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# PERTURBATION PROPERTIES OF THE SPECTRUM OF PERIODIC SCHRÖDINGER OPERATORS

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We prove an estimate of the relative distance of spectra of periodic Schrödinger operators, with distinct periodicity lattices and unbounded potentials. The proof is based on pointwise estimates of the Bloch waves and uses considerations in the nonseparable Hilbert space $B^2(\mathbf{R}^\nu)$.

## 0. Introduction

The aim of this note is to obtain perturbation results for spectra of periodic Schrödinger operators $l = -\Delta + V$, where the periodic potential $V$ may have singularities. Perturbation theory has so far been developed only for the case where the perturbation has the same periodicity lattice as the perturbed potential [1] (or for localized perturbation, describing impurities in the crystal). Comparison of spectra of operators with distinct periodicity lattices requires a different approach. We use pointwise estimates of the Bloch functions, the almost periodic eigenfunctions of $l$, together with simple considerations in the nonseparable Hilbert space $B^2(\mathbf{R}^\nu)$.

The use of nonseparable Hilbert spaces in the spectral analysis of periodic and almost periodic operators was initiated by Burnat [3–6] (see also Herczyński [8] for an up-to-date exposition). The definition, notation and basic properties of $B^2(\mathbf{R}^\nu)$ can be found in [8].

The main technical tool used here is the estimate of $G(x, y, \lambda^2)$, the integral kernel of $(l-\lambda^2)^{-1}$, called the Green function. We establish the exponential decay of $G(x, y, \lambda^2)$ for $|x-y| \to \infty$, for $\lambda^2$ away from the spectrum. A variety of weaker analogous results was obtained previously. We stress that most complete results in this direction, obtained by Shubin [11], rely on the theory of pseudodifferential operators and thus are applicable only to smooth potentials. In order to deal with unbounded potentials we have to consider integral operators in an auxiliary Banach space. This approach is well known, see for instance papers on potential scattering [9, 12]. When the potential is almost periodic, however, the resulting integral operators are not compact, and the solutions can be obtained only for $\lambda^2$ away from the spectrum.

An important feature of our analysis is the use of various Hilbert spaces for the study of one differential operator. We therefore define the Green function through an integral equation ("second resolvent formula"), independent of any particular Hilbert space. We then show its solution exists and is unique, and prove that it is indeed the integral kernel of the resolvent of our operator in the Hilbert spaces considered.

## 1. The Green function

The second resolvent formula

(1) $$(l-\lambda^2)^{-1} = (l_0-\lambda^2)^{-1} - (l_0-\lambda^2)^{-1} V (l-\lambda^2)^{-1}$$

where $l_0 = -\Delta$, leads to the integral equation

(2) $$G(x, y, \lambda^2) = G_0(x, y, \lambda^2) - \int G_0(x, s, \lambda^2) V(s) G(s, y, \lambda^2) ds$$

where the free Green function $G_0(x, y, \lambda^2)$ is well known and can be expressed in terms of Bessel functions (see [2]). Assuming $\nu \geqslant 3$ we have the estimate

(3) $$|G_0(x, y, \lambda^2)| \leqslant C_\nu \frac{\exp(-\frac{1}{2}\operatorname{Im}\lambda |x-y|)}{|x-y|^{\nu-2}}$$

where $C_\nu$ depends only on the dimension $\nu$, for $\lambda^2$ in the domain $\mathcal{G} = \{\lambda^2 \in \boldsymbol{C}\colon \operatorname{Im}\lambda > 0, |\operatorname{Re}\lambda| < 2\operatorname{Im}\lambda\}$. We introduce

DEFINITION. For $\lambda^2 \in \mathcal{G}$, $G(x, y, \lambda^2)$ is the *Green function* of the operator $l = -\Delta + V$ if it is continuous in $x$ and $y$ for $x \neq y$, satisfies (2) and satisfies the estimate

(4) $$|G(x, y, \lambda^2) - G_0(x, y, \lambda^2)| \leqslant C \frac{\exp(-\omega |x-y|)}{|x-y|^{\beta}}$$

for some $\omega > 0$, $\beta < \nu - 2$.

