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Various categorical approaches to statistical spaces

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Introduction

In common statistical practice we often deal with reducing statistical data. Usually this is done by means of functions which are either random variables or statistics dependently on whether the phenomenon is described in terms of a probabilistic or a statistical space (compare Barra (1971)). More refined approach to data reduction involves randomization. Intuitively, to every result of experiment we assign a probability distribution which governs the choice of reduced data. This is done by means of probability transition functions (called also stochastic transformations, Blackwell (1953) or Markov morphisms, Čencov (1972)), here, for brevity, referred to as transition functions. Transition functions appear also in investigations concerning sufficiency and regular conditional probability. If the conditional probability is not regular, then we have to use a more general notion of almost transition function (called also statistical operation, Morse and Sacksteder (1966)).

It is natural to treat various kinds of functions mentioned above as morphisms in some categories with statistical (or probabilistic) spaces as objects. The language of category theory has been used in statistical considerations by several authors: Čencov (1972), Morse and Sacksteder (1966), Martin, Petit and Petit-Littaye (1971). However, these papers are fragmentary and it is hard to discover a connection between them; moreover, the terminology is not unified.

The purpose of the present paper is to give a consistent survey of various categories of statistical spaces and to establish the relationships between these categories. We start with concrete categories of measurable, probabilistic, and statistical spaces, Mes, Pr, and St (Section 2). Morphisms in these categories are measurable functions, random variables, and statistics, respectively. Then we pass to categories of measurable and statistical spaces with transition functions as morphisms, $\mathrm{Mes}_{\mathrm{trans}}$ and $\mathrm{St}_{\mathrm{trans}}$ (Section 3). The relationships between Mes and $\mathrm{Mes}_{\mathrm{trans}}$ and between St and $\mathrm{St}_{\mathrm{trans}}$ are reflected by Theorems 3.1.4 and 3.2.3. In the same section we also consider a category of statistical spaces with morphisms being functions from one family of measures to another one, $\mathrm{St}_{\mathrm{dist}}$. The relationship between $\mathrm{St}_{\mathrm{trans}}$ and $\mathrm{St}_{\mathrm{dist}}$ is given by Theorem 3.3.4. Section 4 concerns a category of statistical spaces with morphisms being some classes of almost transition functions, $\mathrm{St}_{\mathrm{a.trans}}$. Its connections with $\mathrm{St}_{\mathrm{dist}}$ and $\mathrm{St}_{\mathrm{trans}}$ are given by Theorems 4.5 and 4.6.

In Final remarks we show the links between the categories considered in the literature and categories introduced in our paper.

It seems evident that the language of category theory gives a deeper insight into the relationships between various notions of mathematical statistics. On the other hand, the examples of categories presented in our paper may occur interesting for category theorists.

1. Preliminaries

1.0. Measurable, probabilistic, and statistical spaces. According to Barra (1971), we use the following terminology and notation concerning statistics.

A *measurable space* is understood as a pair $(\Omega, \mathscr{A})$ consisting of a non-empty set Ω and a σ-field $\mathscr{A} \subset 2^{\Omega}$.

Given two measurable spaces $(\Omega, \mathscr{A})$ and $(\Omega', \mathscr{A}')$, a function $f\colon \Omega \to \Omega'$ is said to be *measurable* if $\{f^{-1}(A'); A' \in \mathscr{A}'\} \subset \mathscr{A}$.

A *probabilistic space* is understood as a triple $(\Omega, \mathscr{A}, P)$ with $(\Omega, \mathscr{A})$ being a measurable space and $P\colon \mathscr{A} \to [0, 1]$ a probability measure, i.e. normed σ-additive measure.

Given two probabilistic spaces $(\Omega, \mathscr{A}, P)$ and $(\Omega', \mathscr{A}', P')$, a function $f\colon \Omega \to \Omega'$ is said to be a *random variable* whenever f is measurable and $Pf^{-1} = P'$.

A *statistical space* is defined as $M = (\Omega, \mathscr{A}, \mathscr{P})$ with $(\Omega, \mathscr{A})$ being a measurable space and $\mathscr{P}$ a family of probability measures.

Given two statistical spaces M and M', a function $f\colon \Omega \to \Omega'$ is said to be a *statistic* whenever f is measurable and $\{Pf^{-1}; P \in \mathscr{P}\} = \mathscr{P}'$.

For basic notions of category theory the reader is referred to Mac Lane (1971). We admit the following notation: for any pair of objects A, B of a category C, the corresponding set of morphisms is denoted by "$\mathrm{C}(A, B)$".

1.1. Transition functions. Given a measurable space $(\Omega, \mathscr{A})$, we use the symbol "$\mathrm{CAP}(\Omega, \mathscr{A})$" to denote, as usually, the set of all the probability measures on $\mathscr{A}$ ("collection of all probabilities").

Take two measurable spaces, $(\Omega, \mathscr{A})$ and $(\Omega', \mathscr{A}')$. Every function $\varphi\colon \Omega \times \mathscr{A}' \to [0, 1]$ induces the following two collections of functions:

$$\varphi_{\omega}\colon \mathscr{A}' \to [0, 1] \quad \text{for all } \omega \in \Omega$$

and

$$\varphi_{A'}\colon \Omega \to [0, 1] \quad \text{for all } A' \in \mathscr{A}',$$

defined by the formula

$$\varphi_\omega(A') = \varphi_{A'}(\omega) = \varphi(\omega, A') \quad \text{for} \quad (\omega, A') \in \Omega \times \mathscr{A}'. \ (^1)$$

A function $\varphi\colon \Omega \times \mathscr{A}' \to [0, 1]$ is said to be a *transition function* (for $(\Omega, \mathscr{A})$, $(\Omega', \mathscr{A}')$) whenever it satisfies the following two conditions:

(i) $\varphi_\omega \in \mathrm{CAP}(\Omega', \mathscr{A}')$ for every $\omega \in \Omega$;

(ii) $\varphi_{A'}$ is measurable for every $A' \in \mathscr{A}'$.

Let P be a probability measure on $(\Omega, \mathscr{A})$ and let φ be a transition function for $(\Omega, \mathscr{A})$, $(\Omega', \mathscr{A}')$. Let $P'\colon \mathscr{A}' \to [0, 1]$ be defined by the formula

$$P'(A') = \int_\Omega \varphi_{A'}(\omega)\, dP \quad \text{for every } A' \in \mathscr{A}'.$$

Then

1.1.1. *P' is a probability measure on $(\Omega', \mathscr{A}')$.*

The measure P' is said to be *generated by the pair* (P, φ).

A family $\mathscr{P}'$ is said to *be generated by* $(\mathscr{P}, \varphi)$ whenever $\mathscr{P}'$ consists of the measures generated by (P, φ) for all $P \in \mathscr{P}$.

We shall need the following

1.1.2. LEMMA. *Let $(\Omega, \mathscr{A})$ and $(\Omega', \mathscr{A}')$ be measurable spaces and let $f\colon \Omega' \to [0, 1]$ be a measurable function. If $\varphi\colon \Omega \times \mathscr{A}' \to [0, 1]$ is a transition function, then*

(i) *the function $g\colon \Omega \to [0, 1]$ defined by the formula*

$$g(\omega) = \int_{\Omega'} f(\omega')\, d\varphi_\omega$$

is measurable;

(ii) *if $P \in \mathrm{CAP}(\Omega, \mathscr{A})$ and P' is generated by (P, φ), then*

$$\int_{\Omega'} f(\omega')\, dP' = \int_\Omega \int_{\Omega'} f(\omega')\, d\varphi_\omega\, dP.$$

Proof is routine; we first assume f to be a simple function, and then we approximate an arbitrary measurable function $f\colon \Omega' \to [0, 1]$ by an increasing sequence of simple functions. ∎

We shall also use the following well-known proposition:

1.1.3. *Let $(\Omega, \mathscr{A})$ be a measurable space and let*

$$P_\omega(A) = \chi_A(\omega) = \begin{cases} 1 & \text{for} \quad \omega \in A, \\ 0 & \text{for} \quad \omega \notin A, \end{cases} \qquad A \in \mathscr{A}.$$

Then for any measurable function $f\colon \Omega \to [0, 1]$

$$\int_\Omega f(\bar{\omega})\, dP_\omega = f(\omega) \quad \text{for every } \omega \in \Omega.$$

(^{1}) We may assume that $\Omega \cap \mathscr{A}' = \emptyset$; then our notation does not lead to a confusion.

1.2. Linear space of $\mathscr{P}$-bounded measurable real functions. Let $M = (\Omega, \mathscr{A}, \mathscr{P})$ be a statistical space. For any two measurable functions f_1, f_2: $\Omega \to R$, let

$$f_1 \underset{\mathscr{P}}{\leqslant} f_2 \underset{\mathrm{Df}}{\Leftrightarrow} \forall P \in \mathscr{P} \quad P\{\omega \in \Omega;\ f_1(\omega) > f_2(\omega)\} = 0$$

and

$$f_1 \underset{\mathscr{P}}{=} f_2 \underset{\mathrm{Df}}{\Leftrightarrow} f_1 \underset{\mathscr{P}}{\leqslant} f_2 \wedge f_2 \underset{\mathscr{P}}{\leqslant} f_1 .$$

Clearly,

$$f_1 \underset{\mathscr{P}}{=} f_2 \Leftrightarrow \forall P \in \mathscr{P} \quad P\{\omega;\ f_1(\omega) \neq f_2(\omega)\} = 0.$$

The relation $\underset{\mathscr{P}}{\leqslant}$ is a quasi-order, and thus $\underset{\mathscr{P}}{=}$ is an equivalence.

