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On submeasures I

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Introduction

In several papers, this being the first, we want to show that most of the theory of countably additive measures, Lebesgue type integration and Riesz type representation theorems remain valid for far more general non-additive set functions, non-linear integrals and non-linear operators, respectively. A preliminary result in this direction, asserting that a wide class of subadditive operators $U: C_0(T) \to Y$, Y being a Banach space, transform weakly Cauchy sequences into convergent ones, was announced in [8].

In this paper we present a non-additive generalization of the theory of finite non-negative countably additive measures. The basic concept is that of a submeasure, introduced in Definition 1. As Examples 1, 2, 3, and 4 show, this generalization is quite natural. Examples 1 and 3 also show that there are four different classes of submeasures decreasing in generality: general, uniform, subadditive and additive.

The paper is divided into three sections. § 1 collects the basic properties of submeasures. Thus analogues of Lebesgue and Saks decompositions are valid for submeasures, see Theorems 6 and 8. Further, a non-atomic submeasure has the Darboux property, see Theorem 10. Considering submeasures in locally compact Hausdorff topological spaces, in Theorem 11 we prove that every Baire submeasure is regular. In general, every submeasure μ on a σ-ring $\mathcal{S}$ induces a complete pseudometrizable uniform space $(\mathcal{S}, \mathcal{U}_\mu)$, see Theorem 14. For a subadditive submeasure μ the uniformity $\mathcal{U}_\mu$ is generated by the pseudometric $\varrho(A, B) = \mu(A \Delta B)$, $A, B \in \mathcal{S}$. Further, if the σ-ring $\mathcal{S}$ is generated by a ring $\mathcal{R}$, then $\mathcal{R}$ is dense in $(\mathcal{S}, \mathcal{U}_\mu)$, see Theorem 15.

In § 2, Theorem 18, we give necessary and sufficient conditions for extending a submeasure from a ring to the generated σ-ring. The idea of the proof is motivated by [12]. Further, in Definition 4 we introduce for submeasures an interesting property (p), see also [8]. Whether there is a submeasure on a σ-ring which does not have the property (p) remains an open problem.

In § 3, Definition 5, we introduce the concepts of Baire and Borel subcontents in locally compact Hausdorff topological spaces, and in Theorem 23 we give necessary and sufficient conditions for extending a Baire or Borel subcontent to a regular Borel submeasure. As a corollary we find that every Baire submeasure can be uniquely extended to a regular Borel submeasure. Finally, in the Remarks after Theorems 19 and 23 we show how submeasures induce Choquet capacities, see [14].

Subadditive set functions, via outer maesures, occurred in the theory of measure from its beginning. Their absolute continuity is studied in [15] and [16]. The concept of a quasimeasure, which is equivalent to our concept of a subadditive submeasure, was introduced and investigated by V. N. Aleksjuk in [1], [2] and [3], see also [13]. These papers and part I of this paper coincide only in three theorems. Namely, for subadditive submeasures with finite variations the extension Theorem 18 and assertion a) of Theorem 1 are proved in § 1 of [3]. For general subadditive submeasures the extension theorem has recently been proved in [9]. Further, for subadditive submeasures Theorem 15 is proved in [2], Theorem 1.

Let us note that, except Examples 1, 2 and 3, we do not use results from additive measures; on the contrary, they are consequences of our general results. The proofs are motivated by the usual ones; however, for the important extension theorems in § 2 and § 3 we have had to find new approaches. In our opinion these approaches in the additive case are simpler and more direct than usual ones, based on the Caratheodory extension procedure, see [4] and [10].

§ 1. On submeasures

DEFINITION 1. Let $\mathscr{R}$ be a ring of subsets of a set T. We say that a set function $\mu\colon \mathscr{R}\to[0,+\infty)$ is a *submeasure* iff:

1) μ is monotone,

2) μ is continuous; if $A_n\in\mathscr{R}$, $n=1,2,\ldots$ and $A_n\searrow\emptyset$, then $\lim_{n\to\infty}\mu(A_n)=0$, and

3) μ is subadditively continuous; for every $A\in\mathscr{R}$ and $\varepsilon>0$ there is a $\delta>0$ such that $B\in\mathscr{R}$, $\mu(B)<\delta$ implies: a) $\mu(A\cup B)\leqslant\mu(A)+\varepsilon$, and b) $\mu(A)\leqslant\mu(A-B)+\varepsilon$.

If instead of 3) we have:

3u) μ is uniformly subadditively continuous; for every $\varepsilon>0$ there is a $\delta>0$ such that $B\in\mathscr{R}$, $\mu(B)<\delta$ implies $\mu(A\cup B)\leqslant\mu(A)+\varepsilon$ for every $A\in\mathscr{R}$,

then we say that μ is a *uniform submeasure*. Clearly a uniform submeasure is a submeasure for which the δ in condition 3) is uniform with respect to $A\in\mathscr{R}$.

If instead of 3) we have $\mu(A\cup B)\leqslant\mu(A)+\mu(B)$ for every $A, B\in\mathscr{R}$, or $\mu(A\cup B)=\mu(A)+\mu(B)$ for every $A, B\in\mathscr{R}$, $A\cap B=\emptyset$, than we say that μ is a *subadditive* or an *additive submeasure*, respectively. Obviously subadditive, and particularly additive submeasures are uniform.

The following theorem follows immediately.

THEOREM 1. *Let $\mathscr{R}$ be a ring of subsets of a set T and let $\mu: \mathscr{R} \to [0, +\infty)$ be a submeasure. Then:*

a) *μ is monotonely continuous on $\mathscr{R}$; if $A_n \in \mathscr{R}$, $n = 1, 2, \ldots$ is a monotone sequence with $\lim_{n\to\infty} A_n \in \mathscr{R}$, then $\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$,*

b) *if $\mathscr{R} = \mathscr{P}$ is a δ-ring (a ring closed with respect to the forming of countable intersections), if $A_n \in \mathscr{P}$, $n = 1, 2, \ldots$ and $\limsup_n A_n \in \mathscr{P}$, then:*

$$\mu(\liminf_n A_n) \leqslant \liminf_n \mu(A_n) \leqslant \limsup_n \mu(A_n) \leqslant \mu(\limsup_n A_n),$$

c) *if $A_n \in \mathscr{R}$, $n = 1, 2, \ldots$ is a sequence of pairwise disjoint sets and $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$, then $\lim_{n\to\infty} \mu(A_n) = 0$, and*

d) *if μ is subadditive or additive, then μ is countably subadditive, or countably additive on $\mathscr{R}$, respectively. Thus the notion of additive submeasure is equivalent to the usual notion of the countably additive finite non-negative measure.*

EXAMPLE 1. Let $\mathscr{R}$ be a ring of subsets of a set T and let $\lambda: \mathscr{R} \to [0, +\infty)$ be a countably additive measure, or a general submeasure. Further, let $F: [0, +\infty) \to [0, +\infty)$ be an increasing continuous function. Then, clearly, the set function μ on $\mathscr{R}$ defined by the equality $\mu(A) = F[\lambda(A)]$, $A \in \mathscr{R}$ is a submeasure. Moreover, if λ is a uniform submeasure and the function F is uniformly continuous on the range of λ, then μ is a uniform submeasure. Particularly suppose that $\mathscr{R} = \mathscr{S}$ is a σ-ring and that $\lambda: \mathscr{S} \to [0, +\infty)$ is a countably additive measure. Then, since the range of λ is a compact subset of $[0, +\infty)$, see Exercise 3, § 17 and Exercise 4 after § 41 in [10], and since the continuous function F is uniformly continuous on compacts, the μ defined above is a uniform submeasure.

Suppose now that $\lambda: \mathscr{S} \to [-\infty, +\infty]$ is the usual Lebesgue measure on the line, put $\mathscr{P} = \{A \in \mathscr{S}, \lambda(A) < +\infty\}$ and $\mathscr{P}_1 = \{A \in \mathscr{P}, A \subset [0, 1]\}$. Then $\mathscr{P}$ is a δ-ring, $\mathscr{P}_1$ is a σ-ring, the submeasure μ_1 on $\mathscr{P}$ defined by the equality $\mu_1(A) = [\lambda(A)]^2$ is not uniform, while its restriction to $\mathscr{P}_1$ is uniform but not subadditive. Finally, the submeasure μ_2 on $\mathscr{P}$ defined by equality $\mu_2(A) = \lambda(A)/[1+\lambda(A)]$ is subadditive but not additive.

EXAMPLE 2. Important examples of subadditive submeasures are the continuous semivariation $\hat{m}$ of an operator-valued measure m, see [6], § 1. 1, and the continuous L_p-norm $\hat{m}_p(f, \cdot)$ of a measurable function f, see [7], § 1.

EXAMPLE 3. In this example we construct a submeasure on a σ-algebra which is not uniform.

Let $X_0 = \{0, 1\}$, $\Sigma_0 = 2^{X_0}$, $\nu_0(\{0\}) = \nu_0(\{1\}) = \frac{1}{2}$, $(X_i, \Sigma_i, \nu_i) = (X_0, \Sigma_0, \nu_0)$ for every $i = 1, 2, \ldots$ and let $(X, \Sigma, \nu) = (\mathop{\times}_{i=1}^{\infty} X_i, \mathop{\times}_{i=1}^{\infty} \Sigma_i, \mathop{\times}_{i=1}^{\infty} \nu_i)$, see Exercise 5, after § 38 in [10]. Further let $Y = [0,1]$, let $\mathscr{B}$ be the σ-algebra of Borel measurable subsets of Y and let λ be the Lebesgue measure on $\mathscr{B}$.

For $n = 1, 2, \ldots$ put $A_n = \{x = [x_1, \ldots, x_n, \ldots] \in X, x_n = 0\}$ and $\mathscr{U}_n = \{A \in \Sigma, \nu(A \Delta A_n) \leqslant \frac{1}{12}\}$. Then obviously $A_n \in \Sigma$, $\nu(A_n) = \frac{1}{2}$, and $\nu(A_n \Delta A_k) = \frac{1}{2}$ and $\mathscr{U}_n \cap \mathscr{U}_k = \emptyset$ for $n \neq k$, $n, k = 1, 2, \ldots$

We now define $(T, \mathscr{S}, \mu)$ as follows: $T = X \cup Y$, $\mathscr{S} = \Sigma \oplus \mathscr{B} = \{C \subset T, C = A \cup B$, where $A \in \Sigma$ and $B \in \mathscr{B}\}$, and for such $C \in \mathscr{S}$ we put

$$\mu(C) = \nu(A) + \lambda(B) + \sum_{n=1}^{\infty} \big(0 \vee [\tfrac{1}{12} - \nu(A \Delta A_n)]\big)[n\lambda(B) \wedge 1].$$

(i) *For $E, F \in \Sigma$ we write ERF iff $E \subset F$ or $E \supset F$. Then $n \neq k$, $E \in \mathscr{U}_n$, $F \in \mathscr{U}_k$ implies $E\,\mathrm{non}R F$.*

Proof. Suppose that $E \supset F$. Then $\nu[(A_k - A_n) \cap E] \geqslant \nu[(A_k - A_n) \cap \cap F]$ by the monotonicity of ν. However,

$$\begin{aligned} \nu[(A_k - A_n) \cap F] &= \nu(A_k - A_n) - \nu[(A_k - A_n) - F] \\ &\geqslant \tfrac{1}{4} - \nu(A_k - F) \geqslant \tfrac{1}{4} - \nu(A_k \Delta F) \geqslant \tfrac{1}{4} - \tfrac{1}{12} = \tfrac{2}{12}, \end{aligned}$$

and

$$\begin{aligned} \nu[(A_k - A_n) \cap E] &= \nu\big((A_k - A_n) \cap [A_n \Delta (A_n \Delta E)]\big) \\ &= \nu\big[(A_k - A_n) \cap \big([A_n - (A_n \Delta E)] \cup [(A_n \Delta E) - A_n]\big)\big] \\ &= \nu\big([A_k - A_n] \cap [(A_n \Delta E) - A_n]\big) \\ &\leqslant \nu[(A_k - A_n) \cap (A_n \Delta E)] \leqslant \nu(A_n \Delta E) \leqslant \tfrac{1}{12}, \end{aligned}$$

a contradiction. By symmetry, the inclusion $E \subset F$ is also false.

(ii) *μ is monotone on $\mathscr{S}$.*

Proof. Let $E, F \in \mathscr{S}$ and let $E \supset F$. Then $E = E_1 \cup E_2$, $F = F_1 \cup F_2$, where $E_1, F_1 \in \Sigma$, $E_2, F_2 \in \mathscr{B}$, $F_1 \subset E_1$ and $F_2 \subset E_2$.

If $E_1, F_1 \in \Sigma - \bigcup_{n=1}^{\infty} \mathscr{U}_n$, then the inequality $\mu(F) \leqslant \mu(E)$ is clear from the definition of μ.

If $E_1 \in \mathscr{U}_n$ for some n, then by (i) $F_1 \in \Sigma - \bigcup_{k=1, k \neq n}^{\infty} \mathscr{U}_k$. Suppose first that $F_1 \in \Sigma - \bigcup_{k=1}^{\infty} \mathscr{U}_k$. Then obviously

$$\begin{aligned} \mu(E) - \mu(F) = \nu(E_1) - \nu(F_1) + \lambda(E_2) - \lambda(F_2) + \\ + [\tfrac{1}{12} - \nu(E_1 \Delta A_n)][n\lambda(E_2) \wedge 1] \geqslant 0. \end{aligned}$$

Now let $F_1 \in \mathscr{U}_n$. Then

$$\begin{aligned} \mu(E) - \mu(F) &= \nu(E_1) - \nu(F_1) + \lambda(E_2) - \lambda(F_2) + \\ &\quad + [\tfrac{1}{12} - \nu(E_1 \Delta A_n)][n\lambda(E_2) \wedge 1] - [\tfrac{1}{12} - \nu(F_1 \Delta A_n)][n\lambda(F_2) \wedge 1] \\ &\geqslant \nu(E_1) - \nu(F_1) + [n\lambda(E_2) \wedge 1][\tfrac{1}{12} - \nu(E_1 \Delta A_n) - \tfrac{1}{12} + \nu(F_1 \Delta A_n)] \\ &\geqslant \nu(E_1) - \nu(F_1) - \nu(E_1 \Delta F_1) = 0 \end{aligned}$$

(in the last inequality we used the fact that $\nu(E_1 \Delta A_n) \leqslant \nu(E_1 \Delta F_1) + \nu(F_1 \Delta A_n)$).

Finally if $F_1 \in \mathscr{U}_n$ for some n, then by (i) $E_1 \in \Sigma - \bigcup_{\substack{k=1 \\ k \neq n}}^{\infty} \mathscr{U}_k$.

The case $E_1 \in \mathscr{U}_n$ has already been considered above. Thus let $E_1 \in \Sigma - \bigcup_{k=1}^{\infty} \mathscr{U}_k$. Then

$$\begin{aligned} \mu(E) - \mu(F) &= \nu(E_1) - \nu(F_1) + \lambda(E_2) - \lambda(F_2) - [\tfrac{1}{12} - \nu(F_1 \Delta A_n)][n\lambda(F_2) \wedge 1] \\ &\geqslant \nu(E_1) - \nu(F_1) - \tfrac{1}{12} + \nu(F_1 \Delta A_n) \\ &> \nu(E_1) - \nu(F_1) - [\nu(E_1 \Delta A_n) - \nu(F_1 \Delta A_n)] \\ &\geqslant \nu(E_1) - \nu(F_1) - \nu(E_1 \Delta F_1) = 0. \end{aligned}$$

Thus the monotonicity of μ on $\mathscr{S}$ is proved.

