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ON SYMMETRIC FUNCTIONS AND THE SPIN CHARACTERS OF S_n

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0. Introduction

There has been a recent surge of interest in the projective representations of symmetric groups and their associated combinatorial and algebraic structure. The study of these representations began with Schur, who published degree and character formulas in a paper written in 1911 [S]. Unlike the ordinary, linear representations of symmetric groups, there was no subsequent development of machinery to construct, manipulate and study these representations. Indeed, it was not until the 1960's that a series of papers by Morris began to address these problems (see [Mo1–3] and the references cited there).

One of the impediments which prevented this development was the lack, at least initially, of a suitable combinatorial structure analogous to Young tableaux. This difficulty has since been overcome by the advent of *shifted* tableaux. These tableaux first appeared in a 1952 paper by Thrall [T], although it is unclear (but conceivable) that Thrall knew of their representation-theoretic significance. There have since been many advances in the study of projective representations of S_n, including the construction of: a projective analogue of the Murnaghan–Nakayama character recurrence [Mo2]; a version of the Robinson–Schensted–Knuth correspondence for shifted tableaux [W], [Sa]; a Frobenius-type characteristic map [J], [St]; explicit bases for the irreducible representations [N]; a projective analogue of induction from Young subgroups [J], [HH], [St], and a corresponding analogue of the Littlewood–Richardson rule [St].

The purpose of this paper is twofold. Primarily, it is to survey the interconnections between projective (or spin) characters of S_n, shifted tableaux, and the theory of symmetric functions. This survey occupies Sections 1–4. For proofs of most of the results, the reader will be referred to the literature. The remainder of this paper is devoted to providing new proofs of some known

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results in order to illustrate the use of combinatorial methods. Specifically, in § 5 we provide a new proof of the Morris character recurrence. The novelty of this proof lies in its use of shifted tableaux where previous proofs have employed the machinery of Hall–Littlewood functions. In § 6, we give a purely combinatorial proof that Schur's so-called Q-functions are symmetric. This fact is self-evident in Schur's definition, but not so evident if one defines them as tableaux generating functions. The analogous problem for Schur's S-functions was solved by Bender and Knuth [BK].

Conventions

All representations considered here will use the complex field.

The notation $\langle\ ,\ \rangle_G$ is used for the standard Hermitian inner product on the complex vector space spanned by the characters of the (finite) group G.

Let P denote the set of partitions; i.e., nonnegative integer sequences $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots)$. The length $l(\lambda)$ is the number of nonzero parts, and $|\lambda|$ denotes the sum of the parts. We write $\mu \subseteq \lambda$ if $\mu_1 \leqslant \lambda_1, \mu_2 \leqslant \lambda_2, \dots$. If $|\lambda| = n$, z_λ denotes the size of the S_n-centralizer of any permutation of cycle-type λ, as in [Ma].

Let OP denote the set of partitions whose (positive) parts are all odd, and let DP denote the set of all partitions whose (positive) parts are distinct. If X is any of the sets OP, DP or P, we write X_n to denote the subset of partitions of n (i.e., $|\lambda| = n$), and we write X^+ or X^- for the subset consisting of those λ for which $|\lambda| - l(\lambda)$ is even or odd, respectively.

1. Spin characters

A projective representation of a group G on a vector space V is a homomorphism $G \to \mathrm{PGL}(V)$, or equivalently, a map P: $G \to \mathrm{GL}(V)$ such that

$$P(x)P(y) = c_{x,y}P(xy) \qquad (x, y \in G)$$

for suitable (nonzero) scalars $c_{x,y}$. For the symmetric group, the associated Coxeter presentation shows that a representation $S_n \to \mathrm{PGL}(V)$ amounts to a collection of linear transformations $\sigma_1, \dots, \sigma_{n-1} \in \mathrm{GL}(V)$ (representing the adjacent transpositions) such that σ_j^2, $(\sigma_j\sigma_{j+1})^3$, and $(\sigma_j\sigma_k)^2$ (for $|j-k| \geqslant 2$) are all scalars. The possible scalars that arise in this fashion are limited. Of course, one possibility is that the scalars are trivial; this occurs in any ordinary, linear representation of S_n. According to a result of Schur [S], there is only one other possibility (occurring only when $n \geqslant 4$); namely,

(1.1) $$\sigma_j^2 = -1; \quad (\sigma_j\sigma_k)^2 = -1 \quad |j-k| \geqslant 2; \quad (\sigma_j\sigma_{j+1})^3 = -1.$$

All other possibilities can be reduced to this case or the trivial case by a change of scale. See [J], [St] for details.

It is convenient to regard $\sigma_1, \ldots, \sigma_{n-1}$ as elements of an abstract group, and to take (1.1) as a set of defining relations. More precisely, for $n \geqslant 1$ let us define $\tilde{S}_n$ to be the group of order $2 \cdot n!$ generated by $\sigma_1, \ldots, \sigma_{n-1}$ (and -1), subject to the relations (1.1), along with the obvious relations $(-1)^2 = 1$, $(-1)\sigma_j = \sigma_j(-1)$ which force -1 to be a central involution. By Schur's Lemma, an irreducible linear representation of $\tilde{S}_n$ must represent -1 by either of the scalars $+1$ or -1. A representation of the former type is a linear representation of S_n, whereas one of the latter type corresponds to a projective representation of S_n as in (1.1). In general, we will refer to any representation of $\tilde{S}_n$ in which the group element -1 is represented by the scalar -1 as a *spin representation* of $\tilde{S}_n$.

A fundamental construction of spin representations can be obtained from the linear representations of Clifford algebras. Let C_n denote the (complex) Clifford algebra of dimension 2^n generated by $\xi_1, \ldots, \xi_n$, subject to the relations

$$\xi_j^2 = -1; \qquad \xi_j \xi_k = -\xi_k \xi_j \qquad (k \neq j). \tag{1.2}$$

Given any linear representation $C_{n-1} \to \mathrm{End}(V)$, we may regard $\xi_1, \ldots, \xi_{n-1} \in \mathrm{GL}(V)$ and construct a spin representation of $\tilde{S}_n$ via the assignments

$$\sigma_j = \sqrt{\frac{j+1}{2j}}\,\xi_j - \sqrt{\frac{j-1}{2j}}\,\xi_{j-1} \qquad (1 \leqslant j < n).$$

The relations (1.1) follow directly from (1.2). Since the images of $\sigma_1, \ldots, \sigma_{n-1}$ in C_{n-1} generate the full Clifford algebra, it follows that any spin representation of $\tilde{S}_n$ constructed from an irreducible representation $C_{n-1} \to \mathrm{End}(V)$ will also be irreducible. In case $n = 2k+1$, one knows that C_{n-1} is simple and we have $C_{n-1} \cong M(2^k)$, the matrix algebra of order 2^k. Thus $\tilde{S}_{2k+1}$ has an irreducible spin representation of dimension 2^k. In case $n = 2k$, one knows $C_{n-1} \cong M(2^{k-1}) \oplus M(2^{k-1})$, so $\tilde{S}_{2k}$ has two irreducible spin representations of dimension 2^{k-1}. These representations of $\tilde{S}_{2k+1}$ and $\tilde{S}_{2k}$ are known as the *basic spin representations*. See [Mo1], [J], [St] for further details.

To describe the characters of spin representations it is convenient to take advantage of the structure of the conjugacy classes of $\tilde{S}_n$. For each partition μ of n, choose an element $\sigma_\mu \in \tilde{S}_n$ whose S_n-image is of cycle-type μ. Eventually we will be very particular about the choice of σ_μ, but for the moment we merely note that since $\{\pm 1\}$ is the kernel of the epimorphism $\tilde{S}_n \to S_n$, it follows that every $\sigma \in \tilde{S}_n$ is conjugate to either σ_μ or $-\sigma_\mu$ for some μ.

THEOREM 1.1 (Schur). *The elements σ_μ and $-\sigma_\mu$ are not conjugate in $\tilde{S}_n$ iff either* (1) *the parts of μ are all odd* (*i.e.*, $\mu \in \mathrm{OP}_n$), *or* (2) *the parts of μ are distinct and $n - l(\mu)$ is odd* (*i.e.*, $\mu \in \mathrm{DP}_n^-$).

COROLLARY 1.2. *The elements* $\{\sigma_\mu\colon \mu\in \mathrm{P}_n\}\cup\{-\sigma_\mu\colon \mu\in \mathrm{OP}_n\cup \mathrm{DP}_n^-\}$ *form a system of representatives of the distinct conjugacy classes of* $\tilde{S}_n$.