A function f: $\Omega \to R$ is *$\mathscr{P}$-bounded* iff there exists $c \in R$ such that

$$|f| \underset{\mathscr{P}}{\leqslant} c$$

(clearly c may be treated as a constant function).

A sequence $(f_n)_{n \in N}$ is *$\mathscr{P}$-increasing* iff

$$\forall n \ f_n \underset{\mathscr{P}}{\leqslant} f_{n+1} .$$

To any statistical space $M = (\Omega, \mathscr{A}, \mathscr{P})$ we assign the linear space

$$L(M) \underset{\mathrm{Df}}{=} \{f;\ \Omega \to R;\ f \text{ is measurable and } \mathscr{P}\text{-bounded}\},$$

with the linear operations understood in obvious way.

Clearly,

1.2.1. *$\underset{\mathscr{P}}{=}$ is a congruence with respect to linear operations in $L(M)$ as well as with respect to $\underset{\mathscr{P}}{\leqslant}$ (i.e. $\underset{\mathscr{P}}{=}$ is an equivalence relation preserving the linear operations and $\underset{\mathscr{P}}{\leqslant}$).*

For any sequence $(f_n)_{n \in N}$ in $L(M)$ and a function $f \in L(M)$ we define $\mathscr{P}$-convergence of $(f_n)_{n \in N}$ to f as follows:

$$f = \mathscr{P}\text{-}\lim_n f_n \underset{\mathrm{Df}}{\Leftrightarrow} \forall P \in \mathscr{P}\big(P\{\omega;\ (f_n(\omega))_{n \in N} \text{ is not convergent}\} = 0 \wedge$$
$$\wedge\ P\{\omega;\ f(\omega) \neq \lim_n f_n(\omega)\} = 0\big).$$

We have

1.2.2. $(f = \mathscr{P}\text{-}\lim_n f_n \wedge f' \underset{\mathscr{P}}{=} f \wedge f'_n \underset{\mathscr{P}}{=} f_n$ *for every* $n) \Rightarrow f' = \mathscr{P}\text{-}\lim_n f'_n$.

Proof is routine.

By 1.2.1, we can define the quotient linear space $\hat{L}(M)$ as the set of equivalence classes with respect to $\underset{\mathscr{P}}{=}$,

$$\hat{L}(M) \underset{\mathrm{Df}}{=} L(M)\,|_{\underset{\mathscr{P}}{=}} = \{[f];\ f \in L(M)\},$$

with the linear operations $+$ and $\cdot$ defined as follows:

If $f_1, f_2, f \in \hat{L}(M)$, then

$$f_1 + f_2 \underset{\mathrm{Df}}{=} [f_1 + f_2] \quad \text{for some } f_i \in f_i,\ i = 1, 2;$$

If $\alpha \in R$ and $\boldsymbol{f} \in \hat{L}(M)$, then

$$\alpha \cdot \boldsymbol{f} \underset{\mathrm{Df}}{=} [\alpha \cdot f] \quad \text{for some } f \in \boldsymbol{f}.$$

We can also define the following relation $\leqslant$ in $\hat{L}(M)$:

$$\boldsymbol{f}_1 \leqslant \boldsymbol{f}_2 \underset{\mathrm{Df}}{\Leftrightarrow} f_1 \underset{\mathscr{P}}{\leqslant} f_2 \quad \text{for some } f_i \in \boldsymbol{f}_i.$$

Clearly, $\leqslant$ is a partial order.

Further, by 1.2.2, we can define convergence in $\hat{L}(M)$ as follows: Let $\boldsymbol{f}, \boldsymbol{f}_n \in \hat{L}(M)$; then

$$\boldsymbol{f} = \lim_n \boldsymbol{f}_n \underset{\mathrm{Df}}{\Leftrightarrow} f = \mathscr{P}\text{-}\lim_n f_n \quad \text{for some } f \in \boldsymbol{f} \text{ and } f_n \in \boldsymbol{f}_n,\ n = 1, 2, \ldots$$

Finally, since $f \underset{\mathscr{P}}{=} g$ implies $\int\limits_{\Omega} f(\omega)\,dP = \int\limits_{\Omega} g(\omega)\,dP$ for every $P \in \mathscr{P}$, we can define $\int\limits_{\Omega} \boldsymbol{f}\,dP$ by the formula:

$$\int\limits_{\Omega} \boldsymbol{f}\,dP = \int\limits_{\Omega} f(\omega)\,dP \quad \text{for some } f \in \boldsymbol{f}.$$

Let us emphasize that in the above definitions existential quantifiers may be equivalently replaced by universal quantifiers.

We have

1.2.3. *If* $\boldsymbol{f}_n \leqslant \boldsymbol{f}_{n+1} \leqslant \boldsymbol{g}$ *for* $n = 1, 2, \ldots,$ *then there exists an* $\boldsymbol{f} \in \hat{L}(M)$ *such that*

$$\boldsymbol{f} = \lim_n \boldsymbol{f}_n \quad \textit{and} \quad \boldsymbol{f} \leqslant \boldsymbol{g}.$$

1.3. Almost transition functions. The following notion of almost transition function [2] is a natural generalization of the notion of transition function.

1.3.1. DEFINITION. Given two statistical spaces $M = (\Omega, \mathscr{A}, \mathscr{P})$ and $M' = (\Omega', \mathscr{A}', \mathscr{P}')$, we refer to a function $\varphi\colon \Omega \times \mathscr{A}' \to R$ as *almost transition function for* (M, M') if the following conditions are satisfied:

(0) $\forall A' \in \mathscr{A}'\ \ \varphi_{A'}$ is measurable;

(i) $\forall A' \in \mathscr{A}'\ \ 0 \underset{\mathscr{P}}{\leqslant} \varphi_{A'} \underset{\mathscr{P}}{\leqslant} 1$;

(ii) $\varphi_{\Omega'} \underset{\mathscr{P}}{=} 1$;

(iii) $\forall A' \in \mathscr{A}' \big[(\forall P \in \mathscr{P}\ \ P(A') = 0) \Rightarrow \varphi_{A'} \underset{\mathscr{P}}{=} 0\big]$;

(iv) $\forall (A_i')_{i=1,2,\ldots} \big(A_i' \cap A_j' = \emptyset$ for $i \neq j \Rightarrow \varphi_{\bigcup\limits_{i=1}^{\infty} A_i'} = \mathscr{P}\text{-}\lim\limits_n \sum\limits_{i=1}^{n} \varphi_{A_i'}\big)$.

(The consequent of the implication in (iv) will be briefly written in the form:

$$\varphi_{\bigcup\limits_{i=1}^{\infty} A_i'} \underset{\mathscr{P}}{=} \sum_{i=1}^{\infty} \varphi_{A_i'}.)$$

---

(2) Compare *statistical operation* in Morse and Sacksteder (1966).

For any $\varphi, \psi\colon \Omega \times \mathscr{A}' \to R$, let

$$\varphi \underset{\mathscr{P}}{=} \psi \underset{\mathrm{Df}}{\Leftrightarrow} \forall A' \in \mathscr{A}' \ \varphi_{A'} \underset{\mathscr{P}}{=} \psi_{A'}.$$

It is easy to check that

1.3.2. *If $\psi\colon \Omega \times \mathscr{A}' \to R$ is an almost transition function for (M, M'), and $\varphi \underset{\mathscr{P}}{=} \psi$, then φ is an almost transition function as well.*

As a consequence of 1.3.1 (iii) and (iv) we obtain

1.3.3. Lemma. *Let $\varphi\colon \Omega \times \mathscr{A}' \to R$ be an almost transition function for (M, M'). Then for arbitrary reals α_i, β_j, $\alpha_{n,i}$, and $\beta_{n,j}$*

(j) *if $\sum_{i=1}^{k} \alpha_i \chi_{A'_i} = \sum_{j=1}^{l} \beta_j \chi_{B'_j}$ then $\sum_{i=1}^{k} \alpha_i \varphi_{A'_i} \underset{\mathscr{P}}{=} \sum_{j=1}^{l} \beta_j \varphi_{B'_j}$;*

(jj) *if $f_n = \sum_{i=1}^{k_n} \alpha_{n,i} \chi_{A'_{n,i}}$, $f = \sum_{i=1}^{k} \alpha_i \chi_{A'_i}$, $f_n \leqslant f_{n+1}$ for every n, and $\lim_n f_n(\omega') = f(\omega')$ for every ω', then*

$$\sum_{i=1}^{k} \alpha_i \varphi_{A'_i} = \mathscr{P}\text{-}\lim_n \sum_{i=1}^{k_n} \alpha_{n,i} \varphi_{A'_{n,i}};$$

(jjj) *if $f_n = \sum_{i=1}^{k_n} \alpha_{n,i} \chi_{A'_{n,i}}$, $g_n = \sum_{j=1}^{l_n} \beta_{n,j} \chi_{B'_{n,i}}$, $f_n \leqslant f_{n+1}$ and $g_n \leqslant g_{n+1}$ for $n = 1, 2, \ldots$, $(f_n(\omega'))_n$ and $(g_n(\omega'))_n$ are convergent and $\lim_n f_n(\omega') = \lim_n g_n(\omega')$ for every ω', then there is an $f\colon \Omega \to R$ such that*

$$f = \mathscr{P}\text{-}\lim_n \sum_{i=1}^{k_n} \alpha_{n,i} \varphi_{A'_{n,i}} \text{ and } f = \mathscr{P}\text{-}\lim_n \sum_{j=1}^{l_n} \beta_{n,j} \varphi_{B'_{n,i}}.$$

The particular case of Lemma 1.3.3 for $\varphi_{A'} = \text{const}$, $A' \in \mathscr{A}'$, is well known and usually applied in the construction of integral. The proof of general case of 1.3.3 is analogous to that of the particular case.

