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The inverse Laplace transform of the product of two modified Bessel functions $K_{n\nu}[a^{1/2n}p^{1/2n}]\,K_{n\mu}[a^{1/2n}p^{1/2n}]$ where $n=1,2,3,\ldots$

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§ 1. Introductory. In this paper we shall determine the original function whose Laplace transform is

$$p^{k/2-1/2n-1}K_{n\nu}[a^{1/2n}p^{1/2n}]K_{n\mu}[a^{1/2n}p^{1/2n}]\,, \tag{1}$$

where p, as usual, is complex, $R(p)>0$, $R(a)>0$, $K_\mu(x)$ is the modified Bessel function of the second kind and n is any positive integer. The original function whose Laplace transform is the function (1) is not known in the main literature [1]. All that is known is the special case when $n=1$ (see [1], p. 285). When $k=4$, $n=1$, $\mu=\nu-1$, function (1) is

$$p^{1/2}K_\nu[a^{1/2}p^{1/2}]K_{\nu-1}[a^{1/2}p^{1/2}]\,, \tag{2}$$

and the original function is known (see [1], p. 285, equation 61). It is in fact the function

$$2^{-1}a^{-1/2}\pi^{1/2}t^{-1}e^{-a/2t}W_{\frac{1}{2},\nu-\frac{1}{2}}(a/t)\ (^1)\,, \tag{3}$$

where $W_{\frac{1}{2},\nu-\frac{1}{2}}(a/t)$ is the known Whittaker function (see [2], p. 351) defined by

$$W_{\frac{1}{2},\nu-\frac{1}{2}}\left(\frac{a}{t}\right)=\frac{\Gamma(1-2\nu)}{\Gamma(\frac{1}{2}-\nu)}\left(\frac{a}{t}\right)^{\nu}e^{-a/2t}F\left(\begin{matrix}\nu-\frac{1}{2};\ a/t\\ 2\nu\end{matrix}\right)+ \\ +\frac{\Gamma(2\nu-1)}{\Gamma(\nu-\frac{1}{2})}\left(\frac{a}{t}\right)^{1-\nu}e^{-a/2t}F\left(\begin{matrix}\frac{1}{2}-\nu;\ a/t\\ 2-2\nu\end{matrix}\right), \tag{4}$$

where

$${}_pF_q(p;\ a_r;\ q;\ \varrho_s;\ z)=\sum_{r=0}^{\infty}\frac{(a_1;\ r)\ldots(a_p;\ r)}{r!(\varrho_1;\ r)\ldots(\varrho_q;\ r)}z^r\,,$$

$$(a;\ n)=a(a+1)\ldots(a+n-1)\,,\quad n=1,2,3,\ldots,\ (a;\ 0)=1\,.$$

(1) There is an error in $2^{-1/2}$ in equation 61, p. 285 in [1].

In § 2, it will be shown that the original function of function (1) is the function

$$(5)\quad 4^{-1}\pi^{-1/2}(2\pi)^{1-n}n^{nk-3/2}a^{1/2n-k/2}\times$$

$$\times\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}\Delta\left(n;\frac{nk+n\nu+n\mu-1}{2}\right),\ \Delta\left(n;\frac{nk-n\nu+n\mu-1}{2}\right),\\ \Delta\left(n,\frac{nk-n\mu+n\nu-1}{2}\right),\Delta\left(n;\frac{nk-n\mu-n\nu-1}{2}\right):\frac{e^{i\pi}a}{n^{2n}t}\\ \Delta\left(n;\frac{nk-1}{2}\right),\ \Delta\left(n;\frac{nk}{2}\right)\end{matrix}\right]$$

where $\sum\limits_{i,-i}$ means that in the expression following it i is to be replaced by $-i$ and the two expressions are to be added. Also the symbol $\Delta(n;\,\alpha)$ represents the set of parameters

$$\frac{\alpha}{n},\quad\frac{\alpha+1}{n},\quad\frac{\alpha+2}{n},\quad\ldots,\quad\frac{\alpha+n-1}{n}.$$

The function appearing in (5) is MacRobert's E-function, whose definitions and properties are to be found in [2], pp. 348-358. Thus the major result of this paper is a generalization of the Laplace transform relationship between the functions (3) and (2) to expressions in which the parameters of the Bessel functions are different and, moreover, in which the argument of the Bessel function in (2) is $a^{1/2n}p^{1/2n}$ where $n=1,2,3,\ldots$

In § 3, it will be shown how (3) can be derived from (5) and also many other particular cases are deduced. The following formulae are required in the proofs:

([2], p. 352): If $p\leqslant q$

$$(6)\qquad E(p;\,\alpha_r\colon q;\,\varrho_s\colon z)=\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}{\Gamma(\varphi_1)\ldots\Gamma(\varphi_q)}F(p;\,\alpha_r\colon q;\,\varrho_s;\,-1/z)$$

([2], p. 353): If $p\geqslant q+1$

$$(7)\quad E(p;\,\alpha_r\colon q;\,\varrho_s\colon z)=\sum_{r=1}^{p}\prod_{s=1}^{p}{}'\,\Gamma(\alpha_s-\alpha_r)\left\{\prod_{t=1}^{q}\Gamma(\varrho_t-\alpha_r)\right\}^{-1}\Gamma(\alpha_r)z^{\alpha_r}\times$$
$$\times F\left\{\begin{matrix}\alpha_r,\ \alpha_r-\varrho_1+1,\ \ldots,\ \alpha_r-\varrho_q+1;\ (-1)^{p-q}z\\ \alpha_r-\alpha_1+1,\ \ldots *\ldots,\ \alpha_r-\alpha_p+1\end{matrix}\right\},$$

([3], p. 749, equation (2)):

$$(8)\quad E(p;\,m\alpha_r\colon q;\,m\varrho_s\colon ze^{\pm i\pi})$$
$$=(2\pi)^{-(m-1)(p-q-1)/2}m^{m(\Sigma\alpha_r-\Sigma\varphi_s)-\frac{1}{2}(p-q+1)}\sum_{\lambda=0}^{m-1}\left(\frac{m^{p-q-1}}{z}\right)^{\lambda}\times$$

$$\times E\left\{\begin{matrix} a_1+\frac{\lambda}{m}, \ldots, a_1+\frac{\lambda+m-1}{m}; \ldots; a_p+\frac{\lambda}{m}, \ldots, a_p+ \\ \qquad +\frac{\lambda+m-1}{m}: \left(\frac{m^{p-q-1}}{z}\right)^m e^{\pm i\pi} \\ \frac{\lambda+1}{m}, \ldots * \ldots, \frac{\lambda+m}{m}, \varrho_1+\frac{\lambda}{m}, \ldots, \varrho_1+\frac{\lambda+m-1}{m}; \ldots; \\ \qquad \varrho_q+\frac{\lambda}{m}, \ldots, \varrho_q+\frac{\lambda+m-1}{m} \end{matrix}\right\}$$

where $R(a_r) > 0$ and the asterisk denotes that the parameter m/m is omitted. Now if the last notation $\Delta(n, a)$ is introduced and we take $p = 2$ and $q = 3$, write n for m and use the relation between the E and F functions, then (8) in combination with (6) gives

$$\text{(9)} \quad \sum_{\lambda=0}^{n-1} \left(\frac{a^{1/n}}{n^2}\right)^{\lambda} E\left[\begin{matrix} \Delta\left(n; \frac{n\mu+n\nu+1}{2}+\lambda\right), \Delta\left(n; \frac{n\mu+n\nu}{2}+\lambda+1\right): e^{\pm i\pi}\frac{n^{2n}}{ap} \\ \frac{\lambda+1}{n}, \ldots * \ldots, \frac{\lambda+n}{n}, \Delta(n; n\nu+\lambda+1), \Delta(n; n\mu+\lambda+1), \\ \Delta(n; n\mu+n\nu+\lambda+1) \end{matrix}\right]$$

$$= (2\pi)^{1-n}\pi^{1/2}n^{3/2}\frac{n^{n\nu+n\mu}}{\Gamma(1+n\nu)\Gamma(1+n\mu)}\,{}_2F_3\left\{\begin{matrix} \frac{1}{2}(n\nu+n\mu)+\frac{1}{2}, \frac{1}{2}(n\nu+n\mu)+ \\ \qquad +1: a^{1/n}p^{1/n} \\ 1+n\nu, 1+n\mu, 1+n\nu+n\mu \end{matrix}\right\},$$

where the symbol $\Delta(n; a)$ and the asterisk have the same meaning as before. Also the two trignometric identities are required:

$$\text{(10)} \qquad \sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}\ldots\sin\frac{(k+n-1)\pi}{n} = 2^{1-n}\sin k\pi\,,$$

$$\text{(11)} \qquad \sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} = 2^{1-n}n\,.$$