We note that if $V$ is locally $L^2$ and if there exists a set $S \subset \boldsymbol{R}^\nu$, of Lebesgue measure 0, consisting of isolated manifolds of dimension less than $\nu$, such that for any open domain $\Omega$, $\bar{\Omega} \subset \boldsymbol{R}^\nu \backslash S$, $V$ is a Lipschitz function in $\Omega$, then for any bounded function $f$ with $f \in C(\boldsymbol{R}^\nu) \cap C^1(\boldsymbol{R}^\nu \backslash S)$ the function $u(x) = \int G(x, y, \lambda^2) f(y) dy$ is in $C^2(\boldsymbol{R}^\nu \backslash S)$ and satisfies $(-\Delta + V(x) - \lambda^2) u(x) = f(x)$. The above condition on $V$ will be called the *regularity condition*.

While it is necessary to impose this condition when one wants to use the Green function, we will not need it in proving that the Green function exists.

THEOREM 1. *Assume that* $V$ *is uniformly locally* $L^p$, $p > \nu/2$ ($p = 2$ *if* $\nu = 3$). *Then there exists a constant* $K(V)$ *such that for* $\lambda^2 \in \mathscr{D}$ *with* $\operatorname{Im}\lambda > K(V)$ *there exists a unique Green function* $G(x, y, \lambda^2)$ *satisfying* (2) *and* (4), *with* $0 < \omega < \frac{1}{2}(\operatorname{Im}\lambda - K(V))$, $\nu - 4 + \nu/p < \beta < \nu - 2$.

The proof will be performed in the auxiliary Banach space

$$B_{y,\omega,\beta} = \left\{ f \in C(\boldsymbol{R}^\nu \setminus \{y\}) \colon \ |f(x)| \leqslant c \frac{\exp(-\omega|x-y|)}{|x-y|^\beta} \right\}$$

with the norm

$$\|f\|_{y,\omega,\beta} = \sup_{x \neq y} |x-y|^\beta \exp(\omega|x-y|)\, |f(x)|.$$

Define

$$g(x, y, \lambda^2) = G(x, y, \lambda^2) - G_0(x, y, \lambda^2),$$

$$v(x, y, \lambda^2) = -\int G_0(x, s, \lambda^2)\, V(s)\, G_0(s, y, \lambda^2)\, ds,$$

and let $T_{\lambda^2}$ denote the integral operator

$$(T_{\lambda^2} f)(x) = \int G_0(x, s, \lambda^2)\, V(s) f(s)\, ds.$$

Then conditions (2), (4) are equivalent to the integral equation $g = v - T_{\lambda^2} g$ in $B_{y,\omega,\beta}$. We shall show that for appropriate $\lambda^2$, $\omega$ and $\beta$, $v \in B_{y,\omega,\beta}$ and $T_{\lambda^2}$ is a bounded operator in $B_{y,\omega,\beta}$ with norm less than 1. Let $Q_R = \{x \in \boldsymbol{R}^\nu \colon |x_i| < R/2,\ i = 1, \ldots, \nu\}$, and denote

$$\|f\|_p = \sup_{\tau \in \boldsymbol{R}^\nu} \Big( \int_{Q_1} |f(x+\tau)|^p dx \Big)^{1/p}.$$

LEMMA 1. *If* $\alpha > 0$, $\gamma < \nu - \nu/p$, *then*

$$I(x) = \int \frac{\exp(-\alpha|x-y|)}{|x-y|^\gamma} |V(y)|\, dy \leqslant C \|V\|_p (\alpha^{-\nu} + \alpha^{\gamma + \nu/p - \nu})$$

*where the constant* $C$ *depends only on* $\nu$ *and* $\gamma$.

*Proof.* We have

$$I(x) = \sum_{m \in \boldsymbol{Z}^\nu} \int_{m+Q_1} \frac{\exp(-\alpha|y|)}{|y|^\gamma} |V(x+y)|\, dy.$$

If $|m| > 2\sqrt{\nu}$, then for $y \in m + Q_1$ we have

$$\frac{\exp(-\alpha|y|)}{|y|^\gamma} \leqslant \nu^{-\gamma/2} \exp(-\alpha(|m| - \sqrt{\nu}/2)) \leqslant \nu^{-\gamma/2} \exp(-\alpha(|y| - \sqrt{\nu})),$$

which gives

$$\sum_{|m|\geqslant 2\sqrt{v}} \int_{m+Q_1} \frac{\exp(-\alpha|y|)}{|y|^{\gamma}} |V(x+y)|\,dy$$
$$\leqslant C\|V\|_p \sum_{|m|\geqslant 2\sqrt{v}} \exp(-\alpha(|m|-\sqrt{v}/2))$$
$$\leqslant C\|V\|_p \int_{|y|>3\sqrt{v}/2} \exp(-\alpha(|y|-\sqrt{v}))\,dy \leqslant C\|V\|_p \alpha^{-v}.$$