The equivalence class $[\varphi]$ (with respect to $\underset{\mathscr{P}}{=}$) of any almost transition function φ generates an operator

$$T_{[\varphi]}\colon L(M') \to \hat{L}(M),$$

which is defined below in four steps. The steps (ii) and (iii) are justified by 1.3.3 (j) and (jjj), respectively.

1.3.4. Definition. Let $\varphi\colon \Omega \times \mathscr{A}' \to R$ be an almost transition function, and let $f \in L(M')$.

(i) If $f = \chi_{A'}$, then $T_{[\varphi]}(f) \underset{\mathrm{Df}}{=} [\varphi_{A'}]$.

(ii) If $f = \sum_{i=1}^{n} \alpha_i \chi_{A'_i}$ for $A'_i \in \mathscr{A}'$, $\alpha_i \in R$, $i = 1, \ldots, n$, then

$$T_{[\varphi]}(f) \underset{\mathrm{Df}}{=} \sum_{i=1}^{n} \alpha_i T_{[\varphi]}(\chi_{A'_i}).$$

(iii) If $f \geqslant 0$, then there exists a (weakly) increasing sequence of simple functions f_n: $\Omega' \to R$ such that

$$f(\omega') = \lim_n f_n(\omega') \quad \text{for} \quad \omega' \in \Omega;$$

then

$$T_{[\varphi]}(f) \underset{\mathrm{Df}}{=} \lim_n T_{[\varphi]}(f_n).$$

(iv) For arbitrary f, let

$$f_+ = \max(0, f) \quad \text{and} \quad f_- = \max(0, -f);$$

then $f = f_+ - f_-$ and

$$T_{[\varphi]}(f) \underset{\mathrm{Df}}{=} T_{[\varphi]}(f_+) - T_{[\varphi]}(f_-).$$

By 1.3.1 (iii) and (iv) one can prove

1.3.5. $f \underset{\varphi}{=} g \Rightarrow T_{[\varphi]}(f) = T_{[\varphi]}(g)$.

By 1,3.5, the operator $T_{[\varphi]}$ induces the quotient operator

$$\hat{T}_{[\varphi]}\colon \hat{L}(M') \to \hat{L}(M)$$

defined by the obvious formula

1.3.6. $\hat{T}_{[\varphi]}(\mathfrak{f}) \underset{\mathrm{Df}}{=} T_{[\varphi]}(f)$ for arbitrary $f \in \mathfrak{f}$.

Using 1.3.1 and 1.3.3 (ii), (iii), one can prove the following

1.3.7. Proposition. *For every almost transition function φ for (M, M')*

(i) $\hat{T}_{[\varphi]}$ *is a linear operator;*

(ii) $\hat{T}_{[\varphi]}$ *is monotone;*

$$\mathfrak{f} \leqslant \mathfrak{g} \Rightarrow \hat{T}_{[\varphi]}(\mathfrak{f}) \leqslant \hat{T}_{[\varphi]}(\mathfrak{g});$$

(iii) $\hat{T}_{[\varphi]}$ *preserves unit:*

$$\hat{T}_{[\varphi]}[1] = [1];$$

(iv) $\hat{T}_{[\varphi]}$ *is continuous in monotone convergence: if $\mathfrak{f}_n \leqslant \mathfrak{f}_{n+1}$ for $n = 1, 2, \ldots,$ then*

$$\mathfrak{f} = \lim_n \mathfrak{f}_n \Rightarrow \hat{T}_{[\varphi]}(\mathfrak{f}) = \lim_n \hat{T}_{[\varphi]}(\mathfrak{f}_n).$$

(Any operator satisfying (i)–(iv) is called a *statistical operator.*)

As a direct consequence of 1.3.4 we obtain

1.3.8. *Let χ: $\Omega \times \mathscr{A} \to R$ be defined by the formula*

$$\chi(\omega, A) \underset{\mathrm{Df}}{=} \chi_A(\omega) \quad \textit{for} \quad (\omega, A) \in \Omega \times \mathscr{A}.$$

Then

$$\hat{T}_{[\chi]}(f) = f \quad \textit{for every } f \in \hat{L}(M),$$

i.e. $\hat{T}_{[\chi]}$ *is the identity operator.*

We need the following three lemmas concerning the operators $\hat{T}_{[\varphi]}$.

1.3.9. LEMMA. *Let* φ *and* φ' *be almost transition functions for* (M, M') *and* (M', M''), *respectively, and let* φ'': $\Omega \times \mathscr{A}'' \to R$ *be an arbitrary function. If*

$$(*) \qquad [\varphi''_{A''}] = \hat{T}_{[\varphi]}[\varphi'_{A''}] \quad \textit{for every } A'' \in \mathscr{A}'',$$

then

(i) φ'' *is an almost transition function for* (M, M'')

and

(ii) $\hat{T}_{[\varphi'']} = \hat{T}_{[\varphi]}\hat{T}_{[\varphi']}$.

Proof. (i): Since φ' satisfies 1.3.1 (0)–(iv), by 1.3.3 and 1.3.4 the function φ'' satisfies 1.3.1 (0),

by 1.3.7 (i), (ii), and (iii) it satisfies 1.3.1 (i),
by 1.3.7 (iii) it satisfies 1.3.1 (ii),
by 1.3.7 (i) it satisfies 1.3.1 (iii), and
by 1.3.7 (i) and (iv) it satisfies 1.3.1 (iv).

(ii): By 1.3.7 (i) and (iv) combined with 1.3.4 it suffices to show that

$$(1) \qquad \hat{T}_{[\varphi'']}[\chi_{A''}] = \hat{T}_{[\varphi]}(\hat{T}_{[\varphi']}[\chi_{A''}])$$

for every $A'' \in \mathscr{A}''$. By 1.3.4 (i),

$$\hat{T}_{[\varphi'']}[\chi_{A''}] = [\varphi''_{A''}]$$

and

$$\hat{T}_{[\varphi]}\hat{T}_{[\varphi']}[\chi_{A''}] = \hat{T}_{[\varphi]}[\varphi'_{A''}],$$

whence, by (*), condition (1) holds. ■

Lemma 1.3.9 enables us to extend the definition of $\hat{T}_{[\varphi]}[f]$ over classes of $\mathscr{P}$-equal almost transition functions:

1.3.10. DEFINITION. Let φ: $\Omega \times \mathscr{A}' \to R$, φ': $\Omega' \times \mathscr{A}'' \to R$, and φ'': $\Omega \times \mathscr{A}'' \to R$ be almost transition functions (for (M, M'), (M', M''), and (M, M''), respectively); then

$$[\varphi''] = \hat{T}_{[\varphi]}[\varphi'] \underset{\mathrm{Df}}{\Leftrightarrow} \forall A'' \in \mathscr{A}''\, [\varphi''_{A''}] = \hat{T}_{[\varphi]}[\varphi'_{A''}].$$

1.3.11. LEMMA. *Let* φ: $\Omega \times \mathscr{A}' \to R$ *be an almost transition function for* M *and* M' *and let* $P \in \mathscr{P}$. *Then the function* P': $\mathscr{A}' \to R$ *defined by the formula*

$$(**) \qquad P'(A') = \int_{\Omega} \varphi_{A'}(\omega)\, dP$$

satisfies the following two conditions:

(i) $P' \in \mathrm{CAP}(\Omega', \mathscr{A}')$;

(ii) *for any* $f \in L(M')$

$$\int_{\Omega'} f(\omega')\,dP' = \int_{\Omega} \hat{T}_{[\varphi]}[f]\,dP.$$

Proof. Condition (i) follows directly by the properties of almost transition functions (1.3.1). Since both $\hat{T}_{[\varphi]}$ and $\int$ are linear operators continuous in monotone convergence, to prove (ii) it suffices to verify the equality for f being a characteristic function. Take $A' \in \mathscr{A}'$ and let $f = \chi_{A'}$; then $\hat{T}_{[\varphi]}[f] = [\varphi_{A'}]$ and the required equality reduces to

$$(1) \qquad \int_{\Omega'} \chi_{A'}(\omega')\,dP' = \int_{\Omega} \varphi_{A'}(\omega)\,dP.$$

Since

$$\int_{\Omega'} \chi_{A'}(\omega')\,dP' = P'(A'),$$

by the definition of P' we obtain (1). ∎

As for transition functions (compare 1.1), the probability measure P' defined by $(**)$ is referred to as *generated by the pair* (P, φ). The family

$$\mathscr{P}' = \{P';\ P' \text{ is generated by } (P, \varphi) \text{ for some } P \in \mathscr{P}\}$$

is said to be *generated by* $(\mathscr{P}, \varphi)$.

Of course, if $\varphi^{(1)} \underset{\mathscr{P}}{=} \varphi^{(2)}$, then P' is generated by $(P, \varphi^{(1)})$ if and only if it is generated by $(P, \varphi^{(2)})$. Consequently we shall sometimes say that P' is generated by $(P, [\varphi])$ when it is generated by (P, φ).

Finally, let us notice that the operator $\hat{T}_{[\varphi]}$ is a modification of an operator

$$T_{\varphi}^{0}\colon L^0(\Omega', \mathscr{A}') \to L^0(\Omega, \mathscr{A}),$$

where

$$L^0(\Omega, \mathscr{A}) = \{f\colon \Omega \to R;\ f \text{ is measurable and bounded}\}$$

and $\varphi\colon \Omega \times \mathscr{A}' \to [0, 1]$ is a transition function. The operator T_{φ}^{0} is defined by the formula

$$T_{\varphi}^{0}(f) \underset{\mathrm{Df}}{=} \int_{\Omega'} f(\omega')\,d\varphi_{\omega}.$$

The properties of T_{φ}^{0} are analogous to those of $\hat{T}_{[\varphi]}$: clearly T_{φ}^{0} is linear, monotone, continuous, and preserves unit (compare 1.3.7); moreover, Lemmas 1.1.2 and 1.1.3 are counterparts of 1.3.11 (ii) and 1.3.8.