Proofs may be found in [J], [S], [St].

If φ is the character of a spin representation of $\tilde{S}_n$, then $\varphi(-\sigma) = -\varphi(\sigma)$ for all $\sigma\in\tilde{S}_n$. Consequently, $\varphi(\sigma)$ vanishes on classes in which σ and $-\sigma$ are conjugate, and hence, φ is completely determined by the traces $\varphi(\sigma_\mu)$ for $\mu\in \mathrm{OP}_n\cup \mathrm{DP}_n^-$.

Let ε denote the sign character of $\tilde{S}_n$, and let $\tilde{A}_n=\ker\varepsilon$ denote the subgroup of $\tilde{S}_n$ that doubly covers the alternating group A_n. It is convenient to divide the irreducible spin characters of $\tilde{S}_n$ into two classes: those which remain irreducible as $\tilde{A}_n$-characters, and those which split into two irreducible $\tilde{A}_n$-characters. By the standard techniques of Clifford Theory, the latter consist of those characters φ with $\varphi=\varepsilon\varphi$, and are said to be *self-associate*. The former characters occur in pairs $\varphi_\pm$ with $\varphi_-=\varepsilon\varphi_+$, and are said to be *associate* characters.

As further application of Theorem 1.1, we have

COROLLARY 1.3 (cf. [St, § 4], [J, § 3]). *The number of irreducible self-associate (resp., pairs of associate) spin characters of* $\tilde{S}_n$ *is* $|\mathrm{DP}_n^+|$ *(resp.,* $|\mathrm{DP}_n^-|$).

Proof. The space of $\tilde{S}_n$-class functions is of the form $Z_n\oplus Z'_n$, where Z_n and Z'_n denote the subspaces spanned by ordinary and spin characters, respectively. Moreover, we have $Z'_n=Z_n^+\oplus Z_n^-$, where Z_n^+ and Z_n^- denote the subspaces of Z'_n spanned by class functions supported on the even and odd-signed classes of $\tilde{S}_n$, respectively. By Theorem 1.1, we have $\dim Z_n^+=|\mathrm{OP}_n|$ and $\dim Z_n^-=|\mathrm{DP}_n^-|$.

If φ is a self-associate spin character, then $\varphi\in Z_n^+$, whereas if $\varphi_\pm$ is a pair of associate spin characters, then $\varphi_+-\varphi_-\in Z_n^-$ and $\varphi_++\varphi_-\in Z_n^+$. Hence, $\dim Z_n^-$ is the number of associate pairs, and

$$\dim Z_n^+-\dim Z_n^- = |\mathrm{OP}_n|-|\mathrm{DP}_n^-| = |\mathrm{DP}_n^+|$$

is the number of self-associate spin characters, using the well-known fact that $|\mathrm{OP}_n|=|\mathrm{DP}_n|$. ■

We remark that since any pair of associate spin characters differ only by a linear character (namely ε), the corresponding representations are isomorphic in the projective category. Corollary 1.3 therefore shows that the irreducible projective representations of S_n are in one-to-one correspondence with partitions of n with distinct parts.

To explicitly describe particular spin characters, we need to make specific choices for the representatives $\sigma_\mu\in\tilde{S}_n$; subsequent formulae would otherwise only be well-defined up to sign. We therefore define

$$\sigma_\mu=\pi_1\pi_2\ldots\pi_l,$$

where $\pi_j = \sigma_{a+1}\sigma_{a+2}\ldots\sigma_{a+\mu_j-1}$ $(a = \mu_1 + \ldots + \mu_{j-1})$ for $1 \leqslant j \leqslant l = l(\mu)$. For example, if $\mu = (5, 3, 1)$, then $\sigma_\mu = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_6\sigma_7$ and the image of σ_μ in S_9 is the permutation (12345) (678) (9).

With these choices in mind, let us describe the characters of the basic spin representations of $\tilde{S}_n$. We use the notations φ^n $(n = 2k+1)$ or $\varphi^n_\pm$ $(n = 2k)$ for these characters, although the subscript $\pm$ will be dropped whenever the choice of signs is irrelevant. By a straightforward trace calculation, one finds

$$\varphi^n(\sigma_\mu) = 2^{\lfloor (l(\mu)-1)/2 \rfloor} \qquad (\mu \in \mathrm{OP}_n), \tag{1.3}$$

and in case $n = 2k$, $\mu = (2k)$, one finds

$$\varphi^{2k}_\pm(\sigma_\mu) = \pm i^k \sqrt{k}.$$

In all other cases, $\varphi^n(\sigma_\mu) = 0$. In particular, φ^{2k+1} is self-associate and $\varphi^{2k}_\pm$ is a pair of associates. See [Mo1], [J], [S], or [St] for details.

Next we consider an analogue of the outer tensor product for spin characters. The construction we describe below appears in [St], and in an equivalent form in [HH].

Let $\tilde{S}_{(k,n-k)}$ denote the subgroup of $\tilde{S}_n$ generated by $\{\sigma_j : j \neq k\}$ (a double cover of the Young subgroup $S_k \times S_{n-k}$), and let ϱ_1: $\tilde{S}_k \to \mathrm{GL}(V_1)$ and ϱ_2: $\tilde{S}_{n-k} \to \mathrm{GL}(V_2)$ be spin representations, regarding $\tilde{S}_{n-k}$ as the subgroup generated by $\{\sigma_j : k < j < n\}$. It will be convenient to assume below that ϱ_1 and ϱ_2 are irreducible, although this is not strictly necessary. The *spin product* $\varrho_1 \hat{\otimes} \varrho_2$ is a spin representation of $\tilde{S}_{(k,n-k)}$ defined as follows:

CASE 1. If neither ϱ_1 nor ϱ_2 is self-associate, the spin product $\varrho_1 \hat{\otimes} \varrho_2$ represents the generators σ_j $(j \neq k)$ on $\mathrm{GL}(\mathbb{C}^2 \otimes V_1 \otimes V_2)$ as follows:

$$\sigma_j \mapsto \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \varrho_1(\sigma_j) \otimes 1_{V_2} & (1 \leqslant j < k) \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes 1_{V_1} \otimes \varrho_2(\sigma_j) & (k < j < n). \end{cases}$$

Note that $\varrho_1 \hat{\otimes} \varrho_2$ is self-associate.

CASE 2. Assume only one of ϱ_1 or ϱ_2 is self-associate; say ϱ_1. In that case, $\varrho_1 \cong \varepsilon \otimes \varrho_1$, so there must exist $S \in \mathrm{GL}(V_1)$ such that $S^2 = 1$ and

$$S\varrho_1(\sigma_j) = -\varrho_1(\sigma_j)S \qquad (1 \leqslant j < k). \tag{1.4}$$

If ϱ_1 is irreducible, Schur's Lemma implies that $\pm S$ is unique. There are two possible spin products $(\varrho_1 \hat{\otimes} \varrho_2)_\pm$, both defined on $\mathrm{GL}(V_1 \otimes V_2)$, arising from the choice of $+S$ or $-S$:

$$\sigma_j \mapsto \begin{cases} \varrho_1(\sigma_j) \otimes 1_{V_2} & (1 \leqslant j < k) \\ \pm S \otimes \varrho_2(\sigma_j) & (k < j < n). \end{cases} \tag{1.5}$$

In this case, it can be verified that the representations $(\varrho_1 \hat{\otimes} \varrho_2)_{\pm}$ form an associate pair [St]. As above, we will omit the subscript $\pm$ whenever the choice of sign is irrelevant.

CASE 3. If both ϱ_1 and ϱ_2 are self-associate, let $S \in \mathrm{GL}(V_1)$ be the involution as in (1.4). In this case, we define the spin product $\varrho_1 \hat{\otimes} \varrho_2$ on $\mathrm{GL}(V_1 \otimes V_2)$ exactly as in (1.5), but one finds that the same spin representation is produced, regardless of the choice of sign (up to isomorphism). Furthermore, it is easy to check that $\varrho_1 \hat{\otimes} \varrho_2$ is self-associate in this case.

