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TWO OPEN PROBLEMS ON BALANCED BLOCK DESIGNS

SANPEI KAGEYAMA

Department of Mathematics, Hiroshima University
Hiroshima, Japan

We shall state two unsolved problems on balanced block designs, with their brief history of research, which may be interesting also from a combinatorial point of view. The basic designs dealt with here are variance-balanced block designs, in which every elementary treatment contrast is estimated with the same variance. The definition of this design along with a property of resolvability is provided in each section.

1. Symmetric variance-balanced block designs

Consider v treatments arranged in b blocks with the jth block being of size k_j ($j = 1, \ldots, b$) in a block design with the incidence matrix $N = ((n_{ij}))$ such that the ith treatment occurs r_i times ($i = 1, \ldots, v$) and the ith treatment occurs in the jth block n_{ij} times, where n_{ij} can take any of the values $0, 1, \ldots, p-1$. Such a design is called a *p-ary block design*. If $p = 2$, the design is said to be *binary*. When $r_1 = \ldots = r_v$ ($k_1 = \ldots = k_b$), the design is said to be *equireplicate* (*proper*).

A block design is said to be *variance-balanced* if every elementary contrast of treatments is estimated with the same variance (cf. Rao (1958)). Furthermore, by using the C-matrix of a block design, a p-ary variance-balanced block (VBB) design with parameters v ($\geqslant 2$), b (>0), r_i (>0) and k_j (>0) ($i = 1, \ldots, v$; $j = 1, \ldots, b$) can be characterized (cf. Kageyama (1974)) by an incidence matrix N satisfying

$$(C =) \quad R - NK^{-1}N' = \varrho\{I_v - (1/v)J_v\},$$

where $\varrho = \left(n - \sum_{j=1}^{b}\{(1/k_j)\sum_{i=1}^{v} n_{ij}^2\}\right)/(v-1)$ is the only nonzero eigenvalue of C, R is a diagonal matrix with diagonal elements $r_1, \ldots, r_v$, i.e., $R = \operatorname{diag}\{r_1, \ldots, r_v\}$, $K = \operatorname{diag}\{k_1, \ldots, k_b\}$, I_v is the identity matrix of order v, J_v is a $v \times v$ matrix with positive unit elements everywhere, A' is the transpose of the matrix A, and $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$. In particular, for a binary VBB design, $\varrho = (n-b)/(v-1)$.

The literature of block designs contains many articles exclusively related to VBB designs. Kageyama and Tsuji (1980) have shown that for a binary VBB design, barring orthogonal designs, the Fisher inequality $b \geqslant v$ holds. Saturated designs (i.e., with $b = v$) may be important in some statistical and/or combinatorial sense. It is known (Rao (1966)) that an equireplicate and binary VBB design with $b = v$ is a symmetric balanced incomplete block (BIB) design. Here a BIB design is usually defined by the $v \times b$ incidence matrix $N = ((n_{ij}))$ such that (i) $n_{ij} = 0$ or 1 for all i, j; (ii) $\sum_{j=1}^{b} n_{ij} = r$ for all i; (iii) $\sum_{i=1}^{v} n_{ij} = k$ for all j; and (iv) $\sum_{j=1}^{b} n_{ij} n_{i'j} = \lambda$ for all i, i' ($i \neq i' = 1, \ldots, v$). Furthermore, it is also known (cf. Kageyama (1974)) that a proper and binary VBB design is a BIB design. Thus, the existence of a VBB design with $b = v$ having unequal replication numbers r_i and unequal block sizes k_j should be investigated as the next problem. Since such a "non-binary"design exists (Kageyama (1984a)), the existence problem should be considered only for "binary" designs.