2. Concrete categories of measurable, probabilistic, and statistical spaces

We start with the following three concrete categories, Mes, Pr, and St.

The category Mes has measurable spaces as objects and measurable maps as morphisms. (3)

The category Pr has probabilistic spaces as objects and random variables as morphisms; more precisely, $f \in \mathrm{Pr}((\Omega, \mathscr{A}, P), (\Omega, \mathscr{A}', P'))$ iff $f \in \mathrm{Mes}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'))$ and $P' = Pf^{-1}$.

Finally, the category St has statistical spaces as objects and statistics as morphisms. More precisely, objects of St are of the form $M = (\Omega, \mathscr{A}, \mathscr{P})$, where $\mathscr{A}$ is a σ-field of subsets of Ω and $\mathscr{P}$ is a family of probability measures. For any pair of objects in St, $M = (\Omega, \mathscr{A}, \mathscr{P})$ and $M' = (\Omega', \mathscr{A}', \mathscr{P})$, the class of morphisms $\mathrm{St}(M, M')$ consists of functions $f\colon \Omega \to \Omega'$ such that

$$\{f^{-1}(A');\ A' \in \mathscr{A}'\} \subset \mathscr{A}$$

and

$$\mathscr{P}' = \{Pf^{-1};\ P \in \mathscr{P}\}.$$

Clearly, Pr may be treated as a full subcategory of St, with objects of the form $(\Omega, \mathscr{A}, \{P\})$. More precisely, Pr is isomorphic to such a subcategory. The embedding $\nabla\colon \mathrm{Pr} \to \mathrm{St}$ is defined by the formulae

$$\nabla((\Omega, \mathscr{A}, P)) = (\Omega, \mathscr{A}, \{P\})$$

and

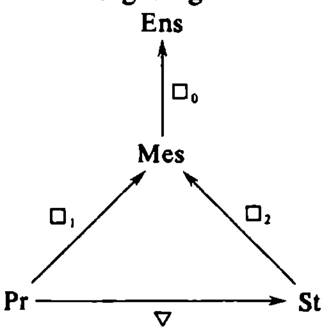
$$\nabla(f) = f.$$

There are also natural functors

$$\Box_1\colon \mathrm{Pr} \to \mathrm{Mes} \quad \text{and} \quad \Box_2\colon \mathrm{St} \to \mathrm{Mes}$$

which assign to every probabilistic space $(\Omega, \mathscr{A}, P)$ or statistical space $(\Omega, \mathscr{A}, \mathscr{P})$ its reduct $(\Omega, \mathscr{A})$, and a functor $\Box_0\colon \mathrm{Mes} \to \mathrm{Ens}$ which assigns to any measurable space $(\Omega, \mathscr{A})$ its universe Ω. We refer to $\Box_i$ $(i = 0, 1, 2)$ as forgetful functors. So, we have the following diagram

(1)



which is, of course, commutative.

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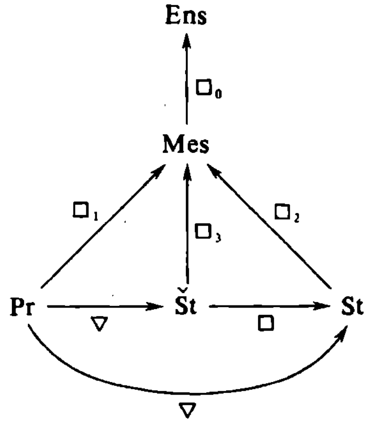
(3) By a map $f\colon X \to Y$ we mean the triple (X, Y, f) consisting of two objects X, Y and a function f. Thus, in fact, morphisms of Mes are of the form $((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'), f)$, where f is a measurable function. The same convention is admitted for other categories.

Let us now define a category $\check{\mathrm{St}}$ of statistical spaces with indexed families of measures. Objects of $\check{\mathrm{St}}$ are of the form $M = (\Omega, \mathscr{A}, (P_\lambda)_{\lambda\in\Lambda})$. For any pair of objects, $M = (\Omega, \mathscr{A}, (P_\lambda)_{\lambda\in\Lambda})$ and $M' = (\Omega', \mathscr{A}', (P'_{\lambda'})_{\lambda'\in\Lambda'})$, $(\alpha, f) \in \check{\mathrm{St}}(M, M')$ iff $\alpha\colon \Lambda \to \Lambda'$ is a surjection and

(2) $\qquad f \in \mathrm{Pr}((\Omega, \mathscr{A}, P_\lambda), (\Omega', \mathscr{A}', P'_{\alpha(\lambda)}))$ for every $\lambda \in \Lambda$.

It is obvious that diagram (1) can be extended to the commutative diagram

(3)

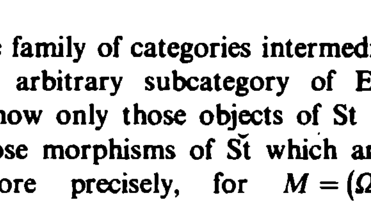


where $\square_3$ is again a forgetful functor, while $\square$, intuitively speaking, allows us to forget of the indexing and of the indexing map, i.e.

$$\square((\Omega, \mathscr{A}, (P_\lambda)_{\lambda\in\Lambda}) = (\Omega, \mathscr{A}, \{P_\lambda;\ \lambda\in\Lambda\}) \quad \text{and} \quad \square((\alpha, f)) = f.$$

When restricting the class of objects in $\check{\mathrm{St}}$ to those with a fixed indexing set Λ, it is natural to distinguish a subcategory $\check{\mathrm{St}}_\Lambda$ of $\check{\mathrm{St}}$ with morphisms preserving the indexing (i.e. with the function α in (2) being identity). Clearly $\check{\mathrm{St}}_\Lambda$ is not a full subcategory of $\check{\mathrm{St}}$. For every Λ there are an embedding $\nabla_\Lambda\colon \check{\mathrm{St}}_\Lambda \to \check{\mathrm{St}}$ and a functor $\square_\Lambda\colon \check{\mathrm{St}}_\Lambda \to \mathrm{St}$ such that the diagram

(4)



is commutative.

There is the whole family of categories intermediate between $\check{\mathrm{St}}_\Lambda$ and $\check{\mathrm{St}}$. Indeed, let C be an arbitrary subcategory of Ens with surjections as morphisms. Consider now only those objects of St which are indexed by a $\Lambda \in \mathrm{Ob}\,\mathrm{C}$ and only those morphisms of $\check{\mathrm{St}}$ which are of the form (α, f) for some $\alpha \in \mathrm{Mor}\,\mathrm{C}$. More precisely, for $M = (\Omega, \mathscr{A}, (P_\lambda)_{\lambda\in\Lambda})$ and $M' = (\Omega', \mathscr{A}', (P'_{\lambda'})_{\lambda'\in\Lambda'})$,

$$(\alpha, f) \in \check{\mathrm{St}}(M, M') \quad \text{iff} \quad \big(\alpha \in \mathrm{C}(\Lambda, \Lambda'),\ f \in \mathrm{Mes}((\Omega, \mathscr{A})(\Omega', \mathscr{A}')),\ \text{and}\ \ P_\lambda f^{-1} = P'_{\alpha(\lambda)} \text{ for every } \lambda\in\Lambda\big).$$

It is clear that we obtain a subcategory, $\check{\mathrm{St}}_{\mathrm{C}}$, of $\check{\mathrm{St}}$. In particular, $\check{\mathrm{St}}_{\Lambda}$ corresponds to C with only one object Λ and only one morphism 1_{Λ}, while $\check{\mathrm{St}}$ corresponds to the subcategory of Ens with all surjections as morphisms. Clearly

$$\mathrm{C} \subset \mathrm{C}' \Rightarrow \check{\mathrm{St}}_{\mathrm{C}} \subset \check{\mathrm{St}}_{\mathrm{C}'},$$

whence

$$\check{\mathrm{St}}_{\Lambda} \subset \check{\mathrm{St}}_{\mathrm{C}} \subset \check{\mathrm{St}} \quad \text{for every C.}$$

These categories of indexed statistical spaces correspond to various kinds of product of statistical spaces (compare Moszyńska (1983)).

3. Categories connected with transition functions

We are now going to define some other categories of measurable, probabilistic, and statistical spaces.

3.1. Category $\mathrm{Mes}_{\mathrm{trans}}$. We define a structure $\mathrm{Mes}_{\mathrm{trans}}$ consisting of objects, morphisms, operation of composition, and identity morphisms, and prove $\mathrm{Mes}_{\mathrm{trans}}$ to be a category. Objects are measurable spaces. Morphisms are defined as follows: for any pair $((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'))$

$$\mathrm{Mes}_{\mathrm{trans}}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'))$$

$$\underset{\mathrm{Df}}{=} \{\varphi\colon \Omega \times \mathscr{A}' \to [0, 1];\ \varphi \text{ is a transition function for } (\Omega, \mathscr{A}), (\Omega', \mathscr{A}')\}.$$

To define composition in $\mathrm{Mes}_{\mathrm{trans}}$, take

$$\varphi \in \mathrm{Mes}_{\mathrm{trans}}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}')) \quad \text{and} \quad \varphi' \in \mathrm{Mes}_{\mathrm{trans}}((\Omega', \mathscr{A}'), (\Omega'', \mathscr{A}'')),$$

and let

(j) $$(\varphi' \circ \varphi)(\omega, A'') \underset{\mathrm{Df}}{=} \int_{\Omega'} \varphi'_{A''}(\omega')\, d\varphi_{\omega}$$

(compare Preliminaries).