Strictly speaking, the spin product is a multi-valued operation (cf. Case 2); modulo this qualification it is commutative (up to isomorphism). There is also a natural way to define multiple products $\varrho_1 \hat{\otimes} \dots \hat{\otimes} \varrho_l$ so that $\hat{\otimes}$ is associative [St]. Finally, we mention the following analogue of a well-known property of outer tensor products:

THEOREM 1.4 ([HH], [St]). *If ϱ_1 and ϱ_2 are irreducible, then so is $\varrho_1 \hat{\otimes} \varrho_2$. Conversely, every irreducible spin representation of $\tilde{S}_{(k,n-k)}$ is of this form.*

2. Symmetric functions

Let $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$ denote the graded algebra of symmetric functions in the variables $x_1, x_2, \dots$ with integer coefficients [Ma]. If F is a field of characteristic 0, let $\Lambda_F = \Lambda \otimes_Z F$ denote the corresponding (graded) F-algebra. There are five fundamental bases for Λ (or Λ_Q) indexed by partitions; the ones we will require here are: the monomial basis m_λ, the power-sum basis p_λ, and the Schur function basis s_λ. For definitions, see [Ma].

Let $\Omega_F = \bigoplus_{n \geq 0} \Omega_F^n$ denote the graded subalgebra of Λ_F generated by 1, $p_1, p_3, p_5, \dots$, and let $\Omega = \Lambda \cap \Omega_F$ denote the integer-coefficient (graded) subring. Note that $\{p_\mu \colon \mu \in \mathrm{OP}_n\}$ is a basis of Ω_Q^n. In particular, $\dim_Q \Omega_Q^n = |\mathrm{OP}_n| = \dim_C Z_n^+$ (cf. the proof of Cor. 1.3). Define an inner product $[\ ,\ ]$ on Ω_C^n by setting

$$[p_\mu, p_\nu] = z_\mu 2^{-l(\mu)} \delta_{\mu\nu} \qquad (\mu, \nu \in \mathrm{OP}). \tag{2.1}$$

The inner product spaces Ω_C^n and Z_n' may be connected by means of a pseudo-isometry $\mathrm{ch}' \colon Z_n' \to \Omega_C^n$ defined as follows:

$$\mathrm{ch}'(f) = \sum_{\mu \in \mathrm{OP}_n} \frac{1}{z_\mu} 2^{l(\mu)/2} f(\sigma_\mu) p_\mu. \tag{2.2}$$

We refer to ch′ as the *spin characteristic*. Note that ch′ is not injective; we have $\ker(\mathrm{ch}') = Z_n^-$, so that $\mathrm{ch}'(f) = 0$ iff $f \downarrow \tilde{A}_n = 0$. This apparent loss of information is not catastrophic since the character table of $\tilde{S}_n$ is very sparse on the odd-signed conjugacy classes (see Theorem 4.1(b)).

For any $f \in Z_n'$, let f_0 denote the $\tilde{S}_n$-class function for which $f_0(\sigma) = f(\sigma)$ $(\sigma \in \tilde{A}_n)$ and $f_0(\sigma) = 0$ $(\sigma \notin \tilde{A}_n)$. Although the spin characteristic is not quite an isometry, a direct application of (2.1) and (2.2) implies

$$[\mathrm{ch}'f, \mathrm{ch}'g] = \langle f_0, g_0\rangle_{\tilde{S}_n} = \tfrac{1}{2}\langle f, g\rangle_{\tilde{A}_n} \tag{2.3}$$

for any $f, g \in Z'_n$. In particular, we do have $[\mathrm{ch}'f, \mathrm{ch}'g] = \langle f, g\rangle_{\tilde{S}_n}$ whenever fg vanishes outside of $\tilde{A}_n$; e.g., whenever f or $g \in Z_n^+$.

By specializing identity (4.1) of [Ma, III] to the case $t = -1$, we obtain

$$\prod_{i,j} \frac{1+x_i y_j}{1-x_i y_j} = \sum_{\mu \in \mathrm{OP}} \frac{1}{z_\mu} 2^{l(\mu)} p_\mu(x) p_\mu(y). \tag{2.4}$$

This identity encodes the structure of the inner product [,], as we shall see in the sequel.

Another set of generators for Ω_Q can be obtained from the symmetric functions q_n defined by the generating function (cf. [Ma, III.3] with $t = -1$)

$$\prod_i \frac{1+x_i t}{1-x_i t} = \sum_{n \geqslant 0} q_n(x) t^n.$$

By specializing (2.4) to the case $y_2 = y_3 = \ldots = 0$, $y_1 = t$, we see that

$$q_n = \sum_{\mu \in \mathrm{OP}_n} \frac{1}{z_\mu} 2^{l(\mu)} p_\mu, \tag{2.5}$$

from which it follows that $q_n \in \Omega^n$. A simple induction can then be used to establish that $q_1, q_3, q_5, \ldots$ are algebraically independent generators (with 1) of Ω_Q. Furthermore, we note that (1.3) and (2.5) imply

$$\mathrm{ch}'\varphi^n = \begin{cases} \dfrac{1}{2} q_n & (n = 2k) \\[2ex] \dfrac{1}{\sqrt{2}} q_n & (n = 2k+1). \end{cases} \tag{2.6}$$

For any partition λ, let $q_\lambda = q_{\lambda_1} q_{\lambda_2} \cdots$ Since the coefficient of $m_\lambda(y)$ in

$$\prod_{i,j} \frac{1+x_i y_j}{1-x_i y_j} = \Big(\sum_{n \geqslant 0} q_n(x) y_1^n\Big)\Big(\sum_{n \geqslant 0} q_n(x) y_2^n\Big) \cdots \; .$$

is clearly $q_\lambda(x)$, it follows that

$$\prod_{i,j} \frac{1+x_i y_j}{1-x_i y_j} = \sum_\lambda q_\lambda(x) m_\lambda(y). \tag{2.7}$$

By comparing this identity with (2.4), we find that for any $f \in \Omega_Q$,

$$f = \sum_{\mu \in \mathrm{OP}} \left[f, \frac{1}{z_\mu} 2^{l(\mu)} p_\mu \right] p_\mu = \sum_\lambda [f, q_\lambda] m_\lambda.$$

In other words, we have proved

PROPOSITION 2.1. *If $f \in \Omega_C$, then $[f, q_\lambda]$ is the coefficient of m_λ in f.*

Using the spin product of § 1, we can define a graded multiplicative structure on $\bigoplus_{n \geqslant 0} Z'_n$ that approximates (via ch′) the multiplication in Ω. Specifically, given irreducible spin characters $\varphi_1 \in Z'_k$, $\varphi_2 \in Z'_{n-k}$, the induced character(s) $(\varphi_1 \hat{\otimes} \varphi_2) \uparrow \tilde{S}_n$ provide a (multi-valued) bilinear operation of the form

$$Z'_k \otimes Z'_{n-k} \to Z'_n.$$

Considering the spin characteristic of this operation, we find

THEOREM 2.2 ([St, § 5]). *If φ_1 and φ_2 are irreducible spin characters of $\tilde{S}_k$ and $\tilde{S}_{n-k}$, then*

$$\mathrm{ch}'((\varphi_1 \hat{\otimes} \varphi_2) \uparrow \tilde{S}_n) = \begin{cases} 2\mathrm{ch}'\varphi_1 \, \mathrm{ch}'\varphi_2 & \text{if } \varphi_1, \varphi_2 \text{ are not self-associate,} \\ \mathrm{ch}'\varphi_1 \, \mathrm{ch}'\varphi_2 & \text{otherwise.} \end{cases}$$

3. Shifted tableaux

For each partition λ with distinct parts there is an associated *shifted diagram* defined via

$$D'_\lambda = \{(i, j) \in \mathbf{Z}^2 \colon \ i \leqslant j < \lambda_i + i - 1, \ 1 \leqslant i \leqslant l(\lambda)\}.$$

We regard the elements of D'_λ as a collection of boxes in the plane with matrix-style coordinates.

Let $\mathbf{P}'$ denote the ordered alphabet $\{1' < 1 < 2' < 2 < 3' < \ldots\}$. The letters $1', 2', 3', \ldots$ are said to be *marked*. We use the notation $|a|$ to refer to the unmarked version of any $a \in \mathbf{P}'$; e.g., $|2'| = |2| = 2$.

A *shifted tableau* of shape λ is an assignment $T \colon D'_\lambda \to \mathbf{P}'$ such that:

(R1) $T(i, j) \leqslant T(i+1, j)$, $T(i, j) \leqslant T(i, j+1)$.

(R2) At most one k appears in each column ($k = 1, 2, 3, \ldots$).

(R3) At most one k' appears in each row ($k' = 1', 2', 3', \ldots$).

An example is given in Fig. 1.

