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SEMIGROUPS OF ORDER-PRESERVING PARTIAL ENDOMORPHISMS ON TREES, I

BY

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1. Introduction. A *rooted tree* is a partially ordered set $(W, \leqslant)$ such that W has a least element w_0 and whenever $a, b, c \in W$ with $a \leqslant c$ and $b \leqslant c$, then either $a \leqslant b$ or $b \leqslant a$. The objective of this paper is to introduce a semigroup of order-preserving partial transformations defined on a tree W and to investigate the structure and properties of this semigroup. In particular, information about trees is obtained *via* information about the associated semigroup. Since rooted trees appear in such diverse areas of mathematics as algebraic linguistics [6], foundations [1], graph theory [2], and semigroups [7], there are several interpretations and applications of our results.

Recall that a subsequence, finite or infinite, of a real-valued sequence on a well-ordered set $(W, \leqslant)$ is a function $s\colon W \to W$ such that $s(x) < s(y)$ whenever $x < y$ in W. A *generalized* (*real-valued*) *sequence* f is a function from a partially ordered set $(W, \leqslant)$ with the least element w_0 to the real numbers such that the domain of f is a totally ordered lower segment of W. Further, a generalized subsequence, finite or infinite, of f is just a strictly increasing function $b\colon W \to W$ such that the domain of b is also a lower segment of W. A second interpretation of this paper is the study of the algebraic properties of generalized subsequences defined on trees. From this point of view, the results form an extension and a generalization of the work of Goetz [5] on real sequences defined on the positive integers N and subsequences with domains either N or initial segments of N.

There is also a connection of the present work with the work [8] of Reilly, in which semigroups of order-preserving partial transformations on totally ordered sets are investigated. Reilly's semigroup is contained in ours and, in fact, is characterized as the subsemigroup of regular elements of our semigroup. We note that Reilly found his semigroup useful in extending certain results on transformation groups. We hope that the same is true of the semigroup introduced here and intend to study these relationships in a subsequent paper.

A brief summary is now given. In Section 2, some basic definitions are presented and the semigroup of generalized subsequences $S(W)$, equivalently, the semigroup of order-preserving partial endomorphisms of a tree W is introduced. Algebraic properties of $S(W)$ are investigated in the third section. In Section 4 we show that the semigroup $S(W)$ characterizes the tree W up to isomorphism. In the next section we return to the properties of $S(W)$ and, in particular, to order properties. The main result here is Proposition 5.5 which shows that if the tree W has an additional property, then the semigroup $S(W)$ can be faithfully represented as a semigroup of endomorphisms of a certain semilattice. In the final section we investigate the problem of obtaining an abstract characterization of $S(W)$. We find necessary and sufficient conditions on a semigroup S such that S is isomorphic to a certain subsemigroup of $S(W)$ for some tree W.

2. Definitions and basic results. Let $W \equiv \langle W, \leqslant \rangle$ be partially ordered set, henceforth called *poset*, with the least element w_0. An *ideal* of W is a subset Y of W such that if $y \in Y$ and $x \leqslant y$, then $x \in Y$. If S is any subset of W, the *ideal generated by* S is the set $\{x \in W \mid x \leqslant y \text{ for some } y \in S\}$ and is denoted by $S]$. In particular, $\emptyset = \emptyset]$. The ideal $\{x \mid x \leqslant y\}$ generated by the singleton $\{y\}$ is called the *principal ideal generated by* y and is denoted by $y]$.

Convention. In this paper, *all* posets will contain a least element.

A poset $(W, \leqslant)$ is said to be *well-ordered by* $\leqslant$ if every non-empty subset of W has a least element. $(W, \leqslant)$ is said to be *totally ordered by* $\leqslant$ if, for $x, y \in W$, either $x \leqslant y$ or $y \leqslant x$. When we say W is a well-ordered (totally ordered) set, we always mean it is well-ordered (totally ordered) by the given ordering. (Note that each well-ordered set is totally ordered.)

A poset $(W, \leqslant)$ is called a *tree* if every principal ideal of W is totally ordered. W is said to be a *strong tree* if every principal ideal of W is well--ordered. It is clear that a strong tree is a tree but not conversely. In the case of finite sets the two concepts do agree. There is also the following relationship which points out that on strong trees well-ordering and total ordering agree:

PROPOSITION 2.1. *Let $(W, \leqslant)$ be a poset. W is well-ordered if and only if W is a strong tree and W is totally ordered.*

Proof. Suppose A is a non-empty subset of W. For $a \in A$, $A \cap a] \neq \emptyset$, and since $A \cap a] \subseteq a]$, $A \cap a]$ has a least element, say a_0. But since all elements of A are comparable to a, a_0 is the least element of A.

Recall from graph theory (see [2]) that a *tree* is a connected (finite) graph without cycles, and a graph is *directed* provided one specifies for each edge the beginning and terminating vertex. A *path* π in a directed graph V is a sequence of vertices $v_1, \ldots, v_m$ such that (v_i, v_{i+1}) is an edge for each i, $1 \leqslant i \leqslant m$. In this case π is said to *start at* v_1 and *end at* v_m. A directed tree is said to be *rooted* if there is a unique vertex v_0 (called the *root*) such that, for any vertex v $(v \neq v_0)$, there is a unique path $\pi(v_0, v)$ starting at v_0 and ending at v. An order relation $\leqslant$ is defined on V by $v_1 \leqslant v_2$ if there exists a path from v_1 to v_2. Since V has no cycles, $\leqslant$ is a partial order on V and the root v_0 is the least element. For $v \in V$, the uniqueness of the path $\pi(v_0, v)$ implies that the set of vertices of this path is the principal ideal $v]$. It is clear that if $v_1, v_2 \in v]$, then $v_1 \leqslant v_2$ or $v_2 \leqslant v_1$, and so we have a tree as defined above in terms of the order relation.