Let us check that

(jj) $$\varphi' \circ \varphi \in \mathrm{Mes}_{\mathrm{trans}}((\Omega, \mathscr{A}), (\Omega'', \mathscr{A}'')).$$

By the assumption, φ_{ω} is a probability measure for every $\omega \in \Omega$, and $\varphi'_{A''}$ is measurable for every $A'' \in \mathscr{A}''$. Since, by (j),

$$(\varphi' \circ \varphi)_{\omega}(A'') = (\varphi' \circ \varphi)_{A''}(\omega) = \int_{\Omega'} \varphi'_{A''}(\omega')\, d\varphi_{\omega},$$

it is easy to check that $(\varphi' \circ \varphi)_{\omega}$ is a probability measure for any $\omega \in \Omega$. Moreover, by Lemma 1.1.2, $(\varphi' \circ \varphi)_{A''}$ is measurable for any $A'' \in \mathscr{A}''$, which proves (jj).

Let $1_{(\Omega,\mathscr{A})}$ be defined by the formula

(jjj) $$1_{(\Omega,\mathscr{A})}(\omega, A) \underset{\mathrm{Df}}{\equiv} \chi_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $1_M \in \mathbf{Mes}_{\mathrm{trans}}((\Omega, \mathscr{A})(\Omega, \mathscr{A}))$.

3.1.1. PROPOSITION. $\mathbf{Mes}_{\mathrm{trans}}$ *is a category.*

Proof. Notice first that for every $\varphi \in \mathbf{Mes}_{\mathrm{trans}}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'))$

$$1_{(\Omega',\mathscr{A}')} \circ \varphi = \varphi \quad \text{and} \quad \varphi \circ 1_{(\Omega,\mathscr{A})} = \varphi.$$

Indeed, according to (j) and (jjj),

$$(1_{M'} \circ \varphi)(\omega, A') = \int_{\Omega'} \chi_{A'}(\omega')\, d\varphi_\omega = \varphi_\omega(A') = \varphi(\omega, A').$$

In turn

$$(\varphi \circ 1_M)(\omega, A') = \int_{\Omega} \varphi_{A'}(\bar{\omega})\, dP_\omega,$$

where $P_\omega(A) = \chi_A(\omega)$; thus, by Lemma 1.1.3,

$$(\varphi \circ 1_M)(\omega, A') = \varphi(\omega, A').$$

It remains to show that the operation $\circ$ is associative. Let $\varphi \in \mathbf{Mes}_{\mathrm{trans}}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'))$, $\varphi' \in \mathbf{Mes}_{\mathrm{trans}}((\Omega', \mathscr{A}'), (\Omega'', \mathscr{A}''))$, and $\varphi'' \in \mathbf{Mes}_{\mathrm{trans}}((\Omega'', \mathscr{A}''), (\Omega''', \mathscr{A}'''))$. We have

$$[\varphi'' \circ (\varphi' \circ \varphi)](\omega, A''') = \int_{\Omega''} \varphi''_{A'''}(\omega'')\, d(\varphi' \circ \varphi)_\omega,$$

where

$$(\varphi' \circ \varphi)_\omega(A'') = \int_{\Omega'} \varphi'_{A''}(\omega')\, d\varphi_\omega \quad \text{for every } A'' \in \mathscr{A}''.$$

On the other hand,

$$[(\varphi'' \circ \varphi') \circ \varphi](\omega, A''') = \int_{\Omega'} (\varphi'' \circ \varphi')_{A'''}(\omega')\, d\varphi_\omega = \int_{\Omega'} \int_{\Omega''} \varphi''_{A'''}(\omega'')\, d\varphi'_{\omega'}\, d\varphi_\omega,$$

whence, by Lemma 1.1.2,

$$\varphi'' \circ (\varphi' \circ \varphi) = (\varphi'' \circ \varphi') \circ \varphi. \ \blacksquare$$

3.1.2. *For any measurable space* $(\Omega, \mathscr{A})$, *let*

(i) $$\Phi(\Omega, \mathscr{A}) = (\Omega, \mathscr{A});$$

for any $f \in \mathbf{Mes}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'))$, *let*

(ii) $$\Phi(f) = \varphi, \quad \textit{where } \varphi(\omega, A') \underset{\mathrm{Df}}{\equiv} \chi_{f^{-1}(A')}(\omega).$$

Formulae (i) *and* (ii) *define a covariant functor*

$$\Phi\colon \text{Mes} \to \text{Mes}_{\text{trans}}.$$

Proof. Notice first that for $f \in \text{Mes}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'))$ the function $\varphi\colon \Omega \times \mathscr{A}' \to [0, 1]$ defined by the formula

$$(1) \qquad \varphi(\omega, A') = \chi_{f^{-1}(A')}(\omega)$$

belongs to $\text{Mes}_{\text{trans}}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'))$. Indeed, for every $\omega \in \Omega$ the function φ_ω is a probability measure; moreover, for every $A' \in \mathscr{A}'$ the function $\varphi_{A'}\colon \Omega \to [0, 1]$ is measurable. Clearly,

$$(2) \qquad \Phi(\text{id}_{(\Omega, \mathscr{A})}) = 1_{(\Omega, \mathscr{A})},$$

since $1_{(\Omega, \mathscr{A})}(\omega, A) = \chi_A(\omega)$. In turn, if $f \in \text{Mes}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'))$ and $f' \in \text{Mes}((\Omega', \mathscr{A}'), (\Omega'', \mathscr{A}''))$, then

$$(3) \qquad \Phi(f'f) = \Phi(f') \circ \Phi(f).$$

Indeed, let $P'_\omega(A') \underset{\text{Df}}{=} \Phi(f)(\omega, A') = \chi_{f^{-1}(A')}(\omega)$; then

$$\begin{aligned}(\Phi(f') \circ \Phi(f))(\omega, A'') &= \int_{\Omega'} \Phi(f')(\omega', A'')\, dP'_\omega \\ &= \int_{\Omega'} \chi_{f'^{-1}(A'')}(\omega')\, dP'_\omega = P'_\omega(f'^{-1}(A'')) = \chi_{f^{-1}f'^{-1}(A'')}(\omega) \\ &= \chi_{(f'f)^{-1}(A'')}(\omega) = \Phi(f'f)(\omega, A''). \quad \blacksquare\end{aligned}$$

Let us consider the following equivalence relation on the class of morphisms of Mes.

3.1.3. Definition. Let $f_1, f_2 \in \text{Mes}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'))$; then

$$f_1 \sim f_2 \underset{\text{Df}}{\Leftrightarrow} f_1^{-1}(A') = f_2^{-1}(A') \quad \text{for every } A' \in \mathscr{A}'.$$

Clearly, $\sim$ is a congruence with respect to the composition in Mes, since

$$f_1 \sim f_2 \Rightarrow gf_1 \sim gf_2 \wedge f_1 f \sim f_2 f$$

whenever $g \in \text{Mes}((\Omega', \mathscr{A}'), (\Omega'', \mathscr{A}''))$ and $f \in \text{Mes}((\Omega'', \mathscr{A}''), (\Omega, \mathscr{A}))$.

Consequently, we can define the quotient category of measurable spaces, $\text{Mes}|_\sim$, with equivalence classes of $\sim$ as morphisms, and with identity morphisms and composition defined by the formulae:

$$\tilde{1}_M \underset{\text{Df}}{=} [1_M]_\sim \quad \text{and} \quad [g]_\sim \circ [f]_\sim \underset{\text{Df}}{=} [g \circ f]_\sim.$$

Let us prove

3.1.4. Theorem. $\text{Mes}|_\sim \underset{\text{iso}}{\subset} \text{Mes}_{\text{trans}}$, *with the embedding being the quotient functor* $\tilde{\Phi}$ *induced by* Φ (comp. 3.1.2).

Proof. Evidently

$$(1) \qquad f_1 \sim f_2 \Rightarrow \Phi(f_1) = \Phi(f_2),$$

thus the functor Φ induces the functor

$$\tilde{\Phi}\colon \mathrm{Mes}|_{\sim} \to \mathrm{Mes}_{\mathrm{trans}}$$

defined by the formulae

$$\tilde{\Phi}((\Omega, \mathscr{A})) = \Phi((\Omega, \mathscr{A})) \quad \text{and} \quad \tilde{\Phi}([f]_{\sim}) = \Phi(f).$$

It remains to prove that $\tilde{\Phi}$ is injective,

$$\Phi(f_1) = \Phi(f_2) \Rightarrow f_1 \sim f_2.$$

Take $f_1, f_2 \in \mathrm{Mes}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}'))$ and let $\Phi(f_1) = \Phi(f_2)$, i.e.

$$\chi_{f_1^{-1}(A')}(\omega) = \chi_{f_2^{-1}(A')}(\omega) \quad \text{for every } (\omega, A') \in \Omega \times \mathscr{A}'.$$

Then $f_1^{-1}(A') = f_2^{-1}(A')$ for every $A' \in \mathscr{A}'$, i.e. $f_1 \sim f_2$. ∎

3.2. Category $\mathrm{St}_{\mathrm{trans}}$. Objects of the structure $\mathrm{St}_{\mathrm{trans}}$ are statistical spaces. Morphisms are defined as follows: for any two statistical spaces, $M = (\Omega, \mathscr{A}, \mathscr{P})$ and $M' = (\Omega', \mathscr{A}', \mathscr{P}')$,

$$\varphi \in \mathrm{St}_{\mathrm{trans}}(M, M')$$

$$\underset{\mathrm{Df}}{\Leftrightarrow} [\varphi \in \mathrm{Mes}_{\mathrm{trans}}((\Omega, \mathscr{A}), (\Omega', \mathscr{A}')) \text{ and } \mathscr{P}' \text{ is generated by } (\mathscr{P}, \varphi)].$$

Composition and identity morphisms in $\mathrm{St}_{\mathrm{trans}}$ are defined by formulae (j) and (jjj) in 3.1.