The remaining terms can be estimated analogously:

$$\sum_{|m|<2\sqrt{v}} \int_{m+Q} \frac{\exp(-\alpha|y|)}{|y|^{\gamma}} |V(x+y)|\,dy \leqslant \int_{|y|<3\sqrt{v}} \frac{\exp(-\alpha|y|)}{|y|^{\gamma}} |V(x+y)|\,dy$$
$$\leqslant \Big(\int_{|y|<3\sqrt{v}} |V(x+y)|^p\,dy\Big)^{1/p} \Big(\int_{|y|<3\sqrt{v}} \frac{\exp(-\alpha|y|\,p/(p-1))}{|y|^{\gamma p/(p-1)}}\,dy\Big)^{(p-1)/p}.$$

In the above, the first integral is bounded above by a constant times $\|V\|_p$, while the second is convergent as $p\gamma/(p-1) < v$ and can be bounded above by the integral over $\boldsymbol{R}^v$. This can be calculated to obtain the assertion of the lemma.

LEMMA 2. *If $\alpha > 0$ and $v-4+v/p < \beta \leqslant \gamma$, $\gamma' < v-2$, then*

$$J(x, y) = \int \frac{\exp(-\alpha|x-s|)}{|x-s|^{\gamma}|y-s|^{\gamma'}} |V(s)|\,ds \leqslant C\|V\|_p \frac{1}{|x-y|^{\beta}} (\alpha^{-v}+\alpha^{\gamma+\gamma'-\beta-v+v/p})$$

*where the constant $C$ depends only on $v$, $\gamma$, $\gamma'$ and $\beta$.*

*Proof.* Note first that under the assumptions of the lemma $\gamma+\gamma'-\beta < v-v/p$. Let $J_1$ denote the integral over $\{s\colon |y-s| < |x-y|/2\}$, and $J_2$ the integral over $\{s\colon |y-s| > |x-y|/2\}$, $J = J_1+J_2$. For $J_1$ we have $|x-s| > |x-y|/2$, $|x-s| > |s-y|$, hence the integrand can be estimated by

$$\frac{\exp(-\alpha|x-s|)}{|x-s|^{\gamma}|y-s|^{\gamma'}} \leqslant \frac{2^{\beta}}{|x-y|^{\beta}} \frac{\exp(-\alpha|y-s|)}{|y-s|^{\gamma+\gamma'-\beta}}$$

and we can apply Lemma 1. Analogously for $J_2$ we have $|y-s| > |x-y|/2$, $|y-s| > |x-s|/3$, hence

$$\frac{\exp(-\alpha|x-s|)}{|x-y|^{\gamma}|y-s|^{\gamma'}} \leqslant \frac{2^{\beta}3^{\gamma'-\beta}}{|x-y|^{\beta}} \frac{\exp(-\alpha|x-s|)}{|x-s|^{\gamma+\gamma'-\beta}}$$

and Lemma 1 is again applicable. This yields the assertion of the lemma.

LEMMA 3. *If $\beta$ satisfies condition* (4) *and $0 < \omega < \frac{1}{2}\operatorname{Im}\lambda$, then $v \in B_{\gamma,\omega,\beta}$, $T_{\lambda 2}$ is a bounded operator in $B_{\gamma,\omega,\beta}$ and the following estimates hold*:

(5) $$\|v\|_{\gamma,\omega,\beta} \leqslant C_1\|V\|_p[(\tfrac{1}{2}\operatorname{Im}\lambda-\omega)^{-v}+(\tfrac{1}{2}\operatorname{Im}\lambda-\omega)^{v+v/p-4-\beta}],$$

(6) $\|T_{\lambda^2}\|_{y,\omega,\beta} \leqslant C_2 \|V\|_p [(\frac{1}{2}\mathrm{Im}\,\lambda-\omega)^{-\nu}+(\frac{1}{2}\mathrm{Im}\,\lambda-\omega)^{\nu/p-2}]$,