We note that a tree is sometimes defined as a poset $(P, \leqslant)$ such that whenever $a \leqslant c$ and $b \leqslant c$, $a, b, c \in P$, then either $a \leqslant b$ or $b \leqslant a$ [7]. This is again equivalent with our original definition under the standing hypothesis that all posets have a least element. With this convention, our trees are "rooted".

For a set K, $P(K)$ denotes the semigroup of partial transformations on K under the operation of function composition. For $a \in P(K)$, $\Delta(a)$ denotes the domain of the transformation a, and $\nabla(a)$ denotes the range.

Let W be a tree (strong tree) and define $S(W)$ to be the collection of $a \in P(W)$ such that

(Di) $\Delta(a)$ is a totally ordered (well ordered) ideal of W;

(Dii) for $x, y \in \Delta(a)$, if $x < y$, then $a(x) < a(y)$.

Since $S(W) \subseteq P(W)$, for $a, b \in S(W)$ we have

$$\Delta(ab) = \{x \in W \mid x \in \Delta(b) \text{ and } b(x) \in \Delta(a)\} \subseteq \Delta(b).$$

From this it is easy to see that $\Delta(ab)$ is a totally ordered (or well-ordered) ideal of W, and $x < y$ in $\Delta(ab)$ implies $ab(x) < ab(y)$. Therefore, $S(W)$ is a semigroup. Since the empty set is considered as a well-ordered ideal, the empty map is a two-sided zero for $S(W)$.

If W is a strong tree, then the elements of $S(W)$ are called generalized subsequences; for, when W is the well-ordered set of positive integers, $S(W)$ is the semigroup studied by Goetz [5]. For $a \in S(W)$, W a strong tree, it may be the case that x and $a(x)$ are not comparable $(x \in \Delta(a))$. However, if they are comparable, then, as is the case for subsequences, $a(x) \geqslant x$. Otherwise, $\{y \in \Delta(a) \mid y > a(y)\}$ is a non-empty subset of $\Delta(a)$, and thus has a least element, say y_0. But $y_0 > a(y_0)$ implies $a(y_0) \in \Delta(a)$; consequently, $y_0 > a(y_0) > a(a(y_0))$, contradicting the choice of y_0.

$S(W)$ can also be considered as a certain subsemigroup of the semigroup of partial endomorphisms of the poset $(W, \leqslant)$. In either interpretation, we investigate the structure of $S(W)$ in order to obtain information about W.

3. Semigroup properties of $S(W)$. In this section, several results about the semigroup $S(W)$ are established, where, unless otherwise indicated, W is a tree.

First, if e is an idempotent of $S(W)$, then $ee(x) = e(x)$ for each x in $\Delta(e)$, i.e., $e(x) \in \Delta(e)$. In fact, $e(x) = x$; for if $e(x) > x$, then $e(x) = ee(x) > e(x)$, and similarly if $e(x) < x$. Hence $e = 1_{\Delta(e)}$, the identity function on $\Delta(e)$. Conversely, it is clear that the identity function on any totally ordered ideal of W is an idempotent. For $e, f \subset E(S(W))$, the set of idempotents in $S(W)$, we have $\Delta(ef) = \Delta(e) \cap \Delta(f)$. Thus $ef = 1_{\Delta(ef)} = 1_{\Delta(fe)} = fe$ which shows that $E(S(W))$ is a semilattice. Recall [4] that the natural partial ordering $\leqslant$ on $E(S(W))$ is given by $e \leqslant f \Leftrightarrow ef = e$. We collect the above results in

PROPOSITION 3.1. *The idempotents of $S(W)$ are the identity functions on totally ordered ideals of W. The set $E(S(W))$ of idempotents of $S(W)$ is a semilattice.*

As one indication of how information about $S(W)$ yields information about W, we present the following easy result:

PROPOSITION 3.2. *W is totally ordered if and only if $S(W)$ has an identity.*

PROPOSITION 3.3. *If $Z(S(W))$ denotes the center of $S(W)$, then $Z(S(W)) \subseteq E(S(W))$ and $1 \leqslant |Z(S(W))| \leqslant 2$* (1).

Proof. The empty function $\emptyset$ is in the center. Suppose $\emptyset \neq a \in Z(S(W))$. If $a \notin E(S(W))$, then there exists a $k \in \Delta(a)$, $a(k) \neq k$. If $e = e_{k]}$, then $\Delta(ae) = \Delta(e)$. Thus if $a(k)$ and k are not comparable or if $a(k) > k$, then $k \in \Delta(ae)$, but $k \notin \Delta(ea)$ which implies $ae \neq ea$. If $a(k) < k$, then for $f = e_{a(k)]}$ we find that $af \neq fa$. In either case $a \notin Z(S(W))$.

If $e \in Z(S(W))$ and $\emptyset \subsetneq \Delta(e) \subsetneq W$, then let y_0 be an arbitrary but fixed element in $W - \Delta(e)$ and define $b \in S(W)$ by $\Delta(b) = \{w_0\}$ and $b(w_0) = y_0$. Then $\Delta(eb) = \emptyset$ but $\Delta(be) = \{w_0\}$ which contradicts the fact that $e \in Z(S(W))$. Hence $\Delta(e) = \emptyset$ or $\Delta(e) = W$. If W is not totally ordered, then, of course, the second case is impossible. Thus, if W is totally ordered, $Z(S(W)) = \{\emptyset, 1_W\}$; otherwise, $Z(S(W)) = \{\emptyset\}$.

We now turn to a characterization of the regular elements in $S(W)$.