3.2.1. Proposition. $\mathrm{St}_{\mathrm{trans}}$ *is a category.*

Proof. Since, by 3.1.1, $\mathrm{Mes}_{\mathrm{trans}}$ is a category, it suffices to check that:

(1) if $\mathscr{P}'$ is generated by $(\mathscr{P}, \varphi)$ and $\mathscr{P}''$ is generated by $(\mathscr{P}', \varphi')$, then $\mathscr{P}''$ is generated by $(\mathscr{P}, \varphi' \circ \varphi)$

and

(2) $\mathscr{P}$ is generated by $(\mathscr{P}, 1_M)$.

Condition (2) is obvious; taking $\varphi'_{A''}$ for f in Lemma 1.1.2, we obtain (1). ∎

3.2.2. *For any statistical space M let*

(i) $$\Phi'(M) = M;$$

for any $f \in \mathrm{St}(M, M')$ *let*

(ii) $$\Phi'(f) = \varphi, \quad \text{where} \quad \varphi(\omega, A') \underset{\mathrm{Df}}{=} \chi_{f^{-1}(A')}(\omega).$$

Formulae (i) *and* (ii) *define a covariant functor*

$$\Phi'\colon \mathrm{St} \to \mathrm{St}_{\mathrm{trans}}.$$

Proof. Let us notice that $\Phi'(f) = \Phi(f)$ for every $f \in \mathrm{St}(M, M')$. Thus, by 3.1.2, it suffices to show that $\mathscr{P}'$ is generated by $(\mathscr{P}, \Phi'(f))$. Indeed, for every $P \in \mathscr{P}$

$$\int_{\Omega} \Phi'(f)(\omega, A')\,dP = \int_{\Omega} \chi_{f^{-1}(A')}(\omega)\,dP = \int_{f^{-1}(A')} dP = Pf^{-1}(A').$$

On the other hand, $\mathscr{P}' = \{Pf^{-1};\ P \in \mathscr{P}\}$. ∎

The relation $\sim$ defined by 3.1.3 for all the measurable functions from $(\Omega, \mathscr{A})$ to $(\Omega', \mathscr{A}')$ is also a congruence in $\mathrm{St}(M, M')$. The connection between the quotient category $\mathrm{St}|_\sim$ and the category $\mathrm{St}_{\mathrm{trans}}$ is given by

3.2.3. Theorem. $\mathrm{St}|_\sim \underset{\mathrm{iso}}{\subset} \mathrm{St}_{\mathrm{trans}}$ *with the embedding being the quotient functor* $\tilde{\Phi}'$ *induced by* Φ' (comp. 3.2.2).

Proof. The functor Φ' defined in 3.2.2 satisfies the conditions

(1) $$f_1 \sim f_2 \Rightarrow \Phi'(f_1) = \Phi'(f_2)$$

and

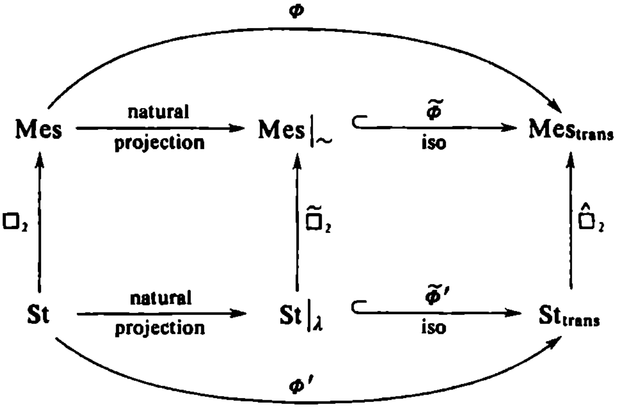
(2) $$\Phi'(f_1) = \Phi'(f_2) \Rightarrow f_1 \sim f_2.$$

By (1), Φ' induces the quotient functor

$$\tilde{\Phi}'\colon \mathrm{St}|_\sim \to \mathrm{St}_{\mathrm{trans}}.$$

By (2), this quotient functor is injective. This completes the proof. ∎

The relationships between Mes, $\mathrm{Mes}_{\mathrm{trans}}$, St, and $\mathrm{St}_{\mathrm{trans}}$ can be now expressed by the following diagram:



It is clear how to define the functors $\tilde{\square}_2$ and $\hat{\square}_2$.

3.3. Category $\mathrm{St}_{\mathrm{dist}}$. Objects of the structure $\mathrm{St}_{\mathrm{dist}}$ are statistical spaces. Morphisms are defined by the formula

$$\mathrm{St}_{\mathrm{dist}}(M, M') \underset{\mathrm{Df}}{=} \{\psi\colon \mathscr{P} \xrightarrow{\mathrm{onto}} \mathscr{P}'\}.$$

Identity morphism 1_M is simply the identity function on $\mathscr{P}$. Composition is the ordinary composition of functions. Thus, obviously, $\mathrm{St}_{\mathrm{dist}}$ is a category (isomorphic to a subcategory of Ens).

3.3.1. *For any statistical space M, let*

(i) $$\Psi(M) = M;$$

for any $\varphi \in \mathrm{St}_{\mathrm{trans}}(M, M')$ *let*

(ii) $$\Psi(\varphi) = \psi\colon \mathscr{P} \to \mathrm{CAP}(\Omega', \mathscr{A}').$$

where

$$\psi(P)(A') \underset{\mathrm{Df}}{\equiv} \int_{\Omega} \varphi(\omega, A')\,dP \quad \textit{for every } A' \in \mathscr{A}', P \in \mathscr{P}.$$

Formulae (i) *and* (ii) *define a covariant functor*

$$\Psi\colon \mathrm{St}_{\mathrm{trans}} \to \mathrm{St}_{\mathrm{dist}}.$$

Proof. If $\varphi \in \mathrm{St}_{\mathrm{trans}}(M, M')$, then $\mathscr{P}'$ is generated by $(\mathscr{P}, \varphi)$. Therefore, by (ii), $\mathscr{P}' = \{\psi(P);\ P \in \mathscr{P}\}$, whence $\psi \in \mathrm{St}_{\mathrm{dist}}(M, M')$.

If $\psi = \Psi(1_M)$, then for every $A \in \mathscr{A}$

$$\psi(P)(A) = \int_{\Omega} \chi_A(\omega)\,dP = P(A),$$

i.e. $\psi(P) = P$, whence

(1) $$\Psi(1_M) = 1_{\Psi(M)}.$$

Take now $\varphi \in \mathrm{St}_{\mathrm{trans}}(M, M')$ and $\varphi' \in \mathrm{St}_{\mathrm{trans}}(M', M'')$, and let $\psi = \Psi(\varphi)$ and $\psi' = \Psi(\varphi')$. Setting

$$\psi_1 = \Psi(\varphi' \circ \varphi) \quad \text{and} \quad \psi_2 = \Psi(\varphi')\,\Psi(\varphi) = \psi'\psi$$

we have

$$\psi_1(P)(A'') = \int_{\Omega} (\varphi' \circ \varphi)(\omega, A'')\,dP = \int_{\Omega}\int_{\Omega'} \varphi'(\omega', A'')\,d\varphi_\omega\,dP$$

and

$$\psi_2(P)(A'') = \big(\psi'\psi(P)\big)(A'') = \int_{\Omega'} \varphi'(\omega', A'')\,d\psi(P),$$

where $\psi(P)(A') = \int_{\Omega} \varphi(\omega, A')\,dP$, whence by Lemma 1.1.2

$$\psi_2(P)(A'') = \int_{\Omega}\int_{\Omega'} \varphi'(\omega', A'')\,d\varphi_\omega\,dP.$$

Thus $\psi_1 = \psi_2$, i.e.

(2) $$\Psi(\varphi' \circ \varphi) = \Psi(\varphi')\,\Psi(\varphi).$$

By (1) and (2), Ψ is a covariant functor. ∎

Let us consider the following equivalence $\approx$ on the class of morphisms of $\mathrm{St}_{\mathrm{trans}}$.

3.3.2. DEFINITION. For $\varphi_1, \varphi_2 \in \mathrm{St}_{\mathrm{trans}}(M, M')$,

$$\varphi_1 \approx \varphi_2 \underset{\mathrm{Df}}{\Leftrightarrow} \int_\Omega \varphi_1(\omega, A')\,dP = \int_\Omega \varphi_2(\omega, A')\,dP \quad \text{for every } P \in \mathscr{P}$$

(i.e. $\varphi_1 \approx \varphi_2$ iff for every $P \in \mathscr{P}$, the pairs (P, φ_1) and (P, φ_2) generate the same measure $P' \in \mathscr{P}'$).

3.3.3. *$\approx$ is a congruence with respect to the composition in* $\mathrm{St}_{\mathrm{trans}}$.