*where the constants $C_1$ and $C_2$ depend only on $\nu$, $p$, $\beta$.*

*Proof.* Let $f \in B_{y,\omega,\beta}$, then $|f(x)| \leqslant \|f\|_{y,\omega,\beta} \exp(-\omega|x-y|)/|x-y|^\beta$ for $x \neq y$. Using (3) we obtain

$$|(T_{\lambda^2} f)(x)| \leqslant C_\nu \|f\|_{y,\omega,\beta} \int \frac{\exp(-\frac{1}{2}\mathrm{Im}\,\lambda|x-s|-\omega|s-y|)}{|x-s|^{\nu-2}|s-y|^{\beta-2}} |V(s)|\, ds.$$

Denote $\alpha = \frac{1}{2}\mathrm{Im}\,\lambda-\omega$; then the inequality $\frac{1}{2}\mathrm{Im}\,\lambda|x-s|+\omega|s-y| \geqslant \omega|x-y| +\alpha|x-s|$ gives

$$|(T_{\lambda^2} f)(x)| \leqslant C_\nu \|f\|_{y,\omega,\beta} \exp(-\omega|x-y|) \int \frac{\exp(-\alpha|x-s|)}{|x-s|^{\nu-2}|y-s|^\beta} |V(s)|\, ds$$

whence the estimate of the norm of $T_{\lambda^2}$ follows upon application of Lemma 2. Analogously, the inequality $\frac{1}{2}\mathrm{Im}\,\lambda(|x-s|+|s-y|) \geqslant \omega|x-y|+\alpha|x-s|$ gives

$$|v(x, y, \lambda^2)| \leqslant C_\nu \exp(-\omega|x-y|) \int \frac{\exp(-\alpha|x-s|)}{|x-s|^{\nu-2}|y-s|^{\nu-2}} |V(s)|\, ds$$

so applying Lemma 2 again we obtain the estimate of the norm of $v$. The continuity of the functions $T_{\lambda^2} f$ and $v$ follows from the uniform convergence of the integrals. Lemma 3 is proved.

The proof of Theorem 1 is now clear. For sufficiently large $\alpha$, $T_{\lambda^2}$ has norm less than 1, so the equation $g = v - T_{\lambda^2} g$ has a unique solution in $B_{y,\omega,\beta}$. This solution does not, however, depend on $\omega$ and $\beta$, since for $\beta \leqslant \beta'$, $\omega' \leqslant \omega$ we have $B_{y,\omega,\beta} \subset B_{y,\omega',\beta'}$.

Assume now that the potential $V$ is a many body potential, that is, $x = (x_1, \ldots, x_N) \in \boldsymbol{R}^{N\nu}$, $V(x) = \sum_{i<j} V_{ij}(x_i - x_j)$. Theorem 1 ensures the exponential decay of the Green function in this situation, with, however, increasingly restrictive conditions on $V_{ij}$ as the number $N$ of particles grows. It is therefore of interest to prove a variant of this theorem with conditions on $V_{ij}$ independent of $N$. We shall prove such a result under the condition that for all $i, j$, $V_{ij} \in L^q(\boldsymbol{R}^\nu) + L^\infty(\boldsymbol{R}^\nu)$, $q > \nu/2$. This means that $V_{ij}$ is a sum of an $L^q$ function and a bounded function.

THEOREM 2. *Under the above conditions on $V_{ij}$ there exists a constant $K(V)$ such that the conclusion of Theorem 1 holds.*

From the proof of Theorem 1 it is clear that it is enough to prove the following lemma and then repeat the arguments of Lemmas 2 and 3. The lemma takes care of the singular part of $V_{ij}$; the bounded part can be estimated with the use of Lemma 1.

LEMMA 4. *If $\alpha > 0$, $\gamma < N\nu - \nu/q$, and $V_{ij} \in L^q(\boldsymbol{R}^\nu)$, then*

$$I(x) = \int \frac{\exp(-\alpha|x-y|)}{|x-y|^\gamma} |V(y)|\, dy \leqslant C \sum_{i<j} \|V_{ij}\|_{L^q} \alpha^{\gamma - N\nu + \nu/q}.$$