Other equations which we shall use are

$$\text{(12)} \qquad {}_1F_1\left(\begin{matrix} a; z \\ \varrho \end{matrix}\right) = e^z\,{}_1F_1\left(\begin{matrix} \varrho-a; -z \\ \varrho \end{matrix}\right),$$

$$\text{(13)} \quad K_\mu(z) = \frac{\pi}{2\sin\mu\pi}\left[\frac{(z/2)^{-\mu}}{\Gamma(1-\mu)}\,{}_0F_1\left(;1-\mu; \frac{z^2}{4}\right) - \frac{(z/2)^{\mu}}{\Gamma(1+\mu)}\,{}_0F_1\left(;1+\mu; \frac{z^2}{4}\right)\right],$$

$$\text{(14)} \quad {}_0F_1(\varrho; z)\,{}_0F_1(\sigma; z) = {}_2F_3[\tfrac{1}{2}\varrho+\tfrac{1}{2}\sigma, \tfrac{1}{2}\varrho+\tfrac{1}{2}\sigma-\tfrac{1}{2}; \varrho, \sigma, \varrho+\sigma-1; 4z]\,,$$

and ([2], p. 394, ex. 106): If $R(a_{p+1}) > 0$

$$\text{(15)} \qquad \int_0^\infty e^{-\lambda}\lambda^{a_{p+1}-1}E\left(p; a_r: q; \varrho_s: \frac{z}{\lambda}\right)d\lambda = E(p+1; a_r: q; \varrho_s: z)\,.$$

§ 2. The Laplace transform theorem. We shall now establish the following Laplace transform relationship in which the original function is a function of t, p is the Laplace transform parameter whose real part is greater than 0.

$$(16)\quad \int_0^\infty e^{-pt} f(t)\, dt = 4\pi^{1/2}(2\pi)^{n-1} n^{3/2-nk} a^{k/2-1/2n} p^{k/2-1/2n-1} K_{n\nu}[a^{1/2n}p^{1/2n}] K_{n\mu}[a^{1/2n}p^{1/2n}],$$

where

$$(17)\quad f(t) = \sum_{i,-i} \frac{1}{i} E\left[\begin{matrix} \Delta\left(n; \frac{nk+n\nu+n\mu-1}{2}\right), \Delta\left(n; \frac{nk+n\nu-n\mu-1}{2}\right), \\ \Delta\left(n; \frac{nk+m\mu-n\nu-1}{2}\right), \Delta\left(n; \frac{nk-n\nu-n\mu-1}{2}\right) : \frac{e^{\pm i\pi}a}{n^{2n}t} \\ \Delta\left(n; \frac{nk-1}{2}\right), \Delta\left(n; \frac{nk}{2}\right) \end{matrix}\right]$$

and the symbol Δ and $\sum_{i,-i}$ have the same meanings as before.

To prove (16) put $t = \lambda/p$ and apply (15), in which the value of a_{p+1} is set equal to one. This yields

$$(18)\quad \int_0^\infty e^{-pt} f(t)\, dt = \frac{1}{p} \sum_{i,-i} \frac{1}{i} E\left[\begin{matrix} 1, \Delta\left(n; \frac{nk+n\mu+n\nu-1}{2}\right), \Delta\left(n; \frac{nk+n\nu-n\mu-1}{2}\right), \\ \Delta\left(n; \frac{nk+n\mu-n\nu-1}{2}\right), \Delta\left(n; \frac{nk-n\nu-n\mu-1}{2}\right) : \frac{e^{\pm i\pi}ap}{n^{2n}} \\ \Delta\left(n; \frac{nk-1}{2}\right), \Delta\left(n; \frac{nk}{2}\right) \end{matrix}\right].$$

Now expand each E-function on the right side of (18) in terms of the value given by (7) and combine the two resulting expressions, obtained from $\sum_{i,-i}$, by factoring out common terms and simplyfing the remaining two terms which result from $[(e^{+i\pi}ap)/n^{2n}]^{\beta}$ and $[(e^{-i\pi}ap)/n^{2n}]^{\beta}$ by means of the simple relations

$$\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$$

and

$$e^{i\pi\beta} - e^{-i\pi\beta} = 2i \sin \pi\beta .$$

Here β stands for any one of the above parameters in the E-function appearing in (18). We also use the definitions of the generalized hypergeometric function ${}_pF_q(\ ;z)$ to cancel factors in the sets of the above and lower parameters. Again we use (6) to transform the resulting hypergeometric functions of the type ${}_{2n}F_{4n-1}\left(\ ;\frac{ap}{n^{2n}}\right)$ to E-functions. Lastly we utilize (10) and (11) to substitute for the products of sines, and the right-hand side of (18) becomes

