t_p)$$

$$= \begin{cases} D^p(t_p, v(t_p))\dfrac{dv^p(t_p)}{dt} & \text{for } p\notin I_S, \\ -b^p(t_p)+D^p(t_p, v(t_p))\dfrac{dv^p(t_p)}{dt} & \text{for } p\in I_S, \end{cases}$$

$$\leqslant \begin{cases} 0 & \text{for } p\notin I_S, \\ -b^p(t_p) & \text{for } p\in I_S,\ t_p < T. \end{cases}$$

Hence we have inequality (5.6) or $p\notin I_S$ and $p\notin J\big(A(tp, v(tp))\big)$. However, for $p\notin J\big(A(t_p, v(t_p))\big)$ there exists $j\in J\big(A(t_p, v(t_p))\big)$, (from the assumption 1°) and a sequence of non-zero elements of A of the form

$$A_{pk_1}(t_p, v(t_p)),\ A_{k_1k_2}(t_p, v(t_p)), \ldots, A_{k_rj}(t_p, v(t_p)).$$

Also, for $p\in I_S$ from (5.7) if follows that $v^p(t_p) = v^{k_1}(t_{k_1})$ and we can take $t_{k_1} = t_p$. In this way we obtain

$$(5.8)\qquad v^p(t_p) = v^{k_1}(t_p) = v^{k_2}(t_p) = \quad = v^{k_r}(t_p) = v^j(t_p).$$

Thus, from the assumption 2^0 and inequality (5.7) it follows that

$$v^p(t_p) \leqslant \max\{|c|_\infty, \|P_S b\|, |P_S v(T)|_\infty\}.$$

Since the function $w(t) = -v(t)$ satisfies the following Cauchy problem:

$$D(t, -w)\frac{dw}{dt} = A(t, -w)w + P_S(-b(t)),$$

$$w(0) = -c,$$

therefore by (5.7) we find

$$-v^r(t_r) = \min\min v^i(t) = \max\max w^i(t) \leqslant \max\{|c|_\infty, \|P_S b\|, (P_S v(T))\}$$

and this ends the proof.

An estimation of a solution v in space R^N by all components of the vectors $\vec{b}$, $\vec{c}$ will be called an *extension of the maximum principle.* This principle we formulate as follows:

DEFINITION 5.2. An operator P of form (5.1) satisfies a maximum principle as an *a priori* estimation with respect to a subspace S of the space R^N and with respect to a constant K (in symbols $P\in\mathfrak{M}_S(K)$) if and only if for any b, c every continuous solution $v(t)$ of (5.1), (5.2) satisfies the following inequality:

$$\|v\| \leqslant \max\{|P_S c|_\infty, \|P_S v\|, \|P_S b\|\} + K \max\{|(E-P_S)c|_\infty, \|(E-P_S)b\|\}, \tag{5.9}$$

where E is the unit matrix.

THEOREM 5.2. *Let* $v \in C^1(0, T] \wedge C^0[0, T]$ *be a solution of* (5.1), (5.2) *and let*

$$B(t_1, t_2, \ldots, t_N) = \big(A_{ij}(t_i, v(t_i))\big)$$

for any $t_1, t_2, \ldots, t_N \in [0, T]$. *If the matrix* $B(t_1, t_2, \ldots, t_N)$ *is monotone* (cf. [18]) *for any* $t_1, t_2, \ldots, t_N$ *and if the following conditions hold*:

1^{oo} $B(t_1, t_2, \ldots, t_N)\xi \geqslant P_S\xi$, $\xi = (1, 1, \ldots, 1)^T$,

2^{oo} *there exists a vector* $\alpha \in R^N$ *such that*

$$B(t_1, t_2, \ldots, t_N)\alpha \geqslant (E-P_S)\xi,$$

3^{oo} $D^i(t, v) \leqslant 0$, $i = 1, 2, \ldots, N$,

4^{oo} $|A(0, c)c|_\infty \leqslant |c|_\infty$,

then

$$\|v\| \leqslant \max\{|P_S A(0, c)c|_\infty, \|P_S b\|\} + \|\alpha\| \max\{|(E-P_S)A(0, c)c|_\infty, \|(E-P_S)b\|\}$$

and by condition 4^{oo} *we have* $P \in \mathfrak{M}_S(\|\alpha\|)$.

Proof. Let $g = F_1\xi + F_2\alpha$, where $F_1 = \max\{|P_S A(0, c)c|_\infty, \|P_S b\|\}$, $F_2 = \max\{|(E-P_S)A(0, c)c|_\infty, \|(E-P_S)b\|\}$. From equation (5.1) it follows that

$$A(g \mp v) = Ag \mp D\frac{dv}{dt} \pm b = F_1 A\xi + F_2 A\alpha \pm D\frac{dv}{dt} \pm b. \tag{5.10}$$

Let $t_1^\mp, t_2^\mp, \ldots, t_N^\mp$ be points of the interval $[0, T]$ satisfying the following relations:

$$v^i(t_i^+) = \max_t v^i(t),$$

$$v^i(t_i^-) = \min_t v^i(t),$$

and let $w^\pm = \big(v^1(t_1^\pm), v^2(t_2^\pm), \ldots, v^N(t_N^\pm)\big)$.

If $(t_i^\mp) \in (0, T]$, then from (5.10) we have

$$e_i B^\mp (g \mp w^\mp) \geqslant F_1 e_i P_S \xi + F_2 e_i (E-P_S)\xi \mp D^i \frac{dv^i(t_i^\mp)}{dt} \pm b^i(t_i^\mp) \geqslant 0.$$

If $t_i^\mp = 0$, then from 1^{oo} and 2^{oo} we have

$$e_i B^\mp (g \mp w^\mp) \geqslant F_1 e_1 P_S \xi + F_2 e_i (E-P_S)\xi \mp e_i A(0, c)c \geqslant 0.$$

Since the matrix B is monotone, we have

$$g^i \mp v^i(t_i^{\mp}) \geqslant 0, \quad i = 1, 2, \ldots, N.$$

Hence we have $\|v\| \leqslant \|g\|$. On the other hand, $\|g\| \leqslant F_1 + \|\alpha\| F_2$. Thus we obtain the following estimation:

$$\|v\| \leqslant \max\{|P_S A(0, c)c|_\infty, \|P_S b\|\} + \\ + \|\alpha\| \max\{|(E-P_S)A(0, c)c|_\infty, \|(E-P_S)b\|\}$$

and this ends the proof.

§ 6. The method of lines for nonlinear parabolic equations which can be degenerated to elliptic equations

Using the maximum principle for systems of ordinary differential equations, we prove the convergence of the method of lines for nonlinear parabolic equations which can be degenerated to elliptic equations.

