POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCLXVII

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A stability theorem for foliations with singularities

WARSZAWA 1988
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

ISBN 83-01-08116-3 ISSN 0012-3862

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CONTENTS

Introduction

The notion of foliation was first introduced by Ehresmann and Reeb [ER] in 1944. The next years have brought intense progress in the study of foliations. The theory of foliations developed in two main directions: the quantitative and qualitative theories. The first one concerns the following question: How many geometrically distinct foliations can be constructed on a given manifold? The characteristic classes of foliations are a very useful tool in the investigation of this problem. Several kinds of such classes have been discovered (see for example [B], [H], [KT]). For a comprehensive exposition of the quantitative theory we refer to [L2].

Properties of foliations are the object of the qualitative theory. One of the most important notions here is the holonomy group of a leaf of a foliation. For different definitions of holonomy we refer to [E1], [EMS], [L2], [R], [T]. Basing on the notion of holonomy, in 1947 G. Reeb formulated the Stability Theorem which is one of the most important in the theory of foliations. Imitating Lawson [L2] we adopt the Stability Theorem in the following formulation:

Let $\mathcal{F}$ *be a codimension* q, C^r *foliation* $(0 \leqslant r \leqslant \omega)$ *on a manifold* M, *and suppose* $L \in \mathcal{F}$ *is a compact leaf with trivial holonomy. Then there exists a saturated neighbourhood* U *of* L *in* M *and a* C^r *diffeomorphism* $f\colon L \times D^q \to U$ *such that* $f^*(\mathcal{F}) = \{L \times \{x\}\}_{x \in D^q}$ (D^q *being a disk in* $\mathbf{R}^q$).

For the proof and other formulations see [L2], [R], [T].

Recent years have brought progress in the study of stable foliations, especially in the case of codimension 1 foliations and of foliations with all leaves compact. Epstein [E1] found several conditions equivalent to the stability of foliations with all leaves compact. For various theorems and examples connected with the problem of stability we refer to [E2], [EV], [EMS], [Sl1], [Sl2] and [V]. A generalization of the Reeb Stability Theorem was proved by Thurston [Th] for codimension 1 foliations.

For a more detailed exposition of the qualitative theory of foliations we refer to [HH], [L1] and [Rh].

The notion of foliation with singularities in the sense considered here was introduced by Stefan [S1], [S2] and [S3] in 1973. Stefan's papers cited above have its origins in control theory (see [He], [Lo], [N], [Su1], [Su2] and [SuJ]).

In this paper, we construct the holonomy group of a leaf of a foliation with singularities and prove a theorem analogous to the Reeb Stability Theorem. Chapter 1 contains basic definitions and facts concerning foliations with singularities. Moreover, the notion of germ of a φ-diffeomorphism is introduced. We show that the set of all germs of φ-diffeomorphisms forms a group. In Chapter 2, we introduce the notion of chain of open sets and of adapted charts along a curve in a leaf, and prove some facts of combinatorial nature concerning chains. Chapter 3 is devoted to the study of topological properties of foliations with singularities. In Chapter 4, the holonomy group of a leaf of a singular foliation is defined and an example of a foliation with a singular compact leaf with nontrivial holonomy group is given. Chapter 5 contains the proof of the main theorem of the paper ((5.15)). This theorem states that *if L is a compact leaf of a singular foliation $\mathcal{F}$ of M and the holonomy group of L is trivial, then for every $x \in L$ there exists a neighbourhood V of L in M and an adapted chart φ around x such that $D_\varphi \subset V$ and the inclusion mapping induces a homeomorphism of the quotient spaces*:

$$D_\varphi/(\mathcal{F}\,|\,D_\varphi) \cong V/(\mathcal{F}\,|\,V).$$

In Chapter 6, we deduce the Reeb Stability Theorem from Theorem (5.15).

I would like to express my gratitude to Prof. Dr. Włodzimierz Waliszewski who called my attention to the theory of foliations, and to Dr. Grzegorz Andrzejczak and Dr. Paweł Grzegorz Walczak for their helpful remarks concerning this paper.

1. Preliminaries

Let M be a connected m-dimensional paracompact manifold of class C^∞.

(1.1) DEFINITION ([S1], [S2]). A subset L of M is said to be a *k-leaf* of M (where k is a nonnegative integer) if there exists a differential structure σ of class C^∞ on L such that

(i) (L, σ) is a connected k-dimensional immersed submanifold of M,

(ii) if N is an arbitrary locally connected topological space, and $f\colon N \to M$ is a continuous function such that $f(N) \subset L$, then the function

$$\hat{f}\colon N \ni p \mapsto f(p) \in L$$

is continuous in the topology induced in L by the differential structure σ.

We say that $L \subset M$ is a *leaf* of M if L is a k-leaf of M for some k.

(1.2) Remarks. (a) A leaf L of a nonsingular k-dimensional foliation $\mathcal{F}$ of M is a leaf of M in the sense of the above definition. Indeed, the only condition to be checked is (ii) of (1.1). Let N be an arbitrary locally connected topological space and $f\colon N \to M$ a continuous mapping such that

$f(N) \subset L$. Let $p \in N$ and let V be an arbitrary neighbourhood of $f(p)$ in L. We can find local coordinates $\varphi = (x^1, \ldots, x^m)$ on a neighbourhood of $f(p)$ distinguished by $\mathcal{F}$ such that

$$\{q \in D_\varphi;\ x^{k+1}(q) = \ldots = x^m(q) = 0\} \subset V.$$

If U is a connected neighbourhood of p such that $f(U) \subset D_\varphi$, then $f(U) \subset V$ and condition (ii) is satisfied.

(b) The assumption of the local connectedness of N is essential. If we omit it, then there exist leaves of nonsingular foliations which are not leaves in the sense of (1.1). Indeed, let $M = S^1 \times S^1$ be the two-dimensional torus and let L be a submanifold of M consisting of all points of the form $(e^{it}, e^{i\alpha t})$, where $t \in \boldsymbol{R}$ and α is a fixed irrational number. It is well known that L is a dense leaf of a nonsingular foliation of M. Let $N = \{0\} \cup \{1/n;\ n \in \boldsymbol{N}\}$ with the topology induced from $\boldsymbol{R}$. Let $x \in L$ and let T be an arbitrary one-dimensional submanifold of M transverse to L such that $x \in T$. Owing to the density of L, there exists a sequence (x_n) such that $x_n \in T \cap L$ for every $n \in \boldsymbol{N}$, and $x_n \to x$ as $n \to \infty$. We define $f\colon N \to M$ by $f(1/n) = x_n$ and $f(0) = x$. Then f is continuous and $f(N) \subset L$. Yet, it is easily seen that $\hat{f}\colon N \to L$ is not continuous so L is not a leaf of M in the sense of (1.1).

(1.3) Definition ([S1], [S2]). We say that $\mathcal{F}$ is a *foliation of M with singularities* if $\mathcal{F}$ is a partition of M into leaves such that for every $x \in M$ there exists a local chart φ of M with the following properties:

(i) φ is a surjection $D_\varphi \to U_\varphi \times W_\varphi$, where U_φ, W_φ are open connected neighbourhoods of 0 in $\boldsymbol{R}^k$ and $\boldsymbol{R}^{m-k}$, respectively, and k is the dimension of the leaf containing x,

(ii) $\varphi(x) = (0, 0)$,

(iii) if $L \in \mathcal{F}$, then $\varphi(L \cap D_\varphi) = U_\varphi \times l$, where $l = \{w \in W_\varphi;\ \varphi^{-1}(0, w) \in L\}$.

A chart φ which fulfils the above conditions is called a *chart distinguished by $\mathcal{F}$* or an *adapted chart* around x.

If $x \in M$, then the leaf of $\mathcal{F}$ containing x is denoted by L_x.

From now on the words "foliation" and "nonsingular foliation" will mean a foliation with singularities and a foliation without singularities, respectively.

(1.4) Examples. (a) Any nonsingular foliation is a foliation.

(b) It is easy to see that a foliation is nonsingular if and only if the set l in Definition (1.3) is totally disconnected (i.e. every connected component of l is a single point).

(c) Let M be an arbitrary manifold and S its closed subset. Let $M \setminus S = \bigcup_\alpha M_\alpha$ be the decomposition into connected components. For every α, let $\mathcal{F}'_\alpha$ be a nonsingular foliation of the manifold M_α. Then the family $\mathcal{F} = \bigcup_\alpha \mathcal{F}'_\alpha \cup \{\{x\};\ x \in S\}$ is a foliation of M.

(d) As particular cases of (c), we obtain the trivial examples presented in Figure 1.

(e) Let $M = S^1 \times S^1$ be the two-dimensional torus and $\mathscr{F} = \{S^1 \times \{1\}, M \setminus (S^1 \times \{1\})\}$. Then $\mathscr{F}$ is a foliation of M.

(f) If $\mathscr{F}$ is a foliation of M and V is an open submanifold of M, then the family $\mathscr{F} | V$ of all connected components of the sets $L \cap V$, where $L \in \mathscr{F}$, is a foliation of V.

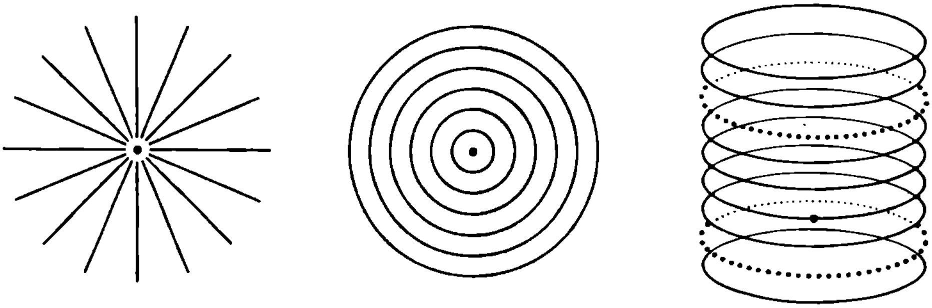

Fig. 1

Let $\mathscr{F}$ be a foliation of M. Consider the function $d_{\mathscr{F}}\colon M \ni x \mapsto \dim L_x \in \boldsymbol{R}$. From Definition (1.3) it follows that this function is lower semicontinuous. Indeed, if $x \in M$, φ is an adapted chart around x, $L \in \mathscr{F}$ and $L \cap D_\varphi \neq \emptyset$, then

$$\dim (L \cap D_\varphi) = \dim (U_\varphi \times l) \geqslant \dim U_\varphi = \dim L_x,$$

so the dimension of leaves cannot decrease in the neighbourhood of x. In connection with this function we make the following definition.

(1.5) Definition. A leaf $L \in \mathscr{F}$ is called *nonsingular* if there exists a neighbourhood V of L in M such that the function $d_{\mathscr{F}} | V$ is constant. Otherwise, we say that L is *singular*.

(1.6) Remark. A foliation $\mathscr{F}$ for which $d_{\mathscr{F}}$ is constant is nonsingular.

Let φ be a chart distinguished by $\mathscr{F}$ around the point x.

(1.7) Definition. A *plaque* of the chart φ is a connected component of the set $L \cap D_\varphi$. Equivalently, a plaque of φ is a set of the form $\varphi^{-1}(U_\varphi \times l_0)$, where l_0 is a connected component of some set l from condition (iii) of (1.3). Equivalently, a plaque of φ is a leaf of the foliation $\mathscr{F} | D_\varphi$.

Let G be an arbitrary open neighbourhood of 0 in W_φ. Then it is easy to see that the mapping

$$\varphi_G = \varphi | \varphi^{-1}(U_\varphi \times G)$$

is an adapted chart around x. We define a relation $\sim_{\varphi_G}$ in G in the following way: if $w, w' \in G$, then $w \sim_{\varphi_G} w'$ if and only if

(1.8) $\varphi^{-1}(0, w)$ and $\varphi^{-1}(0, w')$ belong to the same plaque of φ_G

or, equivalently,

(1.9) w and w' belong to the same connected component of $l \cap G$, where l is a set from (iii) of Definition (1.3).

Obviously, $\sim_{\varphi_G}$ is an equivalence relation in G. If $w \in G$, then the equivalence class of w is denoted by $[w]_{\varphi_G}$.

If $\mathscr{F}$ is a nonsingular foliation, then the relation $\sim_{\varphi_G}$ is trivial for every G (i.e. every equivalence class is a single point).

(1.10) LEMMA. *Let G and G' be neighbourhoods of* 0 *in W_φ such that $G' \subset G$. If $w \in G'$, then $[w]_{\varphi_{G'}}$ is a connected component of the set $[w]_{\varphi_G} \cap G'$.*

Proof. It is obvious that $[w]_{\varphi_{G'}} \subset [w]_{\varphi_G} \cap G'$ and since $[w]_{\varphi_{G'}}$ is connected, it is contained in a connected component of the set $[w]_{\varphi_G} \cap G'$.

Conversely, if w' is a point of a connected component of $[w]_{\varphi_G} \cap G'$, then – in view of (1.8) – $\varphi^{-1}(0, w)$ and $\varphi^{-1}(0, w')$ belong to the same plaque Q of the chart φ_G as well as to the same connected component of the set $Q \cap \varphi^{-1}(U_\varphi \times G')$. Therefore these points lie in the same plaque of $\varphi_{G'}$, and so $w' \in [w]_{\varphi_{G'}}$. ■

Let x and y be points of a leaf $L \in \mathscr{F}$, and φ, ψ charts distinguished by $\mathscr{F}$ around x and y, respectively. The following propositions are simple consequences of the above lemma.

(1.11) PROPOSITION. *Suppose that c: $\langle 0, 1 \rangle \to G$ is a continuous curve in a neighbourhood G of* 0 *in W_φ. If the image of c is contained in some equivalence class of $\sim_{\varphi_G}$ and*

$$c(\langle 0, 1 \rangle) \subset G' \subset G,$$

where G' is a neighbourhood of 0 *in W_φ, then the image of c is contained in some equivalence class of $\sim_{\varphi_{G'}}$.*

(1.12) PROPOSITION. *Let G and H be open neighbourhoods of* 0 *in W_φ and W_ψ, respectively. Let f: $G \to H$ be a continuous mapping compatible with $\sim_{\varphi_G}$ and $\sim_{\psi_H}$* (i.e. *for all $w, w' \in G$, if $w \sim_{\varphi_G} w'$, then $f(w) \sim_{\psi_H} f(w')$*). *If $G' \subset G$ is an arbitrary neighbourhood of* 0 *in W_φ and H' is a neighbourhood of* 0 *in W_ψ such that $f(G') \subset H'$, then the mapping $f|G'$: $G' \to H'$ is compatible with $\sim_{\varphi_{G'}}$ and $\sim_{\psi_{H'}}$.*

Let φ and ψ be as above. We denote by $\mathscr{A}_{\varphi,\psi}$ the family of all diffeomorphisms f satisfying the following conditions:

(i) $f\colon G \to H$ is a surjection, where G and H are open neighbourhoods of 0 in W_φ and W_ψ, respectively,

(ii) $f(0) = 0$,

(iii) f and f^{-1} are compatible with $\sim_{\varphi G}$ and $\sim_{\psi H}$.

If $\varphi = \psi$, then we write $\mathscr{A}_\varphi$ instead of $\mathscr{A}_{\varphi,\varphi}$.

(1.13) DEFINITION. The elements of $\mathscr{A}_\varphi$ are called *φ-diffeomorphisms.*

In $\mathscr{A}_{\varphi,\psi}$ we define a relation $\equiv$ in the following way:

(1.14) DEFINITION. Let $f_i\colon G_i \to H_i$ $(i = 0, 1)$ be two elements of $\mathscr{A}_{\varphi,\psi}$. Then $f_0 \equiv f_1$ if and only if there exists a family $\{\hat{f}_t;\ t \in \langle 0, 1\rangle\}$ such that:

(i) for every $t \in \langle 0, 1\rangle$, $\hat{f}_t\colon \hat{G} \to H_0 \cap H_1$ is an immersion, where $\hat{G}$ is an open neighbourhood of 0 contained in $G_0 \cap G_1$,

(ii) for every $t \in \langle 0, 1\rangle$, $\hat{f}_t$ is compatible with $\sim_{\varphi\hat{G}}$ and $\sim_{\psi H_0 \cap H_1}$,

(iii) $\hat{f}_0 = f_0 \mid \hat{G}$ and $\hat{f}_1 = f_1 \mid \hat{G}$,

(iv) $\langle 0, 1\rangle \times \hat{G} \ni (t, w) \mapsto \hat{f}_t(w) \in H_0 \cap H_1$ is continuous,

(v) for every $w \in \hat{G}$ the image of the curve $\langle 0, 1\rangle \ni t \mapsto \hat{f}_t(w) \in H_0 \cap H_1$ is contained in some equivalence class of $\sim_{\psi H_0 \cap H_1}$.

We sometimes say that the homotopy $\hat{f}_t$ *realizes* the relation $f_0 \equiv f_1$.

We wish to show that $\equiv$ is an equivalence relation. For the proof we need

(1.15) LEMMA. *Let X and Y be topological spaces and $F\colon \langle 0, 1\rangle \times X \to Y$ a continuous mapping. Let $x \in X$ and let V be an open subset of Y such that $F(\langle 0, 1\rangle \times \{x\}) \subset V$. Then there exists a neighbourhood U of x such that $F(\langle 0, 1\rangle \times U) \subset V$.*

Proof. Since F is continuous, $F^{-1}(V)$ is open in $\langle 0, 1\rangle \times X$ and $\langle 0, 1\rangle \times \{x\} \subset F^{-1}(V)$. The projection $\mathrm{pr}_2\colon \langle 0, 1\rangle \times X \to X$ is a closed mapping by the compactness of $\langle 0, 1\rangle$. Consequently, the set

$$A := \mathrm{pr}_2\big((\langle 0, 1\rangle \times X) \setminus F^{-1}(V)\big)$$

is closed in X and $x \notin A$. Let $U := X \setminus A$. Then for $t \in \langle 0, 1\rangle$ and $y \in U$ we have $y \notin A$, and, consequently, $F(t, y) \in V$. ■

From the above lemma, we get the following:

(1.16) PROPOSITION. *Let X, Y, and F be as in Lemma (1.15) and let V be an arbitrary open set of Y. Then the set*

$$A := \bigcap_{t \in \langle 0,1\rangle} F(t, \cdot)^{-1}(V)$$

is open in X.