PROPOSITION 3.4. *a is a regular element of $S(W)$ if and only if $\nabla(a)$ is an ideal of W.*

Proof. If $\nabla(a)$ is an ideal of W, then $\nabla(a)$ is order isomorphic to $\Delta(a)$, and hence a totally ordered ideal of W. Define $b \in S(W)$ by $\Delta(b) = \nabla(a)$ and $b(a(y)) = y$. Then $aba = a$. Conversely, if $a = aba$ for some $b \in S(W)$, then $\nabla(a) \subseteq \Delta(b)$. To complete the proof, we must show that if $x \leqslant y$ for $y \in \nabla(a)$, then $x \in \nabla(a)$. Now, $y \in \nabla(a)$ implies $y = a(y_0)$, $y_0 \in \Delta(a)$,

(1) For a set A, $|A|$ denotes the cardinality of A.

and since $\nabla(a) \subseteq \Delta(b)$, we have $b(x) \leqslant b(y) = ba(y_0) = y_0$ (recall ba is an idempotent). But then $b(x) \in \Delta(a)$ and, consequently, $x = a(b(x)) = ab(x)$, since ab is an idempotent. Thus $x \in \nabla(a)$, as desired.

From the proof of Proposition 3.4 and the fact that $E(S(W))$ is a semilattice, we have the following consequence:

COROLLARY 3.5. *Let* $J(W) = \{a \in S(W) \mid \nabla(a)$ *is an ideal of* $W\}$. *Then* $J(W)$ *is an inverse subsemigroup of* $S(W)$.

Reilly [8] has considered a certain semigroup J_X of order-preserving partial transformations on a totally ordered set X. In the case where W is totally ordered, the semigroup $J(W)$ of Corollary 3.5 is the semigroup studied by Reilly. If W is not totally ordered, then $J(W)$ is a generalization of Reilly's semigroup to trees.

An element c in a semigroup S is called a *right conserver* if, for each $a \in S$, $ac = a$ or $ac = c$. If c is a right conserver, then c is an idempotent since $cc = c$. Right zero-elements and right identity-elements are examples of right conservers. In $S(W)$, $\emptyset$ is a right conserver. For non-trivial right conservers in $E(S(W))$ we have the following result:

PROPOSITION 3.6. *For a tree* W, *the following are equivalent*:

(i) *There exists a totally ordered ideal* K *of* W, $|K| \geqslant 2$, *such that, for each ideal* A *of* W, *either* $A \subseteq K$ *or* $K \subseteq A$.

(ii) $E^{**} = E(S(W)) - \{\emptyset, 0\}$ *is a semigroup with a right conserver.*

Proof. (i) $\Rightarrow$ (ii). Let $e, f \in E^{**}$. If $\Delta(e) \supseteq K$ and $\Delta(f) \supseteq K$, then $\Delta(ef) \supseteq K$ and $ef \in E^{**}$. If $K \supseteq \Delta(e)$ and $K \supseteq \Delta(f)$, then x in $\Delta(e) - \{w_0\}$ and y in $\Delta(f) - \{w_0\}$ are comparable, say $x \leqslant y$. Hence $x \in \Delta(f)$, $ef(x) = x$ and $ef \in E^{**}$. Similar arguments hold for the other cases. Thus E^{**} is a semigroup. Moreover, $e_K = 1_K$ is a right conserver since $\Delta(ee_K) = \Delta(e) \cap K$. If $\Delta(e) \cap K = K$, $ee_K = e_K$ while $K \supseteq \Delta(e)$ implies $ee_K = e$.

(ii) $\Rightarrow$ (i). Let c be a right conserver in E^{**}, $\Delta(c) = K$ and let A be any ideal of W. If $A = \emptyset$, $A \subseteq K$. If $1_{x]}c = 1_{x]}$ for each $x \in A$, then $x] \cap K = x]$ or $x \in K$. On the other hand, if $1_{x]}c = c$ for some $x \in A$, then $x] \cap K = K$ or $K \subseteq x] \subseteq A$.

COROLLARY 3.7. *If* $S(W)$ *has a right conserver* c *such that* $c \neq \emptyset$, *then* $c = 1_K$, *where* K *satisfies* (i) *of Proposition* 3.6.

Let K be any totally ordered ideal of a tree W. Using Zorn's lemma, one establishes the existence of a totally ordered ideal, maximal with respect to containing K. Those totally ordered ideals maximal with respect to containing $\{w_0\}$ are called *branches* of W. The intersection of all branches is called the *trunk* of the tree W and is denoted by $T(W)$. If $|T(W)| \geqslant 2$, W is said to *have a non-trivial trunk*. We use these concepts to illustrate again how information about W can be obtained from $S(W)$.

LEMMA 3.8. *$T(W) = \{x \in W \mid x \leqslant w$ or $w \leqslant x,\ w \in W\}$.*

Proof. By the definition, $T(W) = \bigcap B_a$, where B_a is a branch of W. Let us take

$$\bar{T} = \{x \in W \mid x \leqslant w \text{ or } w \leqslant x,\ w \in W\}.$$

Let $x \in \bigcap B_a$ and w be arbitrary in W. Then $\{w_0\} \subseteq w]$ implies $w]$ is contained in some branch B_{a_0} and, consequently, x and w are comparable. Thus $T(W) \subseteq \bar{T}$. On the other hand, let $x \in \bar{T}$ and consider any B_a. If $x \leqslant w$ for some $w \in B_a$, then $x \in B_a$. If $x > w$ for each $w \in B_a$, then $x] \supsetneqq B_a \supseteq \{w_0\}$ which contradicts the maximality of B_a. Thus $\bar{T} \subseteq B_a$ for each a or $\bar{T} \subseteq T(W)$.

PROPOSITION 3.9. *W has a non-trivial trunk if and only if E^{**} has a right conserver.*

Proof. Suppose $T(W)$ is non-trivial and let A be any non-empty ideal of W. If, for each $a \in A$, there exists an $x \in T(W)$ such that $a \leqslant x$, then $A \subseteq T(W)$. If this is not the case, then $x \leqslant a$ for some $a \in A$ and each $x \in T(W)$. But then $A \supseteq T(W)$. The result now follows from Proposition 3.6.