Proof. Take $\varphi_i \in \mathrm{St}_{\mathrm{trans}}(M, M')$ and let $\varphi_1 \approx \varphi_2$. By Lemma 1.1.2, for arbitrary $\varphi \in \mathrm{St}_{\mathrm{trans}}(M'', M)$ and $P'' \in \mathscr{P}''$, if P is generated by (P'', φ), then

$$\int_{\Omega''} (\varphi_i \circ \varphi)_{A'}(\omega'')\,dP'' = \int_{\Omega''} \int_\Omega \varphi_i(\omega, A')\,d\varphi_\omega\,dP'' = \int_\Omega \varphi_i(\omega, A')\,dP \quad \text{for} \quad i = 1, 2,$$

whence

$$\varphi_1 \circ \varphi \approx \varphi_2 \circ \varphi.$$

On the other hand, for any $\varphi' \in \mathrm{St}_{\mathrm{trans}}(M', M'')$,

$$\varphi' \circ \varphi_1 \approx \varphi' \circ \varphi_2.$$

Indeed, let

$$P_i(A') = \int_\Omega \varphi_i(\omega, A')\,dP, \quad i = 1, 2;$$

then $\varphi_1 \approx \varphi_2$ implies $P_1 \doteq P_2$, and by Lemma 1.1.2

$$\int_\Omega (\varphi' \circ \varphi_i)_{A''}(\omega)\,dP = \int_\Omega \int_{\Omega'} \varphi_{A''}(\omega')\,d(\varphi_i)_\omega\,dP = \int_{\Omega'} \varphi'(\omega', A'')\,dP_i,$$

whence

$$\varphi' \circ \varphi_1 \approx \varphi' \circ \varphi_2. \ \blacksquare$$

We can now define the quotient category $\mathrm{St}_{\mathrm{trans}}|_\approx$, i.e. the category of statistical spaces with equivalence classes of morphisms in $\mathrm{St}_{\mathrm{trans}}$ (with resp. to $\approx$) as morphisms, and with composition and identity morphisms defined in obvious way.

Let us prove

3.3.4. THEOREM. $\mathrm{St}_{\mathrm{trans}}|_\approx \underset{\mathrm{iso}}{\subset} \mathrm{St}_{\mathrm{dist}}$ *with the embedding being the quotient functor $\tilde{\Psi}$ induced by Ψ* (comp. 3.3.1).

Proof. Clearly

(1) $$\varphi_1 \approx \varphi_2 \Rightarrow \Psi(\varphi_1) = \Psi(\varphi_2),$$

thus Ψ induces the functor

$$\tilde{\Psi}\colon \mathrm{St}_{\mathrm{trans}}|_\approx \to \mathrm{St}_{\mathrm{dist}}$$

defined by the formulae

$$\tilde{\Psi}(M) = \Psi(M) \quad \text{and} \quad \tilde{\Psi}([\varphi]_{\approx}) = \Psi(\varphi).$$

It is also clear that

$$\Psi(\varphi_1) = \Psi(\varphi_2) \Rightarrow \varphi_1 \approx \varphi_2, \tag{2}$$

i.e. $\tilde{\Psi}$ is injective. This completes the proof. ∎

The following diagram reflects the relationships between St, $\mathrm{St}_{\mathrm{trans}}$, and $\mathrm{St}_{\mathrm{dist}}$:

$$\mathrm{St} \xrightarrow[\text{proj.}]{\text{nat}} \mathrm{St}|_{\sim} \underset{\text{iso}}{\overset{\Phi'}{\hookrightarrow}} \mathrm{St}_{\mathrm{trans}} \xrightarrow[\text{proj.}]{\text{nat}} \mathrm{St}_{\mathrm{trans}}|_{\approx} \underset{\text{iso}}{\overset{\Psi}{\hookrightarrow}} \mathrm{St}_{\mathrm{dist}}.$$

When replacing statistical spaces by probabilistic spaces, we pass to $\mathrm{Pr}|_{\sim}$ and $\mathrm{Pr}_{\mathrm{trans}}$, and obtain the following analogue of 3.1.4 and 3.2.3.

3.3.5. THEOREM. $\mathrm{Pr}|_{\sim} \underset{\text{iso}}{\subset} \mathrm{Pr}_{\mathrm{trans}}$ *with the embedding being the quotient functor* $\tilde{\Psi}$ *induced by* Ψ.

3.4. Categories of indexed statistical spaces. Similarly as for the category St (see Section 2), for each of the categories considered in this section there are corresponding categories of indexed statistical spaces.

The corresponding categories for $\mathrm{St}_{\mathrm{trans}}$ are $(\check{\mathrm{St}}_{\mathrm{trans}})_{\mathrm{C}}$ for $\mathrm{C} \subset \mathrm{Ens}$ (with surjective morphisms), in particular $\check{\mathrm{St}}_{\mathrm{trans}}$ for C with all surjections as morphisms, and $(\check{\mathrm{St}}_{\mathrm{trans}})_{\Lambda}$ for C with only one object Λ and only one morphism 1_{Λ}. Clearly, for $M = (\Omega, \mathscr{A}, (P_\lambda)_{\lambda\in\Lambda})$ and $M' = (\Omega', \mathscr{A}', (P'_{\lambda'})_{\lambda'\in\Lambda'})$

$$(\alpha, \varphi) \in (\check{\mathrm{St}}_{\mathrm{trans}})_{\mathrm{C}}(M, M')$$
$$\underset{\mathrm{Df}}{\Leftrightarrow} \left[\alpha \in \mathrm{C}(\Lambda, \Lambda') \wedge \forall \lambda \in \Lambda \left(\varphi \in \mathrm{Pr}_{\mathrm{trans}}\left((\Omega, \mathscr{A}, P_\lambda), (\Omega', \mathscr{A}', P'_{\alpha(\lambda)})\right)\right)\right];$$
$$(\alpha, \psi) \in (\check{\mathrm{St}}_{\mathrm{dist}})_{\mathrm{C}}(M, M') \underset{\mathrm{Df}}{\Leftrightarrow} \left[\alpha \in \mathrm{C}(\Lambda, \Lambda') \wedge \forall \lambda \in \Lambda \left(\psi(P_\lambda) = P'_{\alpha(\lambda)}\right)\right].$$

Natural definitions of $(\check{\mathrm{St}}|_{\sim})_{\mathrm{C}}$ and $(\check{\mathrm{St}}_{\mathrm{trans}}|_{\approx})_{\mathrm{C}}$ are left to the reader. Clearly, the relationships between the categories $\check{\mathrm{St}}_{\mathrm{C}}$, $(\check{\mathrm{St}}_{\mathrm{trans}})_{\mathrm{C}}$, and $(\check{\mathrm{St}}_{\mathrm{dist}})_{\mathrm{C}}$ are analogous to those for St, $\mathrm{St}_{\mathrm{trans}}$, and $\mathrm{St}_{\mathrm{dist}}$.

4. Categories connected with almost transition functions

We are now going to define some other category of statistical spaces, $\mathrm{St}_{\mathrm{a.trans}}$. The structure $\mathrm{St}_{\mathrm{a.trans}}$ is defined as follows:

For any two statistical spaces, $M = (\Omega, \mathscr{A}, \mathscr{P})$ and $M' = (\Omega', \mathscr{A}', \mathscr{P}')$, let

$$\mathrm{St}_{\mathrm{a.trans}}(M, M') \underset{\mathrm{Df}}{=} \{[\varphi];\ \varphi \text{ is an almost transition function for } (M, M') \text{ and } \mathscr{P}' \text{ is generated by } (\mathscr{P}, [\varphi])\}$$

(compare 1.3).

The identity morphism 1_M is defined by the formula

$$1_M \underset{\mathrm{Df}}{\equiv} [\chi], \quad \text{where} \quad \chi(\omega, A) = \chi_A(\omega).$$

Let us define composition of morphisms. Take

$$[\varphi] \in \mathrm{St}_{\mathrm{a.trans}}(M, M') \quad \text{and} \quad [\varphi'] \in \mathrm{St}_{\mathrm{a.trans}}(M', M'');$$

then, by 1.3.10, we can set

$$[\varphi'] \circ [\varphi] \underset{\mathrm{Df}}{\equiv} \hat{T}_{[\varphi]} [\varphi'].$$

4.1. PROPOSITION. $\mathrm{St}_{\mathrm{a.trans}}$ *is a category.*

Proof. Let us check that the identity morphisms are neutral elements of the composition. Take $[\varphi] \in \mathrm{St}_{\mathrm{a.trans}}(M, M')$; let $1_M = [\chi]$ and $1_{M'} = [\chi']$, where $\chi(\omega, A) = \chi_A(\omega)$ and $\chi'(\omega', A') = \chi_{A'}(\omega')$. Then, by 1.3.8

$$[\varphi] \circ 1_M = [\varphi] \circ [\chi] = [\varphi],$$

and by 1.3.4 (i)

$$1_M \circ [\varphi'] = [\chi'] \circ [\psi'] = [\varphi'].$$

Further, notice that

$$[\varphi] \in \mathrm{St}_{\mathrm{a.trans}}(M, M') \wedge [\varphi'] \in \mathrm{St}_{\mathrm{a.trans}}(M', M'') \Rightarrow [\varphi'] \circ [\varphi] \in \mathrm{St}_{\mathrm{a.trans}}(M, M''):$$

indeed, taking $\varphi'_{A''}$ for f in 1.3.11 (ii), we infer that if $\mathscr{P}'$ is generated by $(\mathscr{P}, [\varphi])$ and $\mathscr{P}''$ is generated by $(\mathscr{P}', [\varphi'])$, then $\mathscr{P}''$ is generated by $(\mathscr{P}, [\varphi'] \circ [\varphi])$.

It remains to prove associativity of the composition. Let us take

$$[\varphi] \in \mathrm{St}_{\mathrm{a.trans}}(M, M'), \quad [\varphi'] \in \mathrm{St}_{\mathrm{a.trans}}(M', M''),$$

and $[\varphi''] \in \mathrm{St}_{\mathrm{a.trans}}(M'', M''')$. Applying 1.3.9 (ii), we obtain

$$[\varphi''] \circ ([\varphi'] \circ [\varphi]) = \hat{T}_{[\varphi'] \circ [\varphi]} [\varphi''] = \hat{T}_{[\varphi]} \hat{T}_{[\varphi']} [\varphi'']$$
$$= \hat{T}_{[\varphi]} ([\varphi''] \circ [\varphi']) = ([\varphi''] \circ [\varphi']) \circ [\varphi]. \ \blacksquare$$

4.2. *For any statistical space M let*

(i) $$\Psi_a(M) = M;$$

for any $[\varphi] \in \mathrm{St}_{\mathrm{a.trans}}(M, M')$ *let*

(ii) $$\Psi_a[\varphi] = \psi \colon \mathscr{P} \to \mathrm{CAP}(\Omega', \mathscr{A}'),$$

where $\psi(P)$ *is the measure generated by* $(P, [\varphi])$.