Proof. If $x \in A$, then for every $t \in \langle 0, 1\rangle$, $F(t, x) \in V$. Thus, $F(\langle 0, 1\rangle \times \{x\}) \subset V$. By Lemma (1.15), there exists a neighbourhood U of x such that $F(\langle 0, 1\rangle \times U) \subset V$, and then $U \subset A$. ■

(1.17) PROPOSITION. *The relation* $\equiv$ *in* $\mathscr{A}_{\varphi,\psi}$ *is an equivalence relation.*

Proof. The reflexivity and symmetry of $\equiv$ are obvious. We show that $\equiv$ is transitive. Let f_0, f_1, $f_2 \in \mathscr{A}_{\varphi,\psi}$, $f_i\colon G_i \to H_i$ $(i = 0, 1, 2)$, $f_0 \equiv f_1$, $f_1 \equiv f_2$, and let $\hat{g}_t\colon \tilde{G} \to H_0 \cap H_1$ and $\hat{h}_t\colon \tilde{G}' \to H_1 \cap H_2$ be the corresponding homotopies. Set

$$\hat{G} := \bigcap_{t\in\langle 0,1\rangle} \hat{g}_t^{-1}(H_0 \cap H_2) \cap \bigcap_{t\in\langle 0,1\rangle} \hat{h}_t^{-1}(H_0 \cap H_2).$$

$\hat{G}$ is an open neighbourhood of 0 in W_φ because of (iv) of Definition (1.14) and Proposition (1.16). We have

$$\hat{G} \subset \tilde{G} \cap \tilde{G}' \subset G_0 \cap G_2.$$

On $\hat{G}$ we define

$$\hat{f}_t(w) = \begin{cases} \hat{g}_{2t}(w) & \text{for } 0 \leqslant t \leqslant 1/2, \\ \hat{h}_{2t-1}(w) & \text{for } 1/2 \leqslant t \leqslant 1. \end{cases} \tag{1.18}$$

Obviously, this definition is correct (see (iii) of Definition (1.14)). Observe that $\hat{f}_t\colon \hat{G} \to H_0 \cap H_2$. It is easily seen that for every $t \in \langle 0, 1\rangle$, $\hat{f}_t$ is an immersion and, in view of Proposition (1.12), it is compatible with $\sim_{\varphi\hat{G}}$ and $\sim_{\psi_{H_0 \cap H_2}}$. It is clear that $\hat{f}_0 = f_0 | \hat{G}$, $\hat{f}_1 = f_2 | \hat{G}$, and the mapping

$$\langle 0, 1\rangle \times \hat{G} \ni (t, w) \mapsto \hat{f}_t(w) \in H_0 \cap H_2$$

is continuous.

It remains to prove condition (v) of (1.14). Consider the curve

$$c\colon \langle 0, 1\rangle \ni t \mapsto \hat{f}_t(w)$$

for $w \in \hat{G}$. Observe that $c(\langle 0, 1/2\rangle)$ is contained in some equivalence class of $\sim_{\psi_{H_0 \cap H_1}}$ as well as in $H_0 \cap H_1 \cap H_2$, so in view of Proposition (1.11) it lies in some equivalence class of $\sim_{\psi_{H_0 \cap H_1 \cap H_2}}$. Analogously, $c(\langle 1/2, 1\rangle)$ lies in some equivalence class of this relation. Since $c(\langle 0, 1/2\rangle) \cap c(\langle 1/2, 1\rangle) \neq \emptyset$, the set $c(\langle 0, 1\rangle)$ is contained in some equivalence class of $\sim_{\psi_{H_0 \cap H_1 \cap H_2}}$ and moreover in some equivalence class of $\sim_{\psi_{H_0 \cap H_2}}$. Consequently, $\hat{f}_t$ is a homotopy realizing the relation $f_0 \equiv f_2$. ∎

(1.19) **Remarks.** (a) Condition (v) of Definition (1.14) implies the commutativity of the following diagram for every $t \in \langle 0, 1\rangle$:

$$\begin{tikzcd}
G_0/\sim_{\varphi_{G_0}} \arrow[d] & \hat{G}/\sim_{\varphi_{\hat{G}}} \arrow[l] \arrow[r] \arrow[d] & G_1/\sim_{\varphi_{G_1}} \arrow[d] \\
H_0/\sim_{\psi_{H_0}} & H_0 \cap H_1/\sim_{\psi_{H_0 \cap H_1}} \arrow[l] \arrow[r] & H_1/\sim_{\psi_{H_1}}
\end{tikzcd}$$

Here the horizontal arrows are induced by inclusion maps and the vertical arrows are induced by f_0, $\hat{f}_t$, and f_1, respectively.

(b) If $\mathcal{F}$ is a nonsingular foliation, then $f_0 \equiv f_1$ is equivalent to the equality of the germs at 0 of f_0 and f_1.

(1.20) DEFINITION. The elements of the set $\mathcal{A}_\varphi / \equiv$ are called *germs of φ-diffeomorphisms for $\mathcal{F}$*. The germ containing a φ-diffeomorphism f is denoted by $[f]$.

The following is a simple consequence of the definition of $\equiv$:

(1.21) PROPOSITION. *If $(f\colon G \to H) \in \mathcal{A}_\varphi$ satisfies $f \equiv \mathrm{id}_{W_\varphi}$, then there exists an open neighbourhood G' of 0 in G such that $G' \subset H$ and $w \sim_{\varphi_H} f(w)$ for every $w \in G'$.*

Proof. It suffices to take for G' the set $\hat{G}$ which is the domain of the homotopy $\hat{f}_t$ realizing the assumed equivalence. ∎

(1.22) LEMMA. *If $f \in \mathcal{A}_{\varphi,\psi}$, $f\colon G \to H$ and $G' \subset G$ is a neighbourhood of 0, then $(f|G'\colon G' \to f(G')) \in \mathcal{A}_{\varphi,\psi}$ and $f \equiv f|G'$.*

Proof. The mapping $f|G'$ belongs to $\mathcal{A}_{\varphi,\psi}$ owing to Proposition (1.12) and the continuity of f and f^{-1}. It is easy to check that $\hat{f}_t := f|G'\colon G' \to f(G')$ is the desired homotopy. ∎

Let χ be an adapted chart around a point z of the leaf L. To prove the main result of this chapter we need the following:

(1.23) LEMMA. *If $f_0, f_1 \in \mathcal{A}_{\varphi,\psi}$, $f_0 \equiv f_1$, $g_0, g_1 \in \mathcal{A}_{\psi,\chi}$, and $g_0 \equiv g_1$, then $g_0 f_0 \equiv g_1 f_1$ in $\mathcal{A}_{\varphi,\chi}$.*

Proof. If $f_i\colon G_i \to H_i$ and $g_i\colon G'_i \to H'_i$ $(i = 0, 1)$, then $g_i f_i\colon f_i^{-1}(G'_i) \to g_i(H_i)$. Let $\tilde{f}_t\colon \tilde{G} \to H_0 \cap H_1$ and $\tilde{g}_t\colon \tilde{G}' \to H'_0 \cap H'_1$ be homotopies realizing the equivalences $f_0 \equiv f_1$ and $g_0 \equiv g_1$, respectively. Observe that the mapping

$$\langle 0, 1\rangle \times \tilde{G} \ni (t, w) \mapsto \tilde{f}_t(w) \in H_0 \cap H_1$$

is continuous. Thus in view of Proposition (1.16) the set $\bigcap_{t\in\langle 0,1\rangle} \tilde{f}_t^{-1}(\tilde{G}')$ is an open neighbourhood of 0 in W_φ. By the continuity of the mapping

$$\langle 0, 1\rangle \times \bigcap_{t\in\langle 0,1\rangle} \tilde{f}_t^{-1}(\tilde{G}') \ni (t, w) \mapsto \tilde{g}_t \tilde{f}_t(w) \in H'_0 \cap H'_1$$

and Proposition (1.16) the set $\bigcap_{t\in\langle 0,1\rangle} \tilde{f}_t^{-1} \tilde{g}_t^{-1}(g_0(H_0) \cap g_1(H_1))$ is an open neighbourhood of 0 in W_φ. Thus

$$\hat{G} := f_0^{-1}(\tilde{G}') \cap f_1^{-1}(\tilde{G}') \cap \bigcap_{t\in\langle 0,1\rangle} \tilde{f}_t^{-1}(\tilde{G}')$$
$$\cap \bigcap_{t\in\langle 0,1\rangle} \tilde{f}_t^{-1} \tilde{g}_t^{-1}(g_0(H_0) \cap g_1(H_1))$$

is an open neighbourhood of 0 in W_φ contained in $f_0^{-1}(G'_0) \cap f_1^{-1}(G'_1)$, when $\tilde{G}' \subset G'_0 \cap G'_1$. Since

$$\hat{G} \subset \bigcap_{t \in \langle 0,1 \rangle} \tilde{f}_t^{-1}(\tilde{G}') \cap \bigcap_{t \in \langle 0,1 \rangle} \tilde{f}_t^{-1} \tilde{g}_t^{-1}(g_0(H_0) \cap g_1(H_1)),$$

the mappings $\hat{f}_t := \tilde{g}_t \tilde{f}_t$ are defined in $\hat{G}$ and have values in $g_0(H_0) \cap g_1(H_1)$. The mapping $\tilde{f}_t$ is an immersion in

$$\hat{G} \subset \bigcap_{t \in \langle 0,1 \rangle} \tilde{f}_t^{-1}(\tilde{G}') \subset \tilde{G}$$

and $\tilde{g}_t$ is an immersion in $\tilde{G}'$ ($\tilde{f}_t(\hat{G}) \subset \tilde{G}'$) so $\hat{f}_t$ is an immersion in $\hat{G}$ for every $t \in \langle 0, 1 \rangle$. From Proposition (1.12), by the compatibility of $\tilde{f}_t$ with $\sim_{\varphi_{\tilde{G}}}$ and $\sim_{\psi_{H_0 \cap H_1}}$ and the compatibility of $\tilde{g}_t$ with $\sim_{\psi_{\tilde{G}'}}$ and $\sim_{\chi_{H'_0 \cap H'_1}}$, it follows that $\hat{f}_t$ is compatible with $\sim_{\varphi_{\tilde{G}}}$ and $\sim_{\chi_{g_0(H_0) \cap g_1(H_1)}}$. Obviously, $\hat{f}_0 = g_0 f_0 | \hat{G}$, $\hat{f}_1 = g_1 f_1 | \hat{G}$, and the mapping

$$\langle 0, 1 \rangle \times \hat{G} \ni (t, w) \mapsto \hat{f}_t(w) \in g_0(H_0) \cap g_1(H_1)$$

is continuous.

It remains to prove that the image of the curve

$$c\colon \langle 0, 1 \rangle \ni t \mapsto \hat{f}_t(w_0) \in g_0(H_0) \cap g_1(H_1)$$

lies in some equivalence class of $\sim_{\chi_{g_0(H_0) \cap g_1(H_1)}}$ for every $w_0 \in \hat{G}$. To this end, it suffices to show that the image of

$$\text{(1.24)} \qquad \langle 0, 1 \rangle \times \langle 0, 1 \rangle \ni (t, s) \mapsto \tilde{g}_s(\tilde{f}_t(w_0)) \in H'_0 \cap H'_1$$

is contained in some equivalence class of $\sim_{\chi_{H'_0 \cap H'_1}}$, and then to apply Proposition (1.11) to the curve c. For any fixed s the image of the curve

$$\langle 0, 1 \rangle \ni t \mapsto \tilde{g}_s \tilde{f}_t(w_0) \in H'_0 \cap H'_1$$

lies in some equivalence class of $\sim_{\chi_{H'_0 \cap H'_1}}$ by the compatibility of $\tilde{g}_s$ with the corresponding relations and by condition (v) of Definition (1.14) applied to $\tilde{f}_t$. For any fixed t the image of the curve

$$\langle 0, 1 \rangle \ni s \mapsto \tilde{g}_s \tilde{f}_t(w_0) \in H'_0 \cap H'_1$$

lies in some equivalence class of $\sim_{\chi_{H'_0 \cap H'_1}}$ because of condition (v) of Definition (1.14) applied to $\tilde{g}_s$. Consequently, if $(s, t) \in \langle 0, 1 \rangle \times \langle 0, 1 \rangle$, then the image of the curve

$$\langle 0, t+s \rangle \ni t' \mapsto \begin{cases} \tilde{g}_0 \tilde{f}_{t'}(w_0) & \text{for } t' \in \langle 0, t \rangle, \\ \tilde{g}_{t'-t} \tilde{f}_t(w_0) & \text{for } t' \in \langle t, t+s \rangle \end{cases}$$

joining $\tilde{g}_s \tilde{f}_t(w_0)$ to $\tilde{g}_0 \tilde{f}_0(w_0)$ is contained in some equivalence class, which means that all points of the form $\tilde{g}_s \tilde{f}_t(w_0)$ belong to the same equivalence class of $\sim_{\chi_{H'_0 \cap H'_1}}$. We conclude that $\hat{f}_t$ is a homotopy realizing the equivalence $g_0 f_0 \equiv g_1 f_1$. ∎

Now, we can formulate the main result of this chapter.

(1.25) THEOREM. *Let φ be a chart distinguished by $\mathscr{F}$ around x. The set $\mathscr{A}_\varphi / \equiv$ with the multiplication defined by the formula*

$$[f]\cdot[g] = [g \circ f]$$

is a group.

Proof. In view of Lemma (1.23) this multiplication is correctly defined. The associativity is obvious. The unit is $[\mathrm{id}_{W_\varphi}]$. The inverse of $[f] \in \mathscr{A}_\varphi / \equiv$ is $[f^{-1}]$. Indeed, if $f\colon G \to H$ then

$$[f]\cdot[f^{-1}] = [f^{-1} \circ f] = [\mathrm{id}_G] = [\mathrm{id}_{W_\varphi}]$$

in view of Lemma (1.22) and similarly $[f^{-1}]\cdot[f] = [\mathrm{id}_{W_\varphi}]$. This completes the proof. ∎

2. Chains along a curve

Let $\mathscr{F}$ be a foliation of a manifold M, let $L \in \mathscr{F}$, and let $\gamma\colon \langle 0, 1\rangle \to L$ be a continuous curve.

(2.1) DEFINITION. A couple (U, t) is called a *link* on the curve γ if U is an open neighbourhood of $\gamma(t)$ in M.

For a link (U, t) on the curve γ, let $\gamma_t^- U$ and $\gamma_t^+ U$ denote the infimum and supremum of the set $\gamma^{-1}(U)_t$, respectively, where A_t denotes the connected component of a set $A \subset \boldsymbol{R}$ which contains t.

(2.2) DEFINITION. Let (U, t) and (U', t') be two links on the curve γ. We say that these links *overlap* if

$$\gamma^{-1}(U)_t \cap \gamma^{-1}(U')_{t'} \neq \emptyset, \tag{2.3}$$

or equivalently if

$$\gamma(\langle t, t'\rangle) \subset U \cup U' \quad (\text{for } t \leqslant t') \text{ (see Fig. 2).} \tag{2.4}$$

(2.5) DEFINITION. By a *segment of a chain of open sets along* γ we mean a finite sequence

$$\mathscr{C} = (U_1, t_1; \ldots; U_r, t_r) \tag{2.6}$$

of links such that

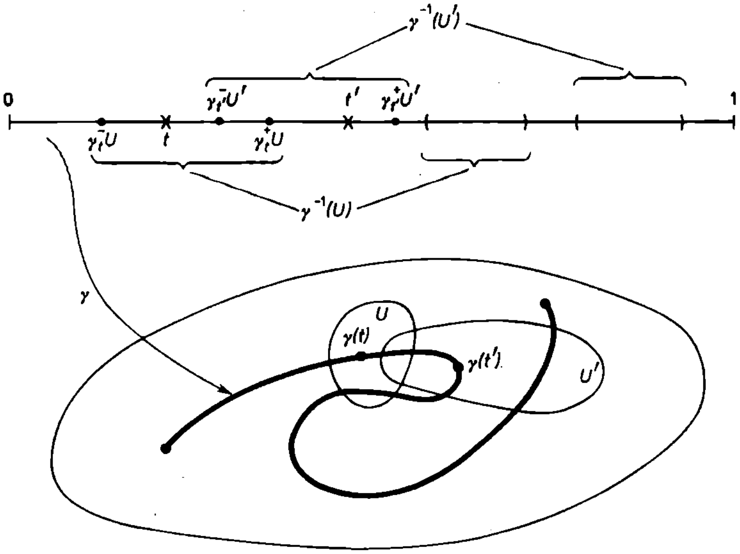

Fig. 2

(i) (U_i, t_i), (U_{i+1}, t_{i+1}) overlap for $i = 1, \ldots, r-1$.

If, moreover,

(ii) $t_1 = 0$ and $t_r = 1$,

then $\mathscr{C}$ is called a *chain of open sets along* γ.

(2.6) will also be denoted by $(U_i, t_i)_{i=1}^r$.

If (2.6) is a segment of a chain along γ, then the number r is denoted by $|\mathscr{C}|$ and called the *length* of $\mathscr{C}$.

Let U and U' be arbitrary neighbourhoods of $\gamma(0)$ and $\gamma(1)$, respectively. We denote by $C^\gamma_{U,U'}$ the family of all chains $(U_i, t_i)_{i=0}^{r+1}$ of open sets along γ such that $U_0 = U$ and $U_{r+1} = U'$. In $C^\gamma_{U,U'}$ we define a relation $\dashv$ in the following way:

(2.7) DEFINITION. If

$$\mathscr{C} = (U_i, t_i)_{i=0}^{r+1}, \qquad \mathscr{C}' = (U_i', t_i')_{i=0}^{r'+1}$$

are elements of $C^\gamma_{U,U'}$, then $\mathscr{C} \dashv \mathscr{C}'$ if and only if

(2.8) $r \leqslant r'$ and there exists an increasing function $\sigma\colon \{1, \ldots, r\} \to \{1, \ldots, r'\}$ such that $U'_{\sigma(i)} = U_i$ and $t'_{\sigma(i)} = t_i$ for every $i \in \{1, \ldots, r\}$.

It is easy to check the following:

(2.9) LEMMA. *The set* $C^\gamma_{U,U'}$ *is partially ordered by* $\dashv$.

Now, we prove the following important fact:

(2.10) PROPOSITION. *The set* $C^\gamma_{U,U'}$ *is directed by* $\dashv$, i.e. *for all* $\mathscr{C}, \mathscr{C}' \in C^\gamma_{U,U'}$ *there exists a chain* $\mathscr{C}'' \in C^\gamma_{U,U'}$ *such that* $\mathscr{C} \dashv \mathscr{C}''$ *and* $\mathscr{C}' \dashv \mathscr{C}''$.

Proof. Let $\mathscr{C} = (U_i, t_i)_{i=0}^{r+1}$ and $\mathscr{C}' = (U'_i, t'_i)_{i=0}^{r'+1}$. For every $i' \in \{0, 1, \ldots, r'+1\}$, we define

$$J_{i'} := \{j \in \{0, \ldots, r+1\};\ (U_j, t_j) \text{ and } (U'_{i'}, t'_{i'}) \text{ overlap}\}.$$

Observe that $J_{i'} \neq \emptyset$ for every i'. Indeed, the family $\{\gamma^{-1}(U_j)_{t_j}\}_{j=0}^{r+1}$ is a covering of $\langle 0, 1\rangle$, so there exists j_0 such that $t'_{i'} \in \gamma^{-1}(U_{j_0})_{t_{j_0}}$. Thus (U_{j_0}, t_{j_0}) and $(U'_{i'}, t'_{i'})$ overlap.