Let us consider the following Fourier problem, (cf. [9]):

$$(6.1) \qquad k(t, x)\frac{\partial u}{\partial t} = G(t, x, u, u_x, u_{xx}), \quad x \in D^p,\ 0 < t \leqslant T,$$

$$(6.2) \qquad u(0, x) = \psi(x), \quad x \in D^p,$$

$$(6.3) \qquad u(t, x) = 0, \quad x \in \partial D^p,\ 0 \leqslant t \leqslant T,$$

where $D^p = \{x = (x_1, x_2, \ldots, x_p) : 0 < x_i < 1,\ i = 1, 2, \ldots, p\}$, ∂D^p – the boundary of D^p, $u = u(t, x)$, $u_x = (u_{x_1}, u_{x_2}, \ldots, u_{x_p})$, $u_{xx} = (u_{x_1x_1}, u_{x_2x_2}, \ldots \ldots, u_{x_px_p})$.

Assumptions. $1^{\circ\circ}$ The function $G(t, x, q, r, s)$, $r = (r_1, r_2, \ldots, r_p)$, $s = (s_1, s_2, \ldots, s_p)$ is defined in $[0, T] \times D^p \times R^{2p+1}$ and continuously defferentiable with respect to variables $q, r_1, \ldots, r_p, s_1, \ldots, s_p$,

$$\frac{\partial G(t, x, q, r, s)}{\partial q} \leqslant 0 \quad \text{for} \quad (t, x, q, r, s) \in [0, T] \times D^p \times R^{2p+1}$$

The functions $k(t, x) \geqslant 0$, $\psi(x)$ are given in $[0, T] \times D^p$ and D^p, respectively.

$2^{\circ\circ}$ There exist functions $\mu_i(q, r, s) > 0$, $L_i(q, r, s)$ and constants $K_i > 0$ $(i = 1, 2, \ldots, p)$ such that

$$\frac{\partial G(t, x, q, r, s)}{\partial s_i} \geqslant \mu_i(q, r, s),$$

$$\left|\frac{\partial G(t, x, q, r, s)}{\partial r_i}\right| \leqslant L_i(q, r, s),$$

$$L_i(q, r, s) \leqslant K_i \mu_i(q, r, s),$$

for $(t, x, q, r, s) \in [0, T] \times D^p \times R^{2p+1}$, $i = 1, 2, \ldots, p$.

An approximation of the Fourier problem. We approximate the (6.1), (6.2), (6.3) by the following system of ordinary equations:

$$-\frac{dv^{m(i)}}{dt} = v^{m(i)}, \quad ih \in \partial\Omega_h,\ 0 < t \leqslant T,$$

(6.4) $$k(t, ih)\frac{dv^{m(i)}}{dt} = G(t, ih, v^i, \bar{\Delta} v^i, \Delta \nabla v^i), \quad ih \in \Omega_h,\ 0 < t < T,$$

$$v^{m(i)}(0) = \psi(ih), \quad ih \in \bar{\Omega}_h,$$

where $\vec{v} = (v^1, v^2, \ldots, v^N)^T$, $v^{m(i)} = v(t, ih)$, the index $m(i)$ follows from the lexicographical ordering of points of the net $\bar{\Omega}_h$ and N is the number of points of $\bar{\Omega}_h$.

Let $u(t, x)$ be a solution of (6.1), (6.2), (6.3) four times continuously differentiable with respect to variables $x_1, x_2, \ldots, x_p$ and twice continuously differentiable with respect to the variable t and let $\vec{v}(t) = (v^1(t), v^2(t), \ldots, v^N)^T$ be a continuous solution in $[0, T]$ of (6.4).

THEOREM 6.1. *If $h_{\max} \leqslant \min(1/K_i)$, then there exists a constant $K_0 > 0$ such that*

$$|u(t, ih) - v^{m(i)}(t) \leqslant K_0 \exp(\gamma) h_{\max}^2, \quad ih \in \Omega_h,\ t \in [0, T],$$

where a constant K_0 is independent of h,

$$\gamma = \min_i (L_i + \sqrt{\mu_i^2 + 2\mu_i})\,\mu_i,$$

$$h_{\max} = \max_i h_i \leqslant \min_i \frac{1}{K_i}.$$

Furthermore, if there exists a constant $\mu_0 > 0$ such that $\mu_i(q, r, s) \geqslant \mu_0$, $(q, r, s) \in R^{2p+1}$ $(i = 1, 2, \ldots, p)$, then $v^{m(i)}(t) \to u(t, ih)$ as fast as $h_{\max}^2 \to 0$.

Proof. From the regularity of $u(t, x)$ and $G(t, x, q, r, s)$ it follows that

$$-\frac{du^{m(i)}}{dt} = u^{m(i)}, \quad ih \in \partial\Omega_h,\ t \in (0, T],$$

(6.5) $$k(t, ih)\frac{du^{m(i)}}{dt} = G(t, ih, u^{m(i)}, \bar{\Delta} u^{m(i)}, \Delta \nabla u^{m(i)}) + w^{m(i)}(h), \quad ih \in \Omega_h,\ t \in (0, T],$$

$$u^{m(i)}(0) = 0,$$

where $u^{m(i)} = u(t, ih)$, $\vec{w}(h) = (w^1(h), w^2(h), \ldots, w^N(h))^T \to 0$ as fast as $h_{\max}^2 \to 0$, $N = (N_1+1)(N_2+1), \ldots, (N_p+1)$.

From (6.4) and (6.5) it follows that

$$\frac{-d\,(u^{m(i)}-v^{m(i)})}{dt} = u^{m(i)}-v^{m(i)}, \quad ih\in\partial\Omega_h,\ t\in(0,T],$$

$$(6.6)\quad k(t,ih)\frac{d\,(u^{m(i)}-v^{m(i)})}{dt} = G(t,ih,u^{m(i)},\bar{\Delta}u^{m(i)},\Delta\nabla u^{m(i)})-$$

$$-G(t,ih,v^{m(i)},\bar{\Delta}v^{m(i)},\Delta\nabla v^{m(i)})+w^{m(i)}(h),$$

$$ih\in\Omega_h,\ t\in(0,T],$$

$$u^{m(i)}(0)-v^{m(i)}(0) = 0, \quad ih\in\bar{\Omega}_h.$$

Substituting in (6.6)

$$G(t,ih,u^{m(i)},\bar{\Delta}u^{m(i)},\Delta\nabla u^{m(i)})-G(t,ih,v^{m(i)},\bar{\Delta}v^{m(i)},\Delta\nabla v^{m(i)})$$

$$= \sum_{m(i)}\frac{\partial G}{\partial u^{m(i)}}(u^{m(i)}-v^{m(i)}), \quad ih\in\Omega_h,\ t\in(0,T],$$

we can write system (6.6) in the following form:

$$D\frac{d\vec{z}}{dt} = A(t,\vec{z})\vec{z}+\vec{w}(h), \quad t\in(0,T],$$

$$\vec{z}(0) = 0,$$

where $\vec{z}(t) = \big(z^1(t), z^2(t), \ldots, z^N(t)\big)^T$, $z^{m(i)}(t) = u^{m(i)}(t)-v^{m(i)}(t)$, D = diagonal $(D_1, D_2, \ldots, D_N)$,

$$D_{m(i)}(t) = \begin{cases} -1 & \text{for} \quad ih\in\partial\Omega_h,\ t\in[0,T], \\ -k(t,ih) & \text{for} \quad ih\in\Omega_h,\ t\in[0,T], \end{cases}$$

a matrix $A(t,\vec{z}) = \big(A_{jk}(t,\vec{z})\big)$,

$$A_{m(i)k} = \begin{cases} 2\sum_{l=1}^{p}\frac{1}{h_l^2}\frac{\partial G^i}{\partial s_l}-\frac{\partial G^i}{\partial q}, & k = m(i),\ ih\in\Omega_h, \\ -\frac{1}{h_l^2}\frac{\partial G^i}{\partial s_l}\mp\frac{1}{2h_l}\frac{\partial G^i}{\partial r_l}, & k = m(i)\mp(N_1+1)\ldots(N_l+1), \\ & l = 1,2,\ldots,p,\ ih\in\Omega_h. \\ 1, \quad k = m(i),\ ih\in\partial\Omega_h, & \end{cases}$$

the remaining elements of A are equal to zero.

Let us observe that the matrix A is of positive type and satisfies the assumptions of Theorem 5.2 for the vector $\vec{\alpha} = (\alpha_1, \alpha_2 \ldots \alpha_N)^T$,

$$\alpha_j = \begin{cases} \exp(\gamma)-(1+\gamma h_1)^j, & j = 1,2,\ldots,N_1, \\ \alpha_{j-N_1}, & j = (N_1+1),(N_1+2),\ldots,N, \end{cases}$$

(cf. the prove of (3.10)).

Since $\vec{z}(0) = 0$, from Theorem 5.2 it follows that

$$\|\vec{z}\| \leqslant K_0 \exp(\gamma) h_{\max}^2 .$$

Hence for $\mu_i(q, r, s) \geqslant \mu_0$, $(i = 1, 2, \ldots, p)$, $(q, r, s) \in R^{2p+1}$ we have $\vec{z}(t) \to 0$ as fast as $h_{\max}^2 \to 0$. Thus the theorem is proved.

§ 7. A geometrical interpretation of a maximum principle for a system of difference equations

1. Let $\Omega \subset E^n$ denote a domain consisting of a finite number of parallepipeds with edges parallel to the hyperplanes $x_s = 0$, $s = 1, 2, \ldots, n$. Let us introduce the net $\Omega_h = \{ih = (i_1 h_1, i_2 h_2, \ldots, i_n h_n) \in \bar{\Omega}\}$, where $i = (i_1, i_2, \ldots, i_n)$ is a set of integers, $h = (h_1 h_2, \ldots, h_n)$, $h_s > 0$, $s = 1, 2, \ldots, n$, $\bar{\Omega} = \partial\Omega \cup \Omega$. $\partial\Omega$ – denotes the boundary of the domain Ω. We shall call two points jh, kh, $j = (j_1, j_2, \ldots, j_n)$, $k = (k_1, k_2, \ldots, k_n)$, adjacent if $\sum_{s=1}^{n} |k_s - j_s| = 1$. Then, an internal point of the net $\bar{\Omega}_h$ may by defined as a point $ih \in \Omega$ whose adjacent points all belong to $\bar{\Omega}$. The set of internal points of the net $\bar{\Omega}_h$ will be denoted by Ω_h. Let $\omega_i = (\omega_i^1, \omega_i^2, \ldots, \omega_i^p) \in E^p$ be a vector-function defined on the net $\bar{\Omega}_h$ and let $\Omega_h \subset E^p$ denote the set of values of the vector-function ω. Below, we shall consider the difference analogue of the maximum principle in the following sense:

Definition 7.1. We shall say that the vector-function ω satisfies the maximum principle if for every point $\omega_i \in \Omega_h$ – such that there exists a hyperplane l_i supporting the set Ω_h at the point ω_i – the inverse image of ω_i belongs to the boundary $\partial\Omega_h$ of the net $\bar{\Omega}_h$.

Definition 7.2. We shall say that the vector-function ω satisfies the maximum principle if the function $R = (\omega, \omega)^{1/2}$ attains its maximum on the boundary $\partial\Omega_h$ of the net $\bar{\Omega}_h$.

The above definitions are not equivalent since if the function ω satisfies the condition of Definition 7.1 it also satisfies the condition from Definition 7.2 but not conversely, in general.

In paper [7] the maximum principle in the sense of Definition 1 is studied for certain systems of differential equations of form (7.1). In papers [5], [11], [13] and [14] the maximum principle in the sense of Definition 7.2 is also considered for systems of differential equations of form (7.1). In the case $A^s(x) = a^s(x)E$, $(a^s(x) > 0, s = 1, 2, \ldots, E$ – denotes the unit matrix) in paper [10] the maximum principle in the sense of Definition 7.1 is considered for any system of difference equations of form (7.2).

2. Let us consider the following system of equations:

$$(7.1)\qquad Lu(x) \equiv \sum_{s=1}^{n} A^s(x)\frac{\partial^2 u}{\partial x_s^2} + \sum_{s=1}^{n} B^s(x)\frac{\partial u}{\partial x_s} + C(x)u = F(x),$$

where $x = (x_1, x_2, \ldots, x_n) \in \Omega$, $u = (u_1, u_2, \ldots, u_p)^T$, $F = (F_1, F_2, \ldots, F_p)^T$, A^T – denotes the transposed matrix A, A^s, B^s, $s = 1, 2, \ldots, n$, and C are matrices of the dimension $p \times p$.

We introduce the following assumptions:

A. We assume that $A^s(x)$, $B^s(x)$, $s = 1, 2, \ldots, n$, $C(x)$ and $F(x)$ are bounded in the $\bar{\Omega}$. Furthermore, we assume that the matrix $B^s(x)$ is the product of some function $b^s(x)$ and the unit matrix E.

B. We assume that for any fixed unit vector $a = (a_1, a_2, \ldots, a_p)$ and an arbitrary vector $\eta = (\eta_1, \eta_2, \ldots, \eta_p)$ the following inequalities are satisfied:

$$aCu \geqslant 0, \quad (aA^s\eta)(a\eta) \geqslant K_s(a\eta)^2,$$

where the constants $K_s > 0$, $s = 1, 2, \ldots, n$ are independent of the vector η.