*Proof.* We shall assume $\gamma > \nu - 1/q$ (for smaller $\gamma$ the lemma is easier to prove). Furthermore, as one can estimate the terms $V_{ij}$ separately, we shall deal with $V_{12}$, so let $x = (x_1, \bar{x})$, $y = (y_1, \bar{y})$, $\bar{x}, \bar{y} \in \boldsymbol{R}^{(N-1)\nu}$. We have

$$I \leqslant \int_{\boldsymbol{R}^{(N-1)\nu}} \exp(-\alpha|\bar{x}-\bar{y}|) \int_{\boldsymbol{R}^\nu} \frac{1}{(|\bar{x}-\bar{y}|^2 + |x_1 - y_1|^2)^{\gamma/2}} |V_{12}(y_1 - y_2)|\, dy_1\, d\bar{y}.$$

Using Hölder's inequality we estimate the inner integral and obtain

$$I \leqslant C \|V_{12}\|_{L^q} \int_{\boldsymbol{R}^{(N-1)\nu}} \frac{\exp(-\alpha|\bar{x}-\bar{y}|)}{|x-y|^{\gamma - \nu + \nu/q}} d\bar{y}.$$

This integral is clearly convergent, its calculation gives the result of the lemma.

We shall now consider perturbation properties of the Green function. Let $V$ and $V'$ be two uniformly locally $L^p(\boldsymbol{R}^\nu)$ potentials, $p > \nu/2$. We shall use subscripts $V$ and $V'$ to denote the Green function, the functions $v$ and $g$ and the operator $T_{\lambda^2}$ for the corresponding Schrödinger operators. The integral equation (2) gives

$$g_V - g_{V'} = v_{V-V'} + T_{\lambda^2, V'-V}\, g_V - T_{\lambda^2, V'}(g_V - g_{V'}).$$

Now estimates (5) and (6) of Lemma 3 give

THEOREM 3. *Let $V$ and $V'$ be as above. Then there exists a constant $K(V, V')$ such that for $\lambda^2 \in \mathcal{G}$ with $\operatorname{Im}\lambda > K(V, V')$ the following estimate holds:*

$$|G_V(x, y, \lambda^2) - G_{V'}(x, y, \lambda^2)| \leqslant C \frac{\exp(-\omega|x-y|)}{|x-y|^\beta} \|V - V'\|_p$$

*where $\omega$ and $\beta$ are as in Theorem 1 and the constant $C$ does not depend on $V$ and $V'$ (and can be shown to tend to 0 as $\operatorname{Im}\lambda \to \infty$).*

If $V$ is a periodic potential, then for fixed $x \neq y$ the function $t \mapsto G(x+t, y+t, \lambda^2)$ is clearly also periodic, with the same periods. To consider almost periodic potentials, let $V'(x) = V(x+t)$, $t \in \boldsymbol{R}^\nu$. Since then $G_{V'}(x, y, \lambda^2) = G_V(x+t, y+t, \lambda^2)$, Theorem 3 implies

$$(7) \qquad |G_V(x, y, \lambda^2) - G_V(x+t, y+t, \lambda^2)| \leqslant C \frac{\exp(-\omega|x-y|)}{|x-y|^\beta} \|V(\cdot) - V(\cdot + t)\|_p.$$

The estimate (7) shows, in particular, that if the potential $V$ is a Stepanov almost periodic function, then the Green function $G(x+t, y+t, \lambda^2)$ is uniformly almost periodic as a function of $t \in \mathbf{R}^\nu$. For smooth potentials this result was obtained by Shubin [11].

## 2. The main result

Assume now that $V$ is a periodic potential, locally in $L^p$, $p > \nu/2$ ($p = 2$ if $\nu = 3$). Suppose $A = -\Delta + V$ is a self-adjoint operator in $L^2(\mathbf{R}^\nu)$ and $\mathfrak{A} = -\Delta + V$ is self-adjoint in the Hilbert space of Besicovitch almost periodic functions $B^2(\mathbf{R}^\nu)$ (cf. [8]). Theorem 1 can be used (see [8]) to prove the following

THEOREM 4. *$\mathfrak{A}$ has pure point spectrum, i.e. the eigenelements of $\mathfrak{A}$ span the whole space $B^2(\mathbf{R}^\nu)$. Moreover, if $F$ is an eigenelement of $\mathfrak{A}$ associated with an eigenvalue $E$, $\mathfrak{A}F = EF$, then $F$ can be represented as an orthogonal sum $F = \sum_n F_n$ of elements of $B^2(\mathbf{R}^\nu)$ such that $\mathfrak{A}F_n = EF_n$ and each $F_n$ is represented by a classical almost periodic eigenfunction, called* Bloch wave, *of the form* $\exp(i\langle\gamma_n, x\rangle)v(x)$, *where $v$ is periodic with the same periodicity lattice as $V$ and $\gamma_n$ are in the Brillouen zone.*