The rank of C $(= R - NK^{-1}N')$ is at most $v-1$. We consider the case where the rank of C is exactly $v-1$. In this case, the design is said to be *connected*. We shall deal only with connected designs throughout this paper. Though in general $\varrho \leqslant r_i$ for all i and $1 \leqslant k_j \leqslant v$ for all j, to exclude the triviality of our problem, we shall consider the present existence problem under

$$\varrho < r_i \text{ for all } i \text{ and } 1 < k_j < v \text{ for all } j.$$

Now, our problem is stated as follows.

PROBLEM 1. *Does there exist a nontrivial block design with parameters $v = b$ (symmetric), r_i ($r_i \neq r_{i'}$ for some i, i') and k_j ($1 < k_j < v$; $k_j \neq k_{j'}$ for some j, j') satisfying*

$$C = \varrho\{I_v - (1/v)J_v\}$$

and $\varrho < r_i$ for $i = 1, \ldots, v$ and $j = 1, \ldots, b$, where $\varrho = (n-v)/(v-1)$?

This problem is still unsolved until now since 1979. But there are some partially solved results, from which we may arrive at a conjecture on the nonexistence of such a VBB design.

It is known (Kageyama (1984a)) that each of the following is a sufficient condition for the nonexistence:

(1) $r_i - \varrho \geqslant 1$ for all i.
(2) $r_i - \varrho \leqslant 1$ for all i.
(3) $v \mid n$ (Hedayat (1981)).
(4) $v-1 \mid n-1$.
(5) v is a prime.
(6) $v-1$ is a prime.

(7) Let $v-1 = p^\alpha$ for a prime p and an integer $\alpha \geqslant 2$. Then $k_j \nmid p^\alpha$ for all j.
(8) $k_j | v$ for all j.

For other conditions, refer to Kageyama (1982, 1984a).

We can also characterize some structure of the incidence matrix of the VBB design, as Kageyama (1984a) shows. Through such observations, some restrictions on parameters r_i and k_j are made as follows:

$$3 \leqslant k_j \leqslant v-2, \quad j = 1, \dots, v;$$
$$4 \leqslant r_i \leqslant v-2, \quad i = 1, \dots, v;$$

in particular,

$$4 \leqslant \min r_i \leqslant v-4.$$

These restrictions are very useful to judge whether or not there exists a VBB design for a given v. In fact, through such an idea it can be shown that there does not exist a VBB design for $v \leqslant 14$.

All the approaches to derive the above-mentioned results are to take one of the following procedures:

(i) determinant,
(ii) trace,
(iii) elementwise comparison

for (some modification of) the relation

$$(C =) \quad R - NK^{-1}N' = \varrho\big(I_v - (1/v)J_v\big).$$

In conclusion, the author believes that the VBB design considered does not exist in general.

2. 1-Resolvable variance-balanced block designs

A block design is said to be $(\mu_1, \dots, \mu_t)$-*resolvable* if the blocks can be separated into t ($\geqslant 2$) sets of m_i ($\geqslant 1$) blocks each such that the set consisting of m_i blocks contains every treatment exactly μ_i ($\geqslant 1$) times ($i = 1, \dots, t$) (cf. Kageyama (1976)). Furthermore, when $\mu_1 = \dots = \mu_t$ ($= \mu$, say), it is said to be μ-*resolvable* for $\mu \geqslant 1$. A μ-resolvable block design is said to be *affine* μ-*resolvable* if any pair of blocks belonging to the same set contain q_1 treatments in common, whereas any pair of blocks belonging to different sets contain q_2 treatments in common. For a $(\mu_1, \dots, \mu_t)$-resolvable block design, it is clear that the design is equireplicate.

The importance of variance-balance and resolvability in the context of experimental planning is well known; the former yields optimal designs apart from ensuring simplicity in the analysis and the latter is helpful, among other respects, in the recovery of interblock information. Also practical situations sometimes demand designs with varying block sizes (Pearce (1964)) or resolvable designs with unequal replication numbers between sets of blocks; for a practical example, see Kageyama (1976). These considerations indicate the importance of $(\mu_1, \ldots, \mu_t)$-resolvable VBB designs with possibly varying block sizes and having $\mu_1, \ldots, \mu_t$ possibly not all equal.