3. Now, we shall deal with the following approximation of system (7.1):

$$(7.2)\qquad L_h v_i \equiv \sum_{s=1}^{n} A_i^s \Delta_s \nabla_s v_i + \sum_{s=1}^{n} B_i^s \bar{\Delta}_s v_i + C_i v_i = F_i,$$

where $ih \in \Omega_h$, $v_i = v(ih)$, $\Delta_s v_i = (v_i^{s+1} - v_i)/h_s$, $\nabla_s v_i = (v_i - v_i^{s-1})/h_s$, $\bar{\Delta}_s = (\Delta_s + \nabla_s)/2$, $v_i^{s\mp 1} = v(i_1 h_1, i_2 h_2, \ldots, (i_s \mp 1)h_s, \ldots, i_n h_n)$,

Let v_i be a solution of system (2) for $F = 0$. We shall prove the following.

THEOREM 7.1. *If assumptions* A *and* B *are satisfied and if for some* i^0 *there exists a hyperplane* l_{i^0} *supporting the set* Ω_h *at* v_{i^0} *such that at least one of the points* $v_{i^0}^{s\mp 1}$, $s = 1, 2, \ldots, n$, *does not lie on the hyperplane* l_{i^0}, *then the inverse image of the point* v_{i^0} *belongs to the boundary* $\partial\Omega_h$ *of the net* $\bar{\Omega}_h$.

Proof. Let us suppose that the inverse image of the point v_{i^0} belongs to the net Ω_h. Suppose that the unit vector $a \perp l_{i^0}$ has inward direction to Ω_h. Multiplying both sides of the kth equation, $k = 1, 2, \ldots, p$, of system (7.2) by the kth component of the vector and taking the sum of all the equations, we obtain the following equation:

$$(7.3)\qquad aL_h v_i \equiv \sum_{s=1}^{n} aA_i^s \Delta_s \nabla_s v_i + \sum_{s=1}^{n} aB_i^s \bar{\Delta}_s v_i + aC_i v_i = 0.$$

Let us substitute

$$\Delta_s \nabla_s v_i = (\omega_i^{s+1} + \omega_i^{s-1})/h_s^2, \quad \bar{\Delta}_s v_i = (\omega_i^{s+1} - \omega_i^{s-1})/2h_s,$$

where $\omega_i^{s\mp 1} = v_i^{s\mp 1} - v_i$, in to equation (7.3)

$$(7.4) \qquad aL_h v_i \equiv \sum_{s=1}^{n} \frac{1}{h_s^2} aA_i^s(\omega_i^{s+1}+\omega_i^{s-1}) + \sum_{s=1}^{n} \frac{1}{2h_s} aB_i^s,$$

$$(\omega_i^{s+1}-\omega_i^{s-1}) + aC_i v_i = 0.$$

Since $v_{i0} \in l_{i0}$ and at least one of the points $v_{i0}^{s\mp 1}$, $s = 1, 2, \ldots, n$, does not lie on the hyperplane l_{i0}, we have $a\omega_{i0}^{s\mp 1} \geqslant 0$ and $\sum_{s=1}^{n} a\omega_{i0}^{s\mp 1} > 0$. From the assumptions A and B it follows that

$$aA_{i0}^s \omega_{i0}^{s\mp 1} \geqslant K_s a\omega_{i0}^{s\mp 11}, \quad aB_{i0}^s \omega_{i0}^{s\mp 1} = b_{i0}^s a\omega_{i0}^{s\mp 1}, \quad aC_{i0} v_{i0} \geqslant 0.$$

Then at the point $i^0 h \in \Omega_h$ we obtain

$$(7.5) \quad aL_h v_{i0} \geqslant \sum_{s=1}^{n} \left(\frac{K_s}{h_s^2} + \frac{b_{i0}^s}{2h_s}\right) \omega_{i0}^{s+1} + \sum_{s=1}^{n} \left(\frac{K_s}{h_s^2} - \frac{b_{i0}^s}{2h_s}\right) \omega_{i0}^{s-1} + aC_{i0} v_{i0}.$$

For sufficiently small h_s inequality (7.5) implies the inequality $al_h v_i > 0$, which contradicts the assumption $L_h v_i = 0$. Then $i^0 h \notin \Omega_h$ and $i^0 h \in \partial\Omega_h$, which ends the proof.

Theorem 1 implies the following

THEOREM 7.2. *If the matrix C is non-positive and the matrices A^s, B^s, $s = 1, 2, \ldots, n$, satisfy the assumptions* A *and* B, *then the function $R = (v, v)^{1/2}$ attains its maximum on the boundary $\partial\Omega_h$ of the net $\bar{\Omega}_h$.*

Proof. Let us suppose that v_i is not constant function and $R_{i0} = \max R_i$. Then there exists a hyperplane l_{i0} supporting the set Ω_h at the point v_{i0}. Furthermore $R_{i0} > 0$ and at least one of the points $v_{i0}^{s\mp 1}$, $s = 1, 2, \ldots, n$, does not lie on the hyperplane l_{i0}. For $a = -v_{i0}/R_{i0}$ we have

$$aC_{i0} v_{s0} = -\frac{1}{R_{i0}} v_{i0} C_{i0} v_{i0} \geqslant 0.$$

Hence the assumptions of Theorem 1 hold and therefore $i^0 h$ belongs to $\partial\Omega_h$. For v_i = constant this theorem is obvious. This ends the proof.

EXAMPLE. Let us consider the following system:

$$(7.6) \qquad A^1 \frac{\partial^2 u}{\partial x_1^2} + A^2 \frac{\partial^2 u}{\partial x_2^2} = 0,$$

where $u = (u_1, u_2)$, $A^1 = E$, $A^2 = (a_{rs})_{r,s=1,2,}$, $a_{11} = a_{22} = 0$, $a_{12} = a_{21} = 1$. It is easily seen that the vector $a = (a_1, a_2)$ has to satisfy, for every $\eta = (\eta_1, \eta_2)$, the inequalities $aA^1\eta a\eta \geqslant K_1(a\eta^2)$, $aA^2\eta a\eta \geqslant K_2(a\eta^2)$ in order to fulfil the assumption B. Since for $a_1 = a_2 = \pm 1/\sqrt{2}$

$$aA^1\eta a\eta = (a\eta^2), \quad aA^2\eta a\eta = (a\eta^2),$$

the vector $a = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ satisfies the assumption B. Let v_i be a solution of the difference equations

$$A^1 \Delta_1 \nabla_1 v_i + A^2 \Delta_2 \nabla_2 v_i = 0 \quad \text{for} \quad ih \in \Omega_h. \tag{7.7}$$

From Theorem 2 it follows that if the function $R = (v, v)^{1/2}$ attains its maximum equal to R_{i0} in the direction of the vector $a = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, then $i_0 h$ belongs to $\partial\Omega_h$.