1	1	1	2′	2	4′
	2′	2	2	5′	
		4	4	5	
			5′		

Fig. 1

Note that if $|T(i, j)| = |T(i+1, j)|$ then (R1) and (R2) will force $T(i, j)$ to be marked; if $|T(i, j-1)| = |T(i, j)|$ then (R1) and (R3) will force $T(i, j)$ to be unmarked. In either of these cases, we say that (i, j) is *forced* in T. Otherwise, if $|T(i, j-1)| < |T(i, j)| < |T(i+1, j)|$ (where defined), then $T(i, j)$ can be marked or unmarked, regardless of the markings of the entries near (i, j). In this case, we say that (i, j) is *free* in T, and let $\mathrm{fr}(T)$ denote the number of free entries in T. In Fig. 1, the (1, 6)-entry and the (3, 5)-entry, as well as the entries on the main diagonal, are free.

A shifted tableau T is said to have *content* $\gamma = (\gamma_1, \gamma_2, \ldots)$, where γ_k denotes the number of boxes $(i, j) \in D'_\lambda$ with $|T(i, j)| = k$. We will write $x^T = x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \ldots$. The example in Fig. 1 has content (3, 5, 0, 3, 3). For each $\lambda \in \mathrm{DP}$, let $Q_\lambda(x)$ denote the content-generating function for tableaux of shape λ; i.e.,

(3.1) $$Q_\lambda(x) = \sum_{T: D'_\lambda \to \mathbf{P}'} x^T,$$

summed over shifted tableaux T.

Let $\mathbf{P}$ denote the ordered alphabet $\{1 < 2 < 3 < \ldots\}$. An assignment T: $D'_\lambda \to \mathbf{P}$ of unmarked letters is said to be *feasible* if it is possible to create a shifted tableau by marking some of the entries of T. We note that T is feasible iff T satisfies rule (R1) as well as

(R4) $$T(i, j) < T(i+1, j+1);$$

ie., T must have increasing diagonals. Since the free entries can be marked independently and arbitrarily, it follows that $2^{\mathrm{fr}(T)}$ shifted tableaux can be obtained by marking a feasible T. Hence, (3.1) can also be written in the form

(3.2) $$Q_\lambda(x) = \sum_{T: D'_\lambda \to \mathbf{P}} 2^{\mathrm{fr}(T)} x^T,$$

summed over feasible T.

Since the main diagonal of a feasible T is always free, it follows that $\mathrm{fr}(T) \geqslant l(\lambda)$. Therefore,

$$P_\lambda(x) := 2^{-l(\lambda)} Q_\lambda(x)$$

has integer coefficients and may be interpreted as the generating function for shifted tableaux with unmarked main diagonals.

The generating functions $Q_\lambda(x)$ coincide with the Hall–Littlewood symmetric functions $Q_\lambda(x; t)$ when t is specialized to -1, although to prove this coincidence is nontrivial (compare (3.2) above with identity (5.11) of [Ma, III]). We also remark that the $Q_\lambda(x)$'s coincide with certain symmetric polynomials defined by Schur (also denoted Q_λ) as the pfaffians of certain skew-symmetric matrices [S] (see also [J]).

The fact that Q_λ is a specialization of a Hall–Littlewood function has many consequences. In particular, Q_λ must be a symmetric function(¹), and so we have the expansion

$$Q_\lambda = \sum_\mu K'_{\lambda\mu} m_\mu,$$

(¹) We give a purely combinatorial proof of this fact in § 6.

where $K'_{\lambda\mu}$ denotes the number of shifted tableaux of shape λ and content μ. A further consequence is the identity [Ma, III.4]

$$\prod_{i,j} \frac{1+x_i y_j}{1-x_i y_j} = \sum_{\lambda \in \mathbf{DP}} P_\lambda(x) Q_\lambda(y), \tag{3.3}$$

from which we obtain

PROPOSITION 3.1 (cf. [St, § 6]). (a) $\{Q_\lambda : \lambda \in \mathrm{DP}_n\}$ *is a basis of* Ω^n_Q.
(b) $[P_\lambda, Q_\mu] = \delta_{\lambda\mu}$.
(c) $q_\mu = \sum_{\lambda \in \mathbf{DP}} K'_{\lambda\mu} P_\lambda$.

Proof. Comparing (2.7) and (3.3), we find

$$\sum_{\mu \in \mathbf{P}_n} q_\mu(x) m_\mu(y) = \sum_{\lambda \in \mathbf{DP}_n} P_\lambda(x) Q_\lambda(y),$$

which yields (c) after the coefficient of y^μ is extracted. Since $\{q_\mu : \mu \in \mathrm{OP}_n\}$ spans Ω^n_Q and $|\mathrm{OP}_n| = |\mathrm{DP}_n|$, it follows that (c) may be inverted over $\mathbf{Q}$, thus expressing P_λ or Q_λ as a linear combination of q_μ's. This proves (a). One may prove (b) from (3.3) by the same technique used to prove (4.6) in [Ma, I]. ∎

We remark that (3.3) has been given a purely combinatorial proof by Worley [W] and Sagan [Sa], starting from the definition in (3.1).

Since the P_λ's form a **Z**-basis of Ω [St, § 6], we may define integers $f^\lambda_{\mu\nu}$ via the expansion

$$P_\mu P_\nu = \sum_{\lambda \in \mathbf{DP}} f^\lambda_{\mu\nu} P_\lambda.$$

There is an explicit combinatorial interpretation of these coefficients analogous to the Littlewood–Richardson rule for the multiplication of Schur functions. To explain this interpretation, we need to introduce skew shifted tableaux and a shifted version of the lattice permutation property (cf. the classical *LR* rule in [Ma]).

A *skew* shifted diagram is an array of boxes of the form $D'_{\lambda/\mu} := D'_\lambda - D'_\mu$ for any $\lambda, \mu \in \mathrm{DP}$ with $\lambda \supseteq \mu$. A shifted tableau of shape λ/μ is defined to be an assignment $T: D'_{\lambda/\mu} \to \mathbf{P}'$ satisfying the usual rules (R1–3). Let

$$Q_{\lambda/\mu} = \sum_{T: D'_{\lambda/\mu} \to \mathbf{P}'} x^T \tag{3.4}$$

denote the associated generating function, and set $Q_{\lambda/\mu} = 0$ if $\lambda \not\supseteq \mu$. By [Ma, III.5], one knows that

$$[Q_{\lambda/\mu}, P_\nu] = [Q_\lambda, P_\mu P_\nu] = f^\lambda_{\mu\nu}, \tag{3.5}$$

so the coefficients $f^\lambda_{\mu\nu}$ also appear in the expansion

$$Q_{\lambda/\mu} = \sum_\nu f^\lambda_{\mu\nu} Q_\nu.$$

In particular, note that $f^\lambda_{\mu\nu} = f^\lambda_{\nu\mu}$ and $f^\lambda_{\mu\nu} = 0$ unless $\mu, \nu \subseteq \lambda$.

Let $w = w_1 \dots w_n$ be any word over the alphabet $\mathbf{P}'$. Let $w^r = w_n \dots w_1$ denote the reverse of w, and let $\hat{w} = \hat{w}_1 \dots \hat{w}_n$ denote the the word obtained by inverting the marks of w, so that $\hat{2} = 2'$ and $\hat{2}' = 2$. Let $n_i(w, j)$ denote the number of occurrence of the letter i (not i') among $w_1 \dots w_j$, with the convention $n_i(w, 0) = 0$. The *word* of a shifted tableau T is the sequence $w(T)$ obtained by reading the successive rows of T from left to right, starting with the bottom row. For example, the word of the first tableau in Fig. 2 is $3123'1^*2'211$. (The notation 1^* serves a later purpose – it indicates that either 1 or $1'$ is allowed.)

The *extended word* of T is the sequence defined by $e(T) = w^r \hat{w}$, where $w = w(T)$. The tableau T is said to satisfy the *shifted lattice property* if the extended word $e = e_1 \dots e_{2n}$ satisfies the following condition for all $i > 1$ and $0 \leqslant j < 2n$:

$$\text{(SLP)} \qquad n_i(e, j) = n_{i-1}(e, j) \quad \text{implies} \quad \begin{cases} e_{j+1} \neq i, i' & (0 \leqslant j < n) \\ e_{j+1} \neq i, (i-1)' & (n \leqslant j < 2n), \end{cases}$$

For example, each of the tableaux in Fig. 2 satisfy (SLP), independently of the markes chosen for the *'s.