Conversely, if E^{**} has a right conserver c, then $c = 1_K$, $|K| \geqslant 2$. For each $w \in W$, either $1_{w]}c = 1_{w]}$ or $1_{w]}c = c$. In the first case $w] \subseteq K$ while in the second case $K \subseteq w]$. However, in either case, each $k \in K$ is comparable with w. Thus $K \subseteq T(W)$ which implies that $T(W)$ is non-trivial. ∎

COROLLARY 3.10. *For a tree W, $T(W) = \bigcup \{K \mid K = \Delta(c),\ c$ a right conserver in $E(S(W))\}$.*

Proof. If $x \in T(W)$, $1_{x]}$ is a right conserver in $E(S(W))$.

COROLLARY 3.11. *If $S(W)$ has a right conserver c such that $c \neq \emptyset$, then W has a non-trivial trunk.*

4. $S(W)$ determines W. In this short section we show that, for trees W, the semigroup $S(W)$ determines W uniquely up to isomorphism.

PROPOSITION 4.1. *Let W_1 and W_2 be trees. W_1 is order isomorphic to W_2 if and only if $S(W_1)$ and $S(W_2)$ are semigroup isomorphic.*

Proof. We note first that the map $\psi_1\colon W_1 \to S(W_1)$ given by $\psi_1(w) = 1_{w]}$ determines an order isomorphism between W_1 and the poset of idempotents whose domains are principal ideals of W_1 where, of course, the idempotents have the natural ordering. An analogous situation holds for W_2. If $\varphi\colon S(W_1) \to S(W_2)$ is a semigroup isomorphism, then φ is also an order isomorphism of $E(S(W_1))$ onto $E(S(W_2))$. Hence to establish that W_1 is order isomorphic to W_2 we need only show that, for each $x \in W_1$, $\Delta(\varphi(1_{x]}))$ is a principal ideal of W_2.

Let $K = \{y \in W_1 \mid y < x\}$ and, for any $w \in W_1$, let e_w denote the idempotent $1_{w]}$. Since $e_x 1_K = 1_K$, $\varphi(e_x)\varphi(1_K) = \varphi(1_K)$ which means that

$\Delta\varphi(1_K) \subseteq \Delta\varphi(e_x)$. Since $1_K \neq e_x$, we have $\varphi(1_K) \neq \varphi(e_x)$, and since $\varphi(1_K)$ and $\varphi(e_x)$ are idempotents, $\Delta\varphi(1_K) \subsetneq \Delta\varphi(e_x)$. Now,

$$\Delta\varphi(1_K) \subsetneq w] \subset \Delta\varphi(e_x) \quad \text{for } w \in \big(\Delta\varphi(e_x) - \Delta\varphi(1_K)\big),$$

which implies $\varphi(1_K)e_w = \varphi(1_K)$ and $e_w\varphi(e_x) = e_w$. There exists an idempotent e such that $\varphi(e) = e_w$, and from this we find that $ee_x = e$, $1_K e = 1_K$ or $K \subseteq \Delta(e) \subseteq x]$. Since $e_w \neq \varphi(1_K)$, we must have $\Delta(e) = x]$ or $e = e_x = 1_{x]}$. Hence $\Delta\varphi(1_{x]}) = \Delta\varphi(e) = w]$, as desired.

Conversely, if $\psi\colon W_1 \to W_2$ is an order isomorphism, then the map which takes $f \in S(W_1)$ to $\psi f \psi^{-1}$ determines a semigroup isomorphism between $S(W_1)$ and $S(W_2)$. This completes the proof of the proposition.

COROLLARY 4.2. *Let W_1 and W_2 be trees and $E\big(S(W_i)\big)$ the semilattice of idempotents of $S(W_i)$, $i = 1, 2$. Then W_1 is order isomorphic to W_2 if and only if $E\big(S(W_1)\big)$ is semigroup isomorphic to $E\big(S(W_2)\big)$. Hence,*

$$S(W_1) \cong S(W_2) \Leftrightarrow E\big(S(W_1)\big) \cong E\big(S(W_2)\big).$$

5. Order properties and a representation of $S(W)$. As mentioned above, it is well known that a semilattice E can be partially ordered by defining, for e, f in E, $e \leqslant f$ if there exists some $h \in E$ such that $e = fh$. In this section we define a relation on $S(W)$, where W is a tree, which, when restricted to $E\big(S(W)\big)$, agrees with this "natural" partial order on $E\big(S(W)\big)$. In the case where W is a strong tree much more can be obtained. In fact, we find that $S(W)$ can be represented as a semigroup of semilattice endomorphisms.

Let W be a tree or a strong tree. For $a, b \in S(W)$, we say $a\varrho b$ if there exists a $c \in S(W)$ such that $a = bc$. The transitivity of ϱ is clear and, since $a = a1_{\Delta(a)}$, so is the reflexivity, whence ϱ is a pre-order. We see from the following characterization that, in general, ϱ is not a partial order:

PROPOSITION 5.1. *Let $a, b \in S(W)$, where W is a tree. $a\varrho b \Leftrightarrow V(a) \subseteq V(b)$.*

Proof. If $a = bc$ for some $c \in S(W)$, then $V(a) \subseteq V(b)$.

For the converse, if $V(a) \subseteq V(b)$, then, for each $x \in \Delta(a)$, there exists a $y_x \in \Delta(b)$ such that $a(x) = b(y_x)$. (Since b is one-one, this y_x is unique.) Define $c \in S(W)$ by $\Delta(c) = \Delta(a)$ and $c(x) = y_x$, $x \in \Delta(a)$. Now $x_1 > x_2$ implies $a(x_1) > a(x_2)$ or, equivalently, $b(y_{x_1}) > b(y_{x_2})$. Since $\Delta(b)$ is totally ordered, $y_{x_1} \leqslant y_{x_2}$ or $y_{x_1} > y_{x_2}$. If $y_{x_1} \leqslant y_{x_2}$, then $b(y_{x_1}) \leqslant b(y_{x_2})$, but this is impossible, which means $y_{x_1} > y_{x_2}$. Hence $c \in S(W)$ and $a = bc$.