For the reader's convenience we recall that the *periodicity lattice* of $V$ is defined as $\Lambda = \{\tau \in \mathbf{R}^\nu\colon V(x+\tau) = V(x)$ for a.e. $x \in \mathbf{R}^\nu\}$. The dual lattice $\Lambda'$ is $\Lambda' = \{y \in \mathbf{R}^\nu\colon \langle y, \tau\rangle \in 2\pi \mathbf{Z}, \tau \in \Lambda\}$. The *basic cell* of the lattice $\Lambda$, uniquely defined, is $Q_{SW} = \{x\colon x$ is closer to 0 than to any other point $\tau$ of the lattice $\Lambda\}$. The basic cell of the dual lattice is called the *Brillouen zone*; we shall denote it by $\Gamma$.

The perturbation result that we will prove is based on the pointwise estimate of the Bloch waves.

LEMMA 5. *Let $R > 0$ be such that $Q_{SW} \subset Q_R$. For $E \in \sigma(\mathfrak{A})$ let $u_E$ be the Bloch wave corresponding to $E$, normalized in $B^2(\mathbf{R}^\nu)$. Then*

$$\|u_E\|_\infty \leqslant K(|E|, \|V\|_p, R)$$
$$= K_1 + (K_2 + K_3\|V\|_p^{\nu p/(2p-\nu)} + K_4|E|^{\nu/4})R^{\nu/2}$$

*where the constants $K_1, \ldots, K_4$ depend only on $\nu$ and can be computed from the proof below.*

*Proof.* All the constants below depend only on $\nu$. Applying the Green function $G(x, y, \lambda^2)$ to the equation $-\Delta u_E + Vu_E = Eu_E$ we obtain

$$u_E(x) = (E-\lambda^2)\int G(x, y, \lambda^2)\, u_E(y)\, dy.$$

For $\lambda^2 \in \mathcal{S}$ we can use the estimates (3) and (4). It is useful to assign to $\beta$ and $\omega$ appearing in Theorem 1 specific values, so let $\beta = \nu - 2 - 1 + \nu/(2p)$, $\omega = \frac{1}{4}\mathrm{Im}\,\lambda \geqslant 1$. By Lemma 3, for $\omega > c_1\|V\|_p^{2p/(2p-\nu)}$ we have $\|v\|_{y,\omega,\beta} \leqslant 1$, $\|T_{\lambda^2}\|_{y,\omega,\beta} < 1/2$, hence $\|g\|_{y,\omega,\beta} \leqslant 2$. If, furthermore, $\mathrm{Im}\,\lambda > (\frac{1}{3}|E|)^{1/2}$ then we can choose $\lambda^2$ so that $E = (\mathrm{Re}\,\lambda)^2 - (\mathrm{Im}\,\lambda^2)$ (we use $\lambda^2 \in \mathscr{S}$ here). Hence

$$|u_E(x)| \leqslant c_2\,\omega^2 \int \frac{\exp(-2\omega|y|)}{|y|^{\nu-2}}\,|u_E(x+y)|\,dy$$
$$+\,c_3\,\omega^2 \int \frac{\exp(-\omega|y|)}{|y|^{\beta}}\,|u_E(x+y)|\,dy.$$

Put $\varepsilon_0 = c_4\,\omega^{-1}$; then for sufficiently small $c_4$ we have

$$c_2\,\omega^2 \int_{|y|<\varepsilon_0} \frac{\exp(-2\omega|y|)}{|y|^{\nu-2}}\,dy < \tfrac{1}{4}, \qquad c_3\,\omega^2 \int_{|y|<\varepsilon_0} \frac{\exp(-\omega|y|)}{|y|^{\beta}}\,dy < \tfrac{1}{4}$$

and thus

$$|u_E(x)| \leqslant \tfrac{1}{2}\|u_E\|_\infty + c_5\,\omega^\nu \int \exp(-\omega|y|)\,|u_E(y+x)|\,dy.$$

We now apply Lemma 1 with $u_E$ playing the role of $V$, $\gamma = 0$, $p = 2$, to obtain $\|u_E\|_\infty \leqslant c_6 + c_7\,\omega^{\nu/2}\|u_E\|_2$. We have imposed three conditions on $\omega$, which are satisfied by

$$\omega_0 = \max\left(1,\ c_1\|V\|_p^{2p/(2p-\nu)},\ (\tfrac{1}{3}|E|)^{1/2}\right),$$

so the result of the lemma follows if we show that $\|u_E\|_2 \leqslant R^{\nu/2}$.