§ 8. A strong maximum principle for an elliptic system of nonlinear equations

1. W. J. Skorobogatko [7] proved the maximum principle for a system of two differential equations of the Monge–Ampere type. Here we extend this result and present a difference analogue of the maximum principle for the system of equations of form (8.1) (cf. [11]). We give certain sufficient conditions for the strong maximum principle. Other sufficient conditions for this principle and for non-linear of equations are presented in article by R. M. Redheffer [4] and in the monograph by W. Walter [15].

Let $D_k \subset R^k$ be a bounded domain and let

$$\begin{aligned}\Omega = \{(x, r, s_x, t_{xx});\ & x = (x_1, x_2, \ldots, x_n) \in D_n;\ r = (r_1, r_2, \ldots, r_p) \in D^p;\\ s_x &= (s_1^1, s_2^1, \ldots, s_p^1, s_1^2, s_2^2, \ldots, s_2^2, \ldots, s_p^n) \in D_{np};\\ t_{xx} &= (t_1^{11}, t_1^{12}; t_1^{1n}, t_2^{11}, t_2^{12}, \ldots, t_2^{1n}, \ldots, t_p^{1n}, \ldots, t_p^{nn}) \in D_{npn}\}.\end{aligned}$$

Let us consider the following system of equations:

$$\begin{aligned}&\sum_{i,k=1}^{n} A^{ik}(x, u, u_x, u_{xx}) \frac{\partial^2 u}{\partial x_1 \partial x_k} + \\ &\quad + \sum_{i=1}^{n} B^i(x, u, u_x, u_{xx}) \frac{\partial u}{\partial x_i} + C(x, u, u_x, u_{xx}) u \\ &\quad = \sum_{i,k=1}^{n} [F^{ik}(x, u, u_x, u_{xx}) v^{ik} + \sum_{\nu=1}^{p} C_\nu^{ik}(x, u, u_x, u_{xx}) u w_\nu^{ik}],\end{aligned} \tag{8.1}$$

where

$$u = [u_1, u_2, \ldots, u_p]^T, \quad v^{ik} = [v_1^{ik}, v_2^{ik}, \ldots, v_p^{ik}]^T,$$

$$v_l^{ik} = \sum_{m=1}^{p} u_m \left(\frac{\partial^2 u_m}{\partial x_1^2} \frac{\partial^2 u_l}{\partial x_k^2} - \frac{\partial^2 u_m}{\partial x_i \partial x_k} \frac{\partial u_l}{\partial x_i \partial x_k} \right),$$

$$w_l^{ik} = \frac{\partial^2 u_l}{\partial x_i^2} \frac{\partial^2 u_l}{\partial x_k^2} - \left(\frac{\partial^2 u_l}{\partial x_i \partial x_k} \right)^2$$

and A^{ik}, B^i, C, F^{ik}, G_ν^{ik} are matrices of the dimension $p \times p$, A^T is the transposed matrix to A.

Assumptions. 1° The coefficients of the system of equations (8.1) are continuous in Ω.

2° $A^{ik}(x, u, u_x, u_{xx}) = a_{ik}(x, u, u_x, u_{xx})E$, where E is the unit matrix, and $A = (a_{ik})$ is a matrix symmetric and positive defined in Ω.

3° The matrices F^{ik}, G^{ik}, $i, k = 1, 2, \ldots, n$, $\nu = 1, 2, \ldots, p$, satisfy the following conditions: $F^{ik}(x, u, u_x, u_{xx}) = f^{ik}(x, u, u_x, u_{xx})E$, the function $f^{ik}(x, u, u_x, u_{xx}) \leqslant 0$ for $(x, u, u_x, u_{xx}) \in \Omega$, and $G_\nu^{ik}(x, u, u_x, u_{xx})$ are asymmetric matrices $i, k = 1, 2, \ldots, n$, $\nu = 1, 2, \ldots, p$.

Let $u(x)$ be a smooth solution of system (8.1) in D_n,

$$R(x) = \left(u_1^2(x) + u_2^2(x) + \ldots + u_n^2(x)\right)^{1/2},$$

and let $e(x)$ be the unit vector in direction of the vector $u(x)$.

We introduce the following notations:

$$\mu_0 = e(x), \quad \mu_i = \frac{\partial e}{\partial x_i}, \quad i = 1, 2, \ldots, n,$$

$$\Phi = \sum_{i,k=0}^{n} (\bar{A}^{ik}\mu_i, \mu_k),$$

where $\bar{A}^{00} = C$, $\bar{A}^{i0} = \frac{1}{2}B^i$, $\bar{A}^{0i} = (A^{i0})^{\Phi}$, $\bar{A}^{ik} = -A^{ik}$, $i, k = 1, 2, \ldots, n$. Now, we prove the following

THEOREM 8.1. *If the function $R(x)$ attains its maximum at the point $x^0 \in D_n$ and in a neighbourhood K of the point x^0 the function Φ is not greater than* 0, *then $R(x) = R(x^0)$ for every $x \in \bar{D}_n$.*

Proof. Substituting $u(x) = e(x)R(x)$ for $R(x) \neq 0$ in (8.1), we have

$$\text{(8.2)} \quad \sum_{i,k=1}^{n} a_{ik} e \frac{\partial^2 R}{\partial x_i \partial x_k} + \sum_{i=1}^{n} \left[B^i e + 2\sum_{k=1}^{n} a_{ik} \frac{\partial e}{\partial x_k}\right] + \frac{\partial R}{\partial x_i} + \left[Ce + \sum_{i=1}^{n} B^i \frac{\partial e}{\partial x_i} + \sum_{i,k=1}^{n} a_{ik} \frac{\partial^2 e}{\partial x_i \partial x_i}\right] R = \sum_{i,k=1}^{n} \left[F^{ik} v^{ik} + R \sum_{\nu=1}^{p} G_\nu^{ik} e w_\nu^{ik}\right].$$

Multiplying both sides of the lth equation, $l = 1, 2, \ldots, p$, of system (8.2) by the lth component of the vector $e(x)$ and taking the sum of all the equations, we obtain the following equation:

$$(8.3)\quad \sum_{i,k=1}^{n} a_{ik}\frac{\partial^2 R}{\partial x_i \partial x_k}+\sum_{i=1}^{n}(B^i e, e)\frac{\partial R}{\partial x_i}+$$
$$+\left[(Ce,e)+\sum_{i=1}^{n}\left(B^i\frac{\partial e}{\partial x_i}, e\right)+\sum_{i,k=1}^{n} a_{ik}\left(e, \frac{\partial^2 e}{\partial x_i \partial x_k}\right)\right]R$$
$$=\frac{1}{R}\sum_{i,k=1}^{n}\left[(F^{ik}u, v^{ik})+\sum_{\nu=1}^{p}(G_\nu^{ik}u, u)w_\nu^{ik}\right].$$