THEOREM 3.2 [St, § 8]. *The coefficient $f^{\lambda}_{\mu\nu}$ is the number of shifted tableaux T of shape λ/μ and content ν such that* (1) *T satisfies* (SLP) *and* (2) *the leftmost i of $|w(T)|$ is unmarked in $w(T)$ $(1 \leqslant i \leqslant l(\nu))$.*

For example, consider $\lambda = (7, 5, 3, 1)$, $\mu = (5, 2)$, $\nu = (4, 3, 2)$. One finds that there are six tableaux satisfying the above properties (see Fig. 2), so we conclude that $f^{\lambda}_{\mu\nu} = 6$.

			1	1					1	1					1	1
	1*	2′	2				1	2	2				1	1	2	
1	2	3′				1	2	3*				2	2	3*		
	3						3						3			

Fig. 2

The representation-theoretic significance of this result will be discussed in the next section.

4. The irreducible spin characters

The Q-functions play a fundamental role in Schur's description of the irreducible spin characters of $\tilde{S}_n$. Recall (Cor. 1.3) that these characters are indexed by $\lambda \in \mathrm{DP}_n$. Let φ^{λ} $(\lambda \in \mathrm{DP}_n^+)$ denote the self-associate spin characters, and let $\varphi^{\lambda}_{\pm}$ $(\lambda \in \mathrm{DP}_n^-)$ denote the pairs of associate spin characters. As usual, the subscript $\pm$ will be dropped whenever the choice is irrelevant. Let us also introduce the notation

$$\varepsilon_\lambda = \begin{cases} \sqrt{2} & \text{if } |\lambda| - l(\lambda) \text{ is odd}, \\ 1 & \text{if } |\lambda| - l(\lambda) \text{ is even}. \end{cases}$$

THEOREM 4.1 (Schur). (a) *For any* $\lambda \in \mathrm{DP}_n$, *we have* $\varepsilon_\lambda \mathrm{ch}' \varphi^\lambda = 2^{-l(\lambda)/2} Q_\lambda$; *i.e.*,

$$\frac{1}{\varepsilon_\lambda} 2^{-l(\lambda)/2} Q_\lambda = \sum_{\mu \in \mathrm{OP}_n} \frac{1}{z_\mu} 2^{l(\mu)/2} \varphi^\lambda(\sigma_\mu) p_\mu. \tag{4.1}$$

The above expansion determines the characters on the even-signed conjugacy classes. For the odd-signed classes, the situation is much simpler.

(b) *If* $\lambda, \mu \in \mathrm{DP}_n^-$, *then* $\varphi_\pm^\lambda(\sigma_\mu) = 0$ *unless* $\lambda = \mu$. *In that case, if* $l = l(\lambda)$, *then*

$$\varphi_\pm^\lambda(\sigma_\lambda) = \pm i^{(n-l+1)/2} \sqrt{\frac{1}{2} \lambda_1 \ldots \lambda_l}.$$

Sketch of proof. Take the above formulas as definitions of certain $\tilde{S}_n$-class functions φ^λ. There are three basic steps in Schur's proof; we discuss these below, and briefly indicate how they can be proved with recently developed combinatorial techniques. For details of the latter, see [St]; for a treatment more faithful to Schur's original techniques, see [J].

STEP 1. *Show that* $\{\varphi^\lambda : \lambda \in \mathrm{DP}_n^+\} \cup \{\varphi_\pm^\lambda : \lambda \in \mathrm{DP}_n^-\}$ *forms an orthonormal basis of* Z_n'.

Via the spin characteristic, this is equivalent to the fact that $[P_\lambda, Q_\mu] = \delta_{\lambda\mu}$ (Prop. 3.1(b)).

STEP 2. *Construct spin characters* η^λ (*resp.*, $\eta_\pm^\lambda$) *indexed by* $\lambda \in \mathrm{DP}_n^+$ (*resp.*, DP_n^-) *whose expansion with respect to* φ^λ *is of the form*

$$\eta^\lambda = \varphi^\lambda + \sum_{\mu > \lambda} c_{\mu\lambda} \varphi^\mu \tag{4.2}$$

for suitable integers $c_{\mu\lambda}$ *and a suitable partial order* $>$ *on* DP_n.

This can be solved by taking η^λ to be the $\tilde{S}_n$-character induced from successive spin products of the basic spin representations of $\tilde{S}_{\lambda_1}, \tilde{S}_{\lambda_2}, \ldots$ By (2.6) and Theorem 2.2, it follows that

$$\mathrm{ch}' \eta^\lambda = \frac{1}{\varepsilon_\lambda} 2^{-l(\lambda)/2} q_\lambda.$$

Via the spin characteristic, the φ^λ-expansion of η^λ is related to the P_λ-expansion of q_λ (cf. Prop. 3.1(c)). In this context, to prove (4.2) amounts to showing that $[2^{-l(\lambda)} K'_{\lambda\mu}]$ is a triangular integer matrix with a unit diagonal.

The expansion (4.2) can be inverted over $\mathbf{Z}$, thereby proving that φ^λ is a $\mathbf{Z}$-linear combination of characters. Since φ^λ is of unit length (Step 1), it follows that either φ^λ or $-\varphi^\lambda$ is an irreducible spin character.

STEP 3. *Show that* $\varphi^\lambda(1) > 0$.

From the expansion (4.1), we have

$$\varphi^\lambda(1) = \frac{1}{\varepsilon_\lambda} 2^{(n-l(\lambda))/2} [Q_\lambda, p_1^n] = \frac{1}{\varepsilon_\lambda} 2^{-(n+l(\lambda))/2} [Q_\lambda, q_1^n],$$

since $q_1 = 2p_1$. Proposition 2.1 therefore implies

$$\varphi^\lambda(1) = \frac{1}{\varepsilon_\lambda} 2^{-(n+l(\lambda))/2} K'_{\lambda,1^n} > 0. \blacksquare$$

Let $g^\lambda = 2^{-n} K'_{\lambda,1^n}$ denote the number of unmarked shifted tableaux of shape λ and content 1^n. A consequence of the above calculation is

COROLLARY 4.2. $\deg \varphi^\lambda = 2^{\lfloor (n-l(\lambda))/2 \rfloor} g^\lambda$.

Since Q_λ is essentially the spin characteristic of φ^λ, Theorem 2.2 shows that, aside from powers of 2, $Q_\mu Q_\nu$ is the spin characteristic of $(\varphi^\mu \hat{\otimes} \varphi^\nu)\uparrow \tilde{S}_n$. Therefore, the shifted Littlewood–Richardson rule (Theorem 3.2) describes the decomposition of $(\varphi^\mu \hat{\otimes} \varphi^\nu)\uparrow \tilde{S}_n$ into irreducible characters. In terms of the coefficients $f^\lambda_{\mu\nu}$ introduced in § 3, one finds

THEOREM 4.3 [St, § 8]. *The multiplicity of* φ^λ *in* $(\varphi^\mu \hat{\otimes} \varphi^\nu)\uparrow \tilde{S}_n$ *is* $\varepsilon_\lambda^{-1} \varepsilon_{\mu\cup\nu}^{-1} 2^{(l(\mu)+l(\nu)-l(\lambda))/2} f^\lambda_{\mu\nu}$ *unless* $\lambda \in \mathrm{DP}_n^-$ *and* $\lambda = \mu \cup \nu$. *In that case, the multiplicity is* 0 *or* 1, *depending on the choice of* φ^λ_+ *or* φ^λ_-.

5. A recurrence for spin characters

Although the irreducible spin characters φ^λ are determined by Theorem 4.1, it is difficult to use (4.1) to calculate $\varphi^\lambda(\sigma_\mu)$ explicitly for $\mu \in \mathrm{OP}$. However, there is a recurrence for the evaluation of these characters due originally to Morris [Mo2], which is quite similar to the well-known Murnaghan–Nakayama rule for ordinary characters [Ma, Ex. I.7.5]. Although Morris' rule has been reformulated several times (e.g. [H], [MY]), all of the proofs have relied heavily on the machinery of Hall–Littlewood functions. Since the rule is essentially combinatorial in nature, it is somewhat surprising that shifted tableaux have not, heretofore, played a role in the proof.

As promised in the introduction, it is our purpose here to provide yet another reformulation of the Morris rule, and to give a new proof which emphasizes combinatorial tableaux methods. Nowhere in the proof will it be necessary to borrow any of the results from the theory of Hall–Littlewood functions. We use (3.1) as our definition of the Q-function, and the properties we subsequently employ can be derived directly from this definition. (Proofs of these properties which substantiate this claim can be found in [St]).