(iii) *It is easy to verify that μ is continuous and subadditively continuous on $\mathscr{S}$. However, μ is not uniformly subadditively continuous on $\mathscr{S}$, since for every $\delta > 0$ and every $B \in \mathscr{B}$, with $\lambda(B) = \delta$, $\mu(A_n \cup B) - \mu(A_n) > \frac{1}{12}$ for $n \geqslant 1/\delta$.*

EXAMPLE 4. Let $\mathscr{R}$ be a ring of subsets of a set T and let $\mu: \mathscr{R} \to [0, +\infty)$ be a submeasure. Then for every $A \in \mathscr{R}$ the set functions μ_{A+} and μ_{A-} defined by the equations $\mu_{A+}(B) = \mu(A \cup B) - \mu(A)$ and $\mu_{A-}(B) = \mu(A) - \mu(A - B)$, $B \in \mathscr{R}$, are submeasures. If μ is uniform or additive, then μ_{A+} and μ_{A-} are also uniform or additive respectively, for every $A \in \mathscr{R}$. It follows from assertion a) of Theorem 1 that for a monotone sequence $A_n \in \mathscr{R}$, $n = 1, 2, \ldots$ with $\lim_{n\to\infty} A_n = A \in \mathscr{R}$, $\lim_{n\to\infty} \mu_{A_n^+}(B) = \mu_{A+}(B)$ and $\lim_{n\to\infty} \mu_{A_n^-}(B) = \mu_{A-}(B)$ for every $B \in \mathscr{R}$.

THEOREM 2. *Let $\mathscr{S}$ be a σ-ring of subsets of a set T and let $\mu: \mathscr{S} \to [0, +\infty)$ be a set function with properties* 1), 2) *and* 3a) (3b)) *from Definition* 1. *Then μ is a submeasure if an only if it has the following property* 3^+) (3^-)): *for every increasing (decreasing) sequence $A_n \in \mathscr{S}$, $n = 1, 2, \ldots$ $\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$.*

Proof. If μ is a submeasure, then 3^+) and 3^-) are consequences of assertion a) of Theorem 1.

First, let μ have the property 3^+) and suppose that μ does not have the property 3b) of a submeasure. Then there is a set $A \in \mathscr{S}$, an $\varepsilon > 0$ and a sequence $B_n \in \mathscr{S}$, $n = 1, 2, \ldots$ such that $\mu(B_n) < 1/n$ and $\mu(A) > \mu(A - B_n) + \varepsilon$ for every $n = 1, 2, \ldots$ According to 3a) we choose a subsequence $\{B_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{B_n\}_{n=1}^{\infty}$ so that $\mu\left(\bigcup_{i=r}^{k} B_{n_i}\right) < 1/r$ for every $r = 1, 2, \ldots, k$. Put $D_r = \bigcup_{i=r}^{\infty} B_{n_i}$ and $D = \bigcap_{r=1}^{\infty} D_r$. Then $\mu(D_r) \leqslant 1/r$ by 3^+); hence $\mu(D) = 0$ by the monotonicity of μ. Since $A - D_r \nearrow A - D$,

$\lim_{r\to\infty} \mu(A-D_r) = \mu(A-D) = \mu(A)$ by 3^+) and 3a). However, $\mu(A)-\varepsilon > \mu(A-D_r)$ for every $r = 1, 2, \ldots$ by the monotonicity of μ, and thus we obtain the false inequality $\mu(A)-\varepsilon \geqslant \mu(A)$. This contradiction proves that μ has the property 3b); hence μ is a submeasure.

Now let μ have the property 3^-) and suppose that μ does not have the property 3a) of a submeasure. Then there is a set $A \in \mathscr{S}$, an $\varepsilon > 0$ and a sequence $B_n \in \mathscr{S}$, $n = 1, 2, \ldots$ such that $\mu(B_n) < 1/n$ and $\mu(A \cup B_n) > \mu(A)+\varepsilon$ for every $n = 1, 2, \ldots$ According to 3b) we choose a subsequence $\{B_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{B_n\}_{n=1}^{\infty}$, so that $\mu\big(A \cup [B_{n_r} - \bigcup_{i=r+1}^{k+1} B_{n_i}]\big) > \mu(A)+\varepsilon$ for every $r = 1, 2, \ldots, k$. Put $D_r = B_{n_r} - \bigcup_{i=r+1}^{\infty} B_{n_i}$ and $E_r = \bigcup_{i=r}^{\infty} D_i$, $r = 1, 2, \ldots$ Since $A \cup \big(B_{n_r} - \bigcup_{i=r+1}^{N} B_{n_i}\big) \searrow A \cup D_r$ for $N \nearrow \infty$, $\mu(A \cup E_r) \geqslant \mu(A \cup D_r) \geqslant \mu(A)+\varepsilon$ by the monotonicity of μ and 3^-). However, $D_r \in \mathscr{S}$, $r = 1, 2, \ldots$ is a sequence of pairwise disjoint sets; hence $E_r \searrow \emptyset$ for $r \nearrow \infty$ and therefore $\lim_{r\to\infty} \mu(A \cup E_r) = \mu(A)$ by 3^-). Thus we obtain the false inequality $\mu(A) \geqslant \mu(A)+\varepsilon$. This contradiction proves that μ has the property 3a); hence μ is a submeasure.

THEOREM 3. a) *Let $\mathscr{R}$ be a ring of subsets of a set T and let $\mu: \mathscr{R} \to [0, +\infty)$ be a set function with properties* 1) *and* 3a) *of a submeasure. Futher let, $A_n, B_n \in \mathscr{R}$, $n = 1, 2, \ldots$ be two decreasing seguences and let $\lim_{n\to\infty} [\mu(A_n)+\mu(B_n)] = 0$. Then $\lim_{n\to\infty} \mu(A_n \cup B_n) = 0$.*

b) *Let $\mathscr{S}$ be a σ-ring of subsets of a set T and let $\mu: \mathscr{S} \to [0, +\infty)$ be a set function with properties* 1), 3a) *of a submeasure and with the property* 3^+) *from Theorem* 2. *Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that $A, B \in \mathscr{S}$, $\mu(A) \vee \mu(B) < \delta$ implies $\mu(A \cup B) < \varepsilon$.*

Proof. a) is clear.

b) Suppose that b) is not valid. Then there is an $\varepsilon > 0$ and two sequences $E_n, F_n \in \mathscr{S}$, $n = 1, 2, \ldots$ such that $\mu(E_n) \vee \mu(F_n) < 1/n$ and $\mu(E_n \cup F_n) > \varepsilon$ for every $n = 1, 2, \ldots$ According to 3a) we choose a subsequence n_k, $k = 1, 2, \ldots$ so that $\mu\big(\bigcup_{i=r}^{k} E_{n_i}\big) \vee \mu\big(\bigcup_{i=r}^{k} F_{n_i}\big) < 1/r$ for every $r = 1, 2, \ldots, k$. Put $A_r = \bigcup_{i=r}^{\infty} E_{n_i}$ and $B_r = \bigcup_{i=r}^{\infty} F_{n_i}$, $r = 1, 2, \ldots$ Then $A_r, B_r \in \mathscr{S}$, $A_r \searrow$, $B_r \searrow$ and $\lim_{r\to\infty} [\mu(A_r)+\mu(B_r)] \leqslant \lim_{r\to\infty} (1/r+1/r) = 0$ by 3^+) and the monotonicity of μ. Thus $\lim_{r\to\infty} \mu(A_r \cup B_r)] = 0$ by a), a contradiction. Thus the theorem is proved.

THEOREM 4. (Exhaustion of a submeasure). *Let $\mathscr{S}$ be a σ-ring of subsets of a set T and let $\mu: \mathscr{S} \to [0, +\infty)$ be a submeasure. Then there is a set $Q \in \mathscr{S}$ such that $\sup_{A\in\mathscr{S}} \mu(A) = \mu(Q) < +\infty$, $\mu(A-Q) = 0$ and $\mu(A) = \mu(A \cap Q)$ for each set $A \in \mathscr{S}$.*

Proof. Define a sequence $Q_n \in \mathcal{S}$, $n = 0, 1, \ldots$ as follows: $Q_0 = \emptyset$ and for $n = 1, 2, \ldots$ let Q_n be a set from $\mathcal{S}$ with the property $\mu(Q_n) = \sup\{\mu(A), A \in \mathcal{S}, A \subset T - \bigcup_{i=0}^{n-1} Q_i\}$ (such Q_n-s obviously exist by the monotone continuity of μ). Since Q_n, $n = 0, 1, \ldots$ is a sequence of pairwise disjoint sets, $\lim_{n\to\infty} \mu(Q_n) = 0$ by assertion c) of Theorem 1. Put $Q = \bigcup_{n=0}^{\infty} Q_n$. Then $Q \in \mathcal{S}$, $\sup_n \mu(A) = \mu(Q_1) = \mu(Q) < +\infty$ and $\mu(A - Q) = \lim_{n\to\infty} \mu\big(A - \bigcup_{i=0} Q_i\big) \leqslant \lim_n \sup \mu(Q_n) = 0$ for each $A \in \mathcal{S}$ by the monotone continuity of μ. But then also $\mu(A) = \mu[(A \cap Q) \cup (A - Q)] = \mu(A \cap Q)$ for each $A \in \mathcal{S}$ by the subadditive continuity of μ, and the theorem is proved.

Let $\mathcal{R}$ be a ring of subsets of a set T and let $\nu, \mu: \mathcal{R} \to [0, +\infty)$ be two submeasures. We say that *ν is absolutely μ continuous* iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that $A \in \mathcal{R}$, $\mu(A) < \delta$ implies $\nu(A) < \varepsilon$. We say that *ν is singular with respect to μ* iff there is a set $Q \in \mathcal{R}$ such that $\mu(Q) = 0$ and $\nu(A - Q) = 0$ for each set $A \in \mathcal{R}$. It follows from Theorem 4 that the notion of singularity of two submeasures on a σ-ring is symmetric.

THEOREM 5. *Let $\mathcal{S}$ be a σ-ring of subsets of a set T and let ν and μ be two submeasures on $\mathcal{S}$. Then ν is absolutely μ continuous if and onl if $A \in \mathcal{S}$, $\mu(A) = 0$ implies $\nu(A) = 0$.*

Proof. Let $A \in \mathcal{S}$, $\mu(A) = 0$, imply $\nu(A) = 0$ and suppose that ν is not absolutely μ continuous. Then there is an $\varepsilon > 0$ and a sequence $A_n \in \mathcal{S}$, $n = 1, 2, \ldots$ such that $\mu(A_n) < 1/n$ and $\nu(A_n) > \varepsilon$ for every $n = 1, 2, \ldots$ According to the subadditive continuity of μ we choose a subsequence $\{A_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{A_n\}_{n=1}^{\infty}$ so that $\mu\big(\bigcup_{i=r}^{k} A_{n_i}\big) < 1/r$ for every $r = 1, 2, \ldots, k$. Put $E_r = \bigcup_{i=r}^{\infty} A_{n_i}$, $r = 1, 2, \ldots$ and $E = \bigcap_{r=1}^{\infty} E_r$. Then $\mu(E) = \lim_{r\to\infty} \mu(E_r) \leqslant \lim_{r\to\infty} 1/r = 0$ by the monotone continuity of μ, while the monotone continuity of ν implies $\nu(E) \geqslant \varepsilon$, a contradiction. Thus the theorem is proved.

THEOREM 6 (Lebesgue decomposition). *Let $\mathcal{S}$ be a σ-ring of subsets of a set T and let μ and ν be two submeasures on $\mathcal{S}$. Then there is a set $Q \in \mathcal{S}$ such that the submeasure $\nu_1(A) = \nu(A - Q)$, $A \in \mathcal{S}$ is absolutely μ continuous and the submeasure $\nu_2(A) = \nu(A \cap Q)$, $A \in \mathcal{S}$ is singular with respect to μ.*

Proof. Put $\mathcal{S}_1 = \{A \in \mathcal{S}, \mu(A) = 0\}$. Then $\mathcal{S}_1$ is a σ-subring of $\mathcal{S}$, and therefore by Theorem 4 there is a set $Q \in \mathcal{S}_1$ such that $\sup_{A \in \mathcal{S}_1} \nu(A) = \nu(Q)$ and $\nu(A - Q) = 0$ for each set $A \in \mathcal{S}_1$. Obviously such a Q has the required properties, and the theorem is proved.

DEFINITION 2. Let $\mathcal{P}$ be a δ-ring of subsets of a set T and let μ: $\mathcal{P}$

$\to[0,\infty)$ be a submeasure. We say that a set $A\in\mathscr{P}$ is an *atom of* μ iff $\mu(A)>0$ and $\mu(A\cap B)\mu(A-B)=0$ for every $B\in\mathscr{P}$. Two atoms A and B of μ are called *different* iff $\mu(A\cap B)=0$. We say that μ is *purely atomic* iff for each set $A\in\mathscr{P}$ there is at most a countable number of atoms $A_i, i\in\mathscr{I}$, of μ such that $\mu(A-\bigcup_{i\in\mathscr{I}}A_i)=0$.

It is easy to verify that a purely atomic submeasure on a σ-ring is uniform.

THEOREM 7. *Let $\mathscr{P}$ be a δ-ring of subsets of a set T and let μ: $\mathscr{P}\to[0,+\infty)$ be a submeasure. Then each set $E\in\mathscr{P}$ contains at most a countable number of different atoms of μ. If each set $E\in\mathscr{P}$ with $\mu(E)>0$ contains an atom of μ, then μ is purely atomic.*

Proof. Clearly we may suppose that $\mu\not\equiv 0$. We prove the second part of the theorem and from here the first part will be obvious. Let $E\in\mathscr{P}$ and let $\mu(E)>0$. If E is an atom of μ, we put $A_1=E$. Otherwise we take $A_1\in\mathscr{P}$, $A_1\subset A$ so that A_1 is an atom of μ and $\mu(A_1)>\frac{1}{2}\sup\{\mu(B)$, $B\subset E$ and B is an atom of $\mu\}$. Put $E_1=E-A_1$. Repeating the procedure step by step we either arrive to an n for which E_n is an atom of μ, and the theorem is proved, or we obtain an infinite sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint atoms of μ contained in E and such that $\mu(A_n)>\frac{1}{2}\sup\{\mu(B)$, $B\subset E-\bigcup_{i=1}^{n-1}A_i$ and B is an atom of $\mu\}$ for every $n=1,2,\ldots$ In that case $\lim_{n\to\infty}\mu(A_n)=0$ by assertion c) of Theorem 1. Suppose that $\mu(\mathrm{E}-\bigcup_{n=1}^{\infty}A_n)>0$. Then there exists an atom $A\subset E-\bigcup_{n=1}^{\infty}A_n$ of μ and for all n we have $\mu(A)<2\mu(A_n)$. Hence $\mu(A)=0$, a contradiction.

THEOREM 8 (Saks decomposition, see [17]). *Let $\mathscr{P}$ be a δ-ring of subsets of a set T, let μ: $\mathscr{P}\to[0,+\infty)$ be a submeasure and let $A\in\mathscr{P}$. Then for every $\varepsilon>0$ there is a finite number $A_0, A_1, \ldots, A_r$ of pairwise disjoint elements of $\mathscr{P}$ such that $A=\bigcup_{i=0}^{r}A_i$ and each $A_i, i=0,1,\ldots,r$ is either an atom of μ with $\mu(A_i)>\varepsilon$, or $\mu(A_i)\leqslant\varepsilon$.*

Proof. Let $\varepsilon>0$. According to assertion c) of Theorem 1 there is at most a finite number $A_1,\ldots,A_k$ of pairwise disjoint atoms of μ in A with $\mu(A_i)>\varepsilon$. If $\mu(A-\bigcup_{i=1}^{k}A_i)=0$, the theorem is proved. Otherwise there are sets $E\in\mathscr{P}$, $E\subset A-\bigcup_{i=1}^{k}A_i$ with $0<\mu(E)\leqslant\varepsilon$. (Indeed, suppose the contrary. Then $A-\bigcup_{i=1}^{k}A_i$ is atomless, and therefore there must be an infinite sequence $F_n\in\mathscr{P}$, $n=1,2,\ldots$ of pairwise disjoint subsets of $A-\bigcup_{i=1}^{k}A_i$ with $\mu(F_n)>\varepsilon$ for every n, a contradiction with assertion c) of Theorem 1.) Take $A_{k+1}\in\mathscr{P}$, $A_{k+1}\subset A-\bigcup_{i=1}^{k}A_i$ so that

$$\varepsilon \geqslant \mu(A_{k+1}) > \tfrac{1}{2}\sup\{\mu(E),\ E\in\mathscr{P},\ E\subset A-\bigcup_{i=1}^{k}A_i \text{ and } 0<\mu(E)\leqslant\varepsilon\}.$$

If $\mu\big(A-\bigcup_{i=1}^{k+1}A_i\big)=0$, we have finished. Otherwise, in a similar way, we take A_{k+2}. Repeating the procedure step by step we either arrive to a p for which $\mu\big(A-\bigcup_{i=1}^{k+p}A_i\big)=0$, and the theorem is proved, or $\mu\big(A-\bigcup_{i=1}^{k+p}A_i\big)>0$ for every $p=1,2,\ldots$ In that case, since $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint sets, $\lim_{n\to\infty}\mu(A_n)=0$ by assertion c) of Theorem 1, and therefore $\sup\{\mu(E),\ E\in\mathscr{P},\ E\subset A-\bigcup_{i=1}^{\infty}A_i \text{ and } 0<\mu(E)\leqslant\varepsilon\}=0$. Hence, by the arguments given above in brackets, $\mu\big(A-\bigcup_{n=1}^{\infty}A_n\big)=0$. Since $\bigcup_{i=n}^{\infty}A_i\searrow\emptyset$ for $n\to\infty$, by the continuity of μ there is a natural r such that $\mu\big(\bigcup_{i=r+1}^{\infty}A_i\big)\leqslant\varepsilon$. Obviously with $A_0=A-\bigcup_{i=1}^{r}A_i$ the theorem is proved.