The literature of block designs contains many articles exclusively related to VBB designs with resolvability. It is well known (cf. Kageyama (1973), Raghavarao (1962, 1971)) that for a μ-resolvable VBB design with parameters v, $b = \sum_{i=1}^{t} m_i$, $r = \mu t$, k_j, the inequality $b \geqslant v+t-1$ holds. Hughes and Piper (1976) showed that for a $(\mu_1, \ldots, \mu_t)$-resolvable BIB design, $b \geqslant v+t-1$ holds and that a $(\mu_1, \ldots, \mu_t)$-resolvable BIB design with $b = v+t-1$ is affine μ-resolvable. Generalizing these results, Kageyama (1984b) established that for a $(\mu_1, \ldots, \mu_t)$-resolvable VBB design with parameters v, $b = \sum_{i=1}^{t} m_i$, $r = \sum_{i=1}^{t} \mu_i$ and k_j $(j = 1, \ldots, b)$, the inequality $b \geqslant v+t-1$ holds. As a characterization of the saturated case of the above result, it is further shown that *in a* $(\mu_1, \ldots, \mu_t)$*-resolvable VBB design with* $b = v+t-1$, *except when* $\mu_1 = \ldots = \mu_t = 1$, *block sizes of blocks belonging to the same set are always equal.* Whether the above holds for the case $\mu_1 = \ldots = \mu_t = 1$ as well, is an open problem. That is,

PROBLEM 2. *Does there exist an incomplete block 1-resolvable VBB design with* $b = v+r-1$, *having unequal block sizes within a set?*

Remark. When $\mu_1 = \ldots = \mu_t = 1$, the design is 1-resolvable. For a 1-resolvable design $t = r$. Furthermore, it is clear that a 1-resolvable VBB design with $b = v+r-1$ is binary and even if there are some complete blocks, the design obtained by deleting these complete blocks will again be a 1-resolvable VBB design with the same property. Therefore, without loss of generality, attention will be restricted to incomplete block designs, so that each set involves at least two blocks, and the above problem will be considered.

This problem makes Mukerjee and Kageyama (1985) consider various characterizations of $(\mu_1, \ldots, \mu_t)$-resolvable VBB designs satisfying $b = v+t-1$. Regarding Problem 2, Mukerjee and Kageyama (1985) stated two equivalent problems in the context of fractional factorial plans as follows.

PROBLEM 2-1. *Does there exist a saturated proportional frequency plan for main effects with unequal replication numbers for the levels of at least one factor?*

PROBLEM 2-2. *Does there exist a saturated orthogonal main effect plan with unequal replication numbers for the levels of at least one factor?*

It should be clarified that "orthogonality" in the last problem is in the sense of Addelman (1963); note that there is another definition of orthogonality (Yamamoto, Shirakura and Kuwada (1975)), which is not being followed here.

Trivially, if v is a prime, then $k_j k_{j'}/v$ cannot be an integer, since incomplete block designs are being considered, and hence nonexistence follows. Also, the existing methods of construction of proportional frequency plans involve the technique of collapsing of levels (Addelman (1963)) and cannot lead to a plan as stated in Problem 2-1. Therefore, in order to find an example, if it exists, satisfying the conditions of Problem 2-1 or equivalently the other problems, one should look for a method for the construction of proportional frequency plans without applying the collapsing technique of Addelman. However, from various experience on combinatorics and statistics, our conjecture is that there does not exist a VBB design envisaged in Problem 2, or, equivalently, a fractional factorial plan as in Problems 2-1 and 2-2.

Usually, combinatorial problems on discrete designs can be tackled by use of number-theoretic approach and/or combinatorial approach including group- or coding-theoretic approach. But sometimes the problems are very deep, even though their representation is simple. It seems that the problems presented here are just such cases.

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