Now, we use the following relations:

$$(8.4)\quad \left(e, \frac{\partial^2 e}{\partial x_i \partial x_i}\right) = -\left(\frac{\partial e}{\partial x_i}, \frac{\partial e}{\partial x_k}\right) \quad \text{for} \quad i, k = 1, 2, \ldots, n.$$

In the neighbourhood K of the point x^0, we have

$$(8.5)\quad \sum_{i,k=1}^{n}\frac{\partial^2 R}{\partial x_i \partial x_k}\lambda_i\lambda_k \leqslant 0, \quad (\lambda_1, \lambda_2, \ldots, \lambda_n)\in R^n.$$

Inequality (8.5) implies the following inequality:

$$(8.6)\quad \sum_{i,k=1}^{n}\sum_{s=1}^{p} u_s\left(\frac{\partial^2 u_s}{\partial x_i \partial x_k}\right)\lambda_i\lambda_k \leqslant 0,$$

for $x\in K$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n)\in R^n$. From inequality (8.6) we have

$$(8.7)\quad \begin{vmatrix} \sum_{s=1}^{p} u_s\frac{\partial^2 u_s}{\partial x_i^2} & \sum_{s=1}^{p} u_s\frac{\partial^2 u_s}{\partial x_i \partial x_k} \\ \sum_{s=1}^{p} u_s\frac{\partial^2 u_s}{\partial x_i \partial x_k} & \sum_{s=1}^{p} u_s\frac{\partial^2 u_s}{\partial x_k^2} \end{vmatrix} \leqslant 0,$$

for $x\in K$, $i, k = 1, 2, \ldots, n$. From assumption 3° and inequality (8.7) it follow that the right side of equation (8.3) is not negative. So, we have

$$(8.8)\quad \sum_{i,k=1}^{n} a_{ik}\frac{\partial^2 R}{\partial x_i \partial x_k}+\sum_{i=1}^{n} b^i\frac{\partial R}{\partial x_i}+\Phi R \geqslant 0,$$

for $x\in K$, where $b^i = (B^i e, e)$. Let D_{max} be a set of points $x\in \bar{D}_n$ in whic the function $R(x)$ attains its maximum $R(x^0)\neq 0$. The set D_{max} is closec Let $\bar{x}\in S_{max}$, where S_{max} denotes the boundary of D_{max}. Let us take $\varepsilon >$ so small that in the sphere $K(\bar{x}, \varepsilon)\subset D_n$ the assumptions of the theorei are satisfied. Then, inequality (8.8) implies the following inequality:

$$(8.9)\quad \sum_{i,k=1}^{n} a_{ik}\frac{\partial^2 R}{\partial x_i \partial x_i}+\sum_{i=1}^{n} b^i\frac{\partial R}{\partial x_i} \geqslant 0, \quad x\in K(\bar{x}, \varepsilon).$$

So, the function $R(x)$ satisfies the assumptions of the strong maximum principle (cf. E. Hopf [2]) in $K(\bar{x}, \varepsilon)$ and $R(x) = R(x^0)$ for $x \in K(\bar{x}, \varepsilon)$. Thus, we conclude that $D_{\max} = \bar{D}_n$. For $R(x^0) = 0$, we have $R(x) = 0$ in $\bar{D}_n$. The end of the proof.

Remark 8.1. The condition $\Phi \leqslant 0$ (cf. [11]) is satisfied if $(C\mathfrak{n}, u) \geqslant -\mu(u, u)$ for $(x, u, u_x, u_{xx}) \in \Omega$, where μ is a sufficiently large number.

2. *A difference analogue of the maximum principle.* Let $D_n \subset R^n$ be a domain consisting of a finite number of parallelepipeds with edges parallel to the hyperplanes of the coordinate system. Below, we consider a difference scheme which approximates the following system of equations:

$$(8.10)\qquad \sum_{s=1}^{n} a^s(x, u, u_x, u_{xx})\frac{\partial^2 u}{\partial x_s^2} + \sum_{s=1}^{n} B^s(x, u, u_x, u_{xx})\frac{\partial u}{\partial x_s} + C(x, u, u_x, u_{xx})u = \sum_{s,l=1}^{n}\sum_{\nu=1}^{p} G_\nu^{sl}(x, u, u_x, u_{xx})uw_\nu^{sl}.$$

We assume that the coefficients of system (8.10) satisfy the conditions 1°, 2°, 3° Let $ih = (i_1 h_1, i_2 h_2, \ldots, i_n h_n)$, where $i = (i_1, i_2, \ldots, i_n)$ is a set of integers, $h = (h_1, h_2, \ldots, h_n)$, $h_s > 0$, $s = 1, 2, \ldots, n$. We introduce $\bar{D}_n^h = \{ih \in \bar{D}_n\}$ ($\bar{D}_n$ denotes the closure of the domain D_n). We shall call two points jh, kh, $j = (j_1, j_2, \ldots, j_n)$, $k = (k_1, k_2, \ldots, k_n)$, adjacent if $\sum_{s=1}^{n} |j_s - k_s| = 1$. Then an internal point of the net $\bar{D}_n^h$ may be defined as a point $ih \in D_n$ whose adjacent points all belong to $\bar{D}_n$. The set of internal points of the net $\bar{D}_n^h$, we denote by D_n^h. The set $S_n^h = D_n^h - D_n^h$ is the boundary of the net $\bar{D}_n^h$. The value of the function v at the point ih is denoted by v_i. The system of equations (8.10) is approximated by the following difference scheme

$$(8.11)\qquad \sum_{s=1}^{n} a_i^s \nabla_s \Delta_s v_i + \sum_{s=1}^{n} B_i^s \bar{\Delta}_s v_i + C_i v_i = \sum_{l,s=1}^{n}\sum_{\nu=1}^{p} G_{\nu i}^{sl} v_i \delta_s w_{\nu i}^{sl},$$

where $ih \in D_n^h$,

$$\Delta_s v_i = \frac{v_i^{s+1} - v_i}{h_s}, \qquad \nabla_s v_i = \frac{v_i - v_i^{s-1}}{h_s},$$

$$\bar{\Delta}_s v_i = \tfrac{1}{2}(\Delta_s + \nabla_s)v_i, \qquad \delta_s w_{\nu i}^{sl} = \nabla_s \Delta_s w_{\nu i}^{sl} \nabla_l \Delta_l w_{\nu i}^{sl} - (\nabla_s \Delta_l w_i^{sl})^2,$$