Set $j_{i'} := \min J_{i'}$ and define

$$\mathscr{C}'' := \big((U_i, t_i)_{i=0}^{j_1}, (U'_1, t'_1), (U_i, t_i)_{i=j_1}^{1}, (U_i, t_i)_{i=0}^{j_2},$$
$$(U'_2, t'_2), \ldots, (U'_{r'}, t'_{r'}), (U_i, t_i)_{i=j_{r'}}^{1}, (U_i, t_i)_{i=0}^{r+1}\big).$$

It is clear that $\mathscr{C}''$ is a chain of open sets from $C^{\gamma}_{U,U'}$ and that $\mathscr{C} \dashv \mathscr{C}''$ and $\mathscr{C}' \dashv \mathscr{C}''$. ∎

Now, if $\tau\colon \{1, \ldots, r\} \to \{1, \ldots, r\}$ is an arbitrary permutation, then for every positive integer $1 \leqslant k \leqslant r$ we define an injection $\tau_k\colon \{1, \ldots, k\} \to \{1, \ldots, r\}$ by

$$\tau_k(1) = \min\{\tau(1), \ldots, \tau(k)\},$$
$$\tau_k(i+1) = \min\big(\{\tau(1), \ldots, \tau(k)\} \setminus \{\tau_k(1), \ldots, \tau_k(i)\}\big).$$

Note that $\tau_r = \mathrm{id}_{\{1,\ldots,r\}}$. Analogously, we can define τ_k for $\mu \leqslant k \leqslant \nu$ if τ is a permutation of $\{\mu, \mu+1, \ldots, \nu-1, \nu\}$.

(2.11) LEMMA. *Let*

$$(U_i, t_i)_{i=0}^{r+1} \tag{2.12}$$

be a segment of a chain of open sets along γ such that the links (U_0, t_0), (U_{r+1}, t_{r+1}) overlap. Then there exists a permutation $\tau\colon \{1, \ldots, r\} \to \{1, \ldots, r\}$ such that for every positive integer $1 \leqslant k \leqslant r$ the sequence

$$\big((U_0, t_0), (U_{\tau_k(i)}, t_{\tau_k(i)})_{i=1}^{k}, (U_{r+1}, t_{r+1})\big)$$

is a segment of a chain of open sets along γ.

Proof. Put $I = \{1, \ldots, r\}$. Observe that since the links (U_0, t_0) and (U_{r+1}, t_{r+1}) overlap, we have

$$\gamma^{-1}(U_0)_{t_0} \cap \gamma^{-1}(U_{r+1})_{t_{r+1}} \neq \emptyset$$

and this intersection is an open interval in $\langle 0, 1\rangle$. We denote this interval by (α, β) and divide the set I into the following parts:

$$I_0 := \{i \in I;\ \gamma^{-1}(U_i)_{t_i} \cap (\alpha, \beta) \neq \emptyset\},$$
$$I_1 := \{i \in I;\ \gamma_{t_i}^{+} U_i \leqslant \alpha\},$$

$$I_2 := \{i \in I;\ \gamma^-_{t_i} U_i \geqslant \beta\}.$$

From the definition of $\gamma^+_t U$ and $\gamma^-_t U$ it is clear that the sets I_0, I_1, I_2 satisfy the conditions

$$I_0 \cup I_1 \cup I_2 = I, \tag{2.13}$$

$$I_j \cap I_{j'} = \emptyset \quad \text{for } j, j' = 0, 1, 2,\ j \neq j'. \tag{2.14}$$

Let $|I_j|$ denote the number of elements in I_j for $j = 0, 1, 2$.

Now we define a permutation $\tau\colon I \to I$ in the following way:

(2.15) $\tau \mid \{1, \ldots, |I_0|\}\colon \{1, \ldots, |I_0|\} \to I_0$ is an increasing mapping,

(2.16) $\tau \mid \{|I_0|+1, \ldots, |I_0|+|I_1|\}\colon \{|I_0|+1, \ldots, |I_0|+|I_1|\} \to I_1$ is such that $\gamma^+_{t_{\tau(i)}} U_{\tau(i)} \geqslant \gamma^+_{t_{\tau(i+1)}} U_{\tau(i+1)}$ for $i \in \{|I_0|+1, \ldots, |I_0|+|I_1|-1\}$,

(2.17) $\tau \mid \{|I_0|+|I_1|+1, \ldots, r\}\colon \{|I_0|+|I_1|+1, \ldots, r\} \to I_2$ is such that $\gamma^-_{t_{\tau(i)}} U_{\tau(i)} \leqslant \gamma^-_{t_{\tau(i+1)}} U_{\tau(i+1)}$ for $i \in \{|I_0|+|I_1|+1, \ldots, r-1\}$.

By (2.13) and (2.14), τ is a well-defined permutation.

We show that this permutation satisfies our condition. Let $k \in I$. We consider three cases:

1° $k \in \{1, \ldots, |I_0|\}$,
2° $k \in \{|I_0|+1, \ldots, |I_0|+|I_1|\}$,
3° $k \in \{|I_0|+|I_1|+1, \ldots, r\}$.

In case 1°, we have $\tau_k = \tau \mid \{1, \ldots, k\}$ because of (2.15), so we have to prove that $\big((U_0, t_0), (U_{\tau(i)}, t_{\tau(i)})_{i=1}^k, (U_{r+1}, t_{r+1})\big)$ is a segment of a chain of open sets along γ. From the definition of I_0 it follows directly that the links (U_0, t_0), $(U_{\tau(1)}, t_{\tau(1)})$ and $(U_{\tau(k)}, t_{\tau(k)})$, (U_{r+1}, t_{r+1}) overlap. So it suffices to show that $(U_{\tau(\varkappa)}, t_{\tau(\varkappa)})$ and $(U_{\tau(\varkappa+1)}, t_{\tau(\varkappa+1)})$ overlap for $\varkappa \in \{1, \ldots, k-1\}$. This is obvious if $\tau(\varkappa+1) = \tau(\varkappa)+1$, by the assumption of the lemma. Thus we assume that

$$\tau(\varkappa+1) > \tau(\varkappa)+1.$$

Note that from the definition of I_0, I_1 and I_2 it follows easily that

$$\{\tau(\varkappa)+1, \tau(\varkappa)+2, \ldots, \tau(\varkappa+1)-1\} \subset I_1 \quad \text{or} \quad \subset I_2. \tag{2.18}$$

In other words, if one of the numbers i and $i+1$ belongs to I_j, then the other cannot lie in I_{3-j} ($j = 1, 2$).

We assume that

$$\{\tau(\varkappa)+1, \ldots, \tau(\varkappa+1)-1\} \subset I_1$$

(otherwise the proof is analogous). We show that

$$\alpha \in \gamma^{-1}(U_{\tau(x)})_{t_{\tau(x)}} \cap \gamma^{-1}(U_{\tau(x+1)})_{t_{\tau(x+1)}}. \tag{2.19}$$

Indeed,

$$\gamma^{-}_{t_{\tau(x)}} U_{\tau(x)} < \gamma^{+}_{t_{\tau(x)+1}} U_{\tau(x)+1} \leqslant \alpha < \gamma^{+}_{t_{\tau(x)}} U_{\tau(x)}$$

since $(U_{\tau(x)}, t_{\tau(x)})$ and $(U_{\tau(x)+1}, t_{\tau(x)+1})$ overlap, $\tau(x)+1 \in I_1$ and $\tau(x) \in I_0$. Thus $\alpha \in \gamma^{-1}(U_{\tau(x)})_{t_{\tau(x)}}$. Similarly,

$$\begin{aligned}\gamma^{-}_{t_{\tau(x+1)}} U_{\tau(x+1)} &< \gamma^{+}_{t_{\tau(x+1)-1}} U_{\tau(x+1)-1} \\ &\leqslant \alpha < \gamma^{+}_{t_{\tau(x+1)}} U_{\tau(x+1)}\end{aligned}$$

and so $\alpha \in \gamma^{-1}(U_{\tau(x+1)})_{t_{\tau(x+1)}}$. Thus we have (2.19), i.e. the links $(U_{\tau(x)}, t_{\tau(x)})$ and $(U_{\tau(x+1)}, t_{\tau(x+1)})$ overlap for $x = 1, 2, \ldots, k-1$. Hence the proposition holds for $k \in \{1, \ldots, |I_0|\}$.

In case 2° we proceed by induction. Let $k = |I_0|+1$ and

$$A_0 := \{(U_i, t_i);\ i \in I_0\} \cup \{(U_0, t_0), (U_{r+1}, t_{r+1})\}.$$

We wish to prove that the sequence

$$((U_0, t_0), (U_i, t_i)_i, (U_{r+1}, t_{r+1})), \qquad \text{where } i = \tau_{|I_0|+1}(1), \ldots, \tau_{|I_0|+1}(|I_0|+1) \tag{2.20}$$

is a segment of a chain along γ. We already know that

$$((U_0, t_0), (U_i, t_i)_i, (U_{r+1}, t_{r+1})), \quad \text{where } i = \tau_{|I_0|}(1), \ldots, \tau_{|I_0|}(|I_0|) \tag{2.21}$$

is a segment of a chain along γ. Note that (2.20) differs from (2.21) in one link, so this is the only point to be checked. Let

$$(U_\varepsilon, t_\varepsilon;\ U_{\tau(k)}, t_{\tau(k)};\ U_\delta, t_\delta) \tag{2.22}$$

be the corresponding subsequence of (2.20). We have to show that $(U_\varepsilon, t_\varepsilon)$, $(U_{\tau(k)}, t_{\tau(k)})$ and $(U_{\tau(k)}, t_{\tau(k)})$, (U_δ, t_δ) are pairs of overlapping links. Let

$$(U_i, t_i)_{i=\varepsilon}^{\delta} \tag{2.23}$$

be the corresponding part of (2.12). Then (2.23) is obviously a segment of a chain of open sets along γ. Now, we prove that $(U_\varepsilon, t_\varepsilon)$ and $(U_{\tau(k)}, t_{\tau(k)})$ overlap (for $(U_{\tau(k)}, t_{\tau(k)})$ and (U_δ, t_δ) the proof is analogous). Since $\tau(k) \in I_1$, in view of (2.18) we have $\varepsilon+1, \ldots, \tau(k)-1 \in I_1$. Since $(U_\varepsilon, t_\varepsilon)$, $(U_{\varepsilon+1}, t_{\varepsilon+1})$ overlap and $(U_\varepsilon, t_\varepsilon) \in A_0$ and $\varepsilon+1 \in I_1$, we get

$$\gamma^{-}_{t_\varepsilon} U_\varepsilon < \gamma^{+}_{t_{\varepsilon+1}} U_{\varepsilon+1}. \tag{2.24}$$

Next, by (2.16) we get

(2.25) $$\gamma_{t_{\varepsilon+1}}^{+} U_{\varepsilon+1} \leqslant \gamma_{t_{\tau(k)}}^{+} U_{\tau(k)},$$

(2.26) $$\gamma_{t_{\tau(k)}}^{+} U_{\tau(k)} \leqslant \alpha$$

because $\tau(k) \in I_1$. Finally, since $(U_\varepsilon, t_\varepsilon) \in A_0$,

(2.27) $$\alpha < \gamma_{t_\varepsilon}^{+} U_\varepsilon.$$

Putting (2.24)–(2.27) together, we get

$$\gamma_{t_\varepsilon}^{-} U_\varepsilon < \gamma_{t_{\tau(k)}}^{+} U_{\tau(k)} < \gamma_{t_\varepsilon}^{+} U_\varepsilon;$$

therefore $(U_\varepsilon, t_\varepsilon)$ and $(U_{\tau(k)}, t_{\tau(k)})$ overlap.

Now, assume that the proposition holds for some $|I_0|+1 \leqslant k \leqslant |I_0|+|I_1|-1$, i.e.

(2.28) $$\big((U_0, t_0), (U_{\tau_k(i)}, t_{\tau_k(i)})_{i=1}^{k}, (U_{r+1}, t_{r+1})\big)$$

is a segment of a chain. Consider the sequence

(2.29) $$\big((U_0, t_0), (U_{\tau_{k+1}(i)}, t_{\tau_{k+1}(i)})_{i=1}^{k+1}, (U_{r+1}, t_{r+1})\big).$$

Then (2.29) differs from (2.28) in one link, namely $(U_{\tau(k+1)}, t_{\tau(k+1)})$.

Let

(2.30) $$(U_\varepsilon, t_\varepsilon;\ U_{\tau(k+1)}, t_{\tau(k+1)};\ U_\delta, t_\delta)$$

be the corresponding subsequence of (2.29). If either $(U_\varepsilon, t_\varepsilon) \in A_0$ or $(U_\delta, t_\delta) \in A_0$, then the required overlapping can be shown as in the case $k = |I_0|+1$. For that reason we can assume that $\varepsilon, \delta \in I_1$. Let

(2.31) $$(U_i, t_i)_{i=\varepsilon}^{\delta}$$

be the corresponding part of (2.12). Note that $\varepsilon+1 \in I_1$. Since $(U_\varepsilon, t_\varepsilon)$, $(U_{\varepsilon+1}, t_{\varepsilon+1})$ overlap and

$$\gamma_{t_\varepsilon}^{+} U_\varepsilon \geqslant \gamma_{t_{\varepsilon+1}}^{+} U_{\varepsilon+1}$$

by (2.16), we have

(2.32) $$\gamma_{t_\varepsilon}^{-} U_\varepsilon < \gamma_{t_{\varepsilon+1}}^{+} U_{\varepsilon+1}.$$

Next, in view of (2.16),

(2.33) $$\gamma_{t_{\varepsilon+1}}^{+} U_{\varepsilon+1} \leqslant \gamma_{t_{\tau(k+1)}}^{+} U_{\tau(k+1)},$$

(2.34) $$\gamma_{t_{\tau(k+1)}}^{+} U_{\tau(k+1)} \leqslant \gamma_{t_\varepsilon}^{+} U_\varepsilon.$$

Putting (2.32)–(2.34) together we obtain the required result – $(U_\varepsilon, t_\varepsilon)$ and $(U_{\tau(k+1)}, t_{\tau(k+1)})$ overlap. In a similar way we find that $(U_{\tau(k+1)}, t_{\tau(k+1)})$ and (U_δ, t_δ) overlap. This completes the induction step in case 2°.

Since in view of (2.18) the links with indices from I_2 cannot overlap with those with indices from I_1, the proof in case 3° is the same as in case 2°. ∎

(2.35) PROPOSITION, *Let* $\mathscr{C}, \mathscr{C}' \in C^{\gamma}_{U,U'}$ and $\mathscr{C} \dashv \mathscr{C}'$. *Then there exists a sequence* $(\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_{|\mathscr{C}'|-|\mathscr{C}|})$ *of chains of* $C^{\gamma}_{U,U'}$ *such that*

$$\mathscr{C}_0 = \mathscr{C}, \quad \mathscr{C}_{|\mathscr{C}'|-|\mathscr{C}|} = \mathscr{C}', \quad \mathscr{C}_{j-1} \dashv \mathscr{C}_j,$$

$$|\mathscr{C}_j| = |\mathscr{C}|+j \quad \textit{for } j = 1, \ldots, |\mathscr{C}'|-|\mathscr{C}|.$$

Proof. Let

$$\mathscr{C} = (U_i, t_i)_{i=0}^{r+1}, \qquad \mathscr{C}' = (U'_i, t'_i)_{i=0}^{r'+1}.$$

By the assumption we have $r \leqslant r'$ and there exists an increasing mapping $\sigma\colon \{1, \ldots, r\} \to \{1, \ldots, r'\}$ such that for every $j \in \{1, \ldots, r\}$, $(U'_{\sigma(j)}, t'_{\sigma(j)}) = (U_j, t_j)$. For the symmetry of notation let $\sigma(0) = 0$ and $\sigma(r+1) = r'+1$. We shall apply Lemma (2.11) to the following segments of the chain $\mathscr{C}'$:

$$(U'_p, t'_p)_{p=0}^{\sigma(1)}, \quad (U'_p, t'_p)_{p=\sigma(1)}^{\sigma(2)}, \quad \ldots, \quad (U'_p, t'_p)_{p=\sigma(j)}^{\sigma(j+1)}, \quad \ldots, \quad (U'_p, t'_p)_{p=\sigma(r)}^{r'+1}.$$

In view of that lemma, there exist permutations:

$$\tau^{(0)}\colon \{1, \ldots, \sigma(1)-1\} \to \{1, \ldots, \sigma(1)-1\},$$

$$\tau^{(1)}\colon \{\sigma(1)+1, \ldots, \sigma(2)-1\} \to \{\sigma(1)+1, \ldots, \sigma(2)-1\},$$

. . . .

$$\tau^{(j)}\colon \{\sigma(j)+1, \ldots, \sigma(j+1)-1\} \to \{\sigma(j)+1, \ldots, \sigma(j+1)-1\},$$

. . . .

$$\tau^{(r)}\colon \{\sigma(r)+1, \ldots, r'\} \to \{\sigma(r)+1, \ldots, r'\}$$

(possibly some of these mappings are empty) which fulfil the assertion of the lemma.

Put $\mathscr{C}_0 := \mathscr{C}$ and let i be the smallest number in $\{0, \ldots, r\}$ such that $\tau^{(i)}$ is a nonempty mapping. Set

$$\mathscr{C}_1 := \big((U_p, t_p)_{p=0}^{i}, (U'_q, t'_q), (U_p, t_p)_{p=i+1}^{r+1}\big), \quad \text{where}$$

$$q = \tau^{(i)}_{\sigma(i)+1}\big(\sigma(i)+1\big).$$

Then by Lemma (2.11) $(U_i, t_i;\ U'_q, t'_q;\ U_{i+1}, t_{i+1})$ is a segment of a chain; hence $\mathscr{C}_1$ is a chain along γ and $|\mathscr{C}_1| = |\mathscr{C}|+1$.

Assume that $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_j$ have already been defined and

$$\mathscr{C}_j = \big((U_p, t_p)_{p=0}^{i}, (U'_{q_i(p)}, t'_{q_i(p)})_{p=\sigma(i)+1}^{\sigma(i+1)-1}, (U_p, t_p)_{p=i+1}^{i'},$$
$$(U'_{q_{i'}(p)}, t'_{q_{i'}(p)})_{p=\sigma(i')+1}^{\sigma(i'+1)-1}, \ldots, (U'_{q'_l(p)}, t'_{q'_l(p)})_{p=\sigma(l)+1}^{\sigma(l)+k}, (U_p, t_p)_{p=l+1}^{r+1}\big),$$

where $q_s(p) = \tau^{(s)}_{\sigma(s+1)-1}(p)$, $q'_l(p) = \tau^{(l)}_{\sigma(l)+k}(p)$ and i' is the next number in $\{0, \ldots, r\}$ such that $\tau^{(i')}$ is a nonempty mapping.