Observe that by the periodicity of $|u_E(x)|$ we have

$$\|u_E\|_2^2 = \sup_{\tau\in \mathbf{R}^\nu} \int_{Q_1} |u_E(x+\tau)|^2\,dx = \sup_{\tau\in Q_{\mathrm{SW}}} \int_{Q_1} |u_E(x+\tau+y)|^2\,dx$$
$$\leqslant \int_{y+Q_{R+1}} |u_E(x)|^2\,dx$$

where $y\in \mathbf{R}^\nu$ is arbitrary. Taking $y\in (R+1)\mathbf{Z}^\nu$ and summing $N^\nu$ terms we find

$$\|u_E\|_2^2 \leqslant N^{-\nu} \int_{Q_{N(R+1)}} |u_E(x)|^2\,dx.$$

We can let $N$ go to infinity in the last inequality and we obtain $\|u_E\|_2 \leqslant R^{\nu/2}$, because

$$\lim_{T\to\infty} T^{-\nu} \int_{Q_T} |u_E(x)|^2\,dx = \|u_E\|_{B^2}^2 = 1.$$

The lemma is proved.

Armed with Lemma 5 we can now prove our main result.

THEOREM 5. *Let $V_i$, $i = 1, 2$, be two periodic locally $L^p$ potentials, p as above. Let N and R be such that $\|V_i\|_p \leqslant N$, $Q_{\mathrm{SW},i} \subset Q_R$, $i = 1, 2$, where $Q_{\mathrm{SW},i}$ is the basic cell of the periodicity lattice of $V_i$. Suppose $A_i = -\Delta + V_i$ are self-adjoint operators in $L^2(\mathbf{R}^\nu)$. Then for k with $(-k, k)\cap\sigma(A_i) \neq \emptyset$ we have*

$$D_k(\sigma(A_1),\ \sigma(A_2)) \leqslant K(k,\ N,\ R)\,\|V_1 - V_2\|_{B^2}$$

*where the relative distance $D_k$ is defined as*

$$D_k(U_1, U_2) = \sup\{\Delta_k(U_1, U_2), \Delta_k(U_2, U_1)\}$$

*with*

$$\Delta_k(U_1, U_2) = \sup_{x \in U_1 \cap (-k,k)} \inf_{y \in U_2} |x-y|$$

*for any two closed subsets $U_1$, $U_2$ of $\mathbf{R}$* (sup *over an empty set is assumed to be* $\infty$).

*Proof.* Using the main result of [8] stating that $\sigma(A_i) = \sigma(\mathfrak{A}_i)$, where $\mathfrak{A}_i$ is $-\Delta + V_i$ in $B^2(\mathbf{R}^\nu)$, we will establish the estimate of the theorem for $\mathfrak{A}_i$. If $(-k, k) \cap \sigma(\mathfrak{A}_1)$ is empty, there is nothing to be proved; so assume $E \in (-k, k) \cap \sigma(\mathfrak{A}_1)$, and let $U_E$ be an eigenelement of $\mathfrak{A}_1$, $\mathfrak{A}_1 U_E = EU_E$, represented by the Bloch wave $u_E$. Such a $u_E$ exists by Theorem 4. By Lemma 5, $\|u_E\|_\infty \leqslant K(k, N, R)$. Therefore

$$\|(\mathfrak{A}_2 - E) U_E\|_{B^2} = \|(V_1 - V_2) U_E\|_{B^2} \leqslant K(k, N, R) \|V_1 - V_2\|_{B^2}.$$

This estimate holds for all $E \in (-k, k) \cap \sigma(\mathfrak{A}_1)$, so we have

$$\Delta_k(\sigma(\mathfrak{A}_1), \sigma(\mathfrak{A}_2)) \leqslant K(k, N, R) \|V_1 - V_2\|_{B^2}$$

and the symmetry in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ yields the assertion of the theorem.