THEOREM 9. *Let $\mathscr{P}$ be a δ-ring of subsets of a set T, let μ and ν be two submeasures on $\mathscr{P}$ and let $E\in\mathscr{P}$, $\mu(E)=0$, imply $\nu(E)=0$. Then, if μ is atomless (purely atomic), then ν is also atomless (purely atomic).*

Proof. Let A be an atom of ν, let $0<\varepsilon<\nu(A)$ and suppose that A contains no atoms of μ. By Theorem 5 ν is absolutely μ continuous on the σ-ring $A\cap\mathscr{P}$. Hence there is a $\delta>0$ such that $E\in A\cap\mathscr{P}$, $\mu(E)\leqslant\delta$, implies $\nu(E)<\varepsilon$. According to Theorem 8 there are sets $A_i\in A\cap\mathscr{P}$, $i=1,2,\ldots,r$ such that $A=\bigcup_{i=1}^{r}A_i$ and $\mu(A_i)\leqslant\delta$ for every $i=1,2,\ldots,r$. Thus for every $i=1,2,\ldots,r$ $\nu(A_i)<\varepsilon<\nu(A)$. But A is an atom of ν; hence $\nu(A_i)=0$ for every $i=1,\ldots,r$. Thus $\nu(A)=0$, a contradiction. This proves the first assertion of the theorem.

Now let μ be purely atomic and let A be an atom of μ. Then from the assumption it is easy to see that either A is also an atom of ν or $\nu(A)=0$. Thus the theorem is proved.

THEOREM 10 (Darboux property). *Let $\mathscr{P}$ be a δ-ring of subsets of a set T and let $\mu\colon\mathscr{P}\to[0,+\infty)$ be a submeasure. Further, let $A\in\mathscr{P}$ contain no atoms of μ and let $\mu(A)>0$. Then for every real number b, $0<b<\mu(A)$ there is a set $B\in\mathscr{P}$, $B\subset A$ such that $\mu(B)=b$.*

Proof. Let $0<b_n<b$, $n=1,2,\ldots$ be a sequence of real numbers increasing to b and let $0<\varepsilon_n<b_{n+1}-b_n$ for every n. We show that there is an increasing sequence of sets $B_n\in\mathscr{P}$, $B_n\subset A$, $n=1,2,\ldots$ such that $b_n\leqslant\mu(B_n)\leqslant b_{n+1}$ for every n, and this will prove the theorem, since then $\mu\big(\bigcup_{n=1}^{\infty}B_n\big)=b$ by the monotone continuity of μ.

According to Theorem 8 there are pairwise disjoint sets $E_i\in\mathscr{P}$, $i=1,\ldots,k_1$ such that $A=\bigcup_{i=1}^{k_1}E_i$ and $0<\mu(E_i)\leqslant b_1$ for each $i=1,\ldots,k_1$.

Since $\mu(A) > b_1$, there is a natural r_1, $1 \leqslant r_1 \leqslant k_1$ such that $\mu(\bigcup_{i=1}^{r_1-1} E_i) \leqslant b_1$ and $\mu(\bigcup_{i=1}^{r_1} E_i) \geqslant b_1$. If $\mu(\bigcup_{i=1}^{r_1} E_i) \leqslant b_2$, we put $B_1 = \bigcup_{i=1}^{r_1} E_i$. Otherwise we put $F_1 = \bigcup_{i=1}^{r_1-1} E_i$ and divide the set $E^{(1)} = E_{r_1}$ into a finite number of pairwise disjoint sets $E_i^{(1)} \in \mathscr{P}$, $i = 1, \ldots, k_2$ such that $0 < \mu(E_i^{(1)}) \leqslant b_1/2$ for every $i = 1, \ldots, k_2$. Since $\mu(F_1) \leqslant b_1$ and $\mu(F_1 \cup E^{(1)}) > b_2$, there is an r_2, $1 \leqslant r_2 \leqslant k_2$ such that $\mu(F_1 \cup \bigcup_{i=1}^{r_2-1} E_i^{(1)}) \leqslant b_1$ and $\mu(F_1 \cup \bigcup_{i=1}^{r_2} E_i^{(1)}) \geqslant b_1$. If $\mu(F_1 \cup \bigcup_{i=1}^{r_2} E_i^{(1)}) \leqslant b_2$, we put $B_1 = F_1 \cup \bigcup_{i=1}^{r_2} E_i^{(1)}$. Otherwise we put $F_2 = F_1 \cup \bigcup_{i=1}^{r_2-1} E_i^{(1)}$ and divide the set $E^{(2)} = E_{r_2}^{(1)}$ into a finite number of pairwise disjoint sets $E_i^{(2)} \in \mathscr{P}$, $i = 1, \ldots, k_3$ so that $0 < \mu(E_i^{(2)}) \leqslant b_1/4$ for every $i = 1, \ldots, k_3$. Continuing in this way we assert that in a finite number of steps we arrive at the required B_1. Indeed, suppose the contrary. Then we obtain an increasing sequence of sets $F_n \in \mathscr{P}$, $F_n \subset A$ with $\mu(F_n) \leqslant b_1$ for every n, a decreasing sequence of sets $E^{(n)} \in \mathscr{P}$, $E^{(n)} \subset A$, $F_n \cap E^{(n)} = \emptyset$ with $\mu(E^{(n)}) \leqslant b_1/2^{n-1}$ for every n, and such that $\mu(F_n \cup E^{(n)}) > b_2$ for every n. Put $F = \bigcup_{n=1}^{\infty} F_n$. Then $\mu(F) \leqslant b_1$ by the monotone continuity of μ. According to the subadditive continuity of μ there is a $\delta > 0$ such that $E \in \mathscr{P}$, $\mu(E) < \delta$ implies $\mu(F \cup E) \leqslant \mu(F) + \varepsilon_1 < b_1 + b_2 - b_1 = b_2$. Take n_0 so that $b_1/2^{n_0-1} < \delta$. Then $\mu(F_{n_0} \cup E^{(n_0)}) \leqslant \mu(F \cup E^{(n_0)}) < b_2$, a contradiction. Thus we have proved the existence of the required B_1. In a similar way we prove the existence of each B_n, $n = 2, 3, \ldots$, and thus the theorem is proved.

The author has been unable to solve the following

PROBLEM 1(*). Let $\mathscr{S}$ be a σ-ring of subsets of a set T and let $\mu\colon \mathscr{S} \to [0, +\infty)$ be a submeasure. Is the set $\{\mu(A), A \in \mathscr{S}\}$ then closed in $[0, +\infty)$?

DEFINITION 3. Let T be a locally compact Hausdorff topological space. By $\mathscr{C}_\wedge$ we denote either the lattice $\mathscr{C}$ of all compact subsets of T or the lattice $\mathscr{C}_0$ of all compact G_δ subsets of T. $\mathscr{B}_\wedge$ will denote the smallest δ-ring containing $\mathscr{C}_\wedge$ and $\mathfrak{S}(B_\wedge)$ will denote the smallest σ-ring containing $\mathscr{C}_\wedge$. Further, $\check{\mathscr{B}}_\wedge = \mathscr{B}_\wedge$ or $\mathfrak{S}(B_\wedge)$ and $\check{\mathscr{U}}_\wedge = \mathscr{U}_\wedge$ or $\mathscr{U}_\wedge^\delta$, where $\mathscr{U}_\wedge$ $(\mathscr{U}_\wedge^\delta)$ denotes the lattice of all open sets belonging to $\mathfrak{S}(\mathscr{B}_\wedge)$ $(\mathscr{B}_\wedge)$. Phrases related to $\check{\mathscr{B}}$ and $\check{\mathscr{U}}$ will have the adjective *Borel*, while phrases related to $\check{\mathscr{B}}_0$ and $\check{\mathscr{U}}_0$ will have the adjective *Baire*. We say that a set function $\mu\colon \check{\mathscr{B}}_\wedge \to [0, +\infty)$ is *regular* iff $\inf\{\mu(V - C),\ C \subset A \subset V,\ C \in \mathscr{C}_\wedge,\ V \in \check{\mathscr{U}}_\wedge\} = 0$ for each set $A \in \check{\mathscr{B}}_\wedge$.

(*) L. Drewnowski has recently solved this problem affirmatively.

THEOREM 11. *Let T be a locally compact Hausdorff topological space. Then every Baire submeasure μ: $\check{\mathscr{B}}_0 \to [0, +\infty)$ is regular.*

Proof. Denote by $\mathscr{P}$ the collection of all sets $A \in \check{\mathscr{B}}_0$ for which $\inf\{\mu(V - C), C \subset A \subset V, C \in \mathscr{C}_0, V \in \check{\mathscr{U}}_0\} = 0$. Then $\mathscr{C}_0 \subset \mathscr{P}$ by the continuity of $\check{\mu}$, and $\mathscr{P}$ is a ring by subadditive continuity of μ. Hence to prove that $\mathscr{P} = \check{\mathscr{B}}_0$ it is enough to prove that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{P}$ for every sequence $A_n \in \mathscr{P}$, $n = 1, 2, \ldots$ of pairwise disjoint sets such that $A_n \subset V$, $n = 1, 2, \ldots$ for some $V \in \check{\mathscr{U}}_0$. Let $A_n \in \mathscr{P}$, $n = 1, 2, \ldots$ be a such sequence and let $\varepsilon > 0$. According to the subadditive continuity of μ we take $V_n \in \check{\mathscr{U}}_0$, $C_n \in \mathscr{C}_0$ so hat $C_n \subset A_n \subset V_n \subset V$ and $\mu[\bigcup_{i=1}^{n} (V_i - C_i)] < \frac{1}{2}\varepsilon$ for every $n = 1, 2, \ldots$ Then $\mu[\bigcup_{n=1}^{\infty} (V_n - C_n)] \leqslant \frac{1}{2}\varepsilon$ by the monotone continuity of μ. Obviously $\bigcup_{n=1}^{\infty} V_n - \bigcup_{n=1}^{\infty} C_n \subset \bigcup_{n=1}^{\infty} (V_n - C_n)$; hence $\mu\left(\bigcup_{n=1}^{\infty} V_n - \bigcup_{n=1}^{\infty} C_n\right) \leqslant \frac{1}{2}\varepsilon$ by the monotonicity of μ. Since $\bigcup_{i=1}^{\infty} V_i - \bigcup_{i=1}^{n} C_i \searrow \bigcup_{i=1}^{\infty} V_i - \bigcup_{i=1}^{\infty} C_i$, by the monotone continuity of μ there is an n_0 such that $\mu\left(\bigcup_{n=1}^{\infty} V_n - \bigcup_{n=1}^{n_0} C_n\right) < \varepsilon$. Clearly $\bigcup_{n=1}^{\infty} V_n$ is an open subset of V and $\bigcup_{n=1}^{\infty} V_n \supset \bigcup_{n=1}^{\infty} A_n \supset \bigcup_{n=1}^{n_0} C_n \in \mathscr{C}_0$. Since $\varepsilon > 0$ was arbitrary, $\bigcup_{n=1}^{\infty} A_n \in \mathscr{P}$ and thus the theorem is proved.

The autor has proved the next theorem for a locally compact metric space T. The following general case is due to L. Drewnowski.

THEOREM 12. *Let T be a locally compact Hausdorff topological space, let $\check{\mu}$: $\mathscr{B} \to [0, +\infty)$ be a regular Borel submeasure and let $A \in \mathscr{B}$ be an atom of μ. Then there is a point $t_0 \in A$ such that $\mu(A - \{t_0\}) = 0$.*

Proof. Let $\mathscr{K}$ be the family of all compact sets $C \subset A$ such that $\mu(A - C) = 0$; obviously each $C \in \mathscr{K}$ is an atom of μ equivalent to A. Since $C_1, C_2 \in \mathscr{K}$ implies $C_1 \cap C_2 \in \mathscr{K}$, the set $K = \bigcap_{C \in \mathscr{K}} C$ is non-empty and compact. Let $C_0 \in \mathscr{K}$. Then $\mu(C_0 - K) = 0$. In fact, otherwise $C_0 - K$ is an atom of μ equivalent to A, hence contains a member of $\mathscr{K}$ which is impossible, since $K \neq \emptyset$. Therefore $K \in \mathscr{K}$, i.e., K is the least compact atom of μ contained in A. We are to show that K is a singleton. Suppose that it is not. Thus let $t_1, t_2 \in K$, $t_1 \neq t_2$, and let U be an open neighbourhood of t_1 such that $\overline{U}$ does not contain t_2. Then $K = K_1 \cup K_2$, where $K_1 = K \cap \overline{U}$, $K_2 = K - U$, $K_1 \neq K$, $K_2 \neq K$, and one of the K_1, K_2 must belong to $\mathscr{K}$. This, however, is impossible.

THEOREM 13. *Let T be a locally compact Hausdorff topological space* *and let* $\mu\colon \check{\mathscr{B}}_{\wedge} \to [0, +\infty)$ *be a monotone and subadditively continuous set function with the following properties*:

(i) $\mu(A) = \sup\{\mu(C), C \in \mathscr{C}_{\wedge}, C \subset A\}$ *for every* $A \in \check{\mathscr{B}}_{\wedge}$, *and*

(ii) *if* $C_n \in \mathscr{C}_{\wedge}$, $n = 1, 2, \ldots$ *is a sequence of pairwise disjoint sets and* $\bigcup_{n=1}^{\infty} C \in \check{\mathscr{B}}_{\wedge}$, *then* $\lim_{n\to\infty} \mu(C_n) = 0$.

Then μ *is a regular Borel (Baire) submeasure on* $\check{\mathscr{B}}_{\wedge}$.

Proof. First we prove the regularity of μ. Suppose the contrary. Then there is a set $A \in \check{\mathscr{B}}_{\wedge}$ and an $\varepsilon > 0$ such that $\mu(V - C) > \varepsilon$ for each sets $C \in \mathscr{C}_{\wedge}$, $V \in \check{\mathscr{U}}_{\wedge}$, $C \subset A \subset V$. Let C_0 and V_0 be such sets. It follows from (i) that there are $D_1, H_1 \in \mathscr{C}_{\wedge}$, $D_1 \subset V_0 - A$, $H_1 \subset A - C_0$ such that $\mu(D_1) \geqslant \frac{1}{2}\mu(V_0 - A)$ and $\mu(H_1) \geqslant \frac{1}{2}\mu(A - C_0)$. Obviously $V_1 = V_0 - D_1 \in \check{\mathscr{U}}_{\wedge}$, $C_1 = C_0 \cup H_1 \in \mathscr{C}_{\wedge}$ and $C_1 \subset A \subset V_1$. Hence, by assumption, $\mu(V_1 - C_1) > \varepsilon$. Repeating the consideration step by step, for $n = 2, 3, \ldots$ we take subsequently D_n, $H_n \in \mathscr{C}_{\wedge}$, $D_n \subset V_{n-1} - A$, $H_n \subset A - C_{n-1}$ so that $\mu(D_n) \geqslant \frac{1}{2}\mu(V_{n-1} - A)$, $\mu(H_n) \geqslant \frac{1}{2}\mu(A - C_{n-1})$, and put $V_n = V_{n-1} - D_n \in \check{\mathscr{U}}_{\wedge}$ and $C_n = C_{n-1} \cup H_n \in \mathscr{C}_{\wedge}$. Then $\{D_n\}_{n=1}^{\infty}$ and $\{H_n\}_{n=1}^{\infty}$ are sequences of pairwise disjoint sets from $\mathscr{C}_{\wedge}$ contained in V_0; hence $\lim_{n\to\infty}\big(\mu(D_n) + \mu(H_n)\big) = 0$ by property (ii). But then $\lim_{n\to\infty} [\mu(V_n - A) + \mu(A - C_n)] = 0$, and since the sequences $\{V_n - A\}_{n=1}^{\infty}$ and $\{A - C_n\}_{n=1}^{\infty}$ are decreasing, by assertion a) of Theorem 3 $\lim_{n\to\infty} \mu(V_n - C_n) = \lim_{n\to\infty} \mu[(V_n - A) \cup (A - C_n)] = 0$, a contradiction. Thus the regularity of μ is proved.