$$2^{2n-2}\pi^{2n}(np)^{-1}\left(\frac{ap}{n^{2n}}\right)^{\frac{k}{2}-\frac{1}{2n}}\times$$

$$\times\sum_{\nu,-\nu}\left[\frac{n^{n\nu+n\mu}[ap/n^{2n}]^{(\mu+\nu)/2}}{\sin n\mu\pi\sin n\nu\pi}\cdot\sum_{\lambda=0}^{n-1}\left[\frac{a^{1/n}p^{1/n}}{n^2}\right]^{\lambda} E\left\{\begin{matrix}\Delta\left(n;\frac{n\nu+n\mu+1}{2}+\lambda\right),\Delta\left(n;\frac{n\nu+n\mu}{2}+\lambda+1\right):e^{\pm i\pi}\frac{n^{2n}}{ap}\\ \frac{\lambda+1}{n},\ldots*\ldots,\frac{\lambda+n}{n},\Delta(n;n\nu+\lambda+1),\Delta(n;n\mu+\lambda+1);\\ \Delta(n;n\nu+n\mu+\lambda+1)\end{matrix}\right\}\right]+$$

$$+2^{2n-2}\pi^{2n}(np)^{-1}\left[\frac{ap}{n^{2n}}\right]^{\frac{k}{2}-\frac{1}{2n}}\times$$

$$\times\sum_{\nu,-\nu}\left[\frac{n^{n\nu-n\mu}[ap/n^{2n}]^{(\nu-\mu)/2}}{\sin(-n\mu\pi)\sin n\pi\nu}\sum_{\lambda=0}^{n-1}\left[\frac{a^{1/n}p^{1/n}}{n^2}\right]^{\lambda} E\left\{\begin{matrix}\Delta\left(n;\frac{n\nu-n\mu+1}{2}+\lambda\right),\Delta\left(n;\frac{n\nu-n\mu}{2}+\lambda+1\right):e^{\pm i\pi}\frac{n^{2n}}{ap}\\ \frac{\lambda+1}{n},\ldots*\ldots,\frac{\lambda+n}{n},\Delta(n;n\nu+\lambda+1),\Delta(n;-n\mu+\lambda+1),\\ \Delta(n;n\nu+\lambda+1-n\mu)\end{matrix}\right\}\right]$$

$$=2^{n-1}\pi^{n+\frac{3}{2}}n^{\frac{1}{2}}p^{-1}\left[\frac{ap}{n^{2n}}\right]^{\frac{k}{2}-\frac{1}{2n}}\times$$

$$\times\sum_{\nu,-\nu}\left[\frac{n^{n\nu+n\mu}(ap/n^{2n})^{(\mu+\nu)/2}}{2^{n\nu+n\mu}\sin(n\mu\pi)\sin(n\nu\pi)\Gamma(1+n\nu)\Gamma(1+n\mu)}\,{}_2F_3\left\{\begin{matrix}\frac{n\nu+n\mu}{2}+\frac{1}{2},\frac{n\mu+n\nu+2}{2};a^{1/n}p^{1/n}\\ 1+n\nu,1+n\mu,1+n\nu+n\mu\end{matrix}\right\}\right]+$$

$$+2^{n-1}\pi^{n+\frac{3}{2}}n^{\frac{1}{2}}p^{-1}\left[\frac{ap}{n^{2n}}\right]^{\frac{k}{2}-\frac{1}{2n}}\times$$

$$\times\sum_{\nu,-\nu}\left[\frac{n^{n\nu-n\mu}(ap/n^{2n})^{(\nu-\mu)/2}}{2^{n\nu-n\mu}\sin(-n\mu\pi)\sin n\nu\pi\,\Gamma(1+n\nu)\Gamma(1-n\mu)}\,{}_2F_3\left\{\begin{matrix}\frac{n\nu-n\mu+1}{2},\frac{n\nu-n\mu+2}{2};a^{1/n}p^{1/n}\\ 1+n\nu,1-n\mu,1+n\nu-n\mu\end{matrix}\right\}\right]\quad\text{by (9).}$$

Now substitute for each ${}_2F_3$ in the last expression from (14). Thus the right side of (18) becomes

$$4\pi^{\frac{1}{2}}(2\pi)^{n-1}n^{\frac{3}{2}-nk}a^{\frac{1}{2}k-1/2n}p^{k/2-1/2n-1}\,\frac{\pi^2}{4\sin n\nu\pi\sin n\mu\pi}\times$$

$$\times\left\{\frac{(\frac{1}{2}a^{1/2n}p^{1/2n})^{-n\mu}}{\Gamma(1-n\mu)}\,{}_0F_1\left(\,;1-n\mu;\frac{a^{1/n}p^{1/n}}{4}\right)-\right.$$

$$\left.-\frac{(\frac{1}{2}a^{1/2n}p^{1/2n})^{n\mu}}{\Gamma(1+n\mu)}\,{}_0F_1\left(\,;1+n\mu;\frac{a^{1/n}p^{1/n}}{4}\right)\right\}\times$$

$$\times\left\{\frac{(\frac{1}{2}a^{1/2n}p^{1/2n})^{-n\nu}}{\Gamma(1-n\nu)}\,{}_0F_1\left(\,;1-n\nu;\frac{a^{1/n}p^{1/n}}{4}\right)-\right.$$

$$\left.-\frac{(\frac{1}{2}a^{1/2n}p^{1/2n})^{n\nu}}{\Gamma(1+n\nu)}\,{}_0F_1\left(\,;1+n\nu;\frac{a^{1/n}p^{1/n}}{4}\right)\right\}$$

$$=4\pi^{1/2}(2\pi)^{n-1}n^{3/2-nk}a^{k/2-1/2n}p^{k/2-1/2n-1}K_{n\nu}[a^{1/2n}p^{1/2n}]K_{n\mu}[a^{1/2n}p^{1/2n}] \text{ by (13).}$$

Thus (16) is proved.

§ 3. Particular cases. To obtain the original function of the function (2) we take in (5) $k=4$, $n=1$, $\mu=\nu-1$. Thus the original function is

$$=4^{-1}\pi^{-1/2}a^{-3/2}\sum_{i,-i}\frac{1}{i}\,E\left(1+\nu,2,1,2-\nu:2,\frac{3}{2}:e^{i\pi}\frac{a}{t}\right)$$

$$=4^{-1}\pi^{-1/2}a^{-1/2}t^{-1}\sum_{i,-i}\frac{1}{i}\,E\left(\nu,1,1-\nu:\frac{1}{2}:e^{i\pi}\frac{a}{t}\right)$$

$$=2^{-1}\pi^{1/2}a^{-1/2}t^{-1}\left\{\frac{\Gamma(1-2\nu)}{\Gamma(\frac{1}{2}-\nu)}\left(\frac{a}{t}\right)^{\nu}{}_1F_1\left(\begin{matrix}\frac{1}{2}+\nu;-\frac{a}{t}\\2\nu\end{matrix}\right)+\right.$$

$$\left.+\frac{\Gamma(2\nu-1)}{\Gamma(\nu-\frac{1}{2})}\left(\frac{a}{t}\right)^{1-\nu}{}_1F_1\left(\begin{matrix}\frac{3}{2}-\nu;-\frac{a}{t}\\2-2\nu\end{matrix}\right)\right\} \quad \text{by (7),}$$

$$=2^{-1}\pi^{1/2}a^{-1/2}t^{-1}e^{-a/2t}\left\{\frac{\Gamma(1-2\nu)}{\Gamma(\frac{1}{2}-\nu)}\left(\frac{a}{t}\right)^{\nu}e^{-a/2t}{}_1F_1\left(\begin{matrix}\nu-\frac{1}{2};\frac{a}{t}\\2\nu\end{matrix}\right)+\right.$$

$$\left.+\frac{\Gamma(2\nu-1)}{\Gamma(\nu-\frac{1}{2})}\left(\frac{a}{t}\right)^{1-\nu}e^{-a/2t}{}_1F_1\left(\begin{matrix}\frac{1}{2}-\nu;\frac{a}{t}\\2-2\nu\end{matrix}\right)\right\} \quad \text{by (12),}$$

$$=2^{-1}\pi^{1/2}a^{-1/2}t^{-1}e^{-a/2t}W_{1/2,\nu-1/2}\left(\frac{a}{t}\right) \quad \text{by (4);}$$

which is the desired result.

In (5) take $n = 1$, write 2ν for ν and take $\mu = 2\nu$ and proceed as before, so getting the original function of the function

$$p^{2\nu}\{K_{2\nu}[a^{1/2}p^{1/2}]\}^2 \tag{19}$$

in the form

$$2^{-1}\pi^{1/2}a^{\nu-1/2}t^{-3\nu-1/2}e^{-a/2t}W_{\nu,\nu}\left(\frac{a}{t}\right), \tag{20}$$

where $R(a) > 0$. This is also a known result (see [1], p. 285, equation 59). We obtain many other particular cases by giving different values to n and to the parameters in (5).

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