For the remainder of this section, we assume W is a strong tree. Recall that the domains of the elements in $S(W)$ are taken as well-ordered ideals of W. The next proposition generalizes results of the work [5] of Goetz to arbitrary well-ordered sets.

PROPOSITION 5.2. *ϱ is a partial order on $S(W)$ if and only if W is well-ordered by $\leqslant$.*

Proof. Suppose $\leqslant$ is a well-ordering on W and let $a, b \in S(W)$ be such that $a\varrho b$ and $b\varrho a$. Since $\Delta(a)$ and $\Delta(b)$ are ideals of the well-ordered set W, either $\Delta(a) \subseteq \Delta(b)$ or $\Delta(b) \subseteq \Delta(a)$, say $\Delta(a) \subseteq \Delta(b)$. Then $a(x) = b(x)$ for each $x \in \Delta(a)$. If not, let x_0 be the first element in $\Delta(a)$ such that $a(x_0) \neq b(x_0)$ where, without loss of generality, we take $a(x_0) > b(x_0)$. Then $a(x) = b(x) < b(x_0)$ for $x < x_0$ and $a(x) \geqslant a(x_0) > b(x_0)$ for $x \geqslant x_0$. Thus $b(x_0) \notin \nabla(a)$. But $a\varrho b$ and $b\varrho a$ imply $\nabla(a) = \nabla(b)$. This contradiction shows that $a(x) = b(x)$ for $x \in \Delta(a)$. If $\Delta(a) \subsetneqq \Delta(b)$ and y_0 is the first element in $\Delta(b) - \Delta(a)$, then $b(y_0) \in \nabla(b)$, but $b(y_0) > b(x) = a(x)$ for all $x \in \Delta(a)$. This again contradicts $\nabla(a) = \nabla(b)$. Therefore, $\Delta(a) = \Delta(b)$ and $a(x) = b(x)$, $x \in \Delta(a)$; i.e., $a = b$.

Conversely, if W is not well-ordered, then W is not totally ordered (Proposition 2.1). Thus, if x and y are non-comparable elements in W,

$$x]\cap y] \subsetneqq x] \quad \text{and} \quad x]\cap y] \subsetneqq y].$$

Let x_0 be the first element in $x] - (x]\cap y])$ and y_0 the first element in $y] - (x]\cap y])$. These elements exist since W is a strong tree. Define $c \in S(W)$ by $\Delta(c) = y_0]$, where $c(w) = w$, $w \in x]\cap y]$ and $c(y_0) = x_0$. Then $\nabla(c) = \nabla(1_{x_0]})$ but $c \neq 1_{x_0]}$. That is, $c\varrho 1_{x_0]}$ and $1_{x_0]}\varrho c$, but $c \neq 1_{x_0]}$.

As with any pre-order σ on a set X, one defines an equivalence relation $\equiv$ on X by $x_1 \equiv x_2$ if $x_1 \sigma x_2$ and $x_2 \sigma x_1$. This leads to a partial order $\leqslant$ on the set X/σ of equivalence classes by defining $[x_1] \leqslant [x_2]$ if $x_1 \sigma x_2$. In our case, for $a, b \in S(W)$, $a \equiv b$ if and only if $\nabla(a) = \nabla(b)$. Since $a \in aS(W)$, we note that the equivalence $\equiv$ on $S(W)$ agrees with the left Green equivalence $\mathcal{R}$. ($a\mathcal{R}b$ if a and b generate the same right ideal of $S(W)$ (see [4])). Thus $S(W)/\mathcal{R}$ is a poset with a well-defined map from $S(W)\times S(W)/\mathcal{R}$ to $S(W)/\mathcal{R}$ given by $(s, [a]) \to s\cdot[a] = [sa]$ such that

$$(s_1\cdot s_2)\cdot[a] = s_1\cdot(s_2\cdot[a]) \quad \text{for } s_1, s_2 \in S(W),\ [a] \in S(W)/\mathcal{R}.$$

In other words, $S(W)/\mathcal{R}$ is an $S(W)$-set.

We now show that $S(W)/\mathcal{R}$ is an $S(W)$-semilattice. That is, there is a binary relation $\wedge$ such that $S(W)/\mathcal{R}$ is a semilattice under $\wedge$ and the function $(s, [a]) \to s\cdot[a]$ has the property

$$s\cdot([a]\wedge[b]) = s\cdot[a]\wedge s\cdot[b] \quad \text{for } s \in S(W),\ [a], [b] \in S(W)/\mathcal{R}.$$

For $[a], [b] \in S(W)/\mathcal{R}$, let $\nabla(a)\cap\nabla(b) = M$. Then M is isomorphic to some subset of $\Delta(a)$, and hence to an ideal A of $\Delta(a)$ (see [3], p. 36, Exercise 2). Let $a\wedge b\colon A \to M$ denote this isomorphism. Since

$$\nabla(a\wedge b) = \nabla(a)\cap\nabla(b),$$

we have $[a\wedge b] \leqslant [a]$ and $[a\wedge b] \leqslant [b]$. If $[c] \leqslant [a]$ and $[c] \leqslant [b]$, then $\nabla(c) \subseteq \nabla(a)\cap\nabla(b)$ which implies $[c] \leqslant [a\wedge b]$. We define $[a]\wedge[b] = [a\wedge b]$ and obtain the following result:

PROPOSITION 5.3. *$\langle S(W)/\mathscr{R}, \wedge\rangle$ is a semilattice.*

We now claim that $a\cdot([b]\wedge[c]) = a\cdot[b]\wedge a\cdot[c]$ for $a\in S(W)$, and $[b], [c]\in S(W)/\mathscr{R}$. Indeed, it suffices to show that $\nabla(a(b\wedge c)) = \nabla(ab\wedge ac)$. To this end we first observe that $x\in \nabla(a(b\wedge c))$ if and only if there exists a $y\in S(W)$ with $x = a(b\wedge c)y$ which is true if and only if $x\in a(\nabla(b\wedge c))$. But $\nabla(b\wedge c) = \nabla b\cap \nabla c$ and since a is a one-one map,

$$a\nabla(b\wedge c) = a(\nabla b\cap\nabla c) = a(\nabla b)\cap a(\nabla c) = \nabla ab\cap\nabla ac = \nabla(ab\wedge ac).$$

PROPOSITION 5.4. *$\langle S(W)/\mathscr{R}, \wedge\rangle$ is an $S(W)$-semilattice.*

From this proposition we see that, for each $s\in S(W)$, the left translation $\lambda_s\colon S(W)/\mathscr{R}\to S(W)/\mathscr{R}$ $(\lambda_s[a] = s\cdot[a] = [sa])$ is a semilattice morphism. Moreover, the map $\lambda\colon S(W)\to \mathrm{End}(S(W)/\mathscr{R})$ $(s\to\lambda_s)$ is a semigroup morphism. Thus $S(W)$ can be represented by a semigroup of semilattice endomorphisms. We will show that this representation is faithful; i.e., for $s, t\in S(W)$, $\lambda(s) = \lambda(t)$ implies $s = t$.

If $\lambda(s) = \lambda(t)$, then $[sa] = [ta]$ for each $a\in S(W)$, which is equivalent to $\nabla(sa) = \nabla(ta)$ for all $a\in S(W)$. Using $a = 1_{\Delta(s)}$ and then $a = 1_{\Delta(t)}$, we find $s = t1_{\Delta(s)}b_1$ and $t = s1_{\Delta(t)}b_2$ for some $b_1, b_2\in S(W)$ which, in turn, implies $\nabla(s) = \nabla(t)$. Using $a = 1_{w_0]}$ (recall w_0 is the least element of W), we find $s(w_0) = t(w_0)$. If $K = \Delta(s)\cap\Delta(t)$, then

$$\nabla(s1_K) = \nabla(t1_K) \quad\text{and}\quad \Delta(s1_K) = \Delta(s)\cap K = \Delta(t)\cap K = \Delta(t1_K).$$

Suppose $\{x\in K\mid s1_K(x) = s(x)\neq t(x) = t1_K(x)\}$ is non-empty and let x_0 be the least element in this subset of $\Delta(s)$. Without loss of generality, take $s(x_0) < t(x_0)$. For $x\in K$, if $x < x_0$,

$$t1_K(x) = t(x) = s(x) = s1_K(x) < s1_K(x_0),$$

while if $x\geqslant x_0$,

$$t1_K(x)\geqslant t1_K(x_0) > s(x_0) = s1_K(x_0).$$

This is a contradiction to $\nabla s1_K = \nabla t1_K$. Hence $s(x) = t(x)$ for all x in K. It remains to show that $\Delta(s) = \Delta(t)$. If this is not the case, then $K\subsetneq\Delta(s)$ or $K\subsetneq\Delta(t)$, say $K\subsetneq\Delta(s)$. Let y_0 be the first element in $\Delta(s)-K$. Then $\nabla s1_{y_0]} = \nabla t1_{y_0]}$ which implies that $s(y_0) = t(w)$ for some $w\in\Delta(t1_{y_0]})\subseteq y_0]$. Since $y_0\notin\Delta(t)$, $w\neq y_0$. But then $w < y_0$ implies that $w\in K$ and, consequently, $t(w) = s(w)\neq s(y_0)$. This means we must have $\Delta(s) = \Delta(t)$ and thus $s = t$.

PROPOSITION 5.5. *If W is a strong tree, then $S(W)$ can be faithfully represented as a semigroup of endomorphisms of a semilattice.*

Suppose W is well-ordered by $\leqslant$. It follows from Proposition 5.2 that the pre-order ϱ on $S(W)$ is a partial order which implies that the equivalence $\equiv$ (or $\mathscr{R}$) is just the equality relation. Therefore, $S(W)$ becomes a semilattice, with the semilattice operation again denoted by $\wedge$, having the property $a(b \wedge c) = ab \wedge ac$, $a, b, c \in S(W)$; i.e., $\langle S(W), \cdot, \wedge \rangle$ is a left distributive semilattice.

COROLLARY 5.6. *If W is well-ordered by $\leqslant$, then $\langle S(W), \cdot, \wedge \rangle$ is a left-distributive semilattice and $\langle S(W), \cdot \rangle$ can be faithfully represented as a semigroup of endomorphisms of $\langle S(W), \wedge \rangle$.*

We remark that a pre-order σ on $S(W)$ can also be obtained by defining $a\sigma b$ if there exists a $c \in S(W)$ such that $a = cb$. An equivalence $\sim$ with $a \sim b$ if $a\sigma b$ and $b\sigma a$ is then obtained. This also gives a partial order on $S(W)/\sim$. Further, the equivalence $\sim$ agrees with the right Green congruence $\mathscr{L}$ on $S(W)$. ($a\mathscr{L}b$ if and only if a and b generate the same left ideal of $S(W)$). We note that if $a \sim b$, then $\Delta(a) = \Delta(b)$, but Example 5.7 shows that the converse is false. However, this is not surprising if one considers the lack of symmetry in the domains and ranges of elements in $S(W)$. We also remark that the Green equivalence $\mathscr{H}$ ($\mathscr{H} = \mathscr{L} \cap \mathscr{R}$) is just the equality relation on $S(W)$. In fact, $a\mathscr{H}b$ implies $\nabla(a) = \nabla(b)$ and $\Delta(a) = \Delta(b)$. As already established above, these two conditions imply that $a = b$.