$$v_i^{s\mp 1} = v(i_1 h_1, i_2 h_2, \ldots, (i_s \mp 1)h_s, \ldots, i_n h_n).$$

It is convenient to introduce the following denotations:

$$A_{i+}^{00} = A_{i-}^{00} = \tfrac{1}{2}C_i, \quad A_{i+}^{k0} = A_{i-}^{k0} = \tfrac{1}{4}B_i, \quad A_{i+}^{0k} = A_{i-}^{0k} = \tfrac{1}{4}(B^k)^T,$$

$$A_{i+}^{kl} = A_{i-}^{kl} = \tfrac{1}{2}a_i^k E\delta_{kl}, \quad (\delta_{lk} = 0 \text{ for } l \neq k \text{ and } \delta_{kl} = 1 \text{ for } l = k).$$

$$\mu_{i+}^0 = \mu_{i-}^0 = e_i \text{ for } R_i \neq 0, \quad \mu_{i+}^0 = \mu_{i-}^0 = 0 \text{ for } R_i = 0,$$

$$\mu_{i+}^s = \Delta_s e_i, \quad \mu_{i-}^s = \nabla_s e_i, \; s = 1, 2, \ldots, n.$$

$$\Phi_i^+ = \sum_{l,k=0}^{n} (A_{i+}^{lk} \mu_{i+}^l, \mu_{i+}^k), \; \Phi_i^- = \sum_{l,k=0}^{n} (A_i^{lk} \mu_{i-}^l, \mu_{i-}^k).$$

Now, we prove the following

THEOREM 8.2. *If the function R_i attains its maximum at a point $i^0 h \in D_n^h$ and $\Phi_{i^0}^+ + \Phi_{i^0}^- \leqslant 0$, then $R_i = \text{const}$ in $\bar{D}_n^h$.*

Proof. Let us perform the following substitutions into system (8.11):

$$\bar{\Delta}_s v_i = \bar{\Delta}_s(e_i R_i) = \tfrac{1}{2}(e_i^{s+1} \Delta_s R_i + R_i \Delta_s e_i + e_i^{s-1} \nabla_s R_i + R_i \nabla_s e_i),$$

$$\nabla_s \Delta_s v_i = \nabla_s \Delta_s (e_i R_i) = e_i \nabla_s \Delta_s R_i + \Delta_s e_i \Delta_s R_i + \nabla_s e_i \nabla_s R_i + R_i \nabla_s \Delta_s e_i.$$

$$(8.12) \quad \sum_{s=1}^{n} e_i a_i^s \nabla_s \Delta_s R_i +$$

$$+ \sum_{s=1}^{n} [a_i^s(\Delta_s e_i \Delta_s R_i + \nabla_s e_i \nabla_s R_i + \tfrac{1}{2} B_i^s (e_i^{s+1} \Delta_s R_i + e_i^{s-1} \nabla_s R_i)] +$$

$$+ \Big[C_i e_i + \tfrac{1}{2} \sum_{s=1}^{n} B_i^s (\Delta_s e_i + \nabla_s e_i) + \sum_{s=1}^{n} a_i^s \nabla_s \Delta_s e_i\Big] R_i = \sum_{l,s=1}^{n} \sum_{\nu=1}^{p} G_{\nu i}^{sl} v_i \delta_s w_{\nu i}^{sl}.$$

Multiplying both sides of the lth equation, $l = 1, 2, \ldots, p$ of system (8.12) by the lth component of the vector e_i and taking the sum of all the equations, we obtain the following equation:

$$(8.13) \quad \sum_{s=1}^{n} a_i^s \nabla_s \Delta_s R_i + \sum_{s=1}^{n} b_i^s \Delta_s R_i + \sum_{s=1}^{n} \bar{b}_i^s \nabla_s R_i +$$

$$+ \Big[(C_i e_i, e_i) + \tfrac{1}{2} \sum_{s=1}^{n} (B_i^s \Delta_s e_i, e_i) + \tfrac{1}{2} \sum_{s=1}^{n} (B_i^s \nabla_s e_i, e_i) + \sum_{s=1}^{n} a_i^s (e_i, \nabla_s \Delta_s e_i)\Big] R_i = 0,$$

where

$$b_i^s = a_i^s(e_i, \Delta_s e_i) + \tfrac{1}{2}(B_i^s e_i^{s+1}, e_i),$$

$$\bar{b}_i^s = a_i^s(e_i, \nabla_s e_i) + \tfrac{1}{2}(B_i^s e_i^{s-1}, e_i),$$

for $ih \in D_n^h$, $s = 1, 2, \ldots, n$. Using the identity $(e_i, e_i) = 1$, we obtain the following equation:

$$(8.14)\quad \sum_{s=1}^{n} a_i^s \nabla_s \Delta_s R_i + \sum_{s=1}^{n} b_i^s \Delta_s R_i + \sum_{s=1}^{n} \bar{b}_i^s \nabla_s R_i + \\ +\Big[(C_i e_i, e_i) + \tfrac{1}{2} \sum_{s=1}^{n} (B_i^s \Delta_s e_i, e_i) + \tfrac{1}{2} \sum_{s=1}^{n} (B_i^s \nabla_s e_i, e_i) - \\ - \tfrac{1}{2} \sum_{s=1}^{n} a_i^s (\Delta_s e_i, \Delta_s e_i) - \tfrac{1}{2} \sum_{s=1}^{n} a_i^s (\nabla_s e_i, \nabla_s e_i)\Big] R_i = 0.$$

So, we have

$$(8.15)\quad L_h R_i \equiv \sum_{s=1}^{n} \left(\frac{a_i^s}{h_s^2} + \frac{b_i^s}{h_s}\right) R_i^{s+1} + \sum_{s=1}^{n} \left(\frac{a_i^s}{h_s^2} - \frac{\bar{b}_i^s}{h_s}\right) R_i^{s-1} + \\ + \sum_{s=1}^{n} \left(\frac{\bar{b}_i^s}{h_s} - \frac{b_i^s}{h_s} - \frac{2a_i^s}{h_s^2}\right) R_i + (\Phi_i^+ + \Phi_i^-) R_i = 0, \qquad ih \in D_n^h.$$

Let

$$D_{\max}^h = \{ih \in D_n^h : \max_{jh \in D_n^h} R_j = R_j\}$$

and let $D_{\max}^h \neq \bar{D}_n^h$; we have $R \neq \text{const}$. Then there exists such a point $jh \in D_{\max}^h$ that, at least at one point $kh \in D_n^h$ adjacent to point jh, $R_k < R_j$. Now, we introduce the following sets:

$$S = \{1, 2, \ldots, n\},$$

$$S_i^+ = \left\{s \in S : \frac{a_i^s}{h_s^2} + \frac{b_i^s}{h_s} > 0\right\},$$

$$S_i^- = \left\{s \in S : \frac{a_i^s}{h_s^2} - \frac{\bar{b}_i^s}{h_s} > 0\right\}.$$