Theorem 5 suggests that if we have a sequence of periodic potentials $V_n$ with $\|V_n - V\|_{B^2} \to 0$ for some periodic $V$, then if the $Q_{\mathrm{SW},n}$ grow too rapidly we need not have $D_k(\sigma(A_n), \sigma(A)) \to 0$, where $A_n = -\Delta + V_n$. We shall show that such a situation is indeed possible and moreover that the polynomial dependence on $R$ in the theorem cannot be improved.

To this end consider a short-range potential $W \in L^p(\mathbf{R}^\nu) \cap L^2(\mathbf{R}^\nu)$, and for any $\tau > 0$ let $V_\tau$ be a periodic potential, $V_\tau(x + \tau m) = V_\tau(x)$ for a.e. $x$ and all $m \in \mathbf{Z}^\nu$, with $V_\tau(x) = W(x)$ for $x$ in the basic cell $Q_{\mathrm{SW},\tau}$ of the lattice $\tau\mathbf{Z}^\nu$. We have $\|V_\tau\|_p \leqslant \|W\|_{L^p}$, $\|V_\tau\|_{B^2} \leqslant C\tau^{-\nu/2}\|W\|_{L^2} \to 0$ as $\tau \to \infty$. Let $A_0 = -\Delta$, $A = -\Delta + W$, $A_\tau = -\Delta + V_\tau$ in $L^2(\mathbf{R}^\nu)$. Clearly $\sigma(A_0) = [0, +\infty)$, $\sigma(A)$ consists of the halfline $[0, +\infty)$ and of discrete eigenvalues in the negative halfline $(-\infty, 0)$, while $\sigma(A)$ has band structure.

Denoting $\Delta(U_1, U_2) = \sup_{k>0} \Delta_k(U_1, U_2)$ for any closed subsets $U_1$, $U_2$ of $\mathbf{R}$ we have

LEMMA 6. *Let $A$, $A_\tau$ be as above. Then $\Delta(\sigma(A), \sigma(A_\tau)) \to 0$ as $\tau \to \infty$.*

*Proof.* Since the lengths of gaps in the spectrum of $A_\tau$ are bounded above by $2\|V_\tau\|_{B^2}$ (cf. [8]), we have $\Delta([0, +\infty), \sigma(A_\tau)) \to 0$. Let $E \in \sigma_{\mathrm{disc}}(A)$, $E < 0$. $A_\tau \to A$ in strong resolvent convergence, so $\Delta(\{E\}, \sigma(A_\tau)) \to 0$ (by Theorems VIII.25, VIII.24 of [10]). This convergence is, however, uniform in $E \in \sigma_{\mathrm{disc}}(A)$, since for any $\varepsilon > 0$ only a finite number of eigenvalues of $A$ lie in $(-\infty, -\varepsilon)$. The result of the lemma follows.

We conclude that for large $k > 0$, $D_k(\sigma(A_\tau), \sigma(A_0))$ does not tend to 0, despite the fact that $\|V_\tau\|_{B^2}$ tends to 0. Theorem 5 in this situation gives

$$D_k(\sigma(A_\tau), \sigma(A_0)) \leqslant K(k, \|V_\tau\|_p, \tau)\|V_\tau\|_{B^2}$$
$$\to \|W\|_{L^2}(C_1 + C_2\|W\|_p^{\nu p/(2p-\nu)} + C_3 k^{\nu/4})$$

as $\tau \to \infty$. This shows that the dependence of $K(k, N, R)$ on $R$ is optimal (the exponent $\nu/2$ cannot be decreased).

Finally, we note that in some situations Theorem 5 can be strengthened. Suppose that $V_i$ are positive potentials. Using the integral kernel $K_t(x, y)$ of $\exp(-tA)$ in place of the Green function we can modify the proof of pointwise estimates of square-integrable eigenfunctions obtained by Davies [7] and prove the following

THEOREM 6. *Let $V_i$ be as in Theorem 5, $V_i$ nonnegative, $i = 1, 2$. Then*

$$D_k(\sigma(A_1), \sigma(A_2)) \leqslant L(k, R)\|V_1 - V_2\|_{B^2}$$

*where $L(k, R) = L_1 + L_2 k^{\nu/4} R^{\nu/2}$, $L_1$, $L_2$ depend only on $\nu$.*

Note that in this theorem we also have optimal dependence on $R$.

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