Example 5.7. Let $\langle W, \leqslant \rangle$ be the tree $w_0 < w_1 < w_2 < w_3 < w_4$. Define a: $\{w_0, w_1\} \to \{w_2, w_3\}$ and b: $\{w_0, w_1\} \to \{w_3, w_4\}$. If there exists a c such that $a = cb$, then $w_2 = a(w_0) = cb(w_0) = c(w_3)$. But no such c exists. Thus we have $\Delta(a) = \Delta(b)$ but $a \not\sim b$.

6. Characterization. In this section an abstract characterization of certain subsemigroups of $S(W)$ is given. In other words, necessary and sufficient conditions are obtained, so that any semigroup S satisfying these conditions is isomorphic to a certain kind of subsemigroup of $S(W)$. In order to simplify the statements of our results we make the following definition. We also remark that unless otherwise specified $(W, \leqslant)$ is a tree.

Definition 6.1. A semigroup S is said to be a *semigroup of tree functions*, abbreviated *tf-semigroup*, if $E(S)$ is a tree under the natural partial ordering $\leqslant$ on the idempotents of S, and if there exist mappings L: $S \to E(S)$ and R: $S \to E(S)$ such that, for $a, b \in S$ and $e \in E(S)$,

D1. $L(a)a = a$ while if $ea = a$, then $L(a) \leqslant e$ and, dually, $aR(a) = a$ while if $ae = a$, then $R(a) \leqslant e$;

D2. $R(a) = R(b)$ and $L(ae) = L(be)$ for $e \leqslant R(a)$ if and only if $a = b$;

D3. $L \circ \lambda_a$ is order-preserving on $R(a)]$, and if $L(b) \leqslant R(a)$, then $L \circ \lambda_a(L(b)) \leqslant L(ab)$;

D4. $e \leqslant R(ab)$ if and only if $e \leqslant R(b)$ and $L \circ \lambda_b(e) \leqslant R(a)$.

We observe from the following proposition that if W is a finite tree, then $S(W)$ is a tf-semigroup:

PROPOSITION 6.2. *Let H be a subsemigroup of $S(W)$ such that*
(a) *for $h \in H$, $\Delta(h)$ is a principal ideal of W, and*
(b) *for $e \in E(S(W))$, if $\Delta(e)$ is a principal ideal of W, then $e \in H$.*
Then H is a tf-semigroup.

Proof. We first show that $E(H)$ is a tree. For $e \in H$, we have $\Delta(e) = x_e]$, thus we define a map $E(H) \to W$ by $e \to x_e$. Recall that the natural ordering in $E(S(W))$ is just the pre-order ϱ restricted to $E(S(W))$. Hence $e_1 \varrho e_2$ is equivalent to $\nabla(e_1) \subseteq \nabla(e_2)$ or $\Delta(e_1) \subseteq \Delta(e_2)$ which, in turn, is equivalent to $x_{e_1} \leqslant x_{e_2}$. Thus $e_1 = e_2$ if and only if $x_{e_1} = x_{e_2}$ which implies that our map is a one-one order-preserving function. Let $x \in W$ and define $e_x \in E(S(W))$ by $\Delta(e_x) = x]$. But then, by (b), $e_x \in H$ and $e_x \to x$. Therefore, $(E(H), \varrho)$ is order isomorphic to $(W, \leqslant)$, and hence is a tree.

For $a \in H$, we have $\Delta(a) = x_a]$. Now we define the mappings $L\colon H \to E(H)$ by $\Delta(L(a)) = a(x_a)]$ and $R\colon H \to E(H)$ by $\Delta(R(a)) = x_a]$. For $y \in \Delta(a)$, we have $a(y) \leqslant a(x_a)$, whence $a(y) \in a(x_a)]$ and $L(a) \cdot a = a$. If $ea = a$, then $\nabla(a) \subseteq \Delta(e)$ and, consequently, $\Delta L(a) = \nabla L(a) \subseteq \Delta(e) = \nabla(e)$ or $L(a) \varrho e$. Clearly, $aR(a) = a$, and if $ae = a$, then $\nabla R(a) = \Delta R(a) = \Delta(a) \subseteq \Delta(e) = \nabla(e)$. That is, $R(a) \varrho e$.

To verify property D2, suppose $R(a) = R(b)$ which is equivalent to $\Delta(a) = \Delta(b)$. For $x \in \Delta(a)$, define $e \in E(H)$ by $\Delta(e) = x]$ and note that $e \varrho R(a)$. Then $\Delta(ae) = \Delta(e) = x]$ and $L(ae) = e_0$, where $\Delta(e_0) = a(x)]$. Similarly, $\Delta(be) = x]$ and $L(be) = e_1$, $\Delta(e_1) = b(x)]$. But $L(ae) = L(be)$ for $e \varrho R(a)$ implies $a(x) = b(x)$. Since x was arbitrary, $a = b$.

For property D3, let $e_1, e_2 \in R(a)]$, $e_1 \subsetneq e_2$. Thus $\Delta(e_1) \subsetneq \Delta(e_2) \subseteq \Delta(a)$, and so there exists an $x_0 \in \Delta(e_2)$ such that $x_0 > y$ for each $y \in \Delta(e_1)$. For $i = 1, 2$, $\Delta(ae_i) = \Delta(e_i) = x_i]$ and $\Delta(L(ae_i)) = a(x_i)]$. For $w \in \Delta(L(ae_1))$, $w \leqslant a(x_1) < a(x_0) \leqslant a(x_2)$. Therefore, $\Delta(L(ae_1)) \subsetneq \Delta(L(ae_2))$ which is equivalent to the fact that $L \circ \lambda_a$ is order-preserving on $R(a)]$. Suppose further $L(b) \varrho R(a)$. If $\Delta(b) = x_b]$, then $b(x_b)] \subseteq \Delta(a)$ which, in turn, implies $\Delta(ab) = \Delta(b)$. For $x \in \Delta(L \circ \lambda_a(L(b)))$, $x \leqslant aL(b)(w)$ for some $w \in \Delta(L(b))$. From this we obtain

$$x \leqslant a(w) \leqslant ab(x_b) \quad \text{or} \quad x \in ab(x_b)] = \Delta L(ab),$$

and so $L \circ \lambda_a(L(b)) \varrho L(ab)$, as desired.