It remains to prove the continuity of μ. Let $A_n \in \check{\mathscr{B}}_{\wedge}$, $n = 1, 2, \ldots$, let $A_n \searrow \emptyset$ and let $\varepsilon > 0$. According to the regularity and subadditive continuity of μ we take $C_n \in \mathscr{C}_{\wedge}$, $n = 1, 2, \ldots$ so that $\mu\big(\bigcup_{i=1}^{n} (A_i - C_i)\big) < \varepsilon$ for every n. Put $D_n = \bigcap_{i=1}^{n} C_i \in \mathscr{C}_{\wedge}$. Then $D_n \subset A_n \searrow \emptyset$; hence $D_n \searrow \emptyset$, and thus by the compactness of D_n there is an n_0 such that $D_n = \emptyset$ for $n \geqslant n_0$. Thus for $n \geqslant n_0$

$$A_n = A_n - D_n = A_n - \bigcap_{i=1}^{n} C_i = \bigcup_{i=1}^{n} (A_n - C_i) \subset \bigcup_{i=1}^{n} (A_i - C_i);$$

hence

$$\mu(A_n) \leqslant \mu\Big(\bigcup_{i=1}^{n} (A_i - C_i)\Big) < \varepsilon$$

by the monotonocity of μ for every $n \geqslant n_0$. Since $\varepsilon > 0$ was arbitrary, μ is continuous, and thus the theorem is proved.

Let $\mathscr{S}$ be a σ-ring of subsets of a set T. Then, clearly, $\mathscr{S}$ is an abelian group with respect to the operation $A \overset{g}{+} B = A \mathbin{\Delta} B = (A - B) \cup$

$\cup(B-A)$. Here $0 = \emptyset$ and $\overset{g}{-}A = A$. If $\mu\colon \mathscr{S}\to[0,+\infty)$ is a subadditive submeasure, then clearly the function ϱ_μ, $\varrho_\mu(A,B) = \mu(A\Delta B)$, $A, B\in\mathscr{S}$ is a pseudo-metric on $\mathscr{S}$. For a general submeasure $\mu\colon \mathscr{S}\to[0,+\infty)$ put $V_{\mu,n} = \{(A,B)\in\mathscr{S}\times\mathscr{S},\ \mu(A\Delta B)<1/n\}$, $n = 1,2,\ldots$ Then using assertion b) of Theorem 3 it is easy to verify that the sets $\{V_{\mu,n}\}_{n=1}^\infty$ form a base of a uniformity $\mathscr{U}_\mu$ on $\mathscr{S}$, see [11], Chap. VI, Theorem 2. Since this base is countable, it follows from [11], Chap. VI, Theorem 13 that the uniform space $(\mathscr{S},\mathscr{U}_\mu)$ is pseudo-metrizable. Let d be a pseudo-metric on $\mathscr{S}$ which generates the uniformity $\mathscr{U}_\mu$, see [11], Chap. VI, Metrization Lemma 12. (For a subadditive μ we may put $d = \varrho_\mu$.) Then it is easy to see that the group operation $(A,B)\to A\overset{g}{-}B = A\overset{g}{+}B = A\Delta B$, $A, B\in\mathscr{S}$ is continuous on the product $(\mathscr{S},\mathscr{U}_\mu)\times(\mathscr{S},\mathscr{U}_\mu)$. Hence $(\mathscr{S}, d)$ is an abelian pseudo-metric topological group. In the next theorem we prove that $(\mathscr{S}, d)$ is complete. Since the uniformity $\mathscr{U}_\mu$ is invariant, we may suppose that the pseudo-metric d is also invariant, see [11], Chap. VI, Problem N. Let us note that introducing the equivalence relation $A\sim A'$ iff $\mu(A\Delta A') = 0$, $A, A'\in\mathscr{S}$, and the abelian group $\mathscr{S}_\mu$ of equivalent classes of $\mathscr{S}$, we may replace above the terms pseudo-metric and $\mathscr{S}$ by the terms metric and $\mathscr{S}_\mu$ respectively.

THEOREM 14. *Let $\mathscr{S}$ be a σ-ring of subsets of a set T and let $\mu\colon \mathscr{S}\to[0,+\infty)$ be a submeasure. Then the pseudo-metrizable uniform space $(\mathscr{S},\mathscr{U}_\mu)$ is complete.*

Proof. According to [11], Chap. VI, Theorem 24, it is enough to prove that every Cauchy sequence is $(\mathscr{S},\mathscr{U}_\mu)$ has a limit in $\mathscr{S}$. Obviously a sequence $A_n\in\mathscr{S}$, $n = 1,2,\ldots$ is Cauchy in $(\mathscr{S},\mathscr{U}_\mu)$ if and only if it is ϱ_μ-Cauchy, where $\varrho_\mu(A,B) = \mu(A\Delta B)$. Hence it is enough to prove that every ϱ_μ-Cauchy sequence in $\mathscr{S}$ has a limit in $\mathscr{S}$.

According to assertion b) of Theorem 3 there is a sequence $\{\delta_n\}_{n=1}^\infty$ such that $0<\delta_n<1/n$ and $A, B\in\mathscr{S}$, $\mu(A)\vee\mu(B<\delta_n$ implies $\mu(A\cup B) < \delta_{n-1}$ for every $n = 2,3,\ldots$ Now if $A_n\in\mathscr{S}$, $n = 1,2,\ldots$ and $\mu(A_n) < \delta_n$ for every n, then the monotone continuity of μ implies

$$\begin{aligned}\mu\Big(\bigcup_{i=k}^{\infty} A_i\Big) &= \lim_{n\to\infty}\mu\Big(\bigcup_{i=k}^{n} A_i\Big)\\ &= \lim_{n\to\infty}\mu\big([\ldots[(\emptyset\cup A_n)\cup A_{n-1}]\cup A_{n-2}\cup\ldots\cup A_k]\big)\\ &\leqslant \delta_{k-1} < \frac{1}{k-1}\end{aligned}$$

for every $k = 2,3,\ldots$

Let $\{E_n\}_{n=1}^\infty$ be a ϱ_μ-Cauchy sequence in $\mathscr{S}$. Then there is a subsequence $\{E_{n_k}\}_{k=1}^\infty$ of this sequence such that $\mu(E_{n_k}\Delta E_{n_{k+1}}) < \delta_k$ for every $k = 1,2,\ldots$

By the preceding consideration

$$\mu\Big(\bigcup_{i=k}^{\infty}(E_{n_i}\Delta E_{n_{i+1}})\Big) < \frac{1}{k-1} \quad \text{for every } k = 2, 3, \ldots$$

Put

$$F_k = E_{n_k} - \bigcup_{i=k}^{\infty}(E_{n_i}\Delta E_{n_{i+1}}) = \bigcap_{i=k}^{\infty} E_{n_i} \quad \text{for} \quad k = 1, 2, \ldots$$

Then $\{F_k\}_{k=1}^{\infty}$ is an increasing sequence of sets from $\mathcal{S}$ with the limit $F = \bigcup_{k=1}^{\infty} F_k \in \mathcal{S}$. We prove that $\lim_{n\to\infty}\mu(E_n \Delta F) = 0$, which will prove the theorem.

Let $\varepsilon > 0$. According to assertion b) of Theorem 3 take $\delta, \delta_0 > 0$ so that $A, B \in \mathcal{S}$, $\mu(A) \vee \mu(B) < \delta$ implies $\mu(A \cup B) < \varepsilon$, and $A, B \in \mathcal{S}$, $\mu(A) \vee \mu(B) < \delta_0$ implies $\mu(A \cup B) < \delta$. Since the sequence $\{E_n\}_{n=1}^{\infty}$ is ϱ_μ-Cauchy, there is an n_0 such that $\mu(E_n \Delta E_m) < \delta$ for every $n, m \geqslant n_0$. Owing to the monotone continuity of μ there is a $k \geqslant n_0$s uch that $1/(k-1) \vee$ $\vee\, \mu(F \Delta F_k) < \delta_0$. Clearly

$$\mu(F_k \Delta E_{n_k}) \leqslant \mu\Big(\bigcup_{i=k}^{\infty}(E_{n_i}\Delta E_{n_{i+1}})\Big) < \frac{1}{k-1} < \delta_0.$$

Hence

$$\mu(F\Delta E_n) \leqslant \mu\big([(F\Delta F_k) \cup (F_k \Delta E_{n_k})] \cup [E_{n_k}\Delta E_n]\big) < \varepsilon \quad \text{for} \quad n \geqslant k.$$

Since $\varepsilon > 0$ was arbitrary, the theorem is proved.

For subadditive submeasures the next theorem is proved in [2], Theorem 1.

THEOREM 15. *Let $\mathcal{R}$ be a ring of subsets of a set T, let $\mathfrak{S}(\mathcal{R})$ be the smallest σ-ring containing $\mathcal{R}$ and let $\mu\colon \mathfrak{S}(\mathcal{R}) \to [0, +\infty)$ be a submeasure. Then $\mathcal{R}$ is dense in the pseudo-metrizable uniform space $(\mathfrak{S}(\mathcal{R}), \mathcal{U}_\mu)$.*

Proof. Put $\mathcal{S} = \{A \in \mathfrak{S}(\mathcal{R}), \inf\{\mu(A \Delta E), E \in \mathcal{R}\} = 0\}$. Then to prove the theorem it is enough to prove that $\mathcal{S} = \mathfrak{S}(\mathcal{R})$. Since $\mathcal{R} \subset \mathcal{S}$, it is enough to prove that $\mathcal{S}$ a σ-ring. Let $\varepsilon > 0$ and let $A_1, A_2 \in \mathcal{S}$. Then by the subadditive continuity of μ there are $E_1\, E_2 \in \mathcal{R}$ so that $\mu[(E_1 \Delta A_1) \cup$ $\cup\, (E_2 \Delta A_2)] < \varepsilon$. However, $(A_1 - A_2)\Delta(E_1 - E_2) \subset (A_1 \Delta E_1) \cup (A_2 \Delta E_2)$, and thus $\mu[(A_1 - A_2)\Delta(E_1 - E_2)] < \varepsilon$ by the monotonicity of μ. Since $\varepsilon > 0$ was arbitrary, $A_1 - A_2 \in \mathcal{S}$.

Now let $A_n \in \mathcal{S}$, $n = 1, 2, \ldots$ Using the subadditive continuity of μ we take $E_n \in \mathcal{R}$, $n = 1, 2 \ldots,$. so that $\mu\big(\bigcup_{i=1}^{n}(A_i \Delta E_i)\big) < \frac{1}{2}\varepsilon$ for every n.

Then $\mu\left(\bigcup_{i=1}^{\infty}(A_i \Delta E_i)\right) \leqslant \frac{1}{2}\varepsilon$ by the monotone continuity of μ. Since

$$B = \left(\bigcup_{i=1}^{\infty} A_i\right) \Delta \left(\bigcup_{i=1}^{\infty} E_i\right) \subset \bigcup_{i=1}^{\infty} (A_i \Delta E_i),\ \mu(B) \leqslant \tfrac{1}{2}\varepsilon.$$

Clearly

$$G_n = \left(\bigcup_{i=1}^{\infty} E_i\right) \Delta \left(\bigcup_{i=1}^{n} E_i\right) \searrow \emptyset.$$

Hence by the continuity and subadditive continuity of μ there is an n_0 such that $\mu(B \cup G_{n_0}) < \varepsilon$. But

$$\left(\bigcup_{i=1}^{\infty} A_i\right) \Delta \left(\bigcup_{i=1}^{n_0} E_i\right) \subset B \cup G_{n_0}$$

and therefore

$$\mu\left[\left(\bigcup_{i=1}^{\infty} A_i\right) \Delta \left(\bigcup_{i=1}^{n_0} E_i\right)\right] < \varepsilon$$

by the monotonicity of μ. Since $\bigcup_{i=1}^{n_0} E_i \in \mathscr{R}$ and since $\varepsilon > 0$ was arbitrary, $\bigcup_{i=1}^{\infty} A_i \in \mathscr{S}$. Thus $\mathscr{S}$ is a σ-ring and the theorem is proved.

Since the ring generated by a countable family of sets is itself countable, see [10], § 5, Theorem C, we immediately have the following.

COROLLARY 1. *Let $\mathscr{E}$ be a countable family of subsets of a set T, let $\mathfrak{S}(\mathscr{E})$ be the smallest σ-ring containing $\mathscr{E}$ and let $\mu\colon \mathfrak{S}(\mathscr{E}) \to [0, +\infty)$ be a submeasure. Then the pseudo-metrizable uniform space $\left(\mathfrak{S}(\mathscr{E}), \mathscr{U}_\mu\right)$ is separable.*

Before the next corollary let us note that a locally compact Hausdorff topological space with a countable base is metrizable and separable, see [11], Chap. IV, Urysohn Metrization Theorem 16.

COROLLARY 2. *Let T be a separable locally compact metric space. Then for every Borel (= Baire, see* [10], § 50, *Theorem* E) *submeasure $\mu\colon \mathfrak{S}(\mathscr{B}_0) \to [0, +\infty)$ the pseudo-metrizable uniform space $\left(\mathfrak{S}(\mathscr{B}_0), \mathscr{U}_\mu\right)$ is separable.*

Proof. Let $\{V_n\}_{n=1}^{\infty}$ be a countable base of the topology on T and let $\{V_{n_k}\}_{k=1}^{\infty}$ be the elements of $\{V_n\}_{n=1}^{\infty}$ with compact closures. Then according to [10], § 50, Theorems D and E, $\mathscr{C} = \mathscr{C}_0 \subset \mathfrak{S}(\{V_{n_k}\}_{k=1}^{\infty})$. Hence we may apply the preceding corollary.

COROLLARY 3. *Let $\mathscr{R}$ be a ring of subsets of a set T, let $\mathscr{P}$ be a δ-ring such that $\mathscr{R} \subset \mathscr{P} \subset \mathfrak{S}(\mathscr{R})$ and let μ_1 and μ_2 be two submeasures on $\mathscr{P}$. Then $\mu_1 \equiv \mu_2$ on $\mathscr{P}$ if and only if $\mu_1 \equiv \mu_2$ on $\mathscr{R}$.*

Proof. Let $F \in \mathscr{P}$. Then by Theorem 15 the ring $F \cap \mathscr{R}$ is dense in the pseudo-metrizable uniform space $(\mathfrak{S}(F \cap \mathscr{R}), \mathscr{U}_\mu)$. For $E \in \mathfrak{S}(F \cap \mathscr{R})$ put $\mu(E) = \mu_1(E) + \mu_2(E)$. Then the submeasures μ_1 and μ_2 are absolutely μ continuous on $\mathfrak{S}(F \cap \mathscr{R})$. Now the assertion of the corollary immediately follows from the next.

THEOREM 16. *Let $\mathscr{S}$ be a σ-ring of subsets of a set T, let μ and ν be two submeasures on $\mathscr{S}$ and let ν be absolutely μ continuous. Then ν is a continuous function on the pseudo-metrizable topological space $(\mathscr{S}, \mathscr{T}_\mu)$, where $\mathscr{T}_\mu$ is the topology on $\mathscr{S}$ induced by the uniformity $\mathscr{U}_\mu$. Further, the functions of two variables $u(A, B) = \nu(A \cup B)$ and $v(A, B) = \nu(A \cap B)$, $A, B \in \mathscr{S}$ are continuous on the product $(\mathscr{S}, \mathscr{T}_\mu) \times (\mathscr{S}, \mathscr{T}_\mu)$. If ν is an uniform submeasure, then the functions ν, u and v are uniformly continuous on $(\mathscr{S}, \mathscr{T}_\mu)$ and on $(\mathscr{S}, \mathscr{T}_\mu) \times (\mathscr{S}, \mathscr{T}_\mu)$ respectively.*

Proof. We prove the continuity of ν on $(\mathscr{S}, \mathscr{T}_\mu)$. The remaining assertions may be proved similarly. Let $A \in \mathscr{S}$ and let $\varepsilon > 0$. Evidently $E - (A \Delta E) \subset A \subset E \cup (A \Delta E)$ for each set $E \in \mathscr{S}$. According to the monotonicity and subadditive continuity of ν there is a $\delta_1 > 0$ such that $E \in \mathscr{S}$ $\nu(A \Delta E) < \delta_1$ implies

$$\nu(E) - \varepsilon \leqslant \nu\big(E - (A \Delta E)\big) \leqslant \nu(A) \leqslant \nu\big(E \cup (A \Delta E)\big) \leqslant \nu(E) + \varepsilon.$$

Since ν is absolutely μ continuous, there is a $\delta > 0$ such that $E \in \mathscr{S}$, $\mu(A \Delta E) < \delta$ implies $\nu(A \Delta E) < \delta_1$. Hence the continuity of ν on $(\mathscr{S}, \mathscr{T}_\mu)$ is clear.

Let $\mathscr{S}$ be a σ-ring of subsets of a set T and let μ: $\mathscr{S} \to [0, +\infty)$ be a submeasure. Then, clearly, a monotone set function ν: $\mathscr{S} \to [0, +\infty)$ which is uniformly continuous on the pseudo-metrizable uniform space $(\mathscr{S}, \mathscr{U}_\mu)$ is an absolutely μ continuous submeasure on $\mathscr{S}$.

From Definition 3, Theorem D in § 50 in [10] and from Theorem 14 we immediately have the following

THEOREM 17. *Let T be a locally compact Hausdorff topological space and let μ: $\mathfrak{S}(\check{\mathscr{B}})\to[0, +\infty)$ be a regular Borel or Baire submeasure. Then the lattice $\mathscr{C}_0$ of compact G_δ sets is dense the pseudo-metrizable uniform space $(\mathfrak{S}(\check{\mathscr{B}}), \mathscr{U}_\mu)$. If μ: $\check{\mathscr{B}} \to [0, +\infty)$ is a regular Borel submeasure, then for each Borel set $A \in \mathscr{B}$ there is a Baire set $E \in \check{\mathscr{B}}_0$ such that $\mu(A \Delta E) = 0$.*

From here and from Theorem 16 we easily obtain the next

COROLLARY. *Let T be a locally compact Hausdorff topological space and let μ_1, μ_2: $\check{\mathscr{B}} \to [0, +\infty)$ be two regular Borel or Baire submeasures. Then $\mu_1 \equiv \mu_2$ on $\check{\mathscr{B}}$ if and only if $\mu_1 \equiv \mu_2$ on $\mathscr{C}_0$.*

§ 2. Extension of submeasures

For the special case of subadditive submeasures with finite variations the extension theorem was proved in [3], § 1. For arbitrary subadditive submeasures it has recently been proved in [9]. The next theorem is the extension theorem for arbitrary submeasures. The idea of the proof is motivated by [12]. We note that for uniform submeasures the conditions II and III below are automatic.

THEOREM 18 (The extension theorem for submeasures). *Let $\mathscr{R}$ be a ring of subsets of a set T and let $\mu_0\colon \mathscr{R}\to[0,+\infty)$ be a submeasure. Then μ_0 can be uniquely extended to a submeasure $\mu\colon \mathfrak{S}(\mathscr{R})\to[0,+\infty]$, $\mathfrak{S}(\mathscr{R})$ being the smallest σ-ring containing $\mathscr{R}$, if and only if the following conditions are fulfilled:*

I. *if $A_n\in\mathscr{R}$, $n=1,2,\ldots$ are pairwise disjoint sets, then $\lim\limits_{n\to\infty}\mu_0(A_n)=0$,*

II. *if $A_n\in\mathscr{R}$, $n=1,2,\ldots$ is an increasing sequence of sets, then for every $\varepsilon>0$ there is a $\delta>0$ such that $B\in\mathscr{R}$, $\mu_0(B)<\delta$ implies $\mu_0(A_n\cup B)\leqslant\mu_0(A_n)+\varepsilon$ for every $n=1,2,\ldots$, and*

III. *if $A_{n,k}\in\mathscr{R}$, $n,k=1,2,\ldots$, if $A_{n,k}\nearrow A_n$ for every n and if $A_n\searrow$, then for every $\varepsilon>0$ there is a $\delta>0$ such that $B\in\mathscr{R}$, $\mu_0(B)<\delta$ implies*

$$\mu^+(A_n-B)\geqslant\mu^+(A_n)-\varepsilon\quad \textit{for every } n=1,2,\ldots,$$

where

$$\mu^+(A_n-B)=\lim_{k\to\infty}\mu_0(A_{n,k}-B)\quad \textit{and}\quad \mu^+(A_n)=\lim_{k\to\infty}\mu_0(A_{n,k}).$$

At the same time, if μ_0 is uniform, subadditive or additive, then μ is also uniform, subadditive or additive respectively.

Proof. The necessity of condition I is a consequence of assertion c) of Theorem 1. For uniform submeasures the conditions II and III are automatically fulfilled. Their necessity for general submeasures will be a consequence of the Vitali-Hahn-Saks theorem for submeasures from part II and of Example 4 in § 1. The uniqueness of the extension is stated in Corollary 3 of Theorem 15; however, it will also be clear from the proof of the sufficiency part given below.

We prove the remaining part of the theorem in four lemmas. First we introduce some notations. For a class $\mathscr{C}$ of subsets of a set T we denote by $\mathscr{C}_\sigma$ ($\mathscr{C}_\delta$) the class of limits of increasing (decreasing) sequences of sets of $\mathscr{C}$.

LEMMA A. *The μ_0 on $\mathscr{R}$ can be extended to a set function μ^+ on $\mathscr{R}_\sigma$ having the following properties:*

a) *μ^+ is non-negative and monotone,*

b) *for every $A \in \mathscr{R}_\sigma$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $B \in \mathscr{R}_\sigma$, $\mu^+(B) < \delta$ implies $\mu^+(A \cup B) \leqslant \mu^+(A) + \varepsilon$,*

c) *for every $A \in \mathscr{R}_\sigma$ and $\varepsilon > 0$ there is an $E \in \mathscr{R}$ such that $E \subset A$ and $\mu^+(A - E) < \varepsilon$. Thus by* b) *$\mu^+$ is finite on R_σ,*

d) *if $A_n \in \mathscr{R}_\sigma$, $n = 1, 2, \ldots$ and $A_n \nearrow A$, then $\mu^+(A) = \lim_{n\to\infty} \mu^+(A_n)$,*

e) *if $A_n \in \mathscr{R}_\sigma$, $n = 1, 2, \ldots$ and $A_n \searrow A \in \mathscr{R}_\sigma$, then $\mu^+(A) = \lim_{n\to\infty} \mu^+(A_n)$, and*

f) *if μ_0 is uniform, subadditive or additive, then μ^+ is also uniform, subadditive or additive respectively on $\mathscr{R}_\sigma$.*

Proof. Let $A \in \mathscr{R}_\sigma$ and let $E_n, F_n \in \mathscr{R}$, $n = 1, 2, \ldots$ be two increasing sequences converging to A. Then

$$\lim_{n\to\infty} \mu_0(E_n) = \lim_{n\to\infty} [\lim_{k\to\infty} \mu_0(E_n \cap F_k)] \leqslant \lim_{k\to\infty} \mu_0(F_k)$$
$$= \lim_{k\to\infty} [\lim_{n\to\infty} \mu_0(E_n \cap F_k)] \leqslant \lim_{n\to\infty} \mu_0(E_n)$$

by the monotonicity and monotone continuity of μ_0 on $\mathscr{R}$. Hence for $A \in \mathscr{R}_\sigma$ we may, without ambiquity, define $\mu^+(A) = \lim_{n\to\infty} \mu_0(E_n)$, where $E_n \in \mathscr{R}$, $n = 1, 2, \ldots$ and $E_n \nearrow A$. Obviously μ^+ is an extension of μ_0.

Now assertions a), d) and f) are evident from the definition of μ^+, while b) follows immediately from condition II and c) from condition I. It remains to prove e). Let $\varepsilon > 0$. Suppose first that $A_n \searrow \emptyset$. With the aid of b) and c) we take $E_n \in \mathscr{R}$, $n = 1, 2, \ldots$ so that $E_n \subset A_n$ and $\mu^+[\bigcup_{i=1}^{n} (A_i - E_i)] < \frac{1}{2}\varepsilon$ for every n. Put $D = \bigcup_{n=1}^{\infty} (A_n - E_n)$ and $F_n = \bigcap_{i=1}^{n} E_i$, $n = 1, 2, \ldots$ Then $F_n \in \mathscr{R}$, $D \in \mathscr{R}_\sigma$ and $\mu^+(D) \leqslant \frac{1}{2}\varepsilon$ by d). Since $F_n \subset A_n$, $F_n \searrow \emptyset$ and thus by the continuity of μ_0 on $\mathscr{R}$ and by b) there is an n_0 such that $\mu^+(D \cup F_n) < \varepsilon$ for $n \geqslant n_0$. But $A_n \subset (A_n - F_n) \cup F_n \subset D \cup F_n$; hence $\mu^+(A_n) < \varepsilon$ for $n \geqslant n_0$ by the monotonicity of μ^+ on $\mathscr{R}_\sigma$. Since $\varepsilon > 0$ was arbitrary, $\lim_{n\to\infty} \mu^+(A_n) = 0$.

Now let $A \in \mathscr{R}_\sigma$, let $G_n \in \mathscr{R}$, $n = 1, 2, \ldots$ and let $G_n \nearrow A$. Then $A_n - G_n \in \mathscr{R}_\sigma$, $A_n - G_n \searrow \emptyset$; hence $\lim_{n\to\infty} \mu^+(A_n - G_n) = 0$. Thus by b) there is an n_0 such that $\mu^+(A) \leqslant \mu^+(A_n) = \mu^+[A \cup (A_n - G_n)] \leqslant \mu^+(A) + \varepsilon$ for $n \geqslant n_0$. Since $\varepsilon > 0$ was arbitrary, the assertion e) and also the lemma are proved.

LEMMA B. *Denote by $\mathfrak{H}\mathfrak{S}(\mathscr{R})$ the hereditary σ-ring generated by $\mathscr{R}$, i. e., $A \in \mathfrak{H}\mathfrak{S}(\mathscr{R})$ iff $A \subset D$ for some $D \in \mathscr{R}_\sigma$. On $\mathfrak{H}\mathfrak{S}(\mathscr{R})$ we define a set function μ^*, called the outer submeasure induced by μ_0, by the equality $\mu^*(A) = \inf\{\mu^+(D), A \subset D \in \mathscr{R}_\sigma\}, A \in \mathfrak{H}\mathfrak{S}(\mathscr{R})$. Then μ^* extends μ^+ and has the following properties:*

a) *μ^* is non-negative, finite and monotone.*

b) *For every* $A \in \mathfrak{H}\mathfrak{S}(\mathscr{R})$ *and* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $B \in \mathfrak{H}\mathfrak{S}(\mathscr{R})$ $\mu^*(B) < \delta$ *implies* $\mu^*(A \cup B) \leqslant \mu^*(A) + \varepsilon$.

c) *For every* $A \in \mathfrak{H}\mathfrak{S}(\mathscr{R})$ *there is a set* $D \in \mathscr{R}_{\sigma\delta}$ *such that* $A \subset D$ *and* $\mu^*(A) = \mu^*(D)$. *If* $A_n \in \mathscr{R}_\sigma$, $n = 1, 2, \ldots$ *and* $A_n \searrow A$, *then* $\lim_{n\to\infty} \mu^+(A_n) = \mu^*(A)$, *and*

d) *If* μ^+ *is uniform or subadditive on* $\mathscr{R}_\sigma$, *then* μ^* *is also uniform or subadditive on* $\mathfrak{H}\mathfrak{S}(\mathscr{R})$, *respectively.*

Proof. Assertions a) and d) are clear from the definion of μ^*.

b) Take $E \in \mathscr{R}_\sigma$ so that $A \subset E$ and $\mu^+(E) \leqslant \mu^*(A) + \frac{1}{2}\varepsilon$. It follows from assertion b) of Lemma A that there is a $\delta > 0$ such that $F \in \mathscr{R}_\sigma$, $\mu^+(F) < \delta$, implies $\mu^+(E \cup F) \leqslant \mu^+(E) + \frac{1}{2}\varepsilon$. Now if $B \in \mathfrak{H}\mathfrak{S}(\mathscr{R})$ and $\mu^*(B) < \delta$, then by the definition of $\mu^*(B)$ there is a set $F \in \mathscr{R}_\sigma$ such that $B \subset F$ and $\mu^+(F) < \delta$. Hence

$$\mu^*(A \cup B) \leqslant \mu^+(E \cup F) \leqslant \mu^+(E) + \tfrac{1}{2}\varepsilon \leqslant \mu^*(A) + \varepsilon.$$

c) Let $A \in \mathfrak{H}\mathfrak{S}(\mathscr{R})$. Since $\mathscr{R}_\sigma$ is a lattice and since μ^+ is monotone on $\mathscr{R}_\sigma$, there is a decreasing sequence $D_n \in \mathscr{R}_\sigma$, $n = 1, 2, \ldots$, $D_n \searrow D \supset A$ such that $\mu^*(A) = \lim_{n\to\infty} \mu^+(D_n) \geqslant \mu^*(D) \geqslant \mu^*(A)$.

Now if $A_n \in \mathscr{R}_\sigma$, $n = 1, 2, \ldots$, and if $A_n \searrow A$, then

$$\begin{aligned}\lim_{n\to\infty} \mu^+(A_n) &\leqslant \lim_{n\to\infty} [\lim_{k\to\infty} \mu^+(A_n \cup D_k)] = \lim_{k\to\infty} [\lim_{n\to\infty} \mu^+(A_n \cup D_k)] \\ &= \lim_{k\to\infty} \mu^+(D_k) = \mu^*(A) \leqslant \lim_{n\to\infty} \mu^+(A_n)\end{aligned}$$

by the monotonicity of μ^+ and assertion e) of Lemma A. Thus c) and also the lemma is proved.

LEMMA C. *Put* $\mathscr{S} = \{A \in \mathfrak{H}\mathfrak{S}(\mathscr{R})$, *there are* $E \in \mathscr{R}_{\sigma\delta}$ *and* $F \in \mathscr{R}_{\delta\sigma}$ *such that* $F \subset A \subset E$ *and* $\mu^*(E - F) = 0\}$. *Then:*

a) *for every* $A \in \mathscr{S}$ *and* $\varepsilon > 0$, *there are* $D \in \mathscr{R}_\sigma$ *and* $G \in \mathscr{R}_\delta$ *such that* $G \subset A \subset D$ *and* $\mu^+(D - G) < \varepsilon$, *and*

b) $\mathscr{S}$ *is a* σ*-ring containing the ring* $\mathscr{R}$.

Proof. a) Let $A \in \mathscr{S}$ and let $\varepsilon > 0$. By the definition of $\mathscr{S}$ there are $E \in \mathscr{R}_{\sigma\delta}$ and $F \in \mathscr{R}_{\delta\sigma}$ such that $F \subset A \subset E$ and $\mu^*(E - F) = 0$. Let $E_n \in \mathscr{R}_\sigma$, $n = 1, 2, \ldots$ and let $E_n \searrow E$, and let $F_n \in \mathscr{R}_\delta$, $n = 1, 2, \ldots$ and let $F_n \nearrow F$. Then clearly $E_n - F_n \in \mathscr{R}_\sigma$ and $E_n - F_n \searrow E - F$. Hence $\lim_{n\to\infty} \mu^+(E_n - F_n) = \mu^*(E - F) = 0$ by assertion c) of Lemma B. This proves a).

b) It is easy to verify that $\mathscr{S}$ is a ring containing $\mathscr{R}$. Hence to prove that $\mathscr{S}$ is a σ-ring it is enough to prove that $A_n \in \mathscr{S}$, $n = 1, 2, \ldots$ and $A_n \nearrow A$ implies $A \in \mathscr{S}$.

Let $A_n \in \mathscr{S}$, $n = 1, 2, \ldots$ and let $A_n \nearrow A$. According to a) and assertion b) of Lemma A for $n = 1, 2, \ldots$ we take $F_{k,n} \in \mathscr{R}_\delta$ and $E_{k,n} \in \mathscr{R}_\sigma$, $k = 1, 2, \ldots$

so that $F_{k,n} \subset A_n \subset E_{k,n}$ and $\mu^+[\bigcup_{i=1}^{n}(E_{k,i}-F_{k,i})] < 1/k$ for every $n, k = 1, 2, \ldots$ Evidently $E_k = \bigcup_{n=1}^{\infty} E_{k,n} \in \mathscr{R}_\sigma$, $F_k = \bigcup_{n=1}^{\infty} F_{k,n} \in \mathscr{R}_{\delta\sigma}$, $F_k \subset A \subset E_k$ and $E_k - F_k \subset \bigcup_{n=1}^{\infty}(E_{k,n}-F_{k,n})$ for every $k = 1, 2, \ldots$ Hence

$$\mu^*(E_k - F_k) \leqslant \mu^+[\bigcup_{n=1}^{\infty}(E_{k,n}-F_{k,n})] \leqslant \frac{1}{k} \quad \text{for every } k = 1, 2, \ldots$$

by the monotonicity of μ^* and assertion d) of Lemma A. Since $E = \bigcap_{k=1}^{\infty} E_k \in \mathscr{R}_{\sigma\delta}$, $F = \bigcup_{k=1}^{\infty} F_k \in \mathscr{R}_{\delta\sigma}$, $F \subset A \subset E$ and $E - F \subset E_k - F_k$ for every $k = 1, 2, \ldots$, $\mu^*(E-F) = 0$ by the monotonicity of μ^*. Thus b) and also the lemma are proved.

LEMMA D. a) *Let $A_n \in \mathscr{S}$, $n = 1, 2, \ldots$ and let $A_n \searrow A$. Then $\mu^*(A) = \lim_{n\to\infty} \mu^*(A_n)$.*

b) *For every $A \in \mathscr{S}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $B \in \mathfrak{H}\mathfrak{S}(\mathscr{R})$, $\mu^*(B) < \delta$ implies $\mu^*(A - B) \geqslant \mu^*(A) - \varepsilon$.*

c) *$\mu^*(A) = \sup\{\mu^*(E), A \supset E \in \mathscr{R}_\delta\}$ for every $A \in \mathscr{S}$.*

d) *The restriction of μ^* to $\mathscr{S}$, which we denote by μ, is a submeasure extending μ_0 and $\mathscr{S}$ is the completion of $\mathfrak{S}(\mathscr{R})$ with respect to μ. At the same time, if μ_0 is uniform, subadditive or additive, then μ is also uniform, subadditive, or additive on $\mathfrak{S}(\mathscr{R})$ respectively.*

Proof. a) Suppose first that $A_n \searrow \emptyset$ and let $\varepsilon > 0$. According to assertion a) of Lemma C and assertion b) of Lemma B there is a sequence $D_n \in \mathscr{R}_\sigma$, $n = 1, 2, \ldots$ such that $D_n \searrow$, $D_n \supset A_n$ and $\mu^*[\bigcup_{i=1}^{n}(D_i - A_i)] < \varepsilon$ for every n. Further, it follows from the definition of μ^* that there is a sequence $G_n \in \mathscr{R}_\sigma$, $n = 1, 2, \ldots$ such that $G_n \supset D_n - A_n$ and $\mu^+(\bigcup_{i=1}^{n} G_i) < \varepsilon$ for every n. Put $G = \bigcup_{n=1}^{\infty} G_n$. Then $G \in \mathscr{R}_\sigma$, $G \supset \bigcup_{n=1}^{\infty}(D_n - A_n)$ and $\mu^+(G) \leqslant \varepsilon$ by assertion d) of Lemma A. Since $A_n \searrow \emptyset$, $\bigcap_{n=1}^{\infty} D_n = \bigcap_{n=1}^{\infty} D_n - \bigcap_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty}(D_n - A_n) \subset G$. Hence

$$\lim_{n\to\infty} \mu^*(A_n) \leqslant \lim_{n\to\infty} \mu^+(D_n) = \mu^*(\bigcap_{n=1}^{\infty} D_n) \leqslant \mu^+(G) \leqslant \varepsilon$$

by the monotonicity of μ^* and assertion c) of Lemma B. Since $\varepsilon > 0$ was arbitrary, $\lim_{n\to\infty} \mu^*(A_n) = 0$.

Now if $A_n \searrow A$, then $A_n - A \searrow \emptyset$; hence $\lim_{n\to\infty} \mu^*(A_n - A) = 0$. Hence, using the monotonicity of μ^* and assertion b) of Lemma B, we immediately have

$$\mu^*(A) \leqslant \lim_{n\to\infty} \mu^*(A_n) = \lim_{n\to\infty} \mu^*[A \cup (A_n - A)] = \mu^*(A).$$

Thus a) is proved.

b) According to Lemma C there is a set $D \in \mathscr{R}_{\sigma\delta}$ such that $A \subset D$ and $\mu^*(D-A) = 0$. Further, it follows from condition III and assertion c) of Lemma B that there is a $\delta > 0$ such that $E \in \mathscr{R}$, $\mu_0(E) < \delta$ implies $\mu^*(A-E) = \mu^*(D-E) \geqslant \mu^*(D) - \varepsilon = \mu^*(A) - \varepsilon$. Now let $E_n \in \mathscr{R}$, $n = 1, 2, \ldots$, let $E_n \nearrow F$ and let $\mu^+(F) < \delta$. Then $\mu^*(A-F) = \lim_{n\to\infty} \mu^*(A-E_n) \geqslant \mu^*(A) - \varepsilon$. Finally, let $B \in \mathfrak{H}\mathfrak{S}(\mathscr{R})$ and let $\mu^*(B) < \delta$. Then the definition of $\mu^*(B)$ implies that there is a set $F \in \mathscr{R}_\sigma$ such that $B \subset F$ and $\mu^+(F) < \delta$. Hence

$$\mu^*(A-B) \geqslant \mu^*(A-F) \geqslant \mu^*(A) - \varepsilon$$

by the monotonicity of μ^*. Thus b) is proved.

c) follows immediately from b) and assertion a) of Lemma C.

d) Obviously assertions a), b) and d) of Lemma B and assertions c) and b) imply that μ is a submeasure on $\mathfrak{S}(\mathscr{R})$. It is easy to verify that $\mathscr{S}$ is the completion of $\mathfrak{S}(\mathscr{R})$ with respect to μ.

The definition of μ and assertion d) of Lemma B imply that if μ_0 is uniform or subadditive, then μ is also uniform or subadditive on $\mathscr{S}$ respectively. Finally, suppose that μ_0 is additive. Then, by the subadditivity of μ_0, μ is subadditive. Thus by c) μ will be additive if we prove that $\mu(A \cup B) \geqslant \mu(A) + \mu(B)$ for $A, B \in \mathscr{R}_\delta$, $A \cap B = \emptyset$. To see this let $A_n, B_n \in \mathscr{R}$, let $A_n \searrow A$, $B_n \searrow B$ and let $\varepsilon > 0$. Then according to b) and assertions b) and e) of Lemma A there is an n_0 such that

$$\mu(A \cup B) + \tfrac{1}{2}\varepsilon \geqslant \mu(A_{n_0} \cup B_{n_0}) = \mu[A_{n_0} \cup (B_{n_0} - A_{n_0})] \geqslant \mu(A) + \mu(B - A_{n_0})$$
$$= \mu(A) + \mu[B - (A_{n_0} - A)] \geqslant \mu(A) + \mu(B) - \tfrac{1}{2}.$$

Since $\varepsilon > 0$ was arbitrary, $\mu(A \cup B) \geqslant \mu(A) + \mu(B)$, which is, what we wanted to show. Thus the lemma and also the theorem are proved.

From this theorem and from Corollary 3 of Theorem 15 we immediately have the following

COROLLARY. *Let $\mathscr{R}$ be a ring of subsets of a set T, let $\mathfrak{P}(\mathscr{R})$ be the smallest δ-ring containing $\mathscr{R}$ and let μ_0: $\mathscr{R} \to [0, +\infty)$ be a submeasure. Then there is a unique submeasure μ: $\mathfrak{P}(\mathscr{R}) \to [0, +\infty)$ extending μ_0 if and only if for each set $E \in \mathscr{R}$ the submeasure μ_0^E: $\mathscr{R} \to [0, +\infty)$ defined by the equality $\mu_0^E(F) = \mu_0(E \cap F)$, $F \in \mathscr{R}$ satisfies the conditions* I, II *and* III *of the preceding theorem.*

THEOREM 19. *Suppose we have the notations from the preceding theorem and its proof. Then:*

a) *if $A_n \in \mathfrak{H}\mathfrak{S}(\mathscr{R})$, $n = 1, 2, \ldots$ and if $A_n \nearrow A$, then $\mu^*(A) = \lim_{n\to\infty} \mu^*(A_n)$, and*

b) *put $\overline{\mathscr{R}} = \{A \in \mathfrak{H}\mathfrak{S}(\mathscr{R}), \inf\{\mu^*(A \,\Delta\, E), E \in \mathscr{R}\} = 0\}$, then $\overline{\mathscr{R}} = \mathscr{S}$.*

Proof. a) According to assertion c) of Lemma B, for every $n = 1, 2, \ldots$ there is a set $D_n \in \mathscr{R}_{\sigma\delta}$ such that $A_n \subset D_n$ and $\mu^*(A_n) = \mu^*(D)$.

Put $D'_n = \bigcap_{i=n}^{\infty} D_n$, $n = 1, 2, \dots$ Then $D'_n \in \mathscr{R}_{\sigma\delta}$, $A_n \subset D'_n$, $\mu^*(A_n) = \mu^*(D'_n)$, $D'_n \nearrow D \in \mathscr{S}$ and $A \subset D$. Hence

$$\mu^*(A) \geqslant \lim_{n\to\infty} \mu^*(A_n) = \lim_{n\to\infty} \mu^*(D'_n) = \mu^*(D) \geqslant \mu^*(A)$$

by the monotonicity and assertion d), of Lemma D. Thus a), is proved.

b) Since $\mathscr{R} \subset \mathscr{S}$ and since by Theorem 14 the pseudo-metrizable uniform space $(\mathscr{S}, \mathscr{U}_\mu)$ is complete, $\overline{\mathscr{R}} \subset \mathscr{S}$. (By Lemma C, $\mathscr{S}$ contains all μ^*-zero sets.) On the other hand, it follows from assertion c) of Lemma A, assertion a) of Lemma C and from the subadditive continuity of μ on $\mathscr{S}$ that $\mathscr{S} \subset \overline{\mathscr{R}}$. Thus the theorem is proved.

Remark. Let $\mathscr{S}$ be a σ-ring of subsets of a set T and let $\mu: \mathscr{S} \to [0, +\infty)$ be a submeasure. We say that a set $A \subset T$ is *locally $\mathscr{S}$-measurable* iff $A \cap E \in \mathscr{S}$ for every $E \in \mathscr{S}$. Obviously the class $\mathscr{S}_l$ of all locally $\mathscr{S}$-measurable subsetso f T is a σ-algebra. According to Theorem 4 let $Q \in \mathscr{S}$ be an exhausting set for the submeasure μ on $\mathscr{S}$. For every $A \in \mathscr{S}_l$ put $\mu_l(A) = \mu(A \cap Q)$. Then it is easy to verify that μ_l on $\mathscr{S}_l$ is a submeasure extending the submeasure μ on $\mathscr{S}$. Let μ_l^* be the outer submeasure induced by the submeasre μ_l, see Lemma B above. Then by assertion a) of the preceding theorem $\mu_l^* (\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \mu_l^*(A_n)$ for every increasing sequence $A_n \subset T$, $n = 1, 2, \dots$ Hence by the terminology of [14], III, § 2, D 18, μ_l^* is a Choquet $\mathscr{S}$-capacity on T. Moreover, if $\mathscr{S} = \mathfrak{S}(\mathscr{R})$ for some ring $\mathscr{R}$ of subsets of T, then μ_l^* is a Choquet $\mathscr{R}$-capacity on T.

Definition 4 (see [8], [18]) and [19]). Let $\mathscr{A}$ be a class of subsets of a set T. We say that a set function $\mu: \mathscr{A} \to [0, +\infty)$ has the *property* (p) iff for every $\varepsilon > 0$ there is a natural number N_ε such that for any collection $A_1, \dots, A_{N_\varepsilon} \in \mathscr{A}$ of pairwise disjoint sets $\mu(A_n) < \varepsilon$ at least for one $n \in \{1, 2, \dots, N_\varepsilon\}$.

Evidently a non-negative bounded and additive set function on a ring of subsets has the property (p). If $\mu: \mathscr{A} \to [0, +\infty)$ has the property (p) and $\nu: \mathscr{A} \to [0, +\infty)$ is absolutely μ continuous, then clearly ν also has the property (p). Thus, if the semivariation $\hat{m}$ of an operator-valued measure $\boldsymbol{m}: \mathscr{S} \to L(\boldsymbol{X}, \boldsymbol{Y})$, $\mathscr{S}$ being a σ-ring and $\boldsymbol{X}, \boldsymbol{Y}$ being Banach spaces, is continuous on $\mathscr{S}$, then by the *-Theorem in [6], section 1.1, $\hat{m}$ has the property (p) on $\mathscr{S}$. More generally, the same is true for the L_1-norm $\hat{m}(\boldsymbol{g}, \cdot)$ of a measurable function $\boldsymbol{g}$, see Theorem 5 in [7]. Obviously a Baire or a regular Borel submeasure $\mu: \mathfrak{S}(\mathscr{B}_\wedge) \to [0, +\infty)$ has the property (p) on $\mathfrak{S}(\mathscr{B}_\wedge)$ if and only if its restriction to $\mathscr{C}_0$ has the property (p).

Theorem 20. *Let $\mathscr{R}$ be a ring of subsets of a set T, let $\mathfrak{S}(\mathscr{R})$ be the smallest σ-ring containing $\mathscr{R}$ and let $\mu: \mathfrak{S}(\mathscr{R}) \to [0, +\infty)$ be a submeasure. Then μ has the property (p) on $\mathfrak{S}(\mathscr{R})$ if and only if its restriction to $\mathscr{R}$ has the property (p) on $\mathscr{R}$.*

Proof. Suppose that μ has the property (p) on $\mathcal{R}$ and let $\varepsilon > 0$. Then there is a natural $N_{\varepsilon/2}$ such that, for any collection $E_1, \ldots, E_{N_{\varepsilon/2}} \in \mathcal{R}$ of pairwise disjoint sets, $\mu(E_n) < \frac{1}{2}\varepsilon$ for at least one $n \in \{1, \ldots, N_{\varepsilon/2}\}$. To prove that μ has the property (p) on $\mathfrak{S}(\mathcal{R})$ by assertion c) of Lemma D above it is enough to prove that μ has the property (p) on $\mathcal{R}_\delta$. To see this, let $A_1, \ldots, A_{N_{\varepsilon/2}} \in \mathcal{R}_\delta$ be a collection of pairwise disjoint sets. For every $n = 1, \ldots, N_{\varepsilon/2}$ take a sequence $E_{n,k} \in \mathcal{R}$, $k = 1, 2, \ldots$ so that $E_{n,k} \searrow A_n$ and put $E'_{n,k} = E_{n,k} - \bigcup_{i=1}^{n-1} E_{i,k}$ for $n = 1, \ldots, N_{\varepsilon/2}$ and $k = 1, 2, \ldots$ Then $E'_{1,k}, \ldots, E'_{N_{\varepsilon/2},k} \in \mathcal{R}$ are pairwise disjoint for every $k = 1, 2, \ldots$ and $\lim_{k\to\infty} E'_{n,k} = A_n$ for every $n = 1, \ldots, N_{\varepsilon/2}$. Hence by assertion b) of Theorem 1 there is a k_0 such that $|\mu(E'_{n,k_0}) - \mu(A_n)| < \frac{1}{2}\varepsilon$ for every $n = 1, 2, \ldots, N_{\varepsilon/2}$. Since the sets $E'_{1,k_0}, \ldots, E'_{N_{\varepsilon/2},k_0} \in \mathcal{R}$ are pairwise disjoint, there is an $n \in \{1, \ldots, N_{\varepsilon/2}\}$ such that $\mu(E'_{n,k_0}) < \frac{1}{2}\varepsilon$. Thus $\mu(A_n) \leqslant \mu(E'_{n,k_0}) + |\mu(A_n) - \mu(E'_{n,k_0})| < \varepsilon$, and the theorem is proved.

Let $\mathcal{R}$ be a ring of subsets of a set T and let $\mu;\ \mathcal{R} \to [0, +\infty)$ be a submeasure having the property (p). Then obviously μ satisfies the condition I in Theorem 18. Thus from Theorem 18 and from the preceding theorem we immediately have the following

COROLLARY. *Let $\mathcal{R}$ be a ring of subsets of a set T, let $\mathfrak{S}(\mathcal{R})$ be the smallest σ-ring containing $\mathcal{R}$ and let μ_0: $\mathcal{R} \to [0, +\infty)$ be a uniform or subadditive submeasure having the property (p). Then μ_0 can be uniquely extended to a uniform respectively subadditive submeasure μ: $\mathfrak{S}(\mathcal{R}) \to [0, +\infty)$ having the property (p) on $\mathfrak{S}(\mathcal{R})$.*

The author has been unable to solve the following

PROBLEM. Find a submeasure on a σ-ring, if possible subadditive, which does not have the property (p).

§ 3. Extension of subcontents

DEFINITION 5. Let T be a locally compact Hausdorff topological space. We say that a set function $\mu_{0\wedge}: \mathscr{C}_\wedge \to [0, +\infty)$ is a *Borel* (*Baire*) *subcontent* iff:

1) $\mu_{0\wedge}$ is monotone and $\mu_{0\wedge}(\emptyset) = 0$,

2) $\mu_{0\wedge}$ is subadditively continuous from the right; for every $C \in \mathscr{C}_\wedge$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $D \in \mathscr{C}_\wedge$, $\mu_{0\wedge}(D) < \delta$ implies $\mu_{0\wedge}(C \cup D) \leqslant \mu_{0\wedge}(C) + \varepsilon$, and

3) if $C_n \in \mathscr{C}_\wedge$, $n = 1, 2, \ldots$ is a sequence of pairwise disjoint sets, then $\lim_{n\to\infty} \mu_{0\wedge}(C_n) = 0$.

We say that a *Borel* (*Baire*) *subcontent* $\mu_{0\wedge}$ *is regular* iff $\mu_{0\wedge}(C) = \inf\{\mu_{0\wedge}(D),\ C \subset D^0 \subset D \in \mathscr{C}_\wedge\}$ for each set $C \in \mathscr{C}_\wedge$, where D^0 denotes the interior of D.

THEOREM 21. *Let T be a locally compact Hausdorff topological space and let $\mu_{00}\colon \mathscr{C}_0 \to [0, +\infty)$ be a regular Baire subcontent. For every $C \in \mathscr{C}$ put $\mu_0(C) = \inf\{\mu_{00}(D),\ C \subset D^0 \subset D \in \mathscr{C}_0\}$. Then μ_0 is the unique extension of μ_{00} to a regular Borel subcontent. At the same time, if μ_{00} is uniformly subadditively continuous from the right, subadditive or additive, then μ_0 also has the corresponding property on $\mathscr{C}$.*

Proof. Obviously μ_0 is monotone and regular; hence, by Theorem D in § 50 of [10], μ_0 is the unique monotone and regular extension of μ_{00}. Since μ_{00} is monotone and $\mathscr{C}_0$ is a lattice closed with respect to the forming of any intersections, for each set $C \in \mathscr{C}$ there is a set $D \in \mathscr{C}_0$ such that $C \subset D$ and $\mu_0(C) = \mu_{00}(D)$. Hence from the monotonicity of μ_0 we immediately have the subadditive continuity from the right of μ_0. The same arguments imply that if μ_{00} is uniformly subadditively continuous from the right or subadditive, then μ_0 has the same property on $\mathscr{C}$. Suppose now that μ_{00} is additive on $\mathscr{C}_0$, let $C_1, C_2 \in \mathscr{C}$ and let $C_1 \cap C_2 = \emptyset$. Since T is Hausdorff, C_1 and C_2 can be separated by open sets, hence owing to Theorem D in § 50 of [10] there are $D_1, D_2 \in \mathscr{C}_0$ such that $D_1 \cap D_2 = \emptyset$, $C_1 \subset D_1^0$ and $C_2 \subset D_2^0$. Take $\varepsilon > 0$ and $D \in \mathscr{C}_0$ so that $D \supset (C_1 \cup C_2)$ and $\mu_0(C_1 \cup C_2) + \varepsilon \geqslant \mu_{00}(D)$. Since

$$\begin{aligned}\mu_{00}(D) &\geqslant \mu_{00}[(D \cap D_1) \cup (D \cap D_2)] = \mu_{00}(D \cap D_1) + \mu_{00}(D \cap D_2)\\ &\geqslant \mu_0(C_1) + \mu_0(C_2),\end{aligned}$$

and since $\varepsilon > 0$ was arbitrary, this proves the additivity of μ_0 on $\mathscr{C}$.

It remains to prove that μ_0 has the property 3) of a subcontent. Let $C_n \in \mathscr{C}$, $n = 1, 2, \ldots$ be a sequence of pairwise disjoint sets. According to Theorem D in § 50 in [10] for C_1 there is a sequence $C_{1,k} \in \mathscr{C}_0$, $k = 2, 3, \ldots$ such that $C_1 \subset C_{1,k}^0$ and $C_{1,k} \cap C_k = \emptyset$ for every $k = 2, 3, \ldots$ Put $D_1 = \bigcap_{k=2}^{\infty} C_{1,k}$. Then $C_1 \subset D_1 \in \mathscr{C}_0$ and $\{D_1, C_2, \ldots, C_n, \ldots\}$ is a sequence of pairwise disjoint sets. Repeating the procedure step by step, we obtain a sequence of pairwise disjoint sets $D_n \in \mathscr{C}_0$, $n = 1, 2, \ldots$ such that $C_n \subset D_n$ for every n. Hence $\lim_{n\to\infty} \mu_0(C_n) \leqslant \lim_{n\to\infty} \mu_{00}(D_n) = 0$. Thus the theorem is proved.

THEOREM 22. *Let T be a locally compact Hausdorff topological space and let $\mu_{00}\colon \mathscr{C}_0 \to [0, +\infty)$ be a Baire subcontent having the property (p), see Definition 4 in § 2. For $C \in \mathscr{C}$ put $\mu_0(C) = \inf\{\mu_{00}(D),\ C \subset D^0 \subset D \in \mathscr{C}_0\}$. Then μ_0 is a regular Borel subcontent having the property (p) on $\mathscr{C}$ and $\mu_{00}(C) \leqslant \mu_0(C)$ for each set $C \in \mathscr{C}_0$. At the same time, if μ_{00} is uniformly subadditively continuous from the right, subadditive or additive, then μ_0 also has the corresponding property on $\mathscr{C}$.*

Proof. Clearly μ_0 is monotone, $\mu_0(\emptyset) = 0$ and $\mu_{00}(C) \leqslant \mu_0(C)$ for each set $C \in \mathscr{C}_0$. It follows from Theorem D in § 50 of [10] that μ_0 has the property (p) on $\mathscr{C}$. To prove the regularity of μ_0 let $\varepsilon > 0$ and let $C \in \mathscr{C}$. Then there is a $D \in \mathscr{C}_0$ such that $C \subset D^0$ and $\mu_0(C) + \varepsilon \geqslant \mu_{00}(D)$. By Theorem D in § 50 of [10] there is a set $D_1 \in \mathscr{C}_0$ such that $C \subset D_1^0 \subset D_1 \subset D^0 \subset D \in \mathscr{C}_0$. Hence $\mu_0(C) + \varepsilon \geqslant \mu_{00}(D) \geqslant \mu_0(D_1) \geqslant \mu_0(C)$. Since $\varepsilon > 0$ was arbitrary, $\mu_0(C) = \inf\{\mu_0(D_1),\ C \subset D_1^0 \subset D_1 \in \mathscr{C}\}$, which is the regularity of μ_0 on $\mathscr{C}$. Finally, it follows from the definition of μ_0 and from Theorem D in § 50 of [10] that if μ_{00} is uniformly subadditively continuous from the right, subadditive or additive, then μ_0 also has the corresponding property on $\mathscr{C}$. Thus the theorem is proved.

THEOREM 23 (The extension theorem for subcontents). *Let T be a locally compact Hausdorff topological space and let $\mu_{0\wedge}: \mathscr{C}_\wedge \to [0, +\infty)$ be a Borel (Baire) subcontent. Then $\mu_{0\wedge}$ can be uniquely extended to a regular Borel submeasure $\mu: \mathfrak{S}(\mathscr{B}) \to [0, +\infty)$ if and only if the following conditions are fulfilled:*

I. *$\mu_{0\wedge}$ is a regular Borel (Baire) subcontent,*

II. *if $C_n \in \mathscr{C}_0$, $n = 1, 2, \ldots$ is an increasing sequence and if $\varepsilon > 0$ then there is a $\delta > 0$ such that $D \in \mathscr{C}_0$, $\mu_{0\wedge}(D) < \delta$ implies $\mu_{0\wedge}(C_n \cup D) \leqslant \mu_{0\wedge}(C_n) + \varepsilon$ for every $n = 1, 2, \ldots$, and*

III. *for every $C \in \mathscr{C}_0$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $V \in \mathfrak{S}(\mathscr{B}_0)$, V open and $\mu^+(V) = \sup\{\mu_{0\wedge}(D),\ D \in \mathscr{C}_0,\ D \subset V\} < \delta$ implies $\mu_{0\wedge}(C) \leqslant \mu_{0\wedge}(C - V) + \varepsilon$.*

At the same time, if $\mu_{0\wedge}$ is uniformly subadditively continuous from the right, subadditive or additive, then conditions II and III are not needed, and μ is uniform, subadditive or additive respectively on $\mathfrak{S}(\mathscr{B})$.

Proof. The necessity of condition I is clear, while that of III is a consequence of the regularity and subadditive continuity of μ. The necessity of condition II will be a consequence of the Vitali–Hahn–Saks theorem for submeasures from part II and of Example 4 in § 1. The uniqueness of the exension is a consequence of the Corollary of Theorem 17 in § 1. We prove the sufficiency part as well as the second assertion of the theorem in four lemmas. According to Theorem 21 it is enough to consider a Borel subcontent μ_0.

LEMMA A. *For $V \in \mathscr{U}$ put $\mu^+(V) = \sup\{\mu_0(C),\ C \in \mathscr{C},\ C \subset V\}$. Then:*

a) *μ^+ is monotone,*

b) $\inf\{\mu^+(V - C),\ C \in \mathscr{C}_0,\ C \subset V\} = 0$ *for every $V \in \mathscr{U}$,*

c) *μ^+ is subadditively continuous from the right on $\mathscr{U}$,*

d) *if $V_n \in \mathscr{U}$, $n = 1, 2, \ldots$ is an increasing sequence, then $\mu^+(\bigcup_{n=1}^{\infty} V_n) = \lim_{n\to\infty} \mu^+(V_n)$,*

e) $\mu_0(C) = \inf\{\mu^+(V),\ C \subset V \in \mathscr{U}\}$ *for every $C \in \mathscr{C}$, and*

f) *if* μ_0 *is uniformly subadditively continuous from the right, subadditive or additive, then* μ^+ *also has the corresponding property on* $\mathscr{U}$.

Proof. a) and d) are clear from the definitions. b) is a consequence of the property 3) of the subcontent μ_0, while e) is a consequence of the regularity of μ_0. f) is obvious from Theorem A of § 50 in [10]. It remains to prove c). Let $V \in \mathscr{U}$ and let $\varepsilon > 0$. According to b) there is an increasing sequence $C_n \in \mathscr{C}_0$, $n = 1, 2, \ldots$ such that $\bigcup_{n=1}^{\infty} C_n \subset V$ and $\lim_{n\to\infty} \mu^+(V - C_n) = 0$. It follows from condition II that there is a $\delta > 0$ such that $D \in \mathscr{C}_0$, $\mu_0(D) < \delta$ implies $\mu_0(C_n \cup D) \leqslant \mu_0(C_n) + \frac{1}{3}\varepsilon$ for every $n = 1, 2, \ldots$ Let $W \in \mathscr{U}$ and let $\mu^+(W) < \delta$. The definition of μ^+ and Theorem D in § 50 of [10] imply that there is a set $H \in \mathscr{C}_0$ such that $H \subset V \cup W$ and $\mu^+(V \cup W) \leqslant \mu_0(H) + \frac{1}{3}\varepsilon$. Owing to Theorems A and D in § 50 and Theorem D in § 51 of [10] there are $H_1, H_2 \in \mathscr{C}_0$ and $V' \in \mathscr{U}_0$ such that $H_1 \subset V' \subset V$, $H_2 \subset W$ and $H = H_1 \cup H_2$. Since $\mu_0(H_2) < \delta$ and since $\lim_{n\to\infty} \mu^+(V' - C_n) = 0$, by condition III there is an n_0 such that

$$\mu_0(H) \leqslant \mu_0[H - (V' - C_{n_0})] + \tfrac{1}{3}\varepsilon = \mu_0[(H_1 \cup H_2) - (V' - C_{n_0})] + \tfrac{1}{3}\varepsilon$$
$$\leqslant \mu_0(C_{n_0} \cup H_2) + \tfrac{1}{3}\varepsilon \leqslant \mu_0(C_{n_0}) + \tfrac{2}{3}\varepsilon \leqslant \mu^+(V) + \tfrac{2}{3}\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, c) and also the lemma are proved.

LEMMA B. *Denote by* $\mathscr{S}$ *the collection of all relatively* σ*-compact subsets* $A \subset T$ *for which* $\inf\{\mu^+(V - C),\ C \subset A \subset V,\ C \in \mathscr{C},\ V \in \mathscr{U}\} = 0$. *Then* $\mathscr{S}$ *is a* σ*-ring containing* $\mathfrak{S}(\mathscr{B})$.

Proof. First we prove that $\mathscr{S}$ is a ring. Let $A_1, A_2 \in \mathscr{S}$ and let $\varepsilon > 0$. Then by assertion c) of Lemma A there are $C_1, C_2 \in \mathscr{C}$ and $V_1, V_2 \in \mathscr{U}$ such that $C_1 \subset A_1 \subset V_1$, $C_2 \subset A_2 \subset V_2$ and $\mu^+[(V_1 - C_1) \cup (V_2 - C_2)] < \varepsilon$. Hence

$$\mu^+[(V_1 - C_2) - (C_1 - V_2)] \leqslant \mu^+[(V_1 - C_1) \cup (V_2 - C_2)] < \varepsilon$$

by the monotonicity of μ^+. Since $C_1 - V_2 \subset A_1 - A_2 \subset V_1 - C_2$, $C_1 \cup C_2 \subset A_1 \cup A_2 \subset V_1 \cup V_2$ and since $\varepsilon > 0$ was arbitrary, $A_1 - A_2$, $A_1 \cup A_2 \in \mathscr{S}$; hence $\mathscr{S}$ is a ring.

Now to prove that $\mathscr{S}$ is a σ-ring it is enough to prove that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{S}$ for any sequence $A_n \in \mathscr{S}$, $n = 1, 2, \ldots$ of pairwise disjoint sets. Let $A_n \in \mathscr{S}$ $n = 1, 2, \ldots$ be a such sequence and let $\varepsilon > 0$. According to assertion c) of Lemma A we choose two sequences $C_n \in \mathscr{C}$, $V_n \in \mathscr{U}$, $n = 1, 2, \ldots$ in the following way: $C_1 \subset A_1 \subset V_1$ and $\mu^+(V_1 - C_1) < \frac{1}{2}\varepsilon$, and if C_n, V_n are choosen, then we take C_{n+1}, V_{n+1} so that $C_{n+1} \subset A_{n+1} \subset V_{n+1}$ and so that

$$\mu^+\Big[\bigcup_{i=k}^{n+1} (V_i - C_i)\Big] < \frac{\varepsilon}{2k} \quad \text{for every } k = 1, \ldots, n+1.$$

Then by assertion d) of Lemma A

$$\mu^+[\bigcup_{i=k}^{\infty}(V_i - C_i)] \leqslant \frac{\varepsilon}{2k} \quad \text{for every } k = 1, 2, \ldots$$

Since μ^+ is monotone,

$$\mu^+(\bigcup_{n=1}^{\infty} V_n - \bigcup_{n=1}^{N} C_n) \leqslant \mu^+[\bigcup_{n=1}^{\infty}(V_n - C_n) \cup \bigcup_{n=N+1}^{\infty} V_n] \quad \text{for every } N = 1, 2, \ldots$$

The subadditive continuity of μ^+ from the right implies that there is a $\delta > 0$ such that $W \in \mathscr{U}$, $\mu^+(W) \leqslant \delta$ implies $\mu^+[\bigcup_{n=1}^{\infty}(V_n - C_n) \cup W] \leqslant \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$. We now show that there is an N_0 such that $\mu^+(\bigcup_{n=N_0+1}^{\infty} V_n) \leqslant \delta$, and hence that $\mu^+(\bigcup_{n=1}^{\infty} V_n - \bigcup_{n=1}^{N_0} C_n) \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary and since $\bigcup_{n=1}^{N_0} C_n \subset \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} V_n$, this will prove that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{S}$.

For every $N, p = 1, 2, \ldots$

$$\mu^+(\bigcup_{n=N+1}^{N+p} V_n) = \mu^+[\bigcup_{n=N+1}^{N+p}(V_n - C_n) \cup \bigcup_{n=N+1}^{N+p} C_n]$$
$$\leqslant \mu^+[\bigcup_{n=N+1}^{\infty}(V_n - C_n) \cup \bigcup_{n=N+1}^{N+p} C_n]$$

by the monotonicity of μ^+. Since

$$\mu^+[\bigcup_{n=N+1}^{\infty}(V_n - C_n)] \leqslant \frac{\varepsilon}{2(N+1)},$$

there is an N_1 such that

$$\mu^+[\bigcup_{n=N_1+1}^{\infty}(V_n - C_n)] < \tfrac{1}{2}\delta.$$

According to the subadditive continuity of μ^+ from the right on $\mathscr{U}$, there is a $\delta_1 > 0$ such that $W \in \mathscr{U}$, $\mu^+(W) < \delta_1$ implies

$$\mu^+[\bigcup_{n=N_1+1}^{\infty}(V_n - C_n) \cup W] < \tfrac{1}{2}\delta + \tfrac{1}{2}\delta.$$

It is easy to verify that owing to property 3) of a subcontent, see Definition 5, there is an $N_0 \geqslant N_1$ such that $\mu_0(\bigcup_{n=N_0+1}^{N_0+p} C_n) < \delta_1$ for every $p = 1, 2, \ldots$ According to assertion e) of Lemma A for every $p = 1, 2, \ldots$ there is a set $W_{N_0}^p \in \mathscr{U}$ such that $\bigcup_{n=N_0+1}^{N_0+p} C_n \subset W_{N_0}^p$ and $\mu^+(W_{N_0}^p) < \delta_1$. From these considerations we have the inequality

$$\mu^+\Big(\bigcup_{n=N_0+1}^{N_0+p} V_n\Big) \leqslant \mu^+\Big[\bigcup_{n=N_0+1}^{\infty} (V_n - C_n) \cup \bigcup_{n=N_0+1}^{N_0+p} C_n\Big]$$
$$\leqslant \mu^+\Big[\bigcup_{n=N_1+1}^{\infty} (V_n - C_n) \cup W_{N_0}^p\Big] < \tfrac{1}{2}\delta + \tfrac{1}{2}\delta$$

for every $p = 1, 2, \ldots$, and thus $\mu^+\Big(\bigcup_{n=N_0+1}^{\infty} V_n\Big) \leqslant \delta$ by assertion d) of Lemma A, which is what we wanted to show.

Finally, $\mathscr{U} \subset \mathscr{S}$ by assertion b) of Lemma A; hence $\mathfrak{S}(\mathscr{B}) = \mathfrak{S}(\mathscr{U}) \subset \mathscr{S}$, see Exercise 3, after § 51 of [10]. Thus the lemma is proved.

LEMMA C. *For $A \in \mathscr{S}$ put $\mu(A) = \sup\{\mu_0(C),\ C \subset A,\ C \in \mathscr{C}\}$, and let μ_0 be uniformly subadditively continuous from the right, subadditive or additive. Then:*

a) μ is monotone on $\mathscr{S}$ and $\mu(C) = \mu_0(C)$ for each $C \in \mathscr{C}$,

b) $\mu(A) = \inf\{\mu^+(V),\ A \subset V \in \mathscr{U},\}$ for each $A \in \mathscr{S}$; hence $\mu(V) = \mu^+(V)$ for each $V \in \mathscr{U}$, and

c) the restriction of μ to $\mathfrak{S}(\mathscr{B})$ is a uniform, subadditive or additive regular Borel submeasure respectively and $\mathscr{S}$ is the completion of $\mathfrak{S}(\mathscr{B})$ with respect to μ.

Proof. a) is obvious.

b) Let $A \in \mathscr{S}$ and let $\varepsilon > 0$. By assertion f) of Lemma A there is a $\delta > 0$ such that $U \in \mathscr{U}$, $\mu^+(U) < \delta$ implies $\mu^+(W \cup U) \leqslant \mu^+(W) + \frac{1}{3}\varepsilon$ for every $W \in \mathscr{U}$. It follows from the definition of $\mu(A)$ and from Lemma B that there are $C \in \mathscr{C}$, $V \in \mathscr{U}$ such that $C \subset A \subset V$, $\mu^+(V - C) < \delta$ and $\mu(A) \leqslant \mu(C) + \frac{1}{3}\varepsilon$. Since the subcontent μ_0 is regular, there is a $W \in \mathscr{U}$ such that $C \subset W \subset V$ and $\mu^+(W) \leqslant \mu_0(C) + \frac{1}{3}\varepsilon$. Thus

$$\mu^+(V) = \mu^+\big(W \cup (V - C)\big) \leqslant \mu^+(W) + \tfrac{1}{3}\varepsilon \leqslant \mu(C) + \tfrac{2}{3}\varepsilon \leqslant \mu(A) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, b) is proved.

c) Suppose first that μ_0 is uniformly subadditively continuous from the right and let $\varepsilon > 0$. According to assertion f) of Lemma A there is a $\delta > 0$ such that $W \in \mathscr{U}$, $\mu^+(W) < \delta$ implies $\mu^+(V \cup W) \leqslant \mu^+(V) + \frac{1}{2}\varepsilon$ for each $V \in \mathscr{U}$. Let $A, B \in \mathscr{S}$ and let $\mu(B) < \delta$. Then by b) there are $V, W \in \mathscr{U}$ such hat $A \subset V$, $B \subset W$, $\mu^+(V) \leqslant \mu(A) + \frac{1}{2}\varepsilon$ and $\mu^+(W) < \delta$. Thus

$$\mu(A \cup B) \leqslant \mu^+(V \cup W) \leqslant \mu^+(V) + \tfrac{1}{2}\varepsilon \leqslant \mu(A) + \varepsilon.$$

Since $A \in \mathscr{S}$ was arbitrary, μ is uniformly subadditively continuous on $\mathscr{S}$.

If μ_0 is subadditive or additive, then the subadditivity of μ is obvious from b). If μ_0 is additive, then the inequality $\mu(A \cup B) \geqslant \mu(A) + \mu(B)$ for $A \cap B = \emptyset$ is clear from the definition of μ.

Lemma B says that the restriction of μ to $\mathfrak{S}(\mathscr{B})$ is Borel regular; hence by Theorem 13 μ on $\mathfrak{S}(\mathscr{B})$ is a uniform, subadditive respectively additive regular Borel submeasure. From the definitions it is easy to verify that $\mathscr{S}$ is the completion of $\mathfrak{S}(\mathscr{B})$ with respect to μ. Let us note finally that in the cases of μ_0 mentioned in this lemma we do not use in the proofs of Lemmas A, B and C, the conditions II and III of the theorem.

LEMMA D. *The assertions of Lemma C are valid for a general subcontent μ_0 satisfying the conditions* I, II *and* III *of the theorem.*

Proof. a) is obvious.

b) Let $A \in \mathscr{S}$ and let $\varepsilon > 0$. According to Lemma B there are two sequences $C_n \in \mathscr{C}$, $V_n \in \mathscr{U}$, $n = 1, 2, \ldots$ such that $C_n \subset A \subset V_n$ for every n, $C_n \nearrow$, $V_n \searrow$ and $\lim_{n\to\infty} \mu^+(V_n - C_n) = 0$. Since μ_0 is monotone and regular and since $\mathscr{C}_0$ is a lattice closed with respect to the forming of any intersections, by Theorem D in § 50 of [10] for every $n = 1, 2, \ldots$ there s a set $C_n' \in \mathscr{C}_0$ such that $C_n \subset C_n' \subset V_n$ and $\mu_0(C_n) = \mu_0(C_n')$. Put $C_n^* = \bigcap_{k=n}^{\infty} C_k'$ for $n = 1, 2, \ldots$ Then $C_n^* \in \mathscr{C}_0$, $n = 1, 2, \ldots$ is an increasing sequence; hence by condition II there is a $\delta > 0$ such that $D \in \mathscr{C}_0$, $\mu_0(D) < \delta$ implies $\mu_0(C_n^* \cup D) \leqslant \mu_0(C_n^*) + \frac{1}{2}\varepsilon$ for every $n = 1, 2, \ldots$ Obviously $C_n \subset C_n^* \subset V_n$ and $\mu_0(C_n) = \mu_0(C_n^*)$ for every n.

Now take n_0 so that $\mu^+(V_{n_0} - C_{n_0}) < \delta$. According to the definition of $\mu^+(V_{n_0})$ and Theorem D in § 50 of [10] there are $H \in \mathscr{C}_0$ and $V_{n_0}' \in \mathscr{U}_0$ such that $H \subset V_{n_0}' \subset V_{n_0}$ and $\mu^+(V_{n_0}) \leqslant \mu_0(H) + \frac{1}{4}\varepsilon$. It follows from condition III that there is a $\delta_1 > 0$ such that $W \in \mathscr{U}_0$, $\mu^+(W) < \delta_1$ implies $\mu_0(H) \leqslant \mu_0(H - W) + \frac{1}{4}\varepsilon$. By assertion b) of Lemma A and Theorem D in § 50 of [10] there is a set $D \in \mathscr{C}_0$ such that $D \subset V_{n_0}' - C_{n_0}^*$ and $\mu^+\big(V_{n_0}' - (C_{n_0}^* \cup D)\big) < \delta_1$. Thus

$$\begin{aligned}\mu^+(V_{n_0}) &\leqslant \mu_0(H) + \tfrac{1}{4}\varepsilon \leqslant \mu_0\big(H - (V_{n_0}' - [C_{n_0}^* \cup D])\big) + \tfrac{1}{2}\varepsilon \\ &\leqslant \mu_0[H \cap (C_{n_0}^* \cup D)] + \tfrac{1}{2}\varepsilon \leqslant \mu_0(C_{n_0}^* \cup D) + \tfrac{1}{2}\varepsilon \\ &\leqslant \mu_0(C_{n_0}^*) + \varepsilon = \mu_0(C_{n_0}) + \varepsilon \leqslant \mu(A) + \varepsilon,\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, b) is proved.

c) The subadditive continuity of μ on $\mathscr{S}$ from the right may be proved in the same way as in the proof of Lemma C. We now prove that μ on $\mathscr{S}$ is also subadditively continuous from the left. Let $A \in \mathscr{S}$ and let $\varepsilon > 0$. Take $C \in \mathscr{C}$, $C \subset A$ so that $\mu(A) \leqslant \mu_0(C) + \frac{1}{2}\varepsilon$. According to Lemma B, Theorem D in § 50 of [10] and assertion b) there is a set $C' \in \mathscr{C}_0$ such that $C \subset C'$ and $\mu(C' - C) = 0$. Owing to condition III take $\delta > 0$ so that $V' \in \mathscr{U}_0$, $\mu^+(V') < \delta$ implies $\mu_0(C') \leqslant \mu_0(C' - V') + \frac{1}{2}\varepsilon$. Now let $B \in \mathscr{S}$ and $\mu(B) < \delta$. Then by b) there is a set $V \in \mathscr{U}$ such that $B \subset V$ and $\mu^+(V)$

$< \delta$. It follows from assertion b) of Lemma A and from Theorem D in § 50 of [10] that there is a set $V' \in \mathscr{U}_0$ such that $V' \subset V$ and $\mu(V - V') = 0$. Thus

$$\begin{aligned}\mu(A) &\leqslant \mu_0(C) + \tfrac{1}{2}\varepsilon \leqslant \mu_0(C') + \tfrac{1}{2}\varepsilon \leqslant \mu_0(C' - V') + \varepsilon \\ &= \mu_0\big([(C' - V') - (C - V]) \cup [C - V]\big) + \varepsilon \\ &\leqslant \mu_0[(C - V) \cup (C' - C) \cup (V - V')] + \varepsilon \\ &= \mu_0(C - V) + \varepsilon \leqslant \mu(A - B) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have proved the subadditive continuity of μ on $\mathscr{S}$.

The rest of c) follows in the same way as in the proof of Lemma C. Thus the lemma as well as the theorem is proved.

From this theorem and from the Corollary of Theorem 17 we immediately have the following

COROLLARY 1. *Let T be a locally compact Hausdorff topological space and let $\mu_{0\wedge}: \mathscr{C}_\wedge \to [0, +\infty)$ be a set function. Then there is a unique regular Borel submeasure $\mu: \mathscr{B} \to [0, +\infty)$ extending $\mu_{0\wedge}$ if and only if for each set $C \in \mathscr{C}_0$ the set function $\mu_{0\wedge}^C: \mathscr{C}_0 \to [0, +\infty)$ defined by the equality $\mu_{0\wedge}^C(D) = \mu_{0\wedge}(C \cap D)$, $D \in \mathscr{C}_0$ is a Baire subcontent having the properties* I, II *and* III *from the preceding theorem. At the same time, if $\mu_{0\wedge}$ is uniformly subadditively continous from the right, subadditive or additive on $\mathscr{C}_\wedge$, then μ is a uniform subadditive or additive regular Borel submeasure on $\mathscr{B}$ respectively.*

From here and from the preceding theorem we have the next

COROLLARY 2. *Let T be a locally compact Hausdorff topological space. Then every Baire submeasure $\mu_0: \check{\mathscr{B}}_0 \to [0, +\infty)$ can be uniquely extended to a regular Borel submeasure $\mu: \check{\mathscr{B}} \to [0, +\infty)$. At the same time, if μ_0 is uniform, subadditive or additive on $\check{\mathscr{B}}_0$, then μ is also uniform, subadditive or additive on $\check{\mathscr{B}}$ respectively.*

Remark. Let T be a locally compact Hausdorff topological space and let $\mu_{00}: C_0 \to [0, +\infty)$ be a Baire subcontent having the properties I, II and III from Theorem 23. According to this theorem and the Remark following Theorem 19 let μ_l be the unique submeasure induced by μ_{00} on the σ-algebra $\mathfrak{S}_l(\mathscr{B})$ of all locally Borel measurable subsets of T. Clearly $\mathfrak{S}_l(\mathscr{B})$ contains the σ-algebra of all weakly Borel measurable subsets of T, i.e., the σ-algebra generated by the family of all open subsets in T, see [5]. Denote by μ_l^* the outer submeasure induced by μ_l, see Lemma B in the proof of Theorem 18. Then in a similar way as in [14], Chap. III, § 2, T27, μ_l^* is a Choquet $\mathscr{C}$-capacity right continuous (regular in our terminology) on the class $\mathscr{A}(\mathscr{C})$ of all $\mathscr{C}$-analytic subsets of T.

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