To describe the Morris rule, we first introduce strip tableaux.

The *j-th diagonal* of a skew diagram $D'_{\lambda/\mu}$ is the collection of boxes $(1, j)$, $(2, j+1)$, $(3, j+2)$, ... in $D'_{\lambda/\mu}$. The first diagonal (which may be empty) is called the main diagonal.

A skew diagram $D'_{\lambda/\mu}$ is said to be a *strip* if it is rookwise connected and each diagonal has at most one box. The *height* h of a strip is the number of rows it occupies. For example, the strip in the middle of Fig. 5 is of height 4. A *double strip* is a skew diagram formed by the union of two strips which both start on the main diagonal. Note that a double strip can be cut into two nonempty connected pieces – one piece (call it α) consisting of the diagonals of length two, the other piece (call it β) consisting of the strip formed by the diagonals of length one. The *depth* of a double strip is defined to be $|\alpha|/2 + h(\beta)$ ($h(\beta)$ = height of β). For example, the double strip on the right in Fig. 4(b) is of depth 5.

A (shifted) *strip tableau* of shape λ/μ and content $\gamma = (\gamma_1, \ldots, \gamma_l)$ is defined to be a nested sequence of shifted diagrams

$$D'_\mu = D'_{\lambda^0} \subseteq D'_{\lambda^1} \subseteq \ldots \subseteq D'_{\lambda^l} = D'_\lambda$$

with $|\lambda^i| - |\lambda^{i-1}| = \gamma_i$ $(1 \leqslant i \leqslant l)$ such that each intermediate diagram $D'_{\lambda^i/\lambda^{i-1}}$ is either a strip or a double strip. We define the weight of a strip of height h to be $(-1)^{h-1}$, and we define the weight of a double strip of depth d to be $2(-1)^{d-1}$. The *weight* of a strip tableau S, denoted $\mathrm{wt}(S)$, is the product of the weights of the component strips and double strips.

THEOREM 5.1. *For any* $\gamma \in \mathrm{OP}$, *we have*

$$[Q_{\lambda/\mu}, p_\gamma] = \sum_S \mathrm{wt}(S),$$

summed over all strip tableaux S *of shape* λ/μ *and content* γ.

We remark that (4.1) implies

$$\varphi^\lambda(\sigma_\gamma) = \frac{1}{\varepsilon_\lambda} 2^{(l(\gamma) - l(\lambda))/2} [Q_\lambda, p_\gamma],$$

so this rule does provide a recurrence for spin characters. For example, consider $\lambda = (4, 3, 2, 1)$, $\gamma = (1, 3, 3, 3)$. There are three strip tableaux of shape λ and content γ. See Fig. 3. Their weights are each -2, so we conclude that $\varphi^{4321}(\sigma_{3331}) = [Q_{4321}, p_1 p_3^3] = -6$.

```
1 2 2 2     1 2 2 2     1 2 2 3
  3 3 4       3 3 3       2 3 3
    3 4         4 4         4 4
      4           4           4
```

Fig. 3

Proof. Since the P_λ's and Q_λ's are dual bases (Prop. 3.1), it follows that for any (odd, positive) integer r,

$$p_r P_\mu = \sum_{\lambda \in \mathrm{DP}} [p_r P_\mu, Q_\lambda] P_\lambda.$$

By iterating this expansion successively for $r = \gamma_1, \ldots, \gamma_l$, we find

$$[p_\gamma P_\mu, Q_\lambda] = \sum_{\{\lambda^j\}} [p_{\gamma_1} P_{\lambda^0}, Q_{\lambda^1}] \cdots [p_{\gamma_l} P_{\lambda^{l-1}}, Q_{\lambda^l}],$$

where $\mu = \lambda^0$, $\lambda = \lambda^l$. However, since the P_ν's span Ω_Q, (3.5) implies

$$[fP_\mu, Q_\lambda] = [Q_{\lambda/\mu}, f]$$

for any $f \in \Omega_Q$, and therefore,

$$[Q_{\lambda/\mu}, p_\gamma] = \sum_{\{\lambda^j\}} [Q_{\lambda^1/\lambda^0}, p_{\gamma_1}] \cdots [Q_{\lambda^l/\lambda^{l-1}}, p_{\gamma_l}]. \tag{5.1}$$

Since $Q_{\lambda/\mu} = 0$ unless $\lambda \supseteq \mu$, it follows that the only nonzero contributions to $[Q_{\lambda/\mu}, p_\gamma]$ in this expansion occur when $\lambda^0 \subseteq \lambda^1 \subseteq \ldots \subseteq \lambda^l$ and $|\lambda^i| - |\lambda^{i-1}| = \gamma_i$ $(1 \leqslant i \leqslant l)$. Thus, we need only to evaluate $[Q_{\lambda/\mu}, p_r]$ for all skew diagrams λ/μ of size r (r odd).

LEMMA 5.2. *$[Q_{\lambda/\mu}, p_r] = 0$ unless λ/μ is rookwise connected.*

Proof. If λ/μ is disconnected, then the shifted diagram $D'_{\lambda/\mu}$ can be separated into two subdiagrams α and β whose rows and columns do not overlap. In that case it is easy to see that $Q_{\lambda/\mu} = Q_\alpha Q_\beta$. Since Q_α and Q_β are of degree $< r$, their p_γ-expansions cannot involve p_r, so the p_γ-expansion of $Q_\alpha Q_\beta$ cannot involve p_r. ∎

If $f \in \Lambda$ is any symmetric function, use the notation $f(x, y)$ to indicate the two-variable specialization of f (i.e., $x_1 = x$; $x_2 = y$; $x_3, x_4, \ldots = 0$). The following result shows that $[Q_{\lambda/\mu}, p_r]$ depends only on $Q_{\lambda/\mu}(x, y)$.

LEMMA 5.3. *If $f \in \Omega^r$, then $[f, p_r] = \frac{1}{2}\frac{\partial}{\partial x} f(x, -1)|_{x=1}$.*

Proof. By linearity, it is enough to check $f = p_\gamma$ ($\gamma \in \mathrm{OP}_r$). In case $f = p_r$, we have $f(x, y) = x^r + y^r$, and it is trivial to verify that $[f, p_r] = r/2 = \frac{\partial}{\partial x} f(1, -1)/2$. Otherwise, observe that if $g \in \Omega$ has no constant term, then $g(1, -1) = 0$ (consider $g = p_\gamma$). Therefore if $f = gh$, where $g, h \in \Omega$ have no constant terms, then $\partial f/\partial x = g\partial h/\partial x + h\partial g/\partial x$ clearly vanishes at $(x, y) = (1, -1)$. In particular, it follows that $\frac{\partial}{\partial x} p_\gamma(1, -1) = 0$ if $\gamma \neq (r)$. ∎

LEMMA 5.4. *$[Q_{\lambda/\mu}, p_r] = 0$ unless λ/μ is a strip or a double strip.*

Proof. The previous lemma shows that $[Q_{\lambda/\mu}, p_r] = 0$ unless $Q_{\lambda/\mu}(x, y) \neq 0$; i.e., there must exist shifted tableaux of shape λ/μ with entries chosen from $\{1', 1, 2', 2\}$. This forces the diagonals of λ/μ to have length at most two. Since Lemma 5.2 shows that λ/μ must be connected, it remains only to verify that if, say, the jth diagonal of λ/μ has length one and the $(j+1)$th has length two, then $[Q_{\lambda/\mu}, p_r] = 0$.