For D4, if $e \varrho R(ab)$, then $\Delta(e) \subseteq \Delta(ab) \subseteq \Delta(b)$, and so $e \varrho R(b)$. If $\Delta(e) = x]$, then $\Delta(be) = x]$ and $L(be) = 1_{b(x)]}$. Since $b(x) \in \Delta(a)$, $1_{b(x)]} \varrho R(a)$. Conversely, if again we let $\Delta(e) = x]$, then $x] \subseteq \Delta(b)$, $\Delta(be) = x]$, and from $L(be) \varrho R(a)$ we obtain $b(x) \in \Delta(a)$ which, in turn, implies $\Delta(e) \subseteq \Delta(ab)$ or $e \varrho R(ab)$.

Turning to the converse problem, we have

PROPOSITION 6.3. *If S is a tf-semigroup, then, for some tree $(W, \leqslant)$, S is isomorphic to a subsemigroup $\bar{S}$ of $S(W)$ such that $\bar{S}$ satisfies* (a) *and* (b) *of Proposition* 6.2.

Proof. Since S is a tf-semigroup, $E(S)$ is a tree under the natural ordering $\leqslant$. For W, choose $E(S)$ and, for each $a \in S$, define $\bar{a}$: $W \to W$ by

$$\Delta(\bar{a}) = \{e \in E(S) \mid e \leqslant R(a)\} \quad \text{and} \quad \bar{a}(e) = L(ae).$$

Since $E(S)$ is a tree, $\Delta(\bar{a})$ is indeed a totally ordered ideal in W, and the first part of property D3 guarantees that $\bar{a}$ is order-preserving. Thus, if we let $\bar{S} = \{\bar{a} \mid a \in S\}$, then $\bar{S} \subseteq S(W)$.

We now show that $\bar{S}$ is a semigroup under function composition; in particular, we show $\bar{a} \circ \bar{b} = \overline{ab}$. To this end, $\Delta(\overline{ab}) = \{e \mid e \leqslant R(ab)\}$ while

$$\Delta(\bar{a} \circ \bar{b}) = \{e \mid e \in \Delta(\bar{b}) \text{ and } \bar{b}(e) \in \Delta(\bar{a})\} = \{e \mid e \leqslant R(b) \text{ and } L(be) \leqslant R(a)\}.$$

Thus, from property D4, $\Delta(\overline{ab}) = \Delta(\bar{a} \circ \bar{b})$. It remains to verify that $\overline{ab}(e) = \bar{a}(\bar{b}(e))$, $e \in \Delta(\overline{ab})$. For $e \in \Delta(\overline{ab})$, we have $L(be) \leqslant R(a)$, and so $L(\lambda_a(L(be))) \leqslant L(abe)$ by the second part of D3. On the other hand, we know

$$a \cdot L(be) = L(a \cdot L(be))\, a \cdot L(be)$$

which implies

$$abe = a \cdot L(be) \cdot be = L(a \cdot L(be))\, abe.$$

Using D1, $L(abe) \leqslant L(a \cdot L(be))$. Consequently,

$$L(abe) = L(aL(be)) = L \circ \lambda_a(L(be))$$

which is equivalent to

$$\overline{ab}(e) = \bar{a}(\bar{b}(e)).$$

This completes the proof that $\overline{ab} = \bar{a} \circ \bar{b}$ and also establishes that the map $a \to \bar{a}$ is a semigroup epimorphism of S onto $\bar{S}$.

If $\bar{a} = \bar{b}$, then $\Delta(\bar{a}) = \Delta(\bar{b})$ or $R(a) = R(b)$. Also, for each $e \leqslant R(a)$, $L(ae) = \bar{a}(e) = \bar{b}(e) = L(be)$. Using D2, $a = b$ and, consequently, S is isomorphic to $\bar{S}$.

It remains to show that $\bar{S}$ satisfies properties (a) and (b) of Proposition 6.2. Clearly, $\Delta(\bar{a})$ is a principal ideal of W for every $\bar{a}$ in $\bar{S}$. Let $e \in S(W)$, $\Delta(e) = e_0]$, $e_0 \in W = E(S)$. Under the above epimorphism, $e_0 \to \bar{e}_0$ and $\Delta(\bar{e}_0) = \{e \mid e \leqslant R(e_0)\}$. However, $e_0 R(e_0) = e_0$ implies $e_0 \leqslant R(e_0)$, and from $e_0 e_0 = e_0$ we obtain $R(e_0) \leqslant e_0$. Thus $\Delta(\bar{e}_0) = e_0]$ which implies that $e = \bar{e}_0$, and thus $e \in \bar{S}$.

Combining Propositions 6.2 and 6.3, we obtain

COROLLARY 6.4. *A semigroup S is isomorphic to a subsemigroup $\bar{S}$ of $S(W)$ for some tree W, where $a \in \bar{S}$ implies $\Delta(a)$ is a principal ideal of W, and $\bar{S}$ contains all idempotents e such that $\Delta(e)$ is a principal ideal of W if and only if S is a tf-semigroup.*

A subsemigroup H of a semigroup S is said to be a *full idempotent subsemigroup* if $E(H) = E(S)$. Clearly, H is a full idempotent subsemigroup of S if $E(S) \subseteq H$. Of course, if $E(S)$ is a subsemigroup, then it is a full idempotent subsemigroup. For finite semigroups we now have the following special case of the above results:

COROLLARY 6.5. *Let S be a finite semigroup. S is a tf-semigroup if and only if S is isomorphic to a full idempotent subsemigroup of $S(W)$ for some finite tree W.*

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