Let us consider two cases:

1° $\sigma(l)+k=\sigma(l+1)-1$. Then by the definition of τ_k

$$\text{(2.36)}\quad \big(\tau^{(l)}_{\sigma(l)+k}(\sigma(l)+1),\ \ldots,\ \tau^{(l)}_{\sigma(l)+k}(\sigma(l)+k)\big)=\big(\sigma(l)+1,\ \ldots,\ \sigma(l+1)-1\big).$$

Let l' be the smallest number in $\{l+1,\ \ldots,\ r\}$ such that $\tau^{(l')}$ is nonempty. Put

$$\mathscr{C}_{j+1}:=\big((U_p,\ t_p)^{i}_{p=0},\ \ldots,\ (U'_{q'_l(p)},\ t'_{q'_l(p)})^{\sigma(l)+k}_{p=\sigma(l)+1},$$
$$(U_p,\ t_p)^{l'}_{p=l+1},\ (U'_{q''},\ t'_{q''}),\ (U_p,\ t_p)^{r+1}_{p=l'+1}\big),\qquad \text{where } q''=\tau^{(l')}_{\sigma(l')+1}(\sigma(l')+1).$$

(If such a l' does not exist, then $\mathscr{C}_j=\mathscr{C}'$ and we get the assertion.) In view of Lemma (2.11), $\mathscr{C}_{j+1}$ is a chain along γ.

2° $\sigma(l)+k<\sigma(l+1)-1$. In this case we set

$$\mathscr{C}_{j+1}:=\big((U_p,\ t_p)^{i}_{p=0},\ \ldots,\ (U'_{q''_l(p)},\ t'_{q''_l(p)})^{\sigma(l)+k+1}_{p=\sigma(l)+1},\ (U_p,\ t_p)^{r+1}_{p=l+1}\big),$$
$$\text{where } q''_l(p)=\tau^{(l)}_{\sigma(l)+k+1}(p).$$

In view of Lemma (2.11), $\mathscr{C}_{j+1}$ is a chain along γ.

In both cases, the equality

$$|\mathscr{C}_{j+1}|=|\mathscr{C}_j|+1=|\mathscr{C}|+j+1$$

follows directly from the construction of $\mathscr{C}_{j+1}$.

It remains to show that

$$\mathscr{C}_{|\mathscr{C}'|-|\mathscr{C}|}=\mathscr{C}'.$$

From (2.36) with $l=0,\ \ldots,\ r$ it follows that

$$\begin{aligned}\mathscr{C}_{|\mathscr{C}'|-|\mathscr{C}|}&=\big((U_0,\ t_0),\ (U'_{q_0(p)},\ t'_{q_0(p)})^{\sigma(1)-1}_{p=1},\ (U_1,\ t_1),\\&\qquad(U'_{q_1(p)},\ t'_{q_1(p)})^{\sigma(2)-1}_{p=\sigma(1)+1},\ (U_2,\ t_2),\ \ldots,\\&\qquad(U_r,\ t_r),\ (U'_{q_r(p)},\ t'_{q_r(p)})^{r'}_{p=\sigma(r)+1},\ (U_{r+1},\ t_{r+1})\big)\\&=(U'_p,\ t'_p)^{r'+1}_{p=0}=\mathscr{C}',\end{aligned}$$

where $q_s(p)$ is as above. ■

(2.37) **Remark.** Note that $\mathscr{C}_{j+1}$ differs from $\mathscr{C}_j$ only in one link. Thus, if $\mathscr{C} \mathrel{-\!\!\!3} \mathscr{C}'$, then $\mathscr{C}'$ can be obtained from $\mathscr{C}$ by adding a number of links, one link at a time, so that each time we arrive at a chain.

In the sequel, we consider links and chains of charts distinguished by $\mathscr{F}$ along a curve γ on a leaf L.

(2.38) DEFINITION. A *link* on γ is a couple $(\varphi,\ t)$, where φ is an adapted chart around a point of L such that

$$\mathrm{pr}_2\,\varphi(\gamma(t))=0$$

($\mathrm{pr}_2\colon U_\varphi\times W_\varphi\to W_\varphi$ is the natural projection).

(2.39) DEFINITION. A *chain of adapted charts* along γ is a sequence $(\varphi_1, t_1; \ldots; \varphi_r, t_r)$, also denoted by $(\varphi_i, t_i)_{i=1}^r$, of links such that $(D_{\varphi_i}, t_i)_{i=1}^r$ is a chain of open sets along γ.

Just as above, we define the family $C^{\gamma}_{\varphi,\psi}$ for adapted charts φ, ψ around points of L as well as the relation $\dashv$ in this family, additionally requiring in the definition of $\dashv$ that

$$\varphi'_{\sigma(i)} = \varphi_i .$$

The above definitions and the combinatorial nature of Propositions (2.10) and (2.35) permit us to formulate the analogous results for chains of adapted charts along γ.

(2.40) PROPOSITION. *The set* $C^{\gamma}_{\varphi,\psi}$ *is directed by* $\dashv$.

(2.41) PROPOSITION. If $\mathscr{C}, \mathscr{C}' \in C^{\gamma}_{\varphi,\psi}$ *and* $\mathscr{C} \dashv \mathscr{C}'$, *then there exists a sequence* $(\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_{|\mathscr{C}'|-|\mathscr{C}|})$ *of chains from* $C^{\gamma}_{\varphi,\psi}$ *such that*

$$\mathscr{C}_0 = \mathscr{C}, \quad \mathscr{C}_{|\mathscr{C}'|-|\mathscr{C}|} = \mathscr{C}', \quad \mathscr{C}_{j-1} \dashv \mathscr{C}_j,$$
$$|\mathscr{C}_j| = |\mathscr{C}| + j \quad \textit{for } j = 1, \ldots, |\mathscr{C}'| - |\mathscr{C}|.$$

3. Some topological properties of foliations with singularities

Let $\mathscr{F}$ be a foliation on a manifold M and let $L \in \mathscr{F}$.

(3.1) LEMMA. *Let* $x, y \in L$. *Then there exists a chain of adapted charts* $(\varphi_i, t_i)_{i=1}^r$ *such that* φ_1 *and* φ_r *are charts around* x *and* y, *respectively and each leaf* L' *intersects* D_{φ_1} *if it intersects* D_{φ_r}.

Proof. Let $\gamma\colon \langle 0, 1\rangle \to L$ be a curve joining x to y. For every $t \in \langle 0, 1\rangle$ we can find an adapted chart $\varphi^{(t)}$ around $\gamma(t)$ such that $D_{\varphi^{(t)}} \cap \gamma(\langle 0, 1\rangle)$ is connected. The covering $\{\gamma^{-1}(D_{\varphi^{(t)}})_t\}_{t\in\langle 0,1\rangle}$ of $\langle 0, 1\rangle$ has a finite subcovering $\{\gamma^{-1}(D_{\varphi^{(t_i)}})_{t_i}\}_{i=1}^r$ such that $t_1 = 0$, $t_r = 1$ and

$$\gamma^{-1}(D_{\varphi^{(t_i)}})_{t_i} \cap \gamma^{-1}(D_{\varphi^{(t_{i+1})}})_{t_{i+1}} \neq \emptyset \qquad \text{for } i = 1, \ldots, r-1.$$

In this way we obtain the chain $(\varphi^{(t_1)}, t_1; \ldots; \varphi^{(t_r)}, t_r)$ of adapted charts along γ. Define

$$\varphi_1 := \varphi^{(t_1)},$$
$$D_{\varphi_2} := \varphi^{(t_2)-1}\big(U_{\varphi^{(t_2)}} \times \mathrm{pr}_2\, \varphi^{(t_2)}(D_{\varphi_1} \cap D_{\varphi^{(t_2)}})\big).$$

The set D_{φ_2} is an open neighbourhood of $\gamma(t_2)$ and the mapping

$$\varphi_2 := \varphi^{(t_2)} \,|\, D_{\varphi_2}$$

is a chart distinguished by $\mathscr{F}$ around $\gamma(t_2)$. We have

$$\gamma^{-1}(D_{\varphi_1})_{t_1} \cap \gamma^{-1}(D_{\varphi_2})_{t_2} = \gamma^{-1}(D_{\varphi^{(t_1)}})_{t_1} \cap \gamma^{-1}(D_{\varphi^{(t_2)}})_{t_2} \neq \emptyset.$$

Let a leaf L' intersect D_{φ_2} and let $z \in L' \cap D_{\varphi_2}$. Then $z = \varphi^{(t_2)^{-1}}(u, w)$, where $u \in U_{\varphi^{(t_2)}}$ and $w = \mathrm{pr}_2\, \varphi^{(t_2)}(z')$ for $z' \in D_{\varphi_1} \cap D_{\varphi^{(t_2)}}$. Note that $z' \in L'$. Therefore $z' \in D_{\varphi_1} \cap L'$, and so $D_{\varphi_1} \cap L' \neq \emptyset$. Set

$$D_{\varphi_i} := \varphi^{(t_i)^{-1}}\big(U_{\varphi^{(t_i)}} \times \mathrm{pr}_2\, \varphi^{(t_i)}(D_{\varphi_{i-1}} \cap D_{\varphi^{(t_i)}})\big),$$
$$\varphi_i := \varphi^{(t_i)} \mid D_{\varphi_i}.$$

It is clear that φ_i is an adapted chart and that $(\varphi_{i-1}, t_{i-1}), (\varphi_i, t_i)$ overlap. Analogously, we prove that if a leaf L' intersects D_{φ_i}, then it also intersects $D_{\varphi_{i-1}}$. This completes the proof. ∎

Let $M/\mathscr{F}$ denote the quotient space of the topological space M by the equivalence relation determined by the decomposition of M into the leaves of $\mathscr{F}$.

(3.2) PROPOSITON. *The canonical projection* $\pi_M\colon M \to M/\mathscr{F}$ *is open.*

Proof. Let U be an open subset of M. We have to show that $\pi_M^{-1}\pi_M(U)$ is open in M. Let $y \in \pi_M^{-1}\pi_M(U)$, i.e. $L_y \cap U \neq \emptyset$. Let $x \in L_y \cap U$. Applying Lemma (3.1) to the leaf L_y and its points x, y we observe that there exists a chain $(\varphi_1, t_1; \ldots; \varphi_r, t_r)$ such that $x \in D_{\varphi_1} \subset U$, $y \in D_{\varphi_r}$ and each leaf L' intersects D_{φ_1} if it intersects D_{φ_r}. Thus $D_{\varphi_r} \subset \pi_M^{-1}\pi_M(U)$ and $\pi_M^{-1}\pi_M(U)$ is open in M. ∎

(3.3) PROPOSITION. *The union A of all nonsingular leaves of $\mathscr{F}$ is open in M.*

Proof. Let $x \in A$. Then L_x is a nonsingular leaf of $\mathscr{F}$, and so by definition there exists an open neighbourhood V of L_x such that the function $d_{\mathscr{F}} \mid V$ is constant. By Proposition (3.2), $U := \pi_M^{-1}\pi_M(V)$ is an open neighbourhood of x in M. Since U is a neighbourhood of every leaf contained in U and $d_{\mathscr{F}}$ is constant on U, it follows that U is contained in A. ∎

4. Holonomy group of a leaf

Let $\mathscr{F}$ be a foliation of M and let $L \in \mathscr{F}$. Let $\gamma\colon \langle 0, 1\rangle \to L$ be an arbitrary curve and let (φ, s) and (ψ, u) be overlapping links on γ. Thus the set

$$\gamma^{-1}(D_\varphi)_s \cap \gamma^{-1}(D_\psi)_u \tag{4.1}$$

is nonempty; it is also connected being the intersection of intervals. Let P and Q denote the central plaques of φ and ψ, respectively, i.e.

$$P = \varphi^{-1}(U_\varphi \times \{0\}), \qquad Q = \psi^{-1}(U_\psi \times \{0\}).$$

Let x be an arbitrary point of the connected component of $P \cap Q$ containing the set $\gamma[\gamma^{-1}(D_\varphi)_s \cap \gamma^{-1}(D_\psi)_u]$. Denote this component by $(P \cap Q)^{(\gamma)}_{s,u}$; it is uniquely determined because of the connectedness of (4.1).

Put

$$G' := \{w \in W_\varphi;\ \varphi^{-1}(\mathrm{pr}_1 \varphi(x), w) \in D_\psi\}.$$

Then G' is obviously an open neighbourhood of 0 in W_φ. In G' we define the C^∞ mapping

$$f\colon\ G' \ni w \mapsto \mathrm{pr}_2 \psi\varphi^{-1}(\mathrm{pr}_1 \varphi(x), w) \in W_\psi.$$

The differential of f at 0 is a monomorphism. Indeed, if $X \in T_0 W_\varphi$ and $f_* X = 0$, then $(\varphi^{-1}(\mathrm{pr}_1 \varphi(x), \cdot))_* X \in T_x L$, so $X \in T_{\mathrm{pr}_1 \varphi(x)} U_\varphi$. Therefore $X = 0$ since $T_0 W_\varphi \cap T_{\mathrm{pr}_1 \varphi(x)} U_\varphi = \{0\}$. The equality $\dim G' = \dim W_\psi$ implies that the differential f_* is an isomorphism. Thus there exists a neighbourhood G of 0 in G' and a neighbourhood H of 0 in W_ψ such that $f \mid G\colon G \to H$ is a diffeomorphism. This diffeomorphism will be denoted by the same symbol f.

(4.2) LEMMA. $f \in \mathscr{A}_{\varphi,\psi}$.

Proof. The mapping f is a diffeomorphism of an open neighbourhood of 0 in W_φ onto an open neighbourhood of 0 in W_ψ. A simple calculation shows that $f(0) = 0$, so it is sufficient to prove that f and f^{-1} are compatible with $\sim_{\varphi_G}$ and $\sim_{\psi_H}$. By Proposition (1.12), for the proof of the compatibility of f it suffices to show that f is compatible with $\sim_{\varphi_G}$ and $\sim_\psi$. Let $[w] \in G/\sim_{\varphi_G}$. Then the set $V := \varphi^{-1}(\{\mathrm{pr}_1 \varphi(x)\} \times [w])$ is contained in some plaque of φ, i.e. in some leaf of $\mathscr{F}$. Since $G \subset G'$, it follows that V is contained in D_ψ. Since $[w]$ is connected and φ^{-1} is continuous, V is also connected. Thus V is a subset of some plaque of ψ and $f([w])$ is contained in an equivalence class of $\sim_\psi$. Therefore f is compatible with $\sim_{\varphi_G}$ and $\sim_{\psi_H}$.

For the proof of the compatibility of f^{-1}, take $[w] \in H/\sim_{\psi_H}$. We have

$$\begin{aligned}
f^{-1}([w]) &= \{w' \in G;\ \mathrm{pr}_2 \psi\varphi^{-1}(\mathrm{pr}_1 \varphi(x), w') \in [w]\} \\
&= \{w' \in G;\ \psi^{-1}(\mathrm{pr}_1 \psi\varphi^{-1}(\mathrm{pr}_1 \varphi(x), w'), \\
&\qquad\qquad \mathrm{pr}_2 \psi\varphi^{-1}(\mathrm{pr}_1 \varphi(x), w')) \in \psi^{-1}(U_\psi \times [w])\} \\
&= \{w' \in G;\ \varphi^{-1}(\mathrm{pr}_1 \varphi(x), w') \in \psi^{-1}(U_\psi \times [w])\} \\
&\subset \{w' \in G;\ \varphi^{-1}(\mathrm{pr}_1 \varphi(x), w') \in L_{\psi^{-1}(0,w)}\} \\
&= \{w' \in G;\ \varphi^{-1}(0, w') \in L_{\psi^{-1}(0,w)}\}.
\end{aligned}$$

Since $f^{-1}([w])$ is connected, it is contained in some connected component of $\{w'\in G;\ \varphi^{-1}(0, w')\in L_{\psi^{-1}(0,w)}\}$. Consequently, $f^{-1}([w])$ is contained in an equivalence class of $\sim_{\varphi_G}$. ■

Let $x_0, x_1\in(P\cap Q)^{(\gamma)}_{s,u}$ and let $f_i\colon G_i\to H_i$ $(i=0, 1)$ be elements of $\mathscr{A}_{\varphi,\psi}$ obtained as above for the points x_i.

(4.3) LEMMA. *The equivalence* $f_0\equiv f_1$ *holds.*

Proof. The set $(P\cap Q)^{(\gamma)}_{s,u}$ is arcwise connected. Let c be a curve joining x_0 to x_1 in $(P\cap Q)^{(\gamma)}_{s,u}$. Let

$$A := \{(t, w)\in\langle 0, 1\rangle\times W_\varphi;\ \varphi^{-1}(\mathrm{pr}_1\,\varphi c(t), w)\in D_\psi\}.$$

Then A is open in $\langle 0, 1\rangle\times W_\varphi$, being the inverse image of an open set by a continuous function. On A we define

$$F\colon A\ni(t, w)\mapsto\tilde{f}_t(w)\in W_\psi,\qquad F'\colon A\ni(t, w)\mapsto J_w\tilde{f}_t\in\boldsymbol{R}, \tag{4.4}$$

where

$$\tilde{f}_t(w) = \mathrm{pr}_2\,\psi\varphi^{-1}(\mathrm{pr}_1\,\varphi c(t), w)$$

and $J_w\tilde{f}_t$ is the jacobian of $\tilde{f}_t$ at w. F and F' are continuous and $F(t, 0)\in H_0\cap H_1$ and $F'(t, 0)\neq 0$ for every $t\in\langle 0, 1\rangle$. Thus by Lemma (1.15) there exists an open neighbourhood $\tilde{G}$ of 0 in W_φ such that for $t\in\langle 0, 1\rangle$ and $w\in\tilde{G}$

$$F(t, w)\in H_0\cap H_1, \tag{4.5}$$

$$F'(t, w)\neq 0. \tag{4.6}$$

Set

$$\hat{G} = G_0\cap G_1\cap\tilde{G}\quad\text{ and }\quad\hat{f}_t = \tilde{f}_t\,|\,\hat{G}.$$

Then $\hat{G}\subset G_0\cap G_1$ and $\hat{f}_t(\hat{G})\subset H_0\cap H_1$ by (4.5), and $\hat{f}_t$ is an immersion by (4.6). The compatibility of $\hat{f}_t$ with the suitable relations can be shown just as in the proof of (4.2). It is clear that $\hat{f}_0 = f_0\,|\,\hat{G}$ and $\hat{f}_1 = f_1\,|\,\hat{G}$. The mapping

$$\langle 0, 1\rangle\times\hat{G}\ni(t, w)\mapsto\hat{f}_t(w)\in H_0\cap H_1$$

is obviously continuous.

It remains to prove that for every $w\in\hat{G}$ the image of the curve $\langle 0, 1\rangle\ni t\mapsto\hat{f}_t(w)\in H_0\cap H_1$ is contained in an equivalence class of $\sim_{\psi_{H_0\cap H_1}}$. In view of (1.11) it suffices to show that the image of that curve lies in an equivalence class of $\sim_\psi$. But this is obvious, since $\varphi^{-1}(\mathrm{pr}_1\,\varphi c(\langle 0, 1\rangle)\times\{w\})$ is a connected set lying in D_ψ and in a leaf of $\mathscr{F}$. ■

Let $\mathscr{C} = (\varphi_i, t_i)_{i=0}^{r+1}$ be a chain from $C^\gamma_{\varphi,\psi}$. For every $i\in\{0, \dots, r\}$, we choose a point $x_i\in(P_i\cap P_{i+1})^{(\gamma)}_{t_i,t_{i+1}}$, where P_h is the central plaque of φ_h for $h\in\{0, \dots, r+1\}$. Then for every i we get a well-defined diffeomorphism

$f_i \in \mathscr{A}_{\varphi_i, \varphi_{i+1}}$ which depends on φ_i, φ_{i+1} and x_i. The superposition

$$f_{\mathscr{C};x_0,\dots,x_r} = f_r \circ f_{r-1} \circ \dots \circ f_0$$

is a nonempty element of $\mathscr{A}_{\varphi,\psi}$ by (1.23). In view of (1.23) and (4.3) the equivalence class of $f_{\mathscr{C};x_0,\dots,x_r}$ does not depend on $x_0, \dots, x_r$. Thus we can write $[f_{\mathscr{C}}] \in \mathscr{A}_{\varphi,\psi}/\equiv$.

Now we prove that $[f_{\mathscr{C}}]$ does not depend on the choice of $\mathscr{C}$ from $C^{\gamma}_{\varphi,\psi}$. Let $\mathscr{C}_0, \mathscr{C}_1 \in C^{\gamma}_{\varphi,\psi}$. By Proposition (2.40) there exists a chain $\mathscr{C} \in C^{\gamma}_{\varphi,\psi}$ such that $\mathscr{C}_0 \mathrel{-\!\!\ni} \mathscr{C}$ and $\mathscr{C}_1 \mathrel{-\!\!\ni} \mathscr{C}$. Thus it suffices to prove that if $\mathscr{C} \mathrel{-\!\!\ni} \mathscr{C}'$, then $[f_{\mathscr{C}}] = [f_{\mathscr{C}'}]$. In view of Proposition (2.41) it is enough to show the equality in the case when $\mathscr{C}'$ is obtained from $\mathscr{C}$ by adding one link. Finally, according to Lemma (1.23) we only have to prove the following:

(4.7) LEMMA. *Let $\mathscr{C} = (\varphi, s; \psi, u)$ and $\mathscr{C}' = (\varphi, s; \chi, v; \psi, u)$ be segments of chains. Let $f_0\colon G_0 \to H_0$ be a diffeomorphism defined by $\mathscr{C}$ and by a point $x \in (P \cap Q)^{(\gamma)}_{s,u}$, where P and Q are the central plaques of φ and ψ, respectively. Let $f_1\colon G_1 \to H_1$ be a diffeomorphism defined by $\mathscr{C}'$ and by points $y \in (P \cap R)^{(\gamma)}_{s,v}$, $z \in (R \cap Q)^{(\gamma)}_{v,u}$, where R is a central plaque of χ. Then $f_0 \equiv f_1$.*

Proof. By Lemma (4.3) and the relation

$$\gamma^{-1}(D_{\varphi})_s \cap \gamma^{-1}(D_{\chi})_v \cap \gamma^{-1}(D_{\psi})_u \neq \emptyset$$

we can assume that $x = y = z$. Then the diffeomorphisms f_0 and f_1 are defined by

$$f_0(w) = \mathrm{pr}_2\, \psi\varphi^{-1}(\mathrm{pr}_1\, \varphi(x), w), \qquad w \in G_0, \tag{4.8}$$

$$f_1(w) = \mathrm{pr}_2\, \psi\chi^{-1}(\mathrm{pr}_1\, \chi(x), \mathrm{pr}_2\, \chi\varphi^{-1}(\mathrm{pr}_1\, \varphi(x), w)), \qquad w \in G_1. \tag{4.9}$$

Let $K \subset G_0 \cap G_1$ be an open ball centred at 0 such that for every $w \in K$

$$\varphi^{-1}(\mathrm{pr}_1\, \varphi(x), w) \in D_{\chi}. \tag{4.10}$$

Define a continuous mapping α by

$$\begin{aligned} \alpha\colon \langle 0, 1\rangle \times K \ni (t, w) \mapsto \chi^{-1}\big[&\mathrm{pr}_1\, \chi\varphi^{-1}(\mathrm{pr}_1\, \varphi(x), (1-t)w), \\ &\mathrm{pr}_2\, \chi\varphi^{-1}(\mathrm{pr}_1\, \varphi(x), w)\big] \in D_{\chi}. \end{aligned} \tag{4.11}$$

Note that

$$\alpha(\langle 0, 1\rangle \times \{0\}) = \{x\} \subset D_{\varphi} \cap D_{\chi},$$

and by Lemma (1.15) there exists an open neighbourhood $\tilde{G}$ of 0 in K such that

$$\alpha(\langle 0, 1\rangle \times \tilde{G}) \subset D_{\chi} \cap D_{\psi}. \tag{4.12}$$

Set

$$\text{(4.13)} \qquad \tilde{f}_t\colon \tilde{G} \ni w \mapsto \mathrm{pr}_2 \psi\alpha(t, w) \in W_\psi$$

and observe that for every t the mapping $\tilde{f}_t$ is an immersion at 0. Indeed, let $X \in T_0 W_\varphi$ and $(\tilde{f}_t)_{*0} X = 0$. Then $\alpha(t, \cdot)_{*0} X \in T_x L$ and by (4.11), $\tilde{f}_{*0} X = 0$, where $\tilde{f}$ denotes the diffeomorphism defined by (φ, s), (χ, v) and the point x. Thus $X = 0$. Just as in the proof of (4.3) we can show that there exists an open neighbourhood $\hat{G}$ of 0 in $\tilde{G}$ such that for every $t \in \langle 0, 1\rangle$ the mapping $\hat{f}_t := \tilde{f}_t | \hat{G}$ is an immersion and $\hat{f}_t(\hat{G}) \subset H_0 \cap H_1$. Obviously, $\hat{G} \subset G_0 \cap G_1$ and $\hat{f}_t\colon \hat{G} \to H_0 \cap H_1$ is an immersion.

We prove that $\hat{f}_t$ is compatible with $\sim_{\varphi_{\hat{G}}}$ and $\sim_{\psi_{H_0 \cap H_1}}$. In view of the continuity of $\alpha(t, \cdot)$ and the connectedness of $[w] \in \hat{G}/\sim_{\varphi_{\hat{G}}}$ it suffices to show that the image of $[w]$ by $\alpha(t, \cdot)$ is contained in some leaf of $\mathscr{F}$. Since $\mathrm{pr}_2 \chi\varphi^{-1}(\mathrm{pr}_1 \varphi(x), \cdot)$ maps $[w]$ to an equivalence class of $\sim_\chi$, it follows from (4.11) that $\alpha(t, w')$ lies in a plaque of χ for $w' \in [w]$ and consequently in a leaf of $\mathscr{F}$. Therefore $\hat{f}_t$ is compatible with $\sim_{\varphi_{\hat{G}}}$ and $\sim_\psi$, so by (1.12) it is also compatible with $\sim_{\varphi_{\hat{G}}}$ and $\sim_{\psi_{H_0 \cap H_1}}$.

Remark that

$$\begin{aligned}
\hat{f}_0(w) &= \mathrm{pr}_2 \psi\chi^{-1}[\mathrm{pr}_1 \chi\varphi^{-1}(\mathrm{pr}_1 \varphi(x), w),\\
&\qquad\qquad \mathrm{pr}_2 \chi\varphi^{-1}(\mathrm{pr}_1 \varphi(x), w)]\\
&= \mathrm{pr}_2 \psi\varphi^{-1}(\mathrm{pr}_1 \varphi(x), w) = f_0(w),
\end{aligned}$$

i.e.

$$\hat{f}_0 = f_0 | \hat{G},$$

and

$$\begin{aligned}
\hat{f}_1(w) &= \mathrm{pr}_2 \psi\chi^{-1}[\mathrm{pr}_1 \chi\varphi^{-1}(\mathrm{pr}_1 \varphi(x), 0),\\
&\qquad\qquad \mathrm{pr}_2 \chi\varphi^{-1}(\mathrm{pr}_1 \varphi(x), w)]\\
&= \mathrm{pr}_2 \psi\chi^{-1}[\mathrm{pr}_1 \chi(x), \mathrm{pr}_2 \chi\varphi^{-1}(\mathrm{pr}_1 \varphi(x), w)]\\
&= f_1(w),
\end{aligned}$$

i.e.

$$\hat{f}_1 = f_1 | \hat{G}.$$

Obviously, the mapping

$$\langle 0, 1\rangle \times \hat{G} \ni (t, w) \mapsto \hat{f}_t(w) \in H_0 \cap H_1$$

is continuous.

It remains to prove that the image of the curve $\langle 0, 1\rangle \ni t \mapsto \hat{f}_t(w) \in H_0 \cap H_1$ ($w \in \hat{G}$) lies in some equivalence class of $\sim_{\psi_{H_0 \cap H_1}}$. To this end it suffices to show that for every $w \in \hat{G}$ the image of the curve $\langle 0, 1\rangle \ni t \mapsto \alpha(t, w)$ lies in a leaf of $\mathscr{F}$. But this is clear, since only the first coordinate of the curve in the chart χ depends on t. Therefore $f_0 \equiv f_1$. ∎

In this way we have proved

(4.14) PROPOSITION. *If $\mathscr{C}, \mathscr{C}' \in C^{\gamma}_{\varphi,\psi}$, then $[f_{\mathscr{C}}] = [f_{\mathscr{C}'}]$.*

We may write $[f_{\gamma;\varphi,\psi}]$ instead of $[f_{\mathscr{C}}]$.

Now, we prove the following:

(4.15) PROPOSITION. *Let γ_s be a homotopy of curves γ_0 and γ_1 in L such that for every $s \in \langle 0, 1\rangle$, $\gamma_s(0) = \gamma_0(0)$ and $\gamma_s(1) = \gamma_0(1)$. If φ and ψ are adapted charts around $\gamma_0(0)$ and $\gamma_0(1)$, respectively, then*

$$[f_{\gamma_0;\varphi,\psi}] = [f_{\gamma_1;\varphi,\psi}]$$

in $\mathscr{A}_{\varphi,\psi}/\equiv$.

Proof. For every $s \in \langle 0, 1\rangle$ we choose a chain $\mathscr{C}^{(s)} \in C^{\gamma_s}_{\varphi,\psi}$ of the form

$$\mathscr{C}^{(s)} = (\varphi_i^{(s)}, t_i^{(s)})_{i=0}^{r(s)+1}$$

and numbers

$$\tilde{t}_i^{(s)} \in \gamma_s^{-1}(D_{\varphi_i^{(s)}})_{t_i^{(s)}} \cap \gamma_s^{-1}(D_{\varphi_{i+1}^{(s)}})_{t_{i+1}^{(s)}}$$

for $i = 0, \ldots, r(s)$. It is easily seen that for every $s \in (0, 1)$ there exists $\varepsilon_s > 0$ such that if $s' \in \langle s-\varepsilon_s, s+\varepsilon_s\rangle$, then

(4.16) $\mathscr{C}^{(s)}$ is a chain along $\gamma_{s'}$,

(4.17) the curve $\langle s-\varepsilon_s, s+\varepsilon_s\rangle \ni s' \mapsto \gamma_{s'}(\tilde{t}_i^{(s)}) \in L$ lies in the connected component of $P_i^{(s)} \cap P_{i+1}^{(s)}$ containing $\gamma_s(\tilde{t}_i^{(s)})$ for $i \in \{0, \ldots, r(s)\}$,

where $P_i^{(s)}$ denotes the central plaque of $\varphi_i^{(s)}$. Moreover, there exists $\varepsilon_0 > 0$ such that if $s' \in \langle 0, \varepsilon_0\rangle$, then

(4.18) $\mathscr{C}^{(0)}$ is a chain along $\gamma_{s'}$,

(4.19) the curve $\langle 0, \varepsilon_0\rangle \ni s' \mapsto \gamma_{s'}(\tilde{t}_i^{(1)}) \in L$ lies in the connected component of $P_i^{(0)} \cap P_{i+1}^{(0)}$ containing $\gamma_0(\tilde{t}_i^{(0)})$ for $i \in \{0, \ldots, r(0)\}$.

Similarly, there exists $\varepsilon_1 > 0$ such that if $s' \in \langle 1-\varepsilon_1, 1\rangle$, then

(4.20) $\mathscr{C}^{(1)}$ is a chain along $\gamma_{s'}$,

(4.21) the curve $\langle 1-\varepsilon_1, 1\rangle \ni s' \mapsto \gamma_{s'}(\tilde{t}_i^{(1)}) \in L$ lies in the connected component of $P_i^{(1)} \cap P_{i+1}^{(1)}$ containing $\gamma_1(\tilde{t}_i^{(1)})$ for $i \in \{0, \ldots, r(1)\}$.

The open covering $\mathfrak{A}$ of $\langle 0, 1\rangle$ obtained above has a finite subcovering $\{I_\alpha\}_{\alpha\in A}$. Observe that $\langle 0, \varepsilon_0)$ and $(1-\varepsilon_1, 1\rangle$ occur in $\{I_\alpha\}_{\alpha\in A}$ since they are the only elements of $\mathfrak{A}$ containing 0 and 1, respectively. We denote $\langle 0, \varepsilon_0)$ by

I_0. From those I_α which contain ε_0 we choose the one with greatest $\sup I_\alpha$; denote its centre by s_1. Let $I_1 := (s_1 - \varepsilon_{s_1}, s_1 + \varepsilon_{s_1})$. Assume that $I_0, \ldots, I_i$ of $\{I_\alpha\}_{\alpha \in A}$ have already been choosen. From those I_α which contain $s_i + \varepsilon_{s_i}$ we choose the one with greatest $\sup I_\alpha$; denote its centre by s_{i+1}. Let $I_{i+1} := (s_{i+1} - \varepsilon_{s_{i+1}}, s_{i+1} + \varepsilon_{s_{i+1}})$. In a finite number of steps we arrive at s_l such that $1 - \varepsilon_1 \in (s_l - \varepsilon_{s_l}, s_l + \varepsilon_{s_l})$. Let $I_{l+1} := (1 - \varepsilon_1, 1\rangle$.

In this way we get a finite covering $\{I_0, I_1, \ldots, I_l, I_{l+1}\}$ of $\langle 0, 1\rangle$ such that for every $i \in \{1, \ldots, l\}$ the left end-point of I_i belongs to I_{i-1} and the right one to I_{i+1}. Moreover, the right end-point of I_0 belongs to I_1 and the left end-point of I_{l+1} belongs to I_l.

Let $s_0 := 0$ and $s_{l+1} := 1$ for the symmetry of notation. For $i = 0, \ldots, l + 1$ denote by $f_{s_i}^{(i)}$ the diffeomorphism determined by the chain $\mathscr{C}^{(s_i)}$ and the points $\gamma_{s_i + \varepsilon_{s_i}}(\tilde{t}_j^{(s_i)})$ for $j \in \{0, \ldots, r(s_i)\}$ and for $i = 0, \ldots, l$ denote by $f_{s_i}^{(i+1)}$ the diffeomorphism determined by $\mathscr{C}^{(s_{i+1})}$ and $\gamma_{s_i + \varepsilon_{s_i}}(\tilde{t}_j^{(s_{i+1})})$ for $j \in \{0, \ldots, r(s_{i+1})\}$.

Now, we have the following sequence of equivalences:

$$\begin{aligned} f_{\gamma_0;\varphi,\psi} &\equiv f_{s_0}^{(0)} \equiv f_{s_0}^{(1)} \equiv f_{s_1}^{(1)} \\ &\equiv \ldots \equiv f_{s_i}^{(i)} \equiv f_{s_i}^{(i+1)} \\ &\equiv f_{s_{i+1}}^{(i+1)} \equiv \ldots \equiv f_{s_l}^{(l)} \\ &\equiv f_{s_l}^{(l+1)} \equiv f_{\gamma_1;\varphi,\psi} \end{aligned}$$

by (4.3), (4.14) and (4.16)–(4.21). Therefore

$$[f_{\gamma_0;\varphi,\psi}] = [f_{\gamma_1;\varphi,\psi}]. \quad \blacksquare$$

Let $x \in L$ and let φ be a chart distinguished by $\mathscr{F}$ around x. In view of the above proposition, we get a well-defined mapping

$$h_{(x,\varphi)}\colon \pi_1(L, x) \ni [\gamma] \mapsto [f_{\gamma;\varphi,\varphi}] \in \mathscr{A}_\varphi / \equiv,$$

where $\pi_1(L, x)$ is the fundamental group of L at x. It is easily seen that $h_{(x,\varphi)}$ is a group homomorphism.

(4.22) DEFINITION. The homomorphism $h_{(x,\varphi)}\colon \pi_1(L, x) \to \mathscr{A}_\varphi / \equiv$ is called the *holonomy homomorphism* and the group

$$\mathscr{H}_{(x,\varphi)}(L) = \operatorname{im} h_{(x,\varphi)}$$

is called the *holonomy group* of the leaf L with respect to the adapted chart φ around x.

Let y be another point of L and let ψ be a chart distinguished by $\mathscr{F}$ around y. Let $c\colon \langle 0, 1\rangle \to L$ be an arbitrary curve joining x to y.

(4.23) LEMMA. *There exists an isomorphism*

$$\Phi_c\colon \mathscr{A}_\varphi/\equiv \;\to \mathscr{A}_\psi/\equiv$$

such that the diagram

$$\begin{tikzcd} \pi_1(L,x) \arrow[r, "\cong", "\Gamma_c"'] \arrow[d, "h_{(x,\varphi)}"'] & \pi_1(L,y) \arrow[d, "h_{(y,\psi)}"] \\ \mathscr{A}_\varphi/\equiv \arrow[r, "\cong", "\Phi_c"'] & \mathscr{A}_\psi/\equiv \end{tikzcd} \tag{4.24}$$

commutes, with $\Gamma_c\colon [\gamma]\mapsto[c^{-1}\cdot\gamma\cdot c]$.

Proof. We define

$$\Phi_c\colon \mathscr{A}_\varphi/\equiv \;\ni[f]\mapsto[f_{c;\varphi,\psi}\circ f\circ f_{c^{-1};\psi,\varphi}]\in \mathscr{A}_\psi/\equiv .$$

The definition is correct in view of Lemma (1.23). The mapping Φ_c is a homomorphism. Indeed,

$$\begin{aligned} \Phi_c([f]\cdot[g]) &= \Phi_c([g\circ f]) \\ &= [f_{c;\varphi,\psi}\circ g\circ f\circ f_{c^{-1};\psi,\varphi}] \\ &= [f_{c;\varphi,\psi}\circ g\circ f_{c^{-1};\psi,\varphi}\circ f_{c;\varphi,\psi}\circ f\circ f_{c^{-1};\psi,\varphi}] \\ &= [f_{c;\varphi,\psi}\circ f\circ f_{c^{-1};\psi,\varphi}]\cdot[f_{c;\varphi,\psi}\circ g\circ f_{c^{-1};\psi,\varphi}] \\ &= \Phi_c([f])\cdot\Phi_c([g]) \end{aligned}$$

since in view of (4.15),

$$f_{c^{-1};\psi,\varphi}\circ f_{c;\varphi,\psi}\equiv \mathrm{id}_{W_\varphi}.$$

It is clear that

$$\Psi_c\colon \mathscr{A}_\psi/\equiv \;\ni[g]\mapsto[f_{c^{-1};\psi,\varphi}\circ g\circ f_{c;\varphi,\psi}]\in \mathscr{A}_\varphi/\equiv$$

is the inverse homomorphism. Hence we only have to prove the commutativity of (4.24):

$$\begin{aligned} h_{(y,\psi)}\big(\Gamma_c([\gamma])\big) &= h_{(y,\psi)}([c^{-1}\cdot\gamma\cdot c]) \\ &= [f_{c^{-1}\cdot\gamma\cdot c;\psi,\psi}] \\ &= [f_{c;\varphi,\psi}\circ f_{\gamma;\varphi,\varphi}\circ f_{c^{-1};\psi,\varphi}] \\ &= \Phi_c(h_{(x,\varphi)}[\gamma]). \quad \blacksquare \end{aligned}$$

The following proposition is an easy consequence of Lemma (4.23).

(4.25) PROPOSITION. $\mathscr{H}_{(x,\varphi)}(L) \cong \mathscr{H}_{(y,\psi)}(L)$.

Thus the isomorphism class of $\mathscr{H}_{(x,\varphi)}(L)$ depends only on the foliation $\mathscr{F}$ and its leaf L; hence we will write $\mathscr{H}(L)$ instead of $\mathscr{H}_{(x,\varphi)}(L)$.

(4.26) EXAMPLES. (a) From the construction, it is easily understood that if $\mathscr{F}$ is a nonsingular foliation, then $\mathscr{H}(L)$ is the classical holonomy group of the leaf L (see [R] and [T]).

(b) If L is simply connected, then obviously $\mathscr{H}(L)$ is trivial. In particular, if $L = \{x\}$, then $\mathscr{H}(L) = 0$. Thus the holonomy groups of all singular leaves of the foliations presented in Example (1.4d) are trivial.

(c) For the foliation in (1.4e) the holonomy group is trivial although L is not simply connected.

(d) Now, we give an example of a foliation for which a singular compact leaf has a nontrivial holonomy group.

Let

$$\tilde{M} = \{(x, y) \in \mathbf{R}^2;\ -1 < x < 1\}.$$

We take the following subsets in $\tilde{M}$:

$$\begin{aligned} \tilde{L} &= \{(x, y);\ x = 0\}, \\ \tilde{L}_q &= \{(x, y);\ y = 1/x^2 + q\} \qquad \text{for } q \in \mathbf{R}. \end{aligned}$$

Let

$$L' = \{(x, y, z) \in \mathbf{R}^3;\ x^2 + z^2 = 0\}.$$

Let M' arise by rotating $\tilde{M} \times \{0\}$ around the straight line L'; hence

$$M' = \{(x, y, z);\ x^2 + z^2 < 1\}.$$

Let L'_q arise from $\tilde{L}_q$ in the same way; hence

$$L'_q = \{(x, y, z);\ y > q + 1 \text{ and } x^2 + z^2 = 1/(y - q)\}.$$

We show that $\mathscr{F}' = \{L'\} \cup \{L'_q\}_{q\in\mathbf{R}}$ is a foliation of M' with singularities (see Fig. 3). It is easily seen that L' and L'_q are leaves (L' is 1-leaf and L'_q are 2-leaves) and that $\mathscr{F}'$ is a partition of M'. It is clear that $\{L'_q\}_{q\in\mathbf{R}}$ is a nonsingular foliation of the manifold $M'\backslash L'$. Thus we have to prove that there exist adapted charts around points of L'. Let

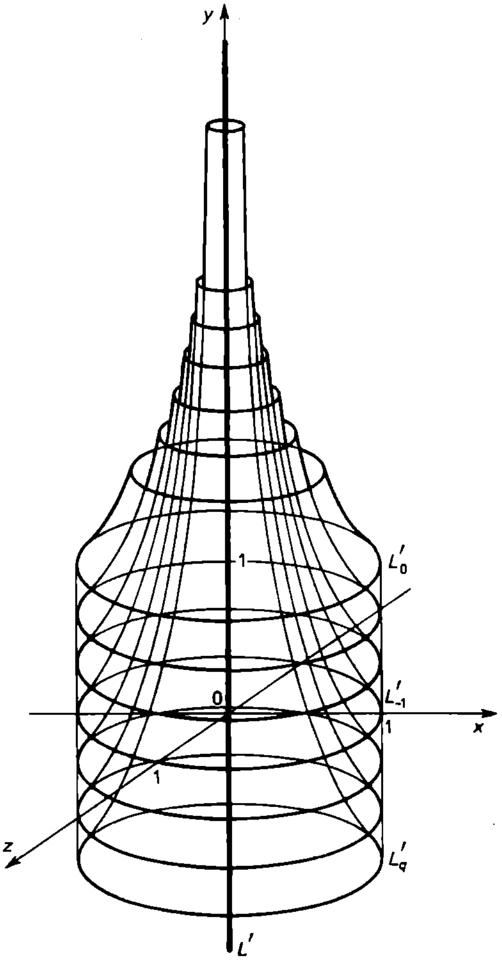

Fig. 3

$$D_{\varphi'_0} = \{(x, y, z);\ -1/2 < y < 1/2 \text{ and } x^2+z^2 < 2/(2y+3)\},$$

$$\varphi'_0(x, y, z) = \left(y,\ x/\sqrt{1-(x^2+z^2)\cdot y},\ z/\sqrt{1-(x^2+z^2)\cdot y}\right), \qquad (x, y, z) \in D_{\varphi'_0}.$$

Obviously, $\varphi'_0(0, 0, 0) = (0, 0, 0)$ and φ'_0 is a diffeomorphism with inverse given by

$$\varphi_0'^{-1}(\xi, \eta, \zeta) = \left(\eta/\sqrt{1+(\eta^2+\zeta^2)\xi},\ \xi,\ \zeta/\sqrt{1+(\eta^2+\zeta^2)\xi}\right).$$

A simple calculation shows that

$$\varphi_0'(D_{\varphi_0'}) = (-1/2, 1/2) \times \{(\eta, \zeta);\ \eta^2+\zeta^2 < 2/3\},$$

$$\varphi_0'(L') = (-1/2, 1/2) \times \{(0, 0)\},$$

$$\varphi_0'(L_q' \cap D_{\varphi_0'}) = (-1/2, 1/2) \times \{(\eta, \zeta);\ \eta^2+\zeta^2 = -1/q\}$$

for $q < -3/2$ (for $q \geqslant -3/2$, $L_q' \cap D_{\varphi_0'} = \emptyset$). If $(0, y_0, 0)$ is an arbitrary point of L', then we put

$$D_{\varphi_{y_0}'} = \{(x, y, z);\ (x, y-y_0, z) \in D_{\varphi_0'}\},$$

$$\varphi_{y_0}'(x, y, z) = \varphi_0'(x, y-y_0, z), \quad (x, y, z) \in D_{\varphi_{y_0}'}.$$

The mapping φ_{y_0}' is a chart satisfying the conditions of Definition (1.3). Thus $\mathscr{F}'$ is a foliation of M'.

Let the group $\boldsymbol{Z}$ act on M' by

$$\theta\colon\ \boldsymbol{Z} \times M' \ni (n, (x, y, z)) \mapsto (x, y+n, z) \in M'.$$

It is easily seen that this action is properly discontinuous, hence $M := M'/\boldsymbol{Z}$ is a manifold (see [KN]). The foliation $\mathscr{F}'$ is invariant under this action, since

$$\theta_n(L') = L', \qquad \theta_n(L_q') = L_{q+n}',$$

and so θ gives a foliation $\mathscr{F}$ of M which consists of

$$L = \{[(x, y, z)];\ (x, y, z) \in L'\}$$

and

$$L_q = \{[(x, y, z)];\ (x, y, z) \in L_q'\}$$

for $q \in \langle 0, 1)$. An adapted chart around $[(0, y_0, 0)]$ is defined in the following way:

$$D_{\varphi_{[y_0]}} = \{[(x, y, z)];\ (x, y, z) \in \bigcup_{n \in \boldsymbol{Z}} D_{\varphi_{y_0+n}'}\},$$

$$\varphi_{[y_0]}[(x, y, z)] = \varphi_{y_0+n}'(x, y, z), \quad (x, y, z) \in D_{\varphi_{y_0+n}'}.$$

Obviously, L is compact since it is a manifold diffeomorphic to a circle.

We wish to compute the holonomy group of L at $[(0, 0, 0)]$. We know that $\pi_1(L) \cong \boldsymbol{Z}$. Consider the curve

$$\gamma\colon\ \langle 0, 1\rangle \ni t \mapsto [(0, t, 0)] \in L.$$

This is a loop at $[(0, 0, 0)]$ which represents a generator of $\pi_1(L)$. The sequence $(\varphi_{[0]}, 0; \varphi_{[1/2]}, 1/2; \varphi_{[0]}, 1)$ is a chain of adapted charts along γ. We now exhibit a $\varphi_{[0]}$-diffeomorphism f for this chain and the points $[(0, 1/4, 0)]$, $[(0, 3/4, 0)]$:

$$f(\eta, \zeta) = \left(\eta/\sqrt{1+\tfrac{1}{4}(\eta^2+\zeta^2)},\ \zeta/\sqrt{1+\tfrac{1}{4}(\eta^2+\zeta^2)}\right),$$

where the domain of f is the disk D of radius $\sqrt{2/3}$ centred at $(0, 0)$ and the range of f is the disk D' of radius $\sqrt{4/7}$ centred at $(0, 0)$. Hence $D' \subset D$. Note that if $(\eta, \zeta) \in D'$ and $(\eta, \zeta) \neq (0, 0)$, then

$$\eta^2+\zeta^2 \neq \frac{\eta^2}{1+\frac{1}{4}(\eta^2+\zeta^2)} + \frac{\zeta^2}{1+\frac{1}{4}(\eta^2+\zeta^2)}.$$

Hence the only point (η, ζ) of D' for which

$$(\eta, \zeta) \sim_{\varphi_{[0]_{D'}}} f(\eta, \zeta)$$

is $(0, 0)$ since the equivalence classes are circles centred at $(0, 0)$. Thus in view of Proposition (1.21)

$$f \not\equiv \mathrm{id}_{W_{\varphi_{[0]}}},$$

and so the holonomy homomorphism is an isomorphism and

$$\mathscr{H}(L) \cong \boldsymbol{Z}.$$

5. A stability theorem

In view of the paracompactness of M, we can choose a Riemannian metric g on M. This metric induces a Riemannian metric g_L on L. Now, we can give the following:

(5.1) DEFINITION. Let L be a k-leaf of a foliation $\mathscr{F}$. A family $\mathscr{M}$ of adapted charts around points of L is called a *family of coherent charts for* L if for every $x \in L$ there exists an $(m-k)$-dimensional submanifold T_x of M containing x such that:

(i) $L \subset \bigcup_{\varphi \in \mathscr{M}} D_\varphi$,

(ii) for every $\varphi \in \mathscr{M}$ the set $D_\varphi \cap L$ is relatively compact and geodesically convex with respect to g_L,

(iii) for every $\varphi \in \mathscr{M}$ there exists an adapted chart $\tilde{\varphi}$ around the same point such that $D_\varphi \subset D_{\tilde{\varphi}}$, $\overline{D_\varphi \cap L} \subset D_{\tilde{\varphi}} \cap L$, $\tilde{\varphi} \mid D_\varphi = \varphi$, and

$$\tilde{\varphi}^{-1}(\{\mathrm{pr}_1\, \tilde{\varphi}(x)\} \times W_{\tilde{\varphi}}) \subset T_x$$

for every $x \in D_{\tilde{\varphi}} \cap L$ (see Fig. 4).

(5.2) **Remark.** *Note that by* (i) *and* (iii)

$$x\in D_\varphi \Rightarrow \varphi^{-1}(\{\mathrm{pr}_1\,\varphi(x)\}\times W_\varphi)\subset T_x$$

for every $x\in L$ *and every* $\varphi\in\mathcal{M}$.

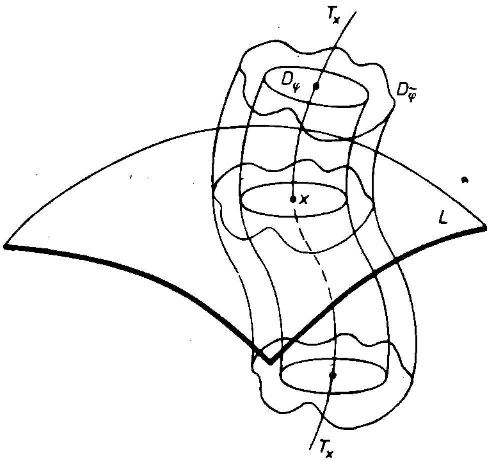

Fig. 4

(5.3) LEMMA. *If* L *is a compact* k*-leaf of* $\mathcal{F}$, *then there exists a family of coherent charts for* L.

Proof. Let (ξ, J) be a tubular neighbourhood of L in M, i.e. $\xi = (E, p, L)$ is a vector bundle over L, $J\colon E\to M$ is an imbedding such that $J\,|\,L=\mathrm{id}_L$ (L on the left side is understood as the zero section of ξ), and $J(E)$ is an open neighbourhood of L in M (see [Hi]). Note that $J(E_x)$ is a submanifold of M transverse to L for every $x\in L$. In view of the compactness of L, we can assume that $J(E_x)$ is transverse to all the leaves of $\mathcal{F}$ for every $x\in L$. We set $T_x := J(E_x)$.

Take an arbitrary point $x_0\in L$ and an arbitrary adapted chart ψ around x_0. Define

$$W := \{x\in J(E);\ JpJ^{-1}(x)\in D_\psi\}\cap D_\psi .$$

Obviously, W is an open subset of M and $x_0\in W$ since $JpJ^{-1}(x)=x$ for all $x\in L$. Consider the mapping

$$F\colon\ W\ni x\longmapsto(\mathrm{pr}_1\,\psi JpJ^{-1}(x),\ \mathrm{pr}_2\,\psi(x))\in \mathbf{R}^k\times\mathbf{R}^{m-k}.$$

We show that F_{*x_0} is an isomorphism. In fact, it suffices to prove that F_{*x_0}

is a monomorphism. Let $X \in T_{x_0} W$ and $F_{*x_0}(X) = 0$. Thus

$$(\mathrm{pr}_1 \psi J p J^{-1})_{*x_0}(X) = 0, \tag{5.4}$$

$$(\mathrm{pr}_2 \psi)_{*x_0}(X) = 0. \tag{5.5}$$

Since $JpJ^{-1} \mid L = \mathrm{id}_L$, by (5.4) we have

$$(\mathrm{pr}_1 \psi)_{*x_0}(X) = 0. \tag{5.6}$$

Owing to (5.5) and (5.6) we get $\psi_{*x_0}(X) = 0$, and consequently $X = 0$.

By the Inverse Function Theorem there exists a neighbourhood $\tilde{W}$ of x_0 such that $F \mid \tilde{W}$ is a chart of M. We take a relatively compact neighbourhood W_0 of x_0 such that $\bar{W}_0 \subset \tilde{W}$ and $W_0 \cap L$ is connected. By the compactness of $\bar{W}_0 \cap L$ we can find an open neighbourhood G of 0 in W_ψ such that for every $x \in \bar{W}_0 \cap L$ we have

$$\psi^{-1}(\{\mathrm{pr}_1 \psi(x)\} \times G) \subset \tilde{W}, \tag{5.7}$$

$$F^{-1}(\{\mathrm{pr}_1 \psi(x)\} \times G) \subset \tilde{W}. \tag{5.8}$$

Set

$$D_{\tilde{\varphi}} := F^{-1}(\mathrm{pr}_1 \psi(L \cap W_0) \times G) \subset \tilde{W}, \qquad \tilde{\varphi} := F \mid D_{\tilde{\varphi}}.$$

The mapping $\tilde{\varphi}$ is a chart distinguished by $\mathscr{F}$ around x_0. Indeed, note that

$$\tilde{\varphi}(D_{\tilde{\varphi}}) = \mathrm{pr}_1 \psi(L \cap W_0) \times G,$$

where $\mathrm{pr}_1 \psi(L \cap W_0)$ is an open neighbourhood of 0 in $\mathbb{R}^k$ and G is an open neighbourhood of 0 in $\mathbb{R}^{m-k}$. It is clear that

$$\tilde{\varphi}(x_0) = F(x_0) = \psi(x_0) = (0, 0).$$

Let L' be an arbitrary leaf of $\mathscr{F}$. We have to show that

$$\tilde{\varphi}(L' \cap D_{\tilde{\varphi}}) = U_{\tilde{\varphi}} \times l' = \mathrm{pr}_1 \psi(L \cap W_0) \times l',$$

where $l' = \{w \in G;\ \tilde{\varphi}^{-1}(0, w) \in L'\}$.

Let $(u, w) \in \tilde{\varphi}(L' \cap D_{\tilde{\varphi}})$. Then $(u, w) = \tilde{\varphi}(y)$ for some $y \in L' \cap D_{\tilde{\varphi}}$. Consequently,

$$(u, w) = \tilde{\varphi}(y) = F(y) \in \mathrm{pr}_1 \psi(L \cap W_0) \times G.$$

On the other hand, $w = \mathrm{pr}_2 \psi(y) \in \{w' \in W_\psi;\ \psi^{-1}(0, w') \in L'\}$ since $y \in L'$ and ψ is an adapted chart. Hence $w \in \{w' \in G;\ \psi^{-1}(0, w') \in L'\}$. Let $z := \tilde{\varphi}^{-1}(0, w)$. Then

$$\mathrm{pr}_2 \psi(z) = w \in \{w' \in G;\ \psi^{-1}(0, w') \in L'\}.$$

Thus $z \in L'$. In other words, $w \in l'$. Therefore $(u, w) \in U_{\tilde{\varphi}} \times l'$.

Conversely, let $(u, w) \in U_{\tilde{\varphi}} \times I'$. Then $u \in \mathrm{pr}_1 \psi(L \cap W_0)$ and $w \in \{w' \in G;\ \tilde{\varphi}^{-1}(0, w') \in L'\}$. Set $z = \tilde{\varphi}^{-1}(u, w)$. Then $z \in L' \cap D_{\tilde{\varphi}}$, hence

$$(u, w) \in \tilde{\varphi}(L' \cap D_{\tilde{\varphi}}).$$

Therefore $\tilde{\varphi}$ is an adapted chart around x_0.

Now, let U be an open neighbourhood of x_0 in L and suppose U is geodesically convex with respect to g_L (see [KN]). We assume that

$$\bar{U} \subset L \cap W_0. \tag{5.9}$$

Set

$$\varphi := \tilde{\varphi} \,|\, \tilde{\varphi}^{-1}(\mathrm{pr}_1 \tilde{\varphi}(U) \times G).$$

We now define $\mathscr{M}$ to be the collection of all charts constructed as above for all points $x_0 \in L$, for all charts ψ distinguished by $\mathscr{F}$ around x_0 and all open geodesically convex neighbourhoods of x_0 in L satisfying the condition (5.9).

The only condition of Definition (5.1) which is nontrivial and should be checked is the last part of (iii). Let $\tilde{\varphi}$ be a chart obtained as above from the adapted chart ψ around x_0 and let $x \in L \cap D_{\tilde{\varphi}}$. Let $y \in \tilde{\varphi}^{-1}(\{\mathrm{pr}_1 \tilde{\varphi}(x)\} \times W_{\tilde{\varphi}})$. Then

$$\mathrm{pr}_1 \tilde{\varphi}(y) = \mathrm{pr}_1 \psi J p J^{-1}(y) = \mathrm{pr}_1 \psi(x)$$

and obviously

$$\mathrm{pr}_2 \psi J p J^{-1}(y) = 0 = \mathrm{pr}_2 \psi(x).$$

Therefore $x = JpJ^{-1}(y)$. In other words,

$$y \in J(E_x) = T_x.$$

Consequently,

$$\tilde{\varphi}^{-1}(\{\mathrm{pr}_1 \tilde{\varphi}(x)\} \times W_{\tilde{\varphi}}) \subset T_x. \quad ■$$

Let L be a compact leaf and $\mathscr{M}$ be a family of coherent charts for L obtained with the help of a tubular neighbourhood (ξ, J) of L.

(5.10) LEMMA. *Let* $\psi, \chi \in \mathscr{M}$, $y, z \in D_\psi \cap D_\chi$, *and* $\mathrm{pr}_1 \psi(y) = \mathrm{pr}_1 \psi(z)$. *Then*

$$\mathrm{pr}_1 \chi(y) = \mathrm{pr}_1 \chi(z).$$

Proof. It suffices to observe that for an arbitrary chart $\varphi \in \mathscr{M}$ and for every $x \in D_\varphi \cap L$

$$\varphi^{-1}(\{\mathrm{pr}_1 \varphi(x)\} \times W_\varphi) = T_x \cap D_\varphi.$$

Therefore both equalities in the lemma are equivalent to

$$JpJ^{-1}(y) = JpJ^{-1}(z). \quad ■$$

(5.11) LEMMA. *Let $\mathcal{M}$ be a family of coherent charts for L, let $\varphi, \psi \in \mathcal{M}$ and $D_\varphi \cap D_\psi \cap L \neq \emptyset$. Then we can find a neighbourhood G of 0 in W_φ such that for every neighbourhood G' of 0 in G and for every plaque P of $\varphi_{G'}$ there exists exactly one plaque Q of ψ such that $P \cap Q \neq \emptyset$.*

Proof. In view of condition (iii) of (5.1) for every $x \in \overline{D_\varphi \cap D_\psi \cap L}$ there exists a neighbourhood G_x of 0 in W_φ such that

$$\tilde{\varphi}^{-1}(\{\mathrm{pr}_1\,\tilde{\varphi}(x)\} \times G_x) \subset \tilde{\psi}^{-1}(\{\mathrm{pr}_1\,\tilde{\psi}(x)\} \times W_\psi).$$

Hence there exists a neighbourhood G of 0 in W_φ such that for every $x \in \overline{D_\varphi \cap D_\psi \cap L}$

$$\tilde{\varphi}^{-1}(\{\mathrm{pr}_1\,\tilde{\varphi}(x)\} \times G) \subset \tilde{\psi}^{-1}(\{\mathrm{pr}_1\,\tilde{\psi}(x)\} \times W_\psi) \tag{5.12}$$

since $\overline{D_\varphi \cap D_\psi \cap L}$ is compact. In particular, for every $x \in D_\varphi \cap D_\psi \cap L$, (5.12) holds. Let $G' \subset G$ be a neighbourhood of 0 and P an arbitrary plaque of the chart $\varphi_{G'}$. By the definition of G it is obvious that $P \cap D_\psi \neq \emptyset$. Hence there exists at least one plaque Q of ψ such that $P \cap Q \neq \emptyset$. We show that

$$P \cap D_\psi \text{ is connected.} \tag{5.13}$$

To this end, we prove that

$$\Xi\colon \varphi^{-1}(U_\varphi \times G') \cap D_\psi \ni y \mapsto (\varphi^{-1}(\mathrm{pr}_1\,\varphi(y), 0), \mathrm{pr}_2\,\varphi(y)) \in (D_\varphi \cap D_\psi \cap L) \times G'$$

is a homeomorphism. Indeed, it is clear that

$$(D_\varphi \cap D_\psi \cap L) \times G' \ni (z, w) \mapsto \varphi^{-1}(\mathrm{pr}_1\,\varphi(z), w) \in \varphi^{-1}(U_\varphi \times G') \cap D_\psi$$

is the inverse and both mappings are continuous. If $P = \varphi^{-1}(U_\varphi \times l)$, where $l = \{w \in G';\ \varphi^{-1}(0, w) \in P\}$, then

$$\Xi(P \cap D_\psi) = (D_\varphi \cap D_\psi \cap L) \times l. \tag{5.14}$$

Indeed, if $(z, w) \in \Xi(P \cap D_\psi)$, then for some $y \in P \cap D_\psi$,

$$z = \varphi^{-1}(\mathrm{pr}_1\,\varphi(y), 0) = \psi^{-1}(\mathrm{pr}_1\,\psi(y), 0) \in D_\varphi \cap D_\psi \cap L$$

since in view of Lemma (5.10) we have

$$\mathrm{pr}_1\,\varphi(\psi^{-1}(\mathrm{pr}_1\,\psi(y), 0)) = \mathrm{pr}_1\,\varphi(y) = \mathrm{pr}_1\,\varphi(z),$$

which implies that $z = \psi^{-1}(\mathrm{pr}_1\,\psi(y), 0)$. On the other hand, $w = \mathrm{pr}_2\,\varphi(y) \in l$, hence

$$(z, w) \in (D_\varphi \cap D_\psi \cap L) \times l.$$

Conversely, if $(z, w) \in (D_\varphi \cap D_\psi \cap L) \times l$, then

$$\Xi^{-1}(z, w) = \varphi^{-1}(\mathrm{pr}_1\,\varphi(z), w) \in \varphi^{-1}(U_\varphi \times l) = P$$

and

$$\varphi^{-1}(\mathrm{pr}_1\,\varphi(z), w)\in D_\psi$$

since $w\in G'\subset G$. This completes the proof of (5.14).

By condition (ii) of (5.1) the set $D_\varphi\cap D_\psi\cap L$ is connected and – by the definition of a plaque – so is l. Hence $P\cap D_\psi$ is connected since Ξ is a homeomorphism. Therefore there exists exactly one plaque Q of ψ such that $P\cap D_\psi\subset Q$. Consequently, $P\cap Q\neq\emptyset$. ■

Now, we can prove the main result of this paper – a stability theorem.

(5.15) THEOREM. *Let L be a compact leaf of a foliation $\mathcal{F}$ such that $\mathcal{H}(L)=0$. Then for every $x\in L$ there exist a neighbourhood V of L in M and a chart φ distinguished by $\mathcal{F}$ around x such that $D_\varphi\subset V$ and the inclusion mapping $\mathcal{I}\colon D_\varphi\hookrightarrow V$ induces a homeomorphism of the quotient spaces*

$$D_\varphi/(\mathcal{F}\,|\,D_\varphi)\cong V/(\mathcal{F}\,|\,V).$$

Proof. First, we construct a neighbourhood V and a chart φ. Let $x\in L$ and let $\mathcal{M}$ be a family of coherent charts for L. Choose a finite subfamily $\mathcal{M}'$ of $\mathcal{M}$ such that

$$L\subset\bigcup_{\varphi\in\mathcal{M}'}D_\varphi.$$

Assume that one of the charts in $\mathcal{M}'$ (say φ_0) is an adapted chart around x. Let

$$((\varphi_0, 0), (\varphi_i, t_i)_{i=1}^{n}, (\varphi_0, 1)) \tag{5.16}$$

be a chain of adapted charts along some loop with origin at x constructed from all charts of $\mathcal{M}'$. For the chain (5.16), choose points $y_{ij}\in D_{\varphi_i}\cap D_{\varphi_j}\cap L$, where i and j are integers such that $0\leqslant i<j\leqslant n$ and $D_{\varphi_i}\cap D_{\varphi_j}\cap L\neq\emptyset$. Let $\varphi_{i_1},\dots,\varphi_{i_{s(0)}}$ be those φ_i for which

$$D_{\varphi_0}\cap D_{\varphi_{i_\alpha}}\cap L\neq\emptyset.$$

For every $\alpha\in\{1,\dots,s(0)\}$, consider the φ_0-diffeomorphism

$$f_\alpha^{(0)}\colon G_\alpha^{(0)}\to H_\alpha^{(0)}$$

defined by the sequence $(\varphi_0,\varphi_1,\dots,\varphi_{i_\alpha},\varphi_0)$ of adapted charts and by the system $(y_{01}, y_{12},\dots,y_{i_\alpha-1,i_\alpha}, y_{0i_\alpha})$. Since $\mathcal{H}(L)=0$, there exist homotopies $\hat{G}_\alpha^{(0)}\times\langle 0, 1\rangle\to H_\alpha^{(0)}$ realizing the equivalences $f_\alpha^{(0)}\equiv\mathrm{id}_{W_{\varphi_0}}$. We set

$$G_0:=\bigcap_{\alpha=1}^{s(0)}\hat{G}_\alpha^{(0)}$$

and

$$\varphi:=\varphi_0\,|\,\varphi_0^{-1}\Big(U_{\varphi_0}\times\bigcup_{\alpha=1}^{s(0)}H_\alpha^{(0)}\Big). \tag{5.17}$$

Assume that we have already constructed neighbourhoods G_i of 0 in W_{φ_i} for $i = 0, 1, \ldots, h-1$. Consider the chart φ_h. Let $\varphi'_{j_1}, \ldots, \varphi'_{j_{r(h)}}$ be those of the charts $\varphi'_j = \varphi_{j_{G_j}}$ $(j = 0, \ldots, h-1)$ for which $D_{\varphi_h} \cap D_{\varphi'_{j_\beta}} \cap L \neq \emptyset$. In view of Lemma (5.11), for every $\beta \in \{1, \ldots, r(h)\}$ there exists a neighbourhood $\tilde{G}^{(h)}_\beta$ of 0 in W_{φ_h} such that every plaque P of $\varphi_{h_{\tilde{G}}}$, with $\tilde{G} - \tilde{G}^{(h)}_\beta$, intersects exactly one plaque of the chart φ'_{j_β}. Set

$$\tilde{G}_h = \bigcap_{\beta=1}^{r(h)} \tilde{G}^{(h)}_\beta.$$

Now take all charts $\varphi_{i_1}, \ldots, \varphi_{i_{s(h)}}$ from $\{\varphi_{h+1}, \ldots, \varphi_n\}$ satisfying $D_{\varphi_h} \cap D_{\varphi_{i_\alpha}} \cap L \neq \emptyset$. For every $\alpha \in \{1, \ldots, s(h)\}$ there exists a φ_h-diffeomorphism $f^{(h)}_\alpha \colon G^{(h)}_\alpha \to H^{(h)}_\alpha$ defined by the sequence $(\varphi_h, \varphi_{h+1}, \ldots, \varphi_{i_\alpha}, \varphi_h)$ of adapted charts and by the system of points $(y_{h,h+1}, \ldots, y_{hi_\alpha})$. The domains of these diffeomorphisms can be chosen in such a way that

$$\bigcup_{\alpha=1}^{s(h)} H^{(h)}_\alpha \subset \tilde{G}_h.$$

Since $\mathscr{H}(L) = 0$, there exist homotopies $\hat{G}^{(h)}_\alpha \times \langle 0, 1\rangle \to H^{(h)}_\alpha$ realizing the equivalences $f^{(h)}_\alpha \equiv \mathrm{id}_{W_{\varphi_h}}$. We put

$$G_h := \bigcap_{\alpha=1}^{s(h)} \hat{G}^{(h)}_\alpha.$$

Thus we have constructed neighbourhoods G_i of 0 in W_{φ_i} for $i = 0, 1, \ldots, n$.

For $j = 0, \ldots, n$ denote by φ'_j the chart $\varphi_{j_{G_j}}$.

(5.18) LEMMA. *If $h < h'$ and $D_{\varphi_h} \cap D_{\varphi_{h'}} \cap L \neq \emptyset$, then every plaque P of the chart $\varphi'_{h'}$ intersects exactly one plaque of the chart φ'_h.*

Proof. Observe that

$$G_{h'} = \bigcap_{\alpha=1}^{s(h')} \hat{G}^{(h')}_\alpha \subset \bigcup_{\alpha=1}^{s(h')} H^{(h')}_\alpha \subset \tilde{G}_{h'} \subset \tilde{G}^{(h')}_{\beta'}, \tag{5.19}$$

where $\beta' \in \{1, \ldots, r(h')\}$ is such that $j_{\beta'} = h$. Since every plaque of the chart $\varphi_{h'_{\tilde{G}}}$, with $\tilde{G} = \tilde{G}^{(h')}_{\beta'}$, intersects exactly one plaque of φ'_h, it follows, in view of (5.19), that every plaque of $\varphi'_{h'}$ intersects exactly one plaque of φ'_h. ■

(5.20) Remark. From (5.19) it follows that every plaque of $\varphi_{h'} | \varphi_{h'}^{-1}(U_{\varphi_{h'}} \times \bigcup_{\alpha=1}^{s(h')} H^{(h')}_\alpha)$ intersects exactly one plaque of φ'_h.

Let $h < h'$, $D_{\varphi_h} \cap D_{\varphi_{h'}} \cap L \neq \emptyset$, and let P be an arbitrary plaque of $\varphi'_{h'}$. Then, by Lemma (5.18), P determines uniquely a plaque $\tilde{Q}_h$ of φ'_h. On the other hand, P determines uniquely a sequence $(Q_{h'-1}, \ldots, Q_h)$, where Q_i is a plaque of φ_i' for $i = h'-1, \ldots, h$.

(5.21) LEMMA. *There exists a plaque Q of the chart $\varphi_h | \varphi_h^{-1}(U_{\varphi_h} \times \bigcup_{\alpha=1}^{s(h)} H_\alpha^{(h)})$ such that $Q_h \cup \tilde{Q}_h \subset Q$.*

Proof. Let $w \in G_{h'}$ be an arbitrary point determining P, i.e. $w \in \mathrm{pr}_2\, \varphi_{h'}(P)$. Then we have

$$\varphi_{h'}^{-1}(\mathrm{pr}_1\, \varphi_{h'}(y_{hh'}), w) \in \tilde{Q}_h.$$

In other words, the point

$$\tilde{w}_h := \mathrm{pr}_2\, \varphi_h \varphi_{h'}^{-1}(\mathrm{pr}_1\, \varphi_{h'}(y_{hh'}), w)$$

determines the plaque $\tilde{Q}_h$, i.e.

$$[\tilde{w}_h]_{\varphi_h'} = \mathrm{pr}_2\, \varphi_h(\tilde{Q}_h).$$

On the other hand, we can define inductively the following sequence of points:

$$w_{h'} := w,$$

$$w_{j-1} := \mathrm{pr}_2\, \varphi_{j-1} \varphi_j^{-1}(\mathrm{pr}_1\, \varphi_j(y_{j-1,j}), w_j), \quad j = h', h'-1, \ldots, h+1.$$

Observe that w_j determines the plaque Q_j. In particular,

$$[w_h]_{\varphi_h'} = \mathrm{pr}_2\, \varphi_h(Q_h).$$

We show that

$$f_{\alpha'}^{(h)}(w_h) = \tilde{w}_h, \tag{5.22}$$

where $\alpha' \in \{1, \ldots, s(h)\}$ is such that $i_{\alpha'} = h'$.

To this end – in view of the definition of $f_{\alpha'}^{(h)}$ – it suffices to prove that

$$w_j = \mathrm{pr}_2\, \varphi_j \varphi_{j-1}^{-1}(\mathrm{pr}_1\, \varphi_{j-1}(y_{j-1,j}), w_{j-1})$$

for every $j \in \{h, h+1, \ldots, h'\}$. We have

$$\begin{aligned} \mathrm{pr}_2\, \varphi_j \varphi_{j-1}^{-1}(\mathrm{pr}_1\, \varphi_{j-1}(y_{j-1,j}), w_{j-1}) &= \mathrm{pr}_2\, \varphi_j \varphi_{j-1}^{-1}\big(\mathrm{pr}_1\, \varphi_{j-1}(y_{j-1,j}), \\ &\qquad \mathrm{pr}_2\, \varphi_{j-1} \varphi_j^{-1}(\mathrm{pr}_1\, \varphi_j(y_{j-1,j}), w_j)\big) \\ &= \mathrm{pr}_2\, \varphi_j \varphi_{j-1}^{-1}\big(\mathrm{pr}_1\, \varphi_{j-1} \varphi_j^{-1}(\mathrm{pr}_1\, \varphi_j(y_{j-1,j}), w_j), \\ &\qquad \mathrm{pr}_2\, \varphi_{j-1} \varphi_j^{-1}(\mathrm{pr}_1\, \varphi_j(y_{j-1,j}), w_j)\big) \\ &= w_j \end{aligned}$$

since by Lemma (5.10)

$$\mathrm{pr}_1\, \varphi_{j-1}(y_{j-1,j}) = \mathrm{pr}_1\, \varphi_{j-1} \varphi_j^{-1}(\mathrm{pr}_1\, \varphi_j(y_{j-1,j}), w_j).$$

In view of (5.22), of the relation $w_h \in G_h \subset \hat{G}^{(h)}_{\alpha'}$, and of Proposition (1.21) we get

$$w_h \sim_{\varphi_{h_{\bar{H}}}} \tilde{w}_h, \quad \text{with } \bar{H} = H^{(h)}_{\alpha'},$$

hence

$$w_h \sim_{\varphi_{h_{\tilde{H}}}} \tilde{w}_h, \quad \text{with } \tilde{H} = \bigcup_{\alpha=1}^{s(h)} H^{(h)}_{\alpha}.$$

Therefore the plaque

$$Q := \Big(\varphi_h | \varphi_h^{-1} \big(U_{\varphi_h} \times \bigcup_{\alpha=1}^{s(h)} H^{(h)}_{\alpha}\big)\Big)^{-1} \big(U_{\varphi_h} \times [w_h]_{\varphi_{h_{\tilde{H}}}}\big)$$

contains both Q_h and $\tilde{Q}_h$. ∎

Now, we define

$$V := D_\varphi \cup \bigcup_{h=1}^{n} \varphi_h^{-1}(U_{\varphi_h} \times G_h).$$

Consider the mapping $\mathscr{I}\colon D_\varphi \hookrightarrow V$. It is clear that $\mathscr{I}$ induces a continuous mapping $\hat{\mathscr{I}}$ on the quotient spaces. Indeed, if $P \in D_\varphi/(\mathscr{F} | D_\varphi)$, then $\mathscr{I}(P)$ is a connected set contained in V and in a leaf L' of $\mathscr{F}$; hence it is also contained in a connected component of $L' \cap V$. Thus $\hat{\mathscr{I}}(P)$ is contained in some leaf of $\mathscr{F} | V$.