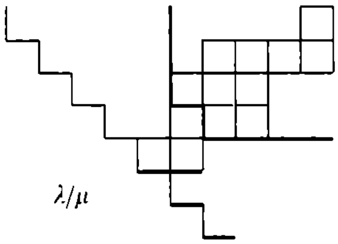

Fig. 4(a)

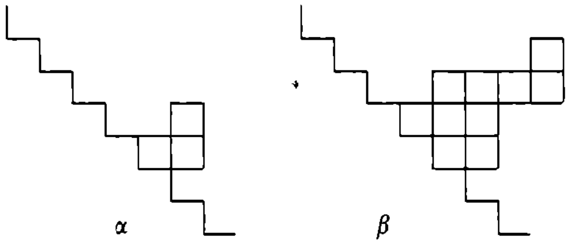

Fig. 4(b)

Assuming of λ/μ to be such a diagram, let α be the subdiagram formed by the first j diagonals, and let β be the remaining subdiagram. See Fig. 4(a). Note that β is not a legal skew diagram (it fails to be diagonally convex), but it becomes legal if it is translated to the main diagonal. See Fig. 4(b). Observe that if $T\colon D'_{\lambda/\mu} \to \{1, 2\}$ is a feasible assignment of 1's and 2's (as in § 3), then the first diagonal of β is necessarily ${}^1{}_2$, and thus does not affect the content of T in α. However, one of these two entries will be free and the other will be forced, depending on $T|_\alpha$, whereas both are free in $T|_\beta$. In summary,

$$Q_{\lambda/\mu}(x, y) = \tfrac{1}{2} Q_\alpha(x, y) Q_\beta(x, y).$$

Hence, by Lemma 5.3 (cf. the consequences of $f = gh$ in the proof), we have $[Q_{\lambda/\mu}, p_r] = 0$. ■

LEMMA 5.5. $[Q_{\lambda/\mu}, p_r] = (-1)^{h-1}$ *if* λ/μ *is a strip of height* h.

Proof. Proceed by induction on h. If $h = 1$, then $Q_{\lambda/\mu} = q_r$ and the result follows from (2.5) or Prop. 2.1. Otherwise, suppose there are $k < r$ boxes in the first (highest) nonempty row of λ/μ. Let α denote the substrip obtained by deleting this row, and let β denote the strip obtained by adding k boxes to the first row of α. We observe that

$$q_k Q_\alpha = Q_{\lambda/\mu} + Q_\beta \tag{5.2}$$

(see Fig. 5), since there is a natural bijection between the shifted tableaux of shape $\alpha \oplus (k)$ (disjoint union of diagrams) and those of shape λ/μ and β – simply compare the top rightmost entry a of α with the leftmost entry b of (k). If $a > b$ (or $a = b$ and both are marked), create a tableau of shape λ/μ. If $a < b$ (or $a = b$ and both are unmarked), create a tableau of shape β.

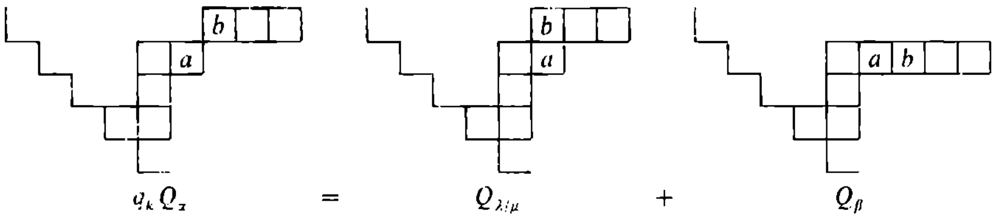

Fig. 5

We have $[q_k Q_\alpha, p_r] = 0$, since p_r cannot be involved in the p_γ-expansion of q_k or Q_α (both are of degree less than r), so (5.2) implies $[Q_{\lambda/\mu}, p_r] = -[Q_\beta, p_r]$. The result now follows by induction since β is a strip of height $h-1$. ∎

LEMMA 5.6. *$[Q_{\lambda/\mu}, p_r] = 2(-1)^{d-1}$ if λ/μ is a double strip of depth d.*

Proof. Let α denote the subdiagram obtained by deleting the main diagonal of λ/μ, and translating the result back to the main diagonal. If λ/μ has only one diagonal of length two, then α is a strip; otherwise, α is a double strip. In the former case note that α has only one (free) box on the main diagonal, whereas λ/μ has two. We therefore have

$$Q_{\lambda/\mu}(x, y) = \begin{cases} 2xyQ_\alpha(x, y) & \text{if } \alpha \text{ is a strip,} \\ xyQ_\alpha(x, y) & \text{if } \alpha \text{ is a double strip.} \end{cases}$$

Note that if $g(1, -1) = 0$ and $f(x, y) = xyg(x, y)$, then $\frac{\partial}{\partial x} f(1, -1) = -\frac{\partial}{\partial x} g(1, -1)$, so Lemma 5.3 implies

$$[Q_{\lambda/\mu}, p_r] = \begin{cases} -2[Q_\alpha, p_{r-2}] & \text{if } \alpha \text{ is a strip,} \\ -[Q_\alpha, p_{r-2}] & \text{if } \alpha \text{ is a double strip.} \end{cases}$$

The result now follows from Lemma 5.5 and induction on the depth. ∎

Using the description of $[Q_{\lambda/\mu}, p_r]$ contained in Lemmas 5.4–5.6 and the recurrence (5.1), the proof of Theorem 5.1 is now complete. ∎

6. An involution for shifted tableaux

Since the Q-functions are specializations of Hall–Littlewood functions, one knows immediately that they are symmetric functions. In particular (cf. (3.1, 4)), this implies that $K'_{\lambda/\mu,\gamma}$, the number of shifted tableaus of shape λ/μ and content $\gamma = (\gamma_1, \gamma_2, \ldots)$, is invariant under permutations of the γ_i's. To deduce such a simple combinatorial fact in this way unfortunately requires considerable machinery; i.e., [Ma, III]. Since the symmetry of Q-functions is fundamental in this theory, it is therefore natural to look for a purely combinatorial explanation of this result, one that requires no machinery *per se.*(²) Our purpose in this section is to provide such an explanation; namely,

(²) A combinatorial explanation also appears in Cor. 6.2(a) of [St], but it requires the machinery of [Sa] or [W].

THEOREM 6.1. *For any shifted shape λ/μ, there is a natural content-reversing involution $T \mapsto \hat{T}$ on the set of shifted tableaux of the form $T: D'_{\lambda/\mu} \to \{1', 1, 2', 2\}$.*

By *content-reversing*, we mean to say that if T has content (a, b), then $\hat{T}$ should have content (b, a).

Given such an involution, it is a simple matter to explain the symmetry of Q-functions. For any shifted tableau T of shape ν, let T_i denote the subtableau formed by the entries $\{i', i, (i+1)', i+1\}$. By applying the above involution to T_i (with the identification $i \leftrightarrow 1$, $i+1 \leftrightarrow 2$), we may deduce that $Q_\nu(x_1, x_2, \ldots)$ is invariant when x_i and x_{i+1} are interchanged, and hence Q_ν is a symmetric function.

Before describing the involution, we first need to analyze the structure of shifted tableaux with restricted entries. In particular, consider the set $R_2(\lambda/\mu)$ of shifted tableaux of the form $T: D'_{\lambda/\mu} \to \{1, 2\}$. Observe that rule (R1–3) of § 3 imply that the boxes occupied by the letter i in any shifted tableau must form a *broken row*; i.e., a skew diagram with at most one box in any column. Hence, there is a one-to-one correspondence between $R_2(\lambda/\mu)$ and the set of partitions $\nu \in \mathrm{DP}$ $(\mu \subseteq \nu \subseteq \lambda)$ such that ν/μ and λ/ν are broken rows.

In the following, it will be necessary to first consider those skew diagrams λ/μ with at most one box on the main diagonal. We call such diagrams *detached* since they can be translated away from the main diagonal without creating an illegitimate skew-diagram. In this context, it is convenient to introduce the following result due to Bender and Knuth [BK]:

LEMMA 6.2. *There is a natural content-reversing involution ϕ_R on $R_2(\lambda/\mu)$, assuming λ/μ is detached.*

Proof. There is nothing to prove unless $R_2(\lambda/\mu)$ is nonempty; i.e., we may assume that the columns of λ/μ have length at most two. Note that in any tableau $T \in R_2(\lambda/\mu)$ a column of length two must be of the form $\begin{smallmatrix}1\\2\end{smallmatrix}$. The remainder of λ/μ can be broken into disconnected rows of boxes whose entries are indeterminate. (This fails if λ/μ is not detached.) Let r_i denote the number of such "indeterminate" boxes in the ith row of λ/μ. The tableau T may be specified by choosing the number n_i $(0 \leqslant n_i \leqslant r_i)$ of these boxes to be assigned the letter 1; the remaining $r_i - n_i$ boxes are to be assigned the letter 2. Since the specifications (n_i) may be chosen independently and arbitrarily, it follows that ϕ_R: $(n_i) \mapsto (r_i - n_i)$ defines a content-reversing involution on $R_2(\lambda/\mu)$. ∎

$$
\begin{array}{cccccccc}
 & & & & & & 1 & 1 \\
 & 1 & 1 & 1 & 1 & 2 & 2 & \\
1 & 2 & 2 & & & & &
\end{array}
\quad \underset{\phi_R}{\longleftrightarrow} \quad
\begin{array}{cccccccc}
 & & & & & & 1 & 2 \\
 & 1 & 1 & 1 & 2 & 2 & 2 & \\
2 & 2 & 2 & & & & &
\end{array}
$$

Fig. 6

An example of the involution ϕ_R appears in Fig. 6. We remark that the above lemma fails if λ/μ is not detached. For example, $\begin{smallmatrix}1&1\\&2\end{smallmatrix}$ is the only shifted tableau in $R_2(21/\emptyset)$. We also remark that this lemma is the unshifted analogue of Theorem 6.1; it provides a purely combinatorial explanation of the fact that the Schur functions $s_{\lambda/\mu}$, as tableaux generating functions, are symmetric [BK].