Formulae (i) *and* (ii) *define a covariant functor* $\Psi_a \colon \mathrm{St}_{\mathrm{a.trans}} \to \mathrm{St}_{\mathrm{dist}}$.

Proof is analogous to that of 3.3.1. (We apply 1.3.11 instead of 1.1.2.) ■

The equivalence relation $\approx$ defined by 3.3.2 for morphisms of $\mathrm{St}_{\mathrm{trans}}$ can be now modified to the following relation for morphisms of $\mathrm{St}_{\mathrm{a.trans}}$.

4.3. DEFINITION. Let $[\varphi_1]$, $[\varphi_2] \in \mathrm{St}_{\mathrm{a.trans}}(M, M')$.

$$[\varphi_1] \approx [\varphi_2] \underset{\mathrm{Df}}{\Leftrightarrow} \forall P \in \mathscr{P} \int_\Omega \varphi_1(\omega, A')\,dP = \int_\Omega \varphi_2(\omega, A')\,dP,$$

i.e. for every $P \in \mathscr{P}$ the pairs $(P, [\varphi_1])$ and $(P, [\varphi_2])$ generate the same measure $P' \in \mathscr{P}'$.

The following statement is an analogue of 3.3.3.

4.4. *$\approx$ is a congruence with respect to the composition in* $\mathrm{St}_{\mathrm{a.trans}}$.

Proof. Take $[\varphi_i] \in \mathrm{St}_{\mathrm{a.trans}}(M, M')$ and let $[\varphi_1] \approx [\varphi_2]$, i.e.

$$\forall P \in \mathscr{P}\, \forall A' \in \mathscr{A}' \int_\Omega \varphi_1(\omega, A')\,dP = \int_\Omega \varphi_2(\omega, A')\,dP.$$

If $[\varphi] \in \mathrm{St}_{\mathrm{a.trans}}(M'', M)$, then $[\varphi_i] \circ [\varphi] = \hat{T}_{[\varphi]}[\varphi_i]$ for $i = 1, 2$, and thus, setting $f = (\varphi_i)_{A'}$ in 1.3.11 (ii), we get

$$\psi_i \in [\varphi_i] \circ [\varphi] \Rightarrow \int_{\Omega''} (\psi_i)_{A'}(\omega'')\,dP'' = \int_\Omega (\varphi_i)_{A'}(\omega)\,dP,$$

whence

$$[\varphi_1] \circ [\varphi] \approx [\varphi_2] \circ [\varphi].$$

If $[\varphi] \in \mathrm{St}_{\mathrm{a.trans}}(M', M'')$, then $[\varphi] \circ [\varphi_i] = \hat{T}_{[\varphi_i]}[\varphi]$. Take $P \in \mathscr{P}$ and let P_i' be generated by $(P, [\varphi_i])$, $i = 1, 2$; then $P_1' = P_2'$. Setting $f = \varphi_{A''}$ in 1.3.11 (ii), we obtain

$$\psi_i \in [\varphi] \circ [\varphi_i] \Rightarrow \int_\Omega \psi_i(\omega, A'')\,dP_i = \int_{\Omega'} \varphi(\omega', A'')\,dP_i',$$

whence

$$\int_\Omega \psi_1(\omega, A'')\,dP_1 = \int_\Omega \psi_2(\omega, A'')\,dP_2, \quad \text{i.e. } [\varphi] \circ [\varphi_1] \approx [\varphi] \circ [\varphi_2]. \ \blacksquare$$

We can now define the quotient category $\mathrm{St}_{\mathrm{a.trans}}|_{\approx}$. Its connection with $\mathrm{St}_{\mathrm{dist}}$ is analogous to that described by Theorem 3.3.4.

4.5. THEOREM. $\mathrm{St}_{\mathrm{a.trans}}|_{\approx} \underset{\mathrm{iso}}{\subset} \mathrm{St}_{\mathrm{dist}}$ *with the embedding being the quotient functor* $\tilde{\Psi}_a$ *induced by* Ψ_a (comp. 4.2).

Proof. It is easy to show that the functor Ψ_a defined in 4.2 satisfies the following condition

$$(1) \qquad [\varphi_1] \approx [\varphi_2] \Leftrightarrow \Psi_a[\varphi_1] = \Psi_a[\varphi_2]$$

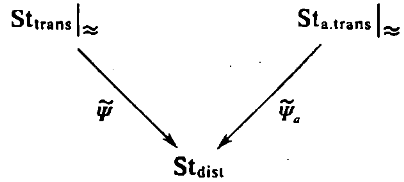
for every $[\varphi_1]$, $[\varphi_2] \in \mathrm{St}_{\mathrm{a.trans}}$.

By (1), the functor Ψ_a induces the quotient functor

$$\tilde{\Psi}_a\colon \mathrm{St}_{\mathrm{a.trans}}|_{\approx} \to \mathrm{St}_{\mathrm{dist}},$$

which is an isomorphic embedding. $\blacksquare$

By 3.3.4 combined with 4.5 we have the following diagram



To complete this diagram, let us first define the following functor

$$\Pi\colon St_{trans} \to St_{a.trans}.$$

For every statistical space M, let

$$\Pi(M) = M;$$

for every transition function $\varphi \in St_{trans}(M, M')$, let

$$\Pi(\varphi) = [\varphi] \in St_{a.trans}.$$

Clearly, Π is a covariant functor. Notice that

$$\varphi_1 \approx \varphi_2 \Rightarrow \Pi(\varphi_1) \approx \Pi(\varphi_2),$$

whence Π induces the quotient functor

$$\tilde{\Pi}\colon St_{trans}|_{\approx} \to St_{a.trans}|_{\approx}.$$

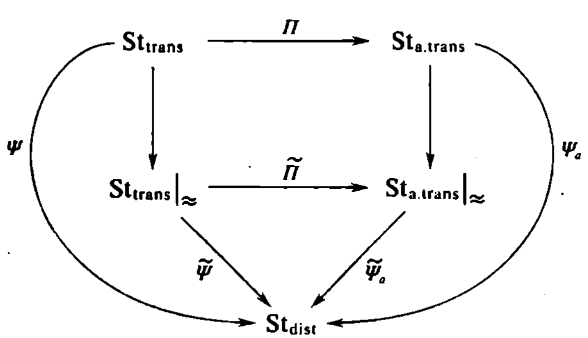
It is easy to check that

$$\Pi(\varphi_1) \approx \Pi(\varphi_2) \Rightarrow \varphi_1 \approx \varphi_2,$$

whence $\tilde{\Pi}$ is injective. So, we have

4.6. THEOREM. $St_{trans}|_{\approx} \underset{iso}{\subset} St_{a.trans}|_{\approx}$ *with* $\tilde{\Pi}$ *as the embedding.*

The following diagram is commutative:



(the vertical arrows denote natural projections).

Let us emphasize that the functor Π is not surjective, since there exist almost transition functions which are not $\mathscr{P}$-equal to any transition functions.

Finally, let us observe that the categories $St_{a.trans}$ and $St_{a.trans}|_{\approx}$ have their "indexed counterparts" $(\check{S}t_{a.trans})_C$ and $(\check{S}t_{a.trans})_C|_{\approx}$. Objects of these categories are indexed statistical spaces; morphisms are defined as follows:

If $M = (\Omega, \mathscr{A}, (P_\lambda)_{\lambda\in\Lambda})$ and $M' = (\Omega', \mathscr{A}', (P'_{\lambda'})_{\lambda'\in\Lambda'})$, then

$$(\alpha, [\varphi]) \in (\check{S}t_{a.trans})_C$$
$$\underset{Df}{\Leftrightarrow} [\alpha \in C(\Lambda, \Lambda') \wedge (\varphi \text{ is an almost transition function for } (M, M')) \wedge$$
$$\wedge (P'_{\alpha(\lambda)} \text{ is generated by } (P_\lambda, \varphi))];$$

$(\check{S}t_{a.trans})_C|_{\approx}$ is the quotient category with respect to the equivalence $\approx$:

$$(\alpha_1, [\varphi_1]) \approx (\alpha_2, [\varphi_2])$$
$$\underset{Df}{\Leftrightarrow} \alpha_1 = \alpha_2 \wedge \int_{\Omega} [\varphi_1] dP_\lambda = \int_{\Omega'} [\varphi_2] dP_\lambda \quad \text{for every } \lambda \in \Lambda.$$

Final remarks

It remains to consider categories introduced in the literature in relation to our categories. The category St was introduced by Moszyńska and Pleszczyńska (1982). Čencov in his book (see Čencov (1972), p. 83) introduced the category of statistical solving rules, which coincides with our category Mes_{trans}. This category is isomorphic to the concrete category CAP (Čencov (1972), Th. 5.2, p. 83). (Our definition is a formal and precise description of the notion introduced sketchily by Čencov.) The category FAM introduced by Čencov (1972), p. 88, is closely related to the category defined by Morse and Sacksteder (1966) (see p. 214). The first one coincides with our category $(\check{S}t_{trans}|_{\approx})_C$, where C is the subcategory of Ens with identities as morphisms. The second one coincides with our category $(\check{S}t_{a.trans}|_{\approx})_\Lambda$. Some related categories are investigated in Martin *et al.* (1971).

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