Now, we show that $\hat{\mathscr{I}}$ is a surjection. Let $(L' \cap V)_y \in V/(\mathscr{F} | V)$. It suffices to prove that $(L' \cap V)_y \cap D_\varphi \neq \emptyset$. By the definition of V, either $(L' \cap V)_y \cap D_\varphi \neq \emptyset$ (and then the assertion holds) or there exists h such that $(L' \cap V)_y \cap \varphi_h^{-1}(U_{\varphi_h} \times G_h) \neq \emptyset$. Let Q_h be an arbitrary plaque of φ'_h contained in this intersection. Then, by Lemma (5.18), there exist plaques Q_i of φ'_i ($i = h-1, h-2, \ldots, 0$) such that $Q_i \cap Q_{i-1} \neq \emptyset$ for $i = 1, \ldots, h$. Consequently, $Q_0 \subset (L' \cap V)_y$. In other words, $(L' \cap V)_y \cap D_\varphi \neq \emptyset$.

Now, we show that $\hat{\mathscr{I}}$ is one-to-one. To this end, we prove that for every $(L' \cap V)_y \in V/(\mathscr{F} | V)$ the intersection $(L' \cap V)_y \cap D_\varphi$ consists of one plaque of φ. Assume that P and P' are plaques of φ contained in $(L' \cap V)_y \cap D_\varphi$. Owing to the connectedness of $(L' \cap V)_y$ there exist plaques Q_i of $\varphi'_{\lambda(i)}$ ($i = 1, \ldots, r$) such that

$$P \cap Q_1 \neq \emptyset,$$

$$Q_i \cap Q_{i+1} \neq \emptyset \quad \text{for } i = 1, \ldots, r-1,$$

$$Q_r \cap P' \neq \emptyset.$$

Obviously, we may assume that $\lambda(i) \neq 0$ for $i = 1, \ldots, r$.

(5.23) LEMMA. *For every* $i \in \{1, \ldots, r\}$ *there exist plaques*

(5.24) P_j *of* φ'_j, $j = 1, \ldots, \lambda(i)$, *such that*

$$P \cap P_1 \neq \emptyset,$$
$$P_j \cap P_{j+1} \neq \emptyset \quad \textit{for } j = 1, \ldots, \lambda(i)-1,$$
$$P_{\lambda(i)} = Q_i.$$

Proof. We use induction on i. Set $P_{\lambda(1)} = Q_1$. Let $P_j, j = \lambda(i) - 1, \ldots, 1, 0$, be the uniquely determined plaque of φ'_j such that $P_j \cap P_{j-1} \neq \emptyset$ for $j = 1, \ldots, \lambda(i)$ (Lemma (5.18)). On the other hand, $P_{\lambda(i)} \cap P = Q_1 \cap P \neq \emptyset$, hence, in view of Lemma (5.21), $P_0 \subset P$. Thus $P \cap P_1 \neq \emptyset$.

Assume that we have constructed a sequence $(P'_1, \ldots, P'_{\lambda(i-1)})$ $(i \leqslant r-1)$ which fulfils the assertion of the lemma. Two cases can occur:

1° $\lambda(i) > \lambda(i-1)$. Set $P_{\lambda(i)} := Q_i$ and let $P_{\lambda(i)-1}, \ldots, P_{\lambda(i-1)}$ be the uniquely determined plaques of the respective charts (Lemma (5.18)). On the other hand, $P'_{\lambda(i-1)}$ is the unique plaque of $\varphi'_{\lambda(i-1)}$ such that $P_{\lambda(i)} \cap P'_{\lambda(i-1)} \neq \emptyset$ since $P'_{\lambda(i-1)} = Q_{i-1}$ and $P_{\lambda(i)} = Q_i$. In view of Lemma (5.21), there exists a plaque $\hat{P}_{\lambda(i-1)}$ of

$$\varphi_{\lambda(i-1)} \,|\, \varphi^{-1}_{\lambda(i-1)}\Big(U_{\varphi_{\lambda(i-1)}} \times \bigcup_{\alpha=1}^{s(\lambda(i-1))} H_\alpha^{(\lambda(i-1))}\Big)$$

such that $P_{\lambda(i-1)} \cup P'_{\lambda(i-1)} \subset \hat{P}_{\lambda(i-1)}$. From Remark (5.20) it follows that the plaque $\hat{P}_{\lambda(i-1)}$ intersects exactly one plaque of the chart $\varphi'_{\lambda(i-1)-1}$. Hence $P_{\lambda(i-1)} \cap P'_{\lambda(i-1)-1} \neq \emptyset$, and we may set

$$P_{\lambda(i-1)-1} = P'_{\lambda(i-1)-1}, \quad \ldots, \quad P_1 = P'_1.$$

2° $\lambda(i) < \lambda(i-1)$. Set $P_{\lambda(i)} := Q_i$. In view of Lemma (5.21), there exists a plaque $\hat{P}_{\lambda(i)}$ of the chart

$$\varphi_{\lambda(i)} \,|\, \varphi^{-1}_{\lambda(i)}\Big(U_{\varphi_{\lambda(i)}} \times \bigcup_{\alpha=1}^{s(\lambda(i))} H_\alpha^{(\lambda(i))}\Big)$$

such that $P_{\lambda(i)} \cup P'_{\lambda(i)} \subset \hat{P}_{\lambda(i)}$. By (5.20), $\hat{P}_{\lambda(i)}$ intersects exactly one plaque of $\varphi'_{\lambda(i)-1}$. Hence $P_{\lambda(i)} \cap P'_{\lambda(i)-1} \neq \emptyset$, and we set

$$P_{\lambda(i)-1} = P'_{\lambda(i)-1} \quad , \ldots, \quad P_1 = P'_1. \ \blacksquare$$

We use Lemma (5.23) for $i = r$ to replace the sequence $(Q_1, \ldots, Q_r)$ by the sequence $(P_1, \ldots, P_{\lambda(r)})$, where $P_{\lambda(r)} = Q_r$. On the one hand, $P_{\lambda(r)}$ determines uniquely a plaque $\tilde{P}' \subset P'$ of the chart φ'_0, and on the other hand – via the sequence (5.24) – a plaque $\tilde{P} \subset P$ of the same chart. In view of (5.21), there exists exactly one plaque of the chart φ which contains $\tilde{P} \cup \tilde{P}'$. Therefore $P = P'$, and consequently $\hat{\mathcal{P}}$ is one-to-one.

The mapping $\hat{\mathscr{I}}$ is open, which follows immediately from (3.2) and from the following obvious fact: if $\hat{U}$ is an arbitrary open set in $D_\varphi/(\mathscr{F} \mid D_\varphi)$, then

$$\hat{\mathscr{I}}(\hat{U}) = \pi_V\left(\pi_{D_\varphi}^{-1}(\hat{U})\right).$$

This completes the proof of Theorem (5.15). ∎

We have a simple

(5.25) COROLLARY. *If L is a compact simply connected leaf of the foliation $\mathscr{F}$, then the assertion of Theorem* (5.15) *holds.*

(5.26) Remark. Observe that if L is a single point x, then the assertion of Theorem (5.15) is trivial. It suffices to set $V = D_\varphi$ for an arbitrary chart φ distinguished by $\mathscr{F}$ around x.

(5.27) EXAMPLE. The assumption of triviality of $\mathscr{H}(L)$ of (5.15) cannot be omitted. Indeed, in the case of the foliation from Example (4.26d) $L \cap D_{\varphi_0}$ has a base of neighbourhoods in $D_{\varphi_0}/(\mathscr{F} \mid D_{\varphi_0})$ consisting of sets homeomorphic to the interval $\langle 0, 1)$, whereas L has a base of neighbourhoods in $V/(\mathscr{F} \mid V)$ consisting of a single set homeomorphic to the topological space $X = (S^1 \cup \{0\}, \tau)$, where S^1 is the circle and $\tau = \{S^1 \cup \{0\}\} \cup \{U \subset S^1;\ U$ is open in $S^1\}$.

6. The case of foliations without singularities

Let $\mathscr{F}$ be a k-dimensional nonsingular foliation. In this particular case, we adopt all the constructions and notation of Chapter 5.

We wish to deduce the classical Reeb Stability Theorem from Theorem (5.15). To this end, we prove the following

(6.1) LEMMA. *Let $L \in \mathscr{F}$, let $\mathscr{M}$ be a family of coherent charts for L and let $\varphi, \psi \in \mathscr{M}$ be such that every plaque of φ intersects exactly one plaque of ψ. Let P and P' be plaques of φ, and let Q be a plaque of ψ. If*

$$P \cap Q \neq \emptyset \quad \text{and} \quad P' \cap Q \neq \emptyset,$$

then $P = P'$.

Proof. Let $P = \varphi^{-1}(U_\varphi \times \{w\})$, $P' = \varphi^{-1}(U_\varphi \times \{w'\})$, and $Q = \psi^{-1}(U_\psi \times \{\tilde{w}\})$. If $x \in P \cap Q$, then

$$x = \varphi^{-1}(\mathrm{pr}_1\, \varphi(x), w) \in P \cap Q$$

and we set

$$\begin{aligned} y &:= \varphi^{-1}\big(\mathrm{pr}_1\, \varphi J p J^{-1}(x),\, w'\big) \\ &\in \varphi^{-1}\big(\mathrm{pr}_1\, \varphi(D_\varphi \cap D_\psi \cap L) \times \{w'\}\big) \\ &= \Xi^{-1}\big((D_\varphi \cap D_\psi \cap L) \times \{w'\}\big) = P' \cap D_\psi \\ &= P' \cap Q. \end{aligned}$$

Observe that

$$\mathrm{pr}_1\, \varphi(y) = \mathrm{pr}_1\, \varphi J p J^{-1}(x) = \mathrm{pr}_1\, \varphi(x),$$

hence, by Lemma (5.10), $\mathrm{pr}_1\, \psi(y) = \mathrm{pr}_1\, \psi(x)$ and, of course, $\mathrm{pr}_2\, \psi(y) = \tilde{w} = \mathrm{pr}_2\, \psi(x)$. Therefore $x = y$ and $P \cap P' \neq \emptyset$, hence $P = P'$. ∎

Obviously, Lemma (5.21) can be reformulated as follows:

(6.2) LEMMA. *There exists a plaque Q of the chart $\varphi_h \,|\, \varphi_h^{-1}\big(U_{\varphi_h} \times \bigcup_{\alpha=1}^{s(h)} H_\alpha^{(h)}\big)$ such that $Q_h = Q = \tilde{Q}_h$.*

Now, we denote by $\psi_0, \psi_1, \ldots, \psi_n$ the charts $\varphi, \varphi_1', \ldots, \varphi_n'$ constructed in the proof of (5.15). We have

(6.3) LEMMA. *If $L' \cap D_{\psi_n} \neq \emptyset$, then $L' \subset V$.*

Proof. Let $y \in L' \cap D_{\psi_n}$ and let Q_n be the plaque of ψ_n containing y. In view of Lemma (5.18), there exist uniquely determined plaques $Q_{n-1}, Q_{n-2}, \ldots, Q_0$ of the charts $\psi_{n-1}, \psi_{n-2}, \ldots, \psi_0$, respectively, such that $Q_i \cap Q_{i+1} \neq \emptyset$ for $i = 0, \ldots, n-1$.

Assume that $L' \not\subset V$; take $z \in L' \setminus V$. Let $c\colon \langle 0, 1\rangle \to L'$ be a curve joining y to z and

$$t_0 := \sup\{t \in \langle 0, 1\rangle;\ c(\langle 0, t\rangle) \subset V\}. \tag{6.4}$$

Observe that

$$c(t_0) \notin V. \tag{6.5}$$

Obviously, $t_0 > 0$, and there exist $\delta > 0$ and $h \in \{0, \ldots, n\}$ such that $c\big((t_0 - \delta, t_0)\big) \subset D_{\psi_h}$. Let $Q_h' = \psi_h^{-1}(U_{\psi_h} \times \{w_h'\})$ be the plaque of ψ_h such that $c\big((t_0 - \delta, t_0)\big) \subset Q_h'$. It is clear that $c(t_0) \in J(E)$. Consider the point $x = JpJ^{-1} c(t_0)$ of L. There exists $j \neq h$ such that

$$x \in D_{\psi_j} \cap L. \tag{6.6}$$

Observe that there exists δ' such that $\delta > \delta' > 0$ and $JpJ^{-1} c(\langle t_0 - \delta', t_0\rangle) \subset D_{\psi_j}$. Two cases can occur:

1° $j < h$. Then Q'_h intersects exactly one plaque Q'_j of ψ_j. Therefore

$$\psi_h^{-1}(\mathrm{pr}_1 \psi_h J p J^{-1} c(t_0-\delta', t_0) \times \{w'_h\}) \subset Q'_h \cap D_{\psi_j} = Q'_h \cap Q'_j,$$

hence by (6.6), $c(t_0) \in Q'_j \subset D_{\psi_j} \subset V$. This contradicts (6.5).

2° $j > h$. Take the plaque $Q_j = \psi_j^{-1}(U_{\psi_j} \times \{w_j\})$. By (6.2), $Q_j \cap Q_h \neq \emptyset$. Therefore $Q'_h \neq Q_h$ since otherwise (see above) $c(t_0) \in Q_j \subset D_{\psi_j} \subset V$. Thus we obtain a contradiction. Observe that Q_h and Q'_h are contained in the same leaf of $\mathscr{F} | V$. Indeed, Q_h is joined to Q_n by the sequence $(Q_{h+1}, \ldots, Q_{n-1})$ of plaques contained in V, and Q'_h is joined to Q_n by the curve $c | \langle 0, t_0-\delta' \rangle$ in V. Therefore, if $Q'_{h-1}, \ldots, Q'_1, Q'_0$ are plaques of the charts $\psi_{h-1}, \ldots, \psi_1, \psi_0$, respectively, and are uniquely determined by Q'_h, then by (5.15), $Q'_0 = Q_0$. Thus there exists $i \in \{0, \ldots, h-1\}$ such that

$$Q_i = Q'_i, \quad Q_{i+1} \neq Q'_{i+1}, \quad Q_i \cap Q_{i+1} \neq \emptyset, \quad Q_i \cap Q'_{i+1} \neq \emptyset.$$

This contradicts Lemma (6.1). ∎

Directly from Lemma (6.3) we get:

(6.7) COROLLARY. $\pi_M^{-1} \pi_M (D_{\psi_n}) \subset V$.

(6.8) COROLLARY. *Every leaf L' for which $L' \cap D_{\psi_n} \neq \emptyset$ has the following property: for every $h \in \{0, \ldots, n\}$ $L' \cap D_{\psi_h}$ is a single plaque of ψ_h.*

Proof. Since $L' \cap D_{\psi_n} \neq \emptyset$, we have $L' \subset V$ in view of Lemma (6.3). Consequently, by Theorem (5.15), $L' \cap D_{\psi_0}$ contains a single plaque P.

Assume that there exists $h \in \{0, \ldots, n\}$ such that $L' \cap D_{\psi_h}$ contains two plaques Q_h and Q'_h. Then Q_h and Q'_h determine uniquely sequences $(Q_{h-1}, \ldots, Q_0)$ and $(Q'_{h-1}, \ldots, Q'_0)$ of plaques of the respective charts. We have $Q_0 = P = Q'_0$, so there exists $j \in \{1, \ldots, h\}$ such that $Q_j \neq Q'_j$ and $Q_{j-1} = Q'_{j-1}$. This contradicts Lemma (6.1). ∎

(6.9) Remark. A simple calculation shows that the space $D_\varphi/(\mathscr{F} | D_\varphi)$ is homeomorphic to W_φ for every adapted chart φ. The homeomorphism is given by

(6.10) $\Phi\colon W_\varphi \ni w \mapsto (L_y \cap D_\varphi)_y \in D_\varphi/(\mathscr{F} | D_\varphi)$, where $y = \varphi^{-1}(0, w)$.

Now, we can prove the Reeb Stability Theorem ([L2]).

(6.11) THEOREM. *If L is a compact leaf of a nonsingular foliation $\mathscr{F}$ and the holonomy group of L is trivial, then there exist a saturated neighbourhood U of L in M, a neighbourhood D of 0 in $\mathbf{R}^{m-k}$, and a diffeomorphism $q\colon U \to L \times D$ such that for every leaf $L' \subset U$*

$$q(L') = L \times \{w\} \tag{6.12}$$

for any $w \in D$.

Proof. Set $U = \pi_M^{-1}\pi_M(D_{\psi_n})$. By Corollary (6.7), $U \subset V \subset J(E)$. Let $D := \{w \in W_{\psi_0};\ \psi_0^{-1}(0, w) \in U\}$. Observe that $\Phi^{-1}\hat{\mathscr{J}}^{-1}\pi_M(z) \in D$ for $z \in U$. Indeed,

$$\hat{\mathscr{J}}^{-1}\pi_M(z) = \hat{\mathscr{J}}^{-1}(L_z) = L_z \cap D_{\psi_0}$$

since $\hat{\mathscr{J}}$ is one-to-one. By (6.10), $\Phi^{-1}(L_z \cap D_{\psi_0}) = w$, where $\psi_0(L_z \cap D_{\psi_0}) = U_{\psi_0} \times \{w\}$. Therefore

$$\psi_0^{-1}(0, w) \in \psi_0^{-1}(U_{\psi_0} \times \{w\}) = L_z \cap D_{\psi_0} \subset U.$$

Now, we define

$$q\colon U \ni z \mapsto (JpJ^{-1}(z), \Phi^{-1}\hat{\mathscr{J}}^{-1}\pi_M(z)) \in L \times D.$$

Observe that q can be expressed locally in the following form: if $z_0 \in U$ and h is an integer such that $z_0 \in D_{\psi_h}$, then for $z \in D_{\psi_h} \cap U$

$$q(z) = \big(\psi_h^{-1}(\mathrm{pr}_1\,\psi_h(z), 0), f_h(\mathrm{pr}_2\,\psi_h(z))\big), \tag{6.13}$$

where $f_h \in \mathscr{A}_{\psi_h, \psi_0}$ is a diffeomorphism determined by the chain consisting of the charts $\psi_h, \psi_{h-1}, \ldots, \psi_0$ and by arbitrary intersection points of central plaques of these charts. Hence q is smooth.

The mapping q is a bijection. Indeed, the inverse mapping is defined in the following way: if $(x, w) \in L \times D$, then, by (6.8), $T_x \cap \hat{\mathscr{J}}\Phi(w)$ contains a single point, say y. Then $q^{-1}(x, w) = y$.

Locally, if $x_0 \in D_{\psi_h}$, then

$$q^{-1}(x, w) = \psi_h^{-1}(\mathrm{pr}_1\,\psi_h(x), f_h^{-1}(w)) \quad \text{for } x \in D_{\psi_h} \cap L,$$

and so q^{-1} is smooth.

Since $\hat{\mathscr{J}}$ is bijective, it follows – in view of the definition of q – that the equality (6.12) holds. More precisely, $q^{-1}(L \times \{w\}) = L_y$, where $y = \psi_0^{-1}(0, w)$. ■

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