Let $C_2(\lambda/\mu)$ denote the set of shifted tableaux T: $D'_{\lambda/\mu} \to \{1', 2'\}$. By transposing ϕ_R, we obtain

COROLLARY 6.3. *There is a natural content-reversing involution* ϕ_C *on* $C_2(\lambda/\mu)$, *assuming* λ/μ *is detached.*

Let $RC(\lambda/\mu)$ denote the set of shifted tableaux of the form T: $D'_{\lambda/\mu} \to \{1, 2'\}$. Similarly, let $CR(\lambda/\mu)$ denote the set of shifted tableaux T: $D'_{\lambda/\mu} \to \{2', 1\}$ satisfying rules (R1–3), but with respect to the nonstandard ordering $2' < 1$. There is a one-to-one correspondence between the tableaux in $RC(\lambda/\mu)$ (resp., $CR(\lambda/\mu)$) and partitions $\nu \in \mathrm{DP}$ $(\mu \subseteq \nu \subseteq \lambda)$ such that ν/μ is a broken row (resp., column) and λ/μ is a broken column (resp., row).

LEMMA 6.4. *There is a natural content-preserving bijection* ψ: $RC(\lambda/\mu) \to CR(\lambda/\mu)$, *assuming* λ/μ *is detached.*

Proof. If λ/μ is detached, both $RC(\lambda/\mu)$ and $CR(\lambda/\mu)$ are empty unless the diagonals of λ/μ are of length at most one. By considering the connected components of λ/μ if necessary, we may therefore assume that λ/μ is a strip (as in § 5).

Linearly order the boxes of λ/μ from southwest to northeast, and label the boxes $b_1, \ldots, b_n$ in this order. If T is a tableau in $RC(\lambda/\mu)$, then we must have

$$T(b_i) = \begin{cases} 1 & \text{if } \operatorname{row}(b_i) = \operatorname{row}(b_{i+1}), \\ 2' & \text{if } \operatorname{col}(b_i) = \operatorname{col}(b_{i+1}). \end{cases}$$

Since λ/μ is assumed to be a strip, this completely determines T except for the choice of $T(b_n)$, which is free to be 1 or 2'. Similarly, if S is a tableau in $CR(\lambda/\mu)$, then

$$S(b_{i+1}) = \begin{cases} 1 & \text{if } \operatorname{row}(b_i) = \operatorname{row}(b_{i+1}) \\ 2' & \text{if } \operatorname{col}(b_i) = \operatorname{col}(b_{i+1}), \end{cases}$$

and the choice of $S(b_1)$ is free. Therefore, for each $T \in RC(\lambda/\mu)$, we may define $\psi T \in CR(\lambda/\mu)$ by setting $\psi T(b_{i+1}) = T(b_i)$ $(1 \leqslant i < n)$ and $\psi T(b_1) = T(b_n)$, and thus obtain a suitable bijection. See Fig. 7. ∎

$$\begin{array}{cccc} & & & 1\\ & 1 & 1 & 2'\\ & 2' & & \\ & 2' & & \\ 1 & 2' & & \end{array} \quad \xrightarrow{\psi} \quad \begin{array}{cccc} & & & 2'\\ & 2' & 1 & 1\\ & 2' & & \\ & 2' & & \\ 1 & 1 & & \end{array}$$

Fig. 7

Proof of Theorem 6.1. First we assume that λ/μ is detached. Let T be a shifted tableau of shape λ/μ with entries $1'$, 1, $2'$, 2. Apply the bijection ψ to the subtableau occupied by 1 and $2'$; this yields a shifted tableau relative to the nonstandard order $1' < 2' < 1 < 2$. Apply the involutions ϕ_C and ϕ_R to the subtableaux of ψT occupied by $1'$, $2'$ and 1, 2, respectively. This reverses the content. Finally, apply ψ^{-1} to the subtableau (of $\phi_C\,\phi_R\,\psi T$) occupied by $2'$ and 1 to obtain the shifted tableau $\hat{T}$. Lemmas 6.2 and 6.4 show that $T \mapsto \hat{T}$ is a suitable content-reversing involution. An example appears in Fig. 8.

$$\begin{array}{cccccc} &&&&&2'\\ &&&1'&1&2\\ &&1&2'&&\\ 1'&2'&2&&& \end{array} \;\xrightarrow[\psi]{}\; \begin{array}{cccccc} &&&&&2'\\ &&&1'&1&2\\ &&2'&1&&\\ 1'&2'&2&&& \end{array} \;\xrightarrow[\phi_{C,R}]{}\; \begin{array}{cccccc} &&&&&1'\\ &&&2'&1&2\\ &&1'&2&&\\ 1'&2'&1&&& \end{array} \;\xrightarrow[\psi^{-1}]{}\; \begin{array}{cccccc} &&&&&1'\\ &&&1&2'&2\\ &&1'&2&&\\ 1'&1&2'&&& \end{array}$$

Fig. 8

Now consider the possibility that λ/μ is not detached. To avoid trivialities, we may assume that the diagonals of λ/μ are of length at most two, and that the kth diagonal is the first of length one ($k > 1$).

Observe that the entries assigned to these first k diagonals (except for the main diagonal) will never be free in any shifted tableau T: $D'_{\lambda/\mu} \to \{1', 1, 2', 2\}$. Therefore, let S be the shifted tableau of detached shape obtained by deleting the first $k-1$ diagonals of T. Since the single entry a in the first nonempty diagonal of S is now free, let us change the mark of a if necessary, so that it is identical to one of the (free) entries on the main diagonal of T. The map $T \mapsto S$ is two-to-one, and so the two tableaux obtained by inverting $\hat{S}$ with respect to this map can be used to define a two-to-two involution $T \to S \to \hat{S} \to \hat{T}$. ■

References

[BK] E. A. Bender and D. E. Knuth, *Enumeration of plane partitions*, J. Combin. Theory Ser. A 13 (1972), 40–54.

[HH] P. N. Hoffman and J. F. Humphreys, *Hopf algebras and the projective representations of $G \wr S_n$ and $G \wr A_n$*, Canad. J. Math. 38 (1986), 1380–1458.

[H] J. F. Humphreys, *Blocks of projective representations of symmetric groups*, J. London Math. Soc. (2) 33 (1986), 441–452.

[J] T. Józefiak, *Characters of projective representations of symmetric groups*, Exposition. Math. 7 (1989), 193–247.

[Ma] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, Oxford, 1979.

[Mo1] A. O. Morris, *The spin representation of the symmetric group*, Proc. London Math Soc. (3) 12 (1962), 55–76.

[Mo2] —, *The spin representation of the symmetric group*, Canad. J. Math. 17 (1965), 543–549.

[Mo3] —, *Projective representations of reflection groups I, II*, Proc. London Math. Soc. (3) 32 (1976) 403–420; 40 (1980), 553–576.

[MY] A. O. Morris and A. K. Yaseen, *Some combinatorial results involving shifted Young diagrams*, Math. Proc. Cambridge Philos. Soc. 99 (1986), 23–31.

[N] M. L. Nazarov, *An orthogonal basis of irreducible projective representations of the symmetric group*, Functional Anal. Appl., 22 (1988), 77–78.

[Sa] B. E. Sagan, *Shifted tableaux, Schur Q-functions and a conjecture of R. Stanley*, J. Combin. Theory Ser. A 45 (1987), 62–103.

[S] I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. 139 (1911), 155–250.

[St] J. R. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*, Adv. in Math. 74 (1989), 87–134.

[T] R. M. Thrall, *A combinatorial problem*, Michigan Math. J. 1 (1952), 81–88.

[W] D. R. Worley, *A Theory of Shifted Young Tableaux*, Ph. D. thesis, M.I.T., 1984.