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On isomorphic classification of tensor products $E_\infty(a) \widehat{\otimes} E'_\infty(b)$

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CONTENTS

Abstract

New linear topological invariants are introduced and utilized to give an isomorphic classification of tensor products of the type $E_\infty(a) \widehat{\otimes} E'_\infty(b)$, where $E_\infty(a)$ is a power series space of infinite type. These invariants are modifications of those suggested earlier by Zahariuta. In particular, some new results are obtained for spaces of infinitely differentiable functions with values in a locally convex space X. These spaces coincide, up to isomorphism, with spaces $L(s', X)$ of all continuous linear operators into X from the dual space of the space s of rapidly decreasing sequences. Most of the results given here with proofs were announced in [12].

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0. Introduction

Let C_X^∞ denote the space of all infinitely differentiable functions defined on the interval $[-1,1]$ with values in a given locally convex space X. Usually C_X^∞ is endowed with the topology of uniform convergence of functions on the interval $[-1,1]$ with all their derivatives in every continous seminorm of X (cf. [14]). We have [14] the isomorphisms

$$C_X^\infty \simeq s \,\widehat{\otimes}\, X \simeq L(s', X).$$

Here and throughout, s is the space of all rapidly decreasing sequences, $X \,\widehat{\otimes}\, Y$ is the complete projective tensor product and $L(X,Y)$ is the space of continuous linear operators from X into Y equipped with the topology of uniform convergence on bounded subsets of X. In particular, X' stands for $L(X,K)$, where K is the scalar field.

Our purpose is to characterize the isomorphism $C_X^\infty \simeq C_Y^\infty$ in terms of the spaces X and Y. Valdivia has shown in [23] that if C_X^∞ is isomorphic to a complemented subspace of C_Y^∞ and C_Y^∞ is in turn isomorphic to a complemented subspace of C_X^∞, then $C_X^\infty \simeq C_Y^\infty$. Using this result a simple application of the decomposition method of Aytuna, Krone and Terzioğlu [1] gives $C_X^\infty \simeq s$ whenever X is a complemented subspace of s.

In contrast, it was shown in [32], [34] that even in the simple case of a nuclear finite type power series space $X = E_0(a)$, the structure of C_X^∞ as a Fréchet space depends on X in a quite delicate way. This case will be treated in Section 2. Those two diverse answers indicate that to study the general case, even if we restrict our attention to the class C_X^∞, with X a nuclear Fréchet space, would not be very promising. Therefore we confine ourselves mainly to some natural classes, such as C_X^∞ for X a nuclear power series space or $X \simeq E'_\infty(a)$. In the latter case we have

$$C_X^\infty \simeq s \,\widehat{\otimes}\, E'_\infty(a) \simeq L(E_\infty(a), s), \tag{0.1}$$

which gives us extra motivation to study this case. Related to this class, we also consider the class

$$s' \,\widehat{\otimes}\, E_\infty(a) \simeq L(s, E_\infty(a)). \tag{0.2}$$

In a more general setting we consider the problem of isomorphic classification of the class of tensor products of the form

$$E_\infty(a) \,\widehat{\otimes}\, E'_\infty(b), \tag{0.3}$$

which covers both classes (0.1) and (0.2).

To classify the spaces (0.3) we introduce new linear topological invariants based on the idea suggested by Zahariuta in [31], [32], [34]. This may be roughly summarized as follows: starting from a given collection of absolutely convex bounded subsets, we construct in a particular invariant manner another collection of absolutely convex sets and to this collection we apply the classical invariants.

This approach yields the invariant characteristics for Köthe spaces, considered earlier in [26], [28], [29], [30] and initiated by Mitiagin's results [18], but in a form more convenient for our purpose.

Here we use as a fundamental one a collection of absolutely convex sets in $X \widehat{\otimes} Y$, which corresponds (in our case) to some basis of equicontinuous sets in $L(Y^*, X)$. It is useful to compare this view with previous results [13] on necessary conditions of isomorphism of spaces $E_\alpha(a) \widehat{\otimes} E'_\beta(b)$ which were based on more traditional considerations, dealing with neighborhoods of zero.

To construct an isomorphism or an isomorphic imbedding for a given pair of spaces $X = E_\infty(a) \widehat{\otimes} E'_\infty(b)$ and $Y = E_\infty(\widetilde{a}) \widehat{\otimes} E'_\infty(\widetilde{b})$, we use here the method suggested in [29] but in a considerably simplified form (in the spirit of [34] in its revised English version).

We also refer to the following results closely connected with our present considerations: [2–11], [15–20], [22], [24], [25].

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1. Preliminaries

1.1. Let $A = (a_{i\lambda})_{i\in I,\, \lambda\in \Lambda}$ be a Köthe matrix, where I is a countable set (often $I = \mathbb{N}$), Λ is a directed set and $a_{i\lambda} \geq 0$. We also have $a_{i\lambda} \leq a_{i\mu}$ if $\lambda \leq \mu$ and $\sup\{a_{i\lambda}: \lambda \in \Lambda\} > 0,\ i \in I$. By $K(A)$ we denote the Köthe space generated by A, i.e., the locally convex space of all sequences $x = (\xi_i)$ such that for every $\lambda \in \Lambda$,

$$|x|_\lambda = \sum_{i\in I} |\xi_i| a_{i\lambda} < \infty, \tag{1.1}$$

equipped with the seminorms (1.1). As usual, (e_i) denotes the canonical basis of $K(A)$. In particular, for $a = (a_i)$, by $E_0(a)$ and by $E_\infty(a)$ we denote the power series space of finite and infinite type, which are Köthe spaces generated by the matrices $(\exp(-p^{-1}a_i))$ and $(\exp(pa_i))$, $p \in \mathbb{N}$, respectively (see, for example, [14]).