From equation (8.15) we obtain the following estimation:

$$(8.16)\quad L_h R_j < R_j \Bigg[\sum_{s \in S_j^+} \left(\frac{a_j^s}{h_s^2} + \frac{b_j^s}{h_s}\right) + \sum_{s \in S_j^-} \left(\frac{a_j^s}{h_s^2} - \frac{\bar{b}_j^s}{h_s}\right) + \\ + \sum_{s \in S} \left(\frac{\bar{b}_j^s}{h_s} - \frac{b_j^s}{h_s} - \frac{2a_j^s}{h_s^2}\right) + \Phi_j^+ + \Phi_j^- \Bigg] \\ = R_j \Bigg[\sum_{s \in S - S_j^-} \left(\frac{\bar{b}_j^s}{h_s} - \frac{a_j^s}{h_s^2}\right) + \sum_{s \in S - S_j^+} \left(\frac{b_j^s}{h_s} - \frac{a_j^s}{h_s^2}\right) + (C_j e_j, e_j) +$$

$$+\tfrac{1}{2}\sum_{s\in S}(B_j^s\Delta_s e_j, e_j)+\tfrac{1}{2}\sum_{s\in S}(B_j^s\nabla_s e_j, e_j)-$$

$$\left.-\tfrac{1}{2}\sum_{s\in S}a_j^s(\Delta_s e_j, \Delta_s e_j)-\tfrac{1}{2}\sum_{s\in S}a_j^s(\nabla_s e_j, \nabla_s e_j)\right]$$

$$= R_j\left\{\sum_{s\in S-S_j^+}\left[-\frac{a_j^s}{h_s}(e_j, \Delta_s e_j)-\frac{1}{2h_s}(B_j^s e_j^{s+1}, e_j)-\frac{a_j^s}{h_s^2}+\right.\right.$$

$$\left.+\tfrac{1}{2}(B_j^s\Delta_s e_j, e_j)-\tfrac{1}{2}a_j^s(\Delta_s e_j, \Delta_s e_j)\right]+$$

$$+\sum_{s\in S-S_j^-}\left[\frac{a_j}{h_s^2}(e_j, \nabla_s e_j)+\frac{1}{2h_s}(B_j^s e_j^{s-1}, e_j)-\frac{a_j^s}{h_s^2}+\right.$$

$$\left.+\tfrac{1}{2}(B_j^s\nabla_s e_j, e_j)-\tfrac{1}{2}a_j^s(\nabla_s e_j, \nabla_s e_j)\right]+(C_j e_j, e_j)+$$

$$+\tfrac{1}{2}\sum_{s\in S_j^+}[(B_j^s\Delta_s e_j, e_j)-a_j^s(\Delta_s e_j, \Delta_s e_j)]+$$

$$\left.+\tfrac{1}{2}\sum_{s\in S_j^-}[(B_j^s\nabla_s e_j, e_j)-a_j^s(\nabla_s e_j, \nabla_s e_j)]\right\}$$

$$\leqslant R_j\left\{\sum_{s\in S-S_j^+}\left[\frac{a_j}{h_s}(\Delta_s e_j, \Delta_s e_j)^{1/2}+\frac{K_s}{h_s}-\frac{a_j^s}{h_s^2}+\right.\right.$$

$$\left.+K_s(\Delta_s e_j, \Delta_s e_j)^{1/2}-\tfrac{1}{2}a_j^s(\Delta_s e_j, \Delta_s e_j)\right]+$$

$$+\sum_{s\in S-S_j^-}\left[\frac{a_j^s}{h_s}(\nabla_s e_j, \nabla_s e_j)^{1/2}+\frac{K_s}{h_s}-\frac{a_j^s}{h_s^2}+\right.$$

$$\left.+K_s(\nabla_s e_j, \nabla_s e_j)^{1/2}-\tfrac{1}{2}a_j^s(\nabla_s e_j, \nabla_s e_j)\right]+K_0+$$

$$+\sum_{s\in S_j^+}[K_s(\Delta_s e_j, \Delta_s e_j)^{1/2}-\tfrac{1}{2}a_j^s(\Delta_s e_j, \Delta_s e_j)]+$$

$$\left.+\sum_{s\in S_j^-}[K_s(\nabla_s e_j, \nabla_s e_j)^{1/2}-\tfrac{1}{2}a_j^s(\nabla_s e_j, \nabla_s e_j)]\right\},$$

where $K_0 = n\sup_{l,k}\sup_{ih\in D_n^h}|C_i^{lk}|$, $K_s = \sup_{l,k}\sup_{ih\in D_n^h}|B_i^{slk}|$. Let us consider the following set of auxiliary functions:

$$G_i^s(t) = \left(K_s+\frac{a_i^s}{h_s}\right)t-\frac{a_i^s}{2}t^2+\frac{K_s}{h_s}-\frac{a_i^s}{h_s^2}, \qquad H_i^s(t) = K_s t-\frac{a_i^s}{2}t^2.$$

Thus, estimation (8.16) implies the inequality

$$(8.17)\qquad L_h R_j < R_j\Big[\sum_{s\in S-S_j^+} G^s_{\max} + \sum_{s\in S-S_j^-} G^s_{\max} + \sum_{s\in S_j} H^s_{\max} - \sum_{s\in S_j} H^s_{\max} + K_0\Big].$$

where

$$G^s_{\max} = \max_{-\infty<t<+\infty} G^s_j(t) = -\frac{a^s_j}{h_s^2} + \frac{K_s}{2a^s_j} + \frac{2K_s}{h_s},$$

$$H_{\max} = \max_{-\infty<t<+\infty} H^s_j(t) = \frac{K_s}{2a^s_j}.$$

For sufficiently small h and $S^+ \neq S$ or $S^- \neq S$ from (8.17) we have

$$(8.18)\qquad L_h R_j < 0.$$

On the other hand, we have the equality

$$(8.19)\qquad L_h R_j = 0,$$

which contradicts (8.18). Thus, we have $D^h_{\max} = D^h_n$ and $R_i = \text{const}$ in $\bar{D}^h_n$. For $S^+ = S^- = S$ equation (8.15) implies the inequality

$$(8.20)\qquad L_h R_j < (\Phi_j^+ + \Phi_j^-) R_j \leqslant 0,$$

which also contradicts (8.18). The end of the proof.

Remark 8.2. The condition $\Phi_i^+ + \Phi_i^- \leqslant 0$ is satisfied if $(C_i v_i, v_i) \geqslant -\mu(v_i, v_i)$ where $\mu \geqslant Kn/4a$, $K = \max K_s$, $a = \inf_s \inf_\Omega a^s(x, r, s_x t_{xx})$.

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