If $A = (a_{i\lambda})_{i\in I,\, \lambda\in\Lambda}$, $B = (b_{j\mu})_{j\in J,\, \mu\in M}$ are two Köthe matrices, then the tensor product can be written as

$$K(A) \widehat{\otimes} K(B) \simeq K(C),$$

where $C = (c_{(i,j),(\lambda,\mu)})$, $c_{(i,j),(\lambda,\mu)} = a_{i\lambda} b_{j\mu}$ with $(i,j) \in I \times J$ and $(\lambda,\mu) \in \Lambda \times M$. The above ismorphism is obtained by identifying the basis sequence $(e_i \otimes e_j)$ of $K(A) \widehat{\otimes} K(B)$ with the natural basis (e_{ij}) of $K(C)$.

1.2. A continuous linear operator $T : K(A) \to K(B)$ is said to be *quasidiagonal* if

$$T(e_i) = t_i e_{\sigma(i)}, \quad i \in I, \tag{1.2}$$

where $\sigma : I \to J$ and t_i is a scalar. In particular, T is *diagonal* if $I = J$ and σ is the identity. If $\sigma : I \to J$ is a bijection and $t_i \equiv 1$, then T is said to be *permutative*. For Köthe spaces $X = K(A)$, $Y = K(B)$ we use the notation $X \overset{\text{qd}}{\simeq} Y$, $X \overset{\text{d}}{\simeq} Y$ and $X \overset{\text{p}}{\simeq} Y$ if there is an isomorphism $T : X \to Y$ which is respectively quasidiagonal, diagonal or permutative. We use the notation $X \overset{\sim}{\to} Y$ if there is an isomorphic imbedding $T : X \to Y$. If T is also quasidiagonal we write $X \overset{\text{qd}}{\overset{\sim}{\to}} Y$. In this context we need the following fact, which was stated in [29] but was considered earlier in [18] in an implicit form.

PROPOSITION 1.1. *If $K(A) \overset{\text{qd}}{\overset{\sim}{\to}} K(B)$ and $K(B) \overset{\text{qd}}{\overset{\sim}{\to}} K(A)$, then $K(A) \overset{\text{qd}}{\simeq} K(B)$.*

1.3. We identify the inductive limit

$$E'_\infty(a) = \operatorname{lim\,ind} l^1(\exp(-pa_i))$$

with the Köthe space $K(A)$, $A = (a_{i\pi})$, where

$$a_{i\pi} = \exp(-\pi_i a_i)$$

and $\pi = (\pi_i)$ runs along the directed set

$$\Pi^\infty = \{\pi = (\pi_i) : \lim \pi_i = \infty\}.$$

The set Π^∞ has the natural order $\lambda \le \mu$ defined by $\mu_i \le \lambda_i$ for all $i \in I$. In case $E_\infty(a)$ is nuclear, $E^*_\infty(a)$ can be naturally identified with $E'_\infty(a)$ ([14]).

1.4. For a given sequence $a = (a_i)$, $a_i \ge 1$, we consider the following counting functions:

$$m_a(\tau, t) = |\{i : \tau < a_i \le t\}|, \tag{1.3}$$

$$m_a(t) = |\{i : a_i \le t\}|, \tag{1.4}$$

where $|A|$ denotes the cardinality of a finite set A and equals $+\infty$ for an infinite A, and $0 < \tau < +\infty$. We also use the following characteristic of lacunarity:

$$n_a(\tau, t) = \begin{cases} 1 & \text{if } m_a(\tau, t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We write $m_a \approx m_b$ or $n_a \approx n_b$ if a constant $c > 0$ exists such that

$$m_a(t) \le m_b(ct), \quad m_a(t) \le m_b(ct), \quad t \ge 1, \tag{1.5}$$

or, respectively,

$$n_a(\tau, t) \le n_b(\tau/c, ct), \quad n_b(\tau, t) \le n_a(\tau/c, ct), \quad 1 \le \tau < t < \infty. \tag{1.6}$$

For non-decreasing sequences $a = (a_i)$, $b = (b_i)$ the relation (1.5) is equivalent to the following condition:

$$b_i/c \le a_i \le cb_i, \quad i \in \mathbb{N}, \tag{1.7}$$

with the same constant c. If for arbitrary sequences a and b the relation (1.7) holds for some constant c, we say a and b are *weakly equivalent* and use the notation $a_i \asymp b_i$ or $a \asymp b$ in this case. If (1.6) holds, we say a and b have the same lacunarities or are *identical in lacunarity.*

The following simple result about the characteristic of lacunarity will be useful.

PROPOSITION 1.2. *Let $a = (a_i)$, $b = (b_i)$ be sequences with $a_i \geq 1$, $b_i \geq 1$. The following statements are equivalent*:

(i) *There is $\Delta > 0$ such that*

$$n_a(\tau, t) \leq n_b(\tau/\Delta, \Delta t), \qquad 1 \leq \tau < t < \infty. \tag{1.8}$$

(ii) *For every $A > 1$ there is $B > 0$ with*

$$n_a(t/A, At) \leq n_b(t/B, Bt), \qquad t \geq 1.$$

(iii) *There exists $A > 1$ and $B > 0$ with*

$$n_a(t/A, At) \leq n_b(t/B, Bt), \qquad t \geq 1.$$

(iv) *There exists $A > 1$ and $B > 0$ with*

$$n_a(A^{2m-1}, A^{2m+1}) \leq n_b(B^{-1}A^{2m-1}, BA^{2m+1}), \qquad m \in \mathbb{Z}.$$

Proof. Since (i)⇒(ii)⇒(iii)⇒(iv) are obvious, we will show that (iv) implies (i). We choose $m, l \in \mathbb{Z}$ such that

$$A^{2m-1} < \tau \leq A^{2m+1}, \qquad A^{2(m+l)-1} < t \leq A^{2(m+l)+1}$$

and use the following chain of inequalities:

$$\begin{aligned} n_a(\tau, t) &\leq n_a(A^{2m-1}, A^{2(m+l)+1}) \leq \sup_{\nu=0,\ldots,l} n_a(A^{2(m+\nu)-1}, A^{2(m+\nu)+1}) \\ &\leq \sup_{\nu=0,\ldots,l} n_b(B^{-1}A^{2(m+\nu)-1}, BA^{2(m+\nu)+1}) \\ &\leq n_b(B^{-1}A^{2m-1}, BA^{2(m+l)+1}) \leq n_b((BA^2)^{-1}\tau, (BA^2)t). \end{aligned}$$

Thus we get (i) with $\Delta = BA^2$. ∎

2. Power series space-valued case

We will consider in detail the isomorphic classification of the spaces C^∞_X when X is a power series space of infinite or finite type. First, we deal with the infinite type, which is quite simple.

In fact we shall consider C^∞_X where X is a complemented subspace of s. It is not known if X has a basis, but if it does, then X is a nuclear power series space of infinite type [1].

PROPOSITION 2.1. *If X is a complemented subspace of s, then $C^\infty_X \simeq s$.*

Proof. $C^\infty_X \simeq s \mathbin{\widehat{\otimes}} X$ is a complemented subspace of $s \simeq s \mathbin{\widehat{\otimes}} s$. Further, the diametral dimensions of C^∞_X and s are equal. We conclude by referring to [1]. ∎

In contrast to the above, the spaces C^∞_X for $X = E_0(a)$ have more intricate topological structure. Here the characteristic of lacunarity distinguishes isomorphic classes.

Proposition 2.2 [32, 34]. *Let $E_0(a)$ be nuclear. Then*

$$s\,\widehat{\otimes}\,E_0(a)\simeq s\,\widehat{\otimes}\,E_0(b)$$

if and only if the following two conditions hold:

(1) $E_0(b)$ *is nuclear*,

(2) $n_a\approx n_b$, *i.e. a and b are identical in lacunarity.*

It is of interest to compare the preceding result with the proposition in [19] on page 309.

3. Main results

Here we state our main results. Most of them will be proved in the sections to come. However, some will be demonstrated here.

Theorem 3.1. *Let $E_\infty(b)$ and $E_\infty(\widetilde{b})$ be two nuclear power series spaces where b and $\widetilde{b}$ are non-decreasing. Then $s\,\widehat{\otimes}\,E'_\infty(b)\simeq s\,\widehat{\otimes}\,E'_\infty(\widetilde{b})$ if and only if the sequences b and $\widetilde{b}$ are identical in lacunarity.*

We say that a sequence $b=(b_i)$ is *non-lacunary* (*shift-stable* [10]) if b and (i) are identical in lacunarity. For a non-decreasing sequence $b=(b_i)$ this is equivalent to

$$\limsup \frac{b_{i+1}}{b_i}<\infty.$$

Hence the following statement is an immediate consequence of our theorem.

Corollary 3.2. *Let b be as in Theorem* 3.1. *Then*

$$s\,\widehat{\otimes}\,E'_\infty(b)\simeq s\,\widehat{\otimes}\,s'$$

if and only if b is shift-stable.

As we shall discuss later, classification of the products $s'\,\widehat{\otimes}\,E_\infty(a)$ depends on a in a more intricate fashion than in the case we have discussed. However, when the non-decreasing positive sequences $a=(a_i)$, $\widetilde{a}=(\widetilde{a}_i)$ satisfy the following stronger condition:

$$\ln i=o(a_i),\qquad \ln i=o(\tilde{a}_i) \tag{3.1}$$

we have an analog of Theorem 3.1.

Theorem 3.3. *Let a and $\widetilde{a}$ satisfy* (3.1). *Then $X=s'\,\widehat{\otimes}\,E_\infty(a)$ is isomorphic to $Y=s'\,\widehat{\otimes}\,E_\infty(\widetilde{a})$ if and only if a and $\widetilde{a}$ have the same lacunarities.*

For $c=(i)$, the space $E_\infty(c)$ is isomorphic to the space of entire functions $\mathcal{O}(\mathbb{C})$, c satisfies the condition (3.1) and is shift-stable. Therefore we have

Corollary 3.4. *Let a satisfy* (3.1) *and be shift-stable. Then*

$$s'\,\widehat{\otimes}\,E_\infty(a)\simeq s'\,\widehat{\otimes}\,E_\infty(c)\simeq s'\,\widehat{\otimes}\,\mathcal{O}(\mathbb{C}).$$

Although $(\ln i)$ is shift-stable, obviously it does not satisfy (3.1). In fact $s'\,\widehat{\otimes}\,s$ has an exceptional position in the class of spaces $s'\,\widehat{\otimes}\,E_\infty(a)$.

THEOREM 3.5. *For the isomorphism* $s' \widehat{\otimes} s \simeq s' \widehat{\otimes} E_\infty(a)$, *it is necessary and sufficient that* $s \simeq E_\infty(a)$.

COROLLARY 3.6. *We have* $L(s,s) \simeq L(s, E_\infty(a))$ *if and only if* $s \simeq E_\infty(a)$.

The preceding results will be derived in Sections 9 and 10 from the following more general and rather technical result, dealing with the isomorphism of spaces $E_\infty(a) \widehat{\otimes} E'_\infty(b)$, which will be proved in sections 6–8.

THEOREM 3.7. *Let* $X = E_\infty(a) \widehat{\otimes} E'_\infty(b)$, $Y = E_\infty(\widetilde{a}) \widehat{\otimes} E'_\infty(\widetilde{b})$ *and* $T : Y \to X$ *be an isomorphism. Then* $\exists \Delta\ \forall \widetilde{\varepsilon}\ \exists \varepsilon\ \forall \delta\ \exists \widetilde{\delta}$ *such that the following inequalities* (3.2)–(3.5) *are true*:

$$\left|\left\{(i,j) : \delta \le \frac{b_j}{a_i + b_j} \le \varepsilon,\ \tau \le a_i + b_j \le t\right\}\right| \le \left|\left\{(k,l) : \widetilde{\delta} \le \frac{\widetilde{b}_l}{\widetilde{a}_k + \widetilde{b}_l} \le \widetilde{\varepsilon},\ \frac{\tau}{\Delta} \le \widetilde{a}_k + \widetilde{b}_l \le \Delta t\right\}\right|, \quad \tau \ge \tau_0, \tag{3.2}$$

$$\left|\left\{(i,j) : \delta \le \frac{b_j}{a_i + b_j},\ \tau \le a_i + b_j \le t\right\}\right| \le \left|\left\{(k,l) : \widetilde{\delta} \le \frac{\widetilde{b}_l}{\widetilde{a}_k + \widetilde{b}_l},\ \frac{\tau}{\Delta} \le \widetilde{a}_k + \widetilde{b}_l \le \Delta t\right\}\right|, \quad \tau \ge \tau_0, \tag{3.3}$$

$$\left|\left\{(i,j) : \frac{b_j}{a_i + b_j} \le \varepsilon,\ \tau \le a_i + b_j \le t\right\}\right| \le \left|\left\{(k,l) : \frac{\widetilde{b}_l}{\widetilde{a}_k + \widetilde{b}_l} \le \widetilde{\varepsilon},\ \frac{\tau}{\Delta} \le \widetilde{a}_k + \widetilde{b}_l \le \Delta t\right\}\right|, \tag{3.4}$$

$$|\{(i,j) : \tau \le a_i + b_j \le t\}| \le \left|\left\{(k,l) : \frac{\tau}{\Delta} \le \widetilde{a}_k + \widetilde{b}_l \le \Delta t\right\}\right|. \tag{3.5}$$

Some quantifiers before absent parameters need to be omitted; the constant τ_0 depends on all participating parameters.

Remark. If t depends on τ, i.e. $t = \varphi(\tau)$, $\tau \ge 1$, then the restriction $\tau \ge \tau_0$ can be removed everywhere in Theorem 3.7. Indeed, let, for example, the relation (3.3) hold with $t = \varphi(\tau)$ and $\tau \ge \tau_0$. Then we choose instead of $\widetilde{\delta}$ some smaller constant $\widetilde{\delta}' > 0$ such that $\widetilde{\delta}' \Delta \varphi(\tau_0) \le 1$ and get the inequality (3.3) with $\widetilde{\delta}'$ instead of $\widetilde{\delta}$ without any restriction on τ.

The last theorem has the following partial converse. The notation Y^k means $Y \times \ldots \ldots \times Y$, k times.

THEOREM 3.8. *Let* X, Y *be as in Theorem* 3.7 *and let the conditions* (3.2), (3.3) *and* (3.4) *be valid. Then* $X \overset{\mathrm{qd}}{\hookrightarrow} Y^9$.

With some restrictions on X, Y we get the following criterion of isomorphism.

THEOREM 3.9. *Let* X, Y *be as in Theorem* 3.7 *and* $X \overset{\mathrm{qd}}{\simeq} X^2$, $Y \overset{\mathrm{qd}}{\simeq} Y^2$. *Then the following statements are equivalent*:

(i) $X \simeq Y$,

(ii) $X \overset{\mathrm{qd}}{\simeq} Y$,

(iii) *the inequalities* (3.2), (3.3) *are true together with the inequalities which are obtained from these inequalities by interchanging* $a, b, (i,j)$ *with* $\widetilde{a}, \widetilde{b}, (k,l)$ *respectively.*

Proof. Because (ii)⇒(i) is obvious and (i)⇒(iii) follows from Theorem 3.7, we need only prove (iii)⇒(ii). Since $X \overset{\mathrm{qd}}{\simeq} X^2$ implies $X \overset{\mathrm{qd}}{\simeq} X^9$, we get by Theorem 3.8 that $Y \overset{\mathrm{qd}}{\hookrightarrow} X^9 \overset{\mathrm{qd}}{\simeq} X$, i.e. $Y \overset{\mathrm{qd}}{\hookrightarrow} X$. By symmetry we get also $X \overset{\mathrm{qd}}{\hookrightarrow} Y$. Hence Proposition 1.1 implies $X \overset{\mathrm{qd}}{\simeq} Y$. ■

COROLLARY 3.10. *Let* X, Y *be the same as in Theorem* 3.7 *and additionally the conditions*

$$a_{2i} \asymp a_i, \quad \widetilde{a}_{2i} \asymp \widetilde{a}_i \tag{3.6}$$

or the conditions

$$b_{2j} \asymp b_j, \quad \widetilde{b}_{2j} \asymp \widetilde{b}_j \tag{3.7}$$

hold. Then $X \overset{\mathrm{qd}}{\simeq} Y$ (*and all the more* $X \simeq Y$).

Proof. Indeed, both (3.6) and (3.7) imply $X \overset{\mathrm{qd}}{\simeq} X^2$ and $Y \overset{\mathrm{qd}}{\simeq} Y^2$, hence we can apply Theorem 3.9. ■

THEOREM 3.11. *Let* $X = s' \,\widehat{\otimes}\, E_\infty(a)$ *and* $Y = s' \,\widehat{\otimes}\, E_\infty(\widetilde{a})$ *be nuclear. Then the following statements are equivalent*:

(i) $X \overset{\mathrm{qd}}{\simeq} Y$,

(ii) $X \simeq Y$,

(iii) $\exists A\ \forall \gamma > 0\ \exists \tau_0$ *such that*

$$m_a(\tau, t) \le (\exp \gamma t) m_{\tilde{a}}(\tau/A, At), \qquad \tau_0 \le \tau \le t, \tag{3.8}$$

$$m_{\tilde{a}}(\tau, t) \le (\exp \gamma t) m_a(\tau/A, At), \qquad \tau_0 \le \tau \le t, \tag{3.9}$$

(iv) $\forall A\ \exists B\ \forall \gamma\ \exists \tau_0$ *such that*

$$m_a(t/A, At) \le (\exp \gamma t) m_{\tilde{a}}(t/B, Bt), \qquad t \ge \tau_0, \tag{3.10}$$

$$m_{\tilde{a}}(t/A, At) \le (\exp \gamma t) m_a(t/B, Bt), \qquad t \ge \tau_0, \tag{3.11}$$

and $\exists E > 1$ *such that*

$$m_a(t) \le (\exp Et) m_{\tilde{a}}(Et), \tag{3.12}$$

$$m_{\tilde{a}}(t) \le (\exp Et) m_a(Et), \qquad t \ge 1. \tag{3.13}$$

4. F- and DF-subspaces

Let $X = E_\infty(a) \,\widehat{\otimes}\, E'_\infty(b)$ and M be an infinite subset of $\mathbb{N}^2$. By X_M we denote the closed subspace of X which is generated by the basic sequence $\{e_i \otimes e_j : (i,j) \in M\}$. We now determine when X_M is an F-space or a DF-space.

LEMMA 4.1. *X_M is an F-space if and only if*

$$\lim_{(i,j)\in M} \frac{b_j}{a_i} = 0. \tag{4.1}$$

X_M is a DF-space if and only if there exists $\delta > 0$ with

$$\frac{b_j}{a_i} \ge \delta, \qquad (i,j) \in M. \tag{4.2}$$

Proof. Identifying X with the Köthe space $K(A)$, where

$$A = \{\exp(pa_i - \pi_j b_j)\}, \qquad p \in \mathbb{N},\ \pi \in \Pi^\infty,$$

we see that X_M can be considered as the space of all doubly indexed sequences $x = (\xi_{i,j})$, $(i,j) \in M$, endowed with the locally convex topology defined by the norms

$$\|x\|_{p,\pi} = \sum_{(i,j)\in M} |\xi_{ij}| \exp(pa_i - \pi_j b_j) < \infty. \tag{4.3}$$

The inequality

$$\|x\|_{p,\pi} \le \sum_{(i,j)\in M} |\xi_{ij}| \exp pa_i$$

is obvious. On the other hand, if (4.1) holds, we set

$$\pi_j^\circ = \inf_i \{a_i/b_j : (i,j) \in M\}.$$

For $\pi \in \Pi^\infty$ satisfying $\pi_j = O(\pi_j^\circ)$ and $p \in \mathbb{N}$ we have

$$\sum_{(i,j)\in M} |\xi_{ij}| \exp(p - \Delta)a_i \le \|x\|_{p,\pi},$$

where $\Delta = \sup\{\pi_j/\pi_j^\circ\}$. Hence X_M is a Fréchet space if (4.1) holds.

Assume (4.2). Then for $p \in \mathbb{N}$ and $\pi \in \Pi^\infty$ we have the inequalities

$$\sum_{(i,j)\in M} |\xi_{ij}| \exp(-\pi_j b_j) \le \|x\|_{p,\pi} \le \sum_{(i,j)\in M} |\xi_{ij}| \exp(-\widetilde{\pi}_j b_j),$$

where $\widetilde{\pi}_j b_j = \pi_j - p/\delta$. Hence the topology of X_M is defined by the system of norms

$$\|x\|_\pi^{(\circ)} = \sum_{(i,j)\in M} |\xi_{ij}| \exp(-\pi_j b_j), \qquad \pi \in \Pi^\infty.$$

Thus the space X_M can be represented as the inductive limit

$$\lim\operatorname{ind} l_M^1(\exp(-qb_j)).$$

Hence X is a DF-space.

Let us consider the converse situation. If X_M is an F-space but (4.1) is not true, then we can find a subsequence $M' \subset M$ for which (4.2) holds. But then $X_{M'}$ is a DF-space by what we have already proved. This is impossible. In exactly the same manner we can prove that if X_M is a DF-space then (4.2) is true. ∎

5. Quasidiagonal isomorphism

We consider a criterion for the existence of quasidiagonal isomorphism between spaces $E_\infty(a)\,\widehat{\otimes}\,E'_\infty(b)$.

Proposition 5.1. *For $E_\infty(a)\,\widehat{\otimes}\,E'_\infty(b)$ to be quasidiagonally isomorphic to $E_\infty(\widetilde{a})\,\widehat{\otimes}\,E'_\infty(\widetilde{b})$ it is necessary and sufficient that a bijection $\sigma=(\sigma_1,\sigma_2):\mathbb{N}^2\to\mathbb{N}^2$ exists such that the following conditions hold*:

(i) $a_i+b_j\asymp\widetilde{a}_k+\widetilde{b}_l$, *where* $(k,l)=\sigma((i,j))$.
(ii) *For any subsequence $M\subset\mathbb{N}^2$, we have*

$$\lim_{(i,j)\in M}\frac{b_j}{a_i}=0\quad\text{if and only if}\quad\lim_{(k,l)\in\sigma(M)}\frac{\widetilde{b}_l}{\widetilde{a}_k}=0.$$

First we note that it is not necessary to consider diagonal isomorphism at all, because the existence of a diagonal isomorphism means automatically that the spaces coincide. By Proposition 1.1 it is enough to show the following:

Proposition 5.2. *Let $\sigma:\mathbb{N}^2\to\mathbb{N}^2$ be an injection. Then the operator $T:E_\infty(a)\,\widehat{\otimes}\,E'_\infty(b)\to E_\infty(\widetilde{a})\,\widehat{\otimes}\,E'_\infty(\widetilde{b})$ defined by*

$$T(e_i\otimes e_j)=e_k\otimes e_l,\qquad (k,l)=\sigma((i,j)),\tag{5.1}$$

is an isomorphic imbedding if and only if the conditions (i) *and* (ii) *of Proposition* 5.1 *are satisfied.*

Proof. Suppose that the operator T defined by (5.1) is an isomorphic imbedding. Since an isomorphism preserves the F- or DF-character of subspaces, the condition (ii) of Proposition 5.1 is true by Lemma 4.1. To obtain the condition (i), we use the continuity of T and its inverse. By Grothendieck's factorization theorem [14], I, p. 16, we choose natural numbers $p_1<r_1<r_2<p_2<q_2<s_2<s_1<q_1$ and a constant c such that the following inequalities hold for every $(i,j)\in\mathbb{N}^2$:

$$\exp(p_1\widetilde{a}_k-q_1\widetilde{b}_l)\le c\exp(r_1a_i-s_1b_j),\qquad \exp(r_2\widetilde{a}_k-s_2\widetilde{b}_l)\le c\exp(p_2a_i-q_2b_j),$$
$$\exp(p_1a_i-q_1b_j)\le c\exp(r_1\widetilde{a}_k-s_1\widetilde{b}_l),\qquad \exp(r_2a_i-s_2b_j)\le c\exp(p_2\widetilde{a}_k-q_2\widetilde{b}_l).$$

Here and in what follows, $(k,l)=\sigma((i,j))$. From the above we get

$$(r_2-r_1)\widetilde{a}_k+(s_1-s_2)\widetilde{b}_l\le(p_2-p_1)a_i+(q_1-q_2)b_j+2\ln c,$$
$$(r_2-r_1)a_i+(s_1-s_2)b_j\le(p_2-p_1)\widetilde{a}_k+(q_1-q_2)\widetilde{b}_l+2\ln c,$$

and so (i) is true.

Conversely, let $\sigma:\ \mathbb{N}^2\to\mathbb{N}^2$ be an injection, so that the conditions (i) and (ii) are satisfied. We want to show that the formula (5.1) generates an isomorphic imbedding.

Suppose T generated by σ is not continuous. This means that $\exists p\ \forall r\ \exists s\ \forall q\ \exists(i_q,j_q)\in\mathbb{N}^2$ such that

$$\exp(p\widetilde{a}_{k_q}-q\widetilde{b}_{l_q})\ge\exp(ra_{i_q}-sb_{j_q}),\tag{5.2}$$

where $(k_q,l_q)=\sigma(i_q,j_q)$.

Further, if

$$\gamma = \inf \frac{a_i + b_j}{\widetilde{a}_k + \widetilde{b}_l},$$

let r be chosen so that $r\gamma > p$.

Without loss of generality, we may assume the existence of the following limits:

$$\lim_{q\to\infty} \frac{\widetilde{b}_{l_q}}{\widetilde{a}_{k_q} + \widetilde{b}_{l_q}} = \alpha, \qquad \lim_{q\to\infty} \frac{b_{j_q}}{a_{i_q} + b_{j_q}} = \beta.$$

Let us take the logarithm of inequality (5.2) and, after dividing by $\widetilde{a}_k + \widetilde{b}_l$, let q tend to infinity. Then we get $\alpha = 0$, since otherwise $-\infty \geq (r(1-\beta) - s\beta)\gamma$, which is impossible. This means that

$$\lim \frac{\widetilde{b}_{l_q}}{\widetilde{a}_{k_q}} = 0$$

and so by condition (ii) we have

$$\lim \frac{b_{j_q}}{a_{i_q}} = 0.$$

Therefore $\beta = 0$ as well. Hence, from (5.2) we have $p \geq r\gamma$, which contradicts the original choice of r. ∎

6. Sufficiency

6.1. Here Theorem 3.8 will be proved. By Theorem 3.7 we can assume that a constant Δ and a non-decreasing function $\varphi : (0,1] \to (0,1]$ exist such that for any $\tau \geq 1$ and $t = \Delta\tau$ the conditions (3.3), (3.4) are valid with Δ, $\forall \widetilde{\varepsilon} \in [0,1]$, $\varepsilon = \varphi(\widetilde{\varepsilon})$, $\forall \delta \in (0,1]$ and $\widetilde{\delta} = \varphi(\delta)$; $\varphi(\delta) \to 0$ if $\delta \to 0$. Define the sequence (ε_k) by $\varepsilon_{-1} = \varepsilon_0 = 1$ and $\varepsilon_k = \varphi(\varepsilon_{k-1})$, $k \in \mathbb{N}$. Let us represent the set $\mathbb{N}^2$ as the union of families of disjoint subsets:

$$\mathbb{N}^2 = \bigcup_{m=0}^{\infty} \bigcup_{s=0}^{\infty} \mathcal{N}_{m,s} = \bigcup_{m=0}^{\infty} \bigcup_{s=0}^{\infty} \mathcal{M}_{m,s},$$

where, for $s, m \in \mathbb{Z}_+$,

$$\mathcal{N}_{m,s} = \left\{ (i,j) \in \mathbb{N}^2 : \varepsilon_{m+1} < \frac{b_j}{a_i + b_j} \leq \varepsilon_m;\ \Delta^s \leq a_i + b_j < \Delta^{s+1} \right\},$$

$$\mathcal{M}_{m,s} = \left\{ (k,l) \in \mathbb{N}^2 : \varepsilon_{m+1} < \frac{\widetilde{b}_l}{\widetilde{a}_k + \widetilde{b}_l} \leq \varepsilon_m;\ \Delta^s \leq \widetilde{a}_k + \widetilde{b}_l < \Delta^{s+1} \right\}.$$

For the sets

$$\begin{aligned}\widetilde{\mathcal{M}}_{m,s} &= \bigcup_{\alpha=0}^{2} \bigcup_{\beta=0}^{2} \mathcal{M}_{m-1+\alpha,s-1+\beta} \\ &= \left\{ (k,l) : \varepsilon_{m+2} < \frac{\widetilde{b}_l}{\widetilde{a}_k + \widetilde{b}_l} \leq \varepsilon_{m-1};\ \Delta^{s-1} \leq \widetilde{a}_k + \widetilde{b}_l < \Delta^{s+2} \right\}, \qquad s, m \in \mathbb{Z}_+,\end{aligned}$$

the following estimates arise from the conditions (3.3), (3.4) with the above choice of parameters:

$$|\mathcal{N}_{m,s}| \le |\widetilde{\mathcal{M}}_{m,s}|, \quad s, m \in \mathbb{Z}_+.$$

It is clear that $Y^9 \overset{\text{qd}}{\simeq} E_\infty(\widetilde{a})^3 \widehat{\otimes} E'_\infty(\widetilde{b})^3 \overset{\text{qd}}{\simeq} E_\infty(c) \widehat{\otimes} E'_\infty(d)$, where $c = (c_\mu)$, $d = (d_\nu)$ and

$$c_\mu = \widetilde{a}_{k,\mu}; \quad \mu = 3k - \alpha,\ \alpha = 0,1,2,\ k \in \mathbb{N},$$
$$d_\nu = \widetilde{b}_{l,\nu}; \quad \nu = 3l - \beta,\ \beta = 0,1,2,\ l \in \mathbb{N}.$$

By construction the sets

$$\mathcal{M}^*_{m,s} := \{(3k - \alpha, 3l - \beta) : (k,l) \in \mathcal{M}_{m-1+\alpha, s-1+\beta},\ \alpha, \beta = 0,1,2\}$$

are disjoint and

$$|\mathcal{M}^*_{m,s}| = |\widetilde{\mathcal{M}}_{m,s}| \ge |\mathcal{N}_{m,s}|, \quad m, s \in \mathbb{Z}_+.$$

Then by construction,

$$\Delta^{-2} \le \frac{\Delta^s}{\Delta^{s-2}} \le \frac{a_i + b_j}{c_\mu + d_\nu} \le \frac{\Delta^{s+1}}{\Delta^{s-1}} = \Delta^2$$

and

$$\frac{c_\mu}{c_\mu + d_\nu} \ge \varepsilon_{m+2} = \varphi^2(\varepsilon_m) \ge \varphi^2\left(\frac{b_j}{a_i + b_j}\right),$$
$$\frac{b_j}{a_i + b_j} \ge \varepsilon_{m+1} \ge \varphi^2(\varepsilon_{m-1}) \ge \varphi^2\left(\frac{c_\mu}{c_\mu + d_\nu}\right),$$

where φ^2 means the composition with itself. Hence, we have

$$\frac{1}{\Delta^2} \le \frac{a_i + b_j}{c_\mu + d_\nu} \le \Delta^2, \qquad \frac{c_\mu}{c_\mu + d_\nu} \ge \varphi^2\left(\frac{b_j}{a_i + b_j}\right), \qquad \frac{b_j}{a_i + b_j} \ge \varphi^2\left(\frac{c_\mu}{c_\mu + d_\nu}\right)$$

for each $(i,j) \in \mathbb{N}^2$ and $(\mu, \nu) = \sigma((i,j))$.

So, the permutation operator $T : X \to E_\infty(c) \widehat{\otimes} E'_\infty(d)$, generated by the injection σ:

$$T(e_i \otimes e_j) = e_\mu \otimes e_\nu, \quad (\mu, \nu) = \sigma((i,j)),\ (i,j) \in \mathbb{N}^2,$$

must be an isomorphic imbedding by Proposition 5.2.

Because of symmetry, using the conditions which can be obtained from the conditions (3.3), (3.4) by interchanging $a, b, (i,j)$ with $\widetilde{a}, \widetilde{b}, (k,l)$, we analogously get $Y \overset{\text{qd}}{\hookrightarrow} X^9$ (under similar assumptions on the choice of parameters).

6.2. Taking into consideration Theorem 3.7 and the proof of Theorem 3.8, we can derive the following refinement of Theorem 3.9.

PROPOSITION 6.1. *Under the conditions of Theorem* 3.9 *the following statements are equivalent*:

(i) $X \simeq Y$.

(ii) $X \overset{\text{qd}}{\simeq} Y$.

(iii) *A strongly decreasing sequence* $\varepsilon_k \to 0$, $\varepsilon_0 = 1$, *and a constant* $\Delta > 0$ *exist such that*

$$\left|\left\{(i,j) \in \mathbb{N}^2 : \varepsilon_{m+1} < \frac{b_j}{a_i + b_j} \le \varepsilon_m;\ \Delta^s \le a_i + b_j \le \Delta^{s+1}\right\}\right| \\ \le \left|\left\{(k,l) \in \mathbb{N}^2 : \varepsilon_{m+2} < \frac{\widetilde{b}_l}{\widetilde{a}_k + \widetilde{b}_l} \le \varepsilon_{m-1};\ \Delta^{s-1} \le \widetilde{a}_k + \widetilde{b}_l < \Delta^{s+2}\right\}\right| \tag{6.1}$$

and

$$\left|\left\{(k,l) \in \mathbb{N}^2 : \varepsilon_{m+1} < \frac{\widetilde{b}_l}{\widetilde{a}_k + \widetilde{b}_l} \le \varepsilon_m;\ \Delta^s \le \widetilde{a}_k + \widetilde{b}_l < \Delta^{s+1}\right\}\right| \\ \le \left|\left\{(i,j) \in \mathbb{N}^2 : \varepsilon_{m+2} < \frac{b_j}{a_i + b_j} \le \varepsilon_{m-1};\ \Delta^{s-1} \le a_i + b_j < \Delta^{s+2}\right\}\right|$$

for every $s, m \in \mathbb{Z}_+$ *(put* $\varepsilon_{-1} = 1$*).*

6.3. The condition (iii) in Proposition 6.1 is only necessary if we consider spaces X, Y from Theorem 3.7 without any additional restriction. It is useful to compare Proposition 6.1 with the following criterion of the quasidiagonal isomorphism $X \overset{\text{qd}}{\simeq} Y$ in the general case; this fact can be proved similarly to [5] by using the Hall–Koenig Lemma about representatives.

Proposition 6.2. *Let* $X = E_\infty(a) \mathbin{\widehat{\otimes}} E'_\infty(b)$ *and* $Y = E_\infty(\widetilde{a}) \mathbin{\widehat{\otimes}} E'_\infty(\widetilde{b})$. *Then the following statements are equivalent*:

(i) $X \overset{\text{qd}}{\simeq} Y$.

(ii) *A strongly decreasing sequence* $\varepsilon_k \to 0$, $\varepsilon_0 = \varepsilon_{-1} = 1$, *and a constant* $\Delta > 1$ *exist such that*

$$\left|\bigcup_{\alpha\in A}\left\{(i,j) : \varepsilon_{m(\alpha)+1} < \frac{b_j}{a_i + b_j} \le \varepsilon_{m(\alpha)};\ \Delta_{s(\alpha)} \le a_i + b_j < \Delta_{s(\alpha)+1}\right\}\right| \\ \le \left|\bigcup_{\alpha\in A}\left\{(k,l) : \varepsilon_{m(\alpha)+2} < \frac{\widetilde{b}_l}{\widetilde{a}_k + \widetilde{b}_l} \le \varepsilon_{m(\alpha)-1};\ \Delta_{s(\alpha)-1} \le \widetilde{a}_k + \widetilde{b}_l < \Delta_{s(\alpha)+2}\right\}\right|$$

for each finite collection $\{(\varepsilon_{m(\alpha)}, s(\alpha)) : \alpha \in A\}$, *with* $m(\alpha), s(\alpha) \in \mathbb{Z}_+$.

7. Linear Topological Invariants (LTI)

7.1. We shall exploit here the idea, suggested in [31], [32], [34], to use some very well known classical invariant characteristics, but considered for special ("synthetic") sets, which should be constructed in some invariant geometric manner from a fixed system of subsets (for instance, a basis of absolutely convex neighbourhoods of zero or a basis of bounded absolutely convex sets in a locally convex space X). As a fundamental characteristic we shall use the following simplest function of a pair of absolutely convex subsets in X:

$$\beta(W_1, W_2) = \sup\{\dim L : W_1 \cap L \subset W_2\}, \tag{7.1}$$

where L stands for a finite-dimensional subspace of X. The function $\beta(W_1, tW_2) = \beta(t^{-1}W_1, W_2)$ is the inverse function (counting function) for the sequence $\{1/b_i(W_2, W_1)\}$, where

$$b_i(W_2, W_1) = \sup_{L \in \mathcal{L}_i} \sup\{\alpha > 0 : \alpha W_1 \cap L \subset W_2\}, \tag{7.2}$$

$\mathcal{L}_i$ being the collection of all i-dimensional subspaces in X. The numbers in (7.2) are the so-called *Bernstein diameters* (we use them instead of more traditional Kolmogorov diameters $d_i(W_2, W_1)$ for our convenience only), and the relation between (7.1) and (7.2) is described by

$$\beta(W_1, tW_2) = \left|\left\{ i : \frac{1}{b_i(W_2, W_1)} \le t \right\}\right|.$$

For a given quadruple of absolutely convex sets W_i, $i = 1, 2, 3, 4$, the following characteristics can be constructed by using the simplest function ([29, 31, 32]):

$$\begin{gathered} \beta(W_1, W_2, W_3, W_4) := \beta(\overline{\text{conv}}(W_1 \cup W_2); W_3 \cap W_4) \\ \beta(W_1, W_2, W_3) := \beta(W_1, W_2, W_3, W_2) \end{gathered} \tag{7.3}$$

If we put some parameters in these characteristics, for example, if we consider the function $\beta(\tau^{-1}W_1, W_2, tW_3)$, we can obtain considerably more information about the space X than the classical one-parameter characteristics could provide.

The following fact, very useful for construction of invariants, is an immediate consequence of the definitions.

Proposition 7.1. *If $W_1 \subset V_1$, $W_2 \subset V_2$, $W_3 \supset V_3$, $W_4 \supset V_4$, then*

$$\beta(V_1, V_2, V_3, V_4) \le \beta(W_1, W_2, W_3, W_4), \qquad \beta(V_1, V_3) \le \beta(W_1, W_3).$$

7.2. Now we describe a way to estimate the general characteristics (7.3) for weighted l^1-balls, generated by a fixed absolute basis in X.

Let X be a locally convex space with an absolute basis $\{e_i\}_I$, I being a countable set, and let $\{e'_i\}$ be a biorthogonal system in X^*. We use the notation

$$B(a) = B^e(a) := \Big\{ x \in X : \sum_{i \in I} |e'_i(x)| a_i \le 1 \Big\},$$

for any sequence $a = (a_i)$ of positive numbers.

Proposition 7.2. *Let $a^{(p)} = (a_{ip})_{i \in I}$, $p = 1, 2, 3, 4$. Then*

$$\beta(B^e(a^{(1)}), B^e(a^{(2)}), B^e(a^{(3)}), B^e(a^{(2)})) \ge \left|\left\{ i : \frac{a_{i3}}{a_{i2}} \le 1;\ \frac{a_{i2}}{a_{i1}} \le 1 \right\}\right|, \tag{7.4}$$

$$\beta(B^e(a^{(1)}), B^e(a^{(2)}), B^e(a^{(3)}), B^e(a^{(4)})) \le \left|\left\{ i : \frac{a_{i3}}{a_{i2}} \le 2;\ \frac{a_{i4}}{a_{i1}} \le 2 \right\}\right|. \tag{7.5}$$

Proof. (a) First we show that for a pair $b^{(1)} = (b_{i1})$, $b^{(2)} = (b_{i2})$,

$$\beta(B^e(b^{(1)}), B^e(b^{(2)})) = \left|\left\{ i \in I : \frac{b_{i2}}{b_{i1}} \le 1 \right\}\right|. \tag{7.6}$$

We put

$$|x|_q = \sum_{i\in I} |e_i'(x)| b_{iq}, \quad q = 1,2; \quad \mathcal{N} = \{i \in I : b_{i2} \le b_{i1}\}, \quad L_0 = \operatorname{span}\{e_i : i \in \mathcal{N}\}.$$

By definition (7.1),

$$\beta(B^e(b^{(1)}), B^e(b^{(2)})) \ge \dim L_0 = |\mathcal{N}|. \tag{7.7}$$

It remains to prove

$$\beta(B^e(b^{(1)}), B^e(b^{(2)})) \le |\mathcal{N}|. \tag{7.8}$$

Consider the natural projection $P : X \to L_0$, defined by $Px = \sum_{i\in\mathcal{N}} e_i'(x) e_i$. Let L be an arbitrary finite-dimensional subspace in X which satisfies the condition

$$L \cap B^e(b^{(1)}) \subset B^e(b^{(2)}). \tag{7.9}$$

To show (7.8) it is enough to prove that the linear operator $T = P|L : L \to L_0$ is an injection, because this implies immediately that $\dim L \le \dim L_0 = |\mathcal{N}|$. Ad absurdum, suppose that an element $z \in L$ exists such that $|z|_2 = 1$ but $Pz = 0$. Then, from (7.9) it follows that $1 = |z|_2 \le |z|_1$, and from $Pz = 0$ we have $|z|_1 = \sum_{i\in I\setminus\mathcal{N}} |e_i'(z)| b_{i1} < \sum_{i\in I\setminus\mathcal{N}} |e_i'(z)| b_{i2} = |z|_2 = 1$, therewith the strong inequality has been realized, because at least one of the coefficients $e_i'(z)$ must be non-zero, since $z \ne 0$. The discovered contradiction proves (7.8) and together with (7.7) implies (7.6).

(b) Now we use the following obvious geometric relations:

$$\overline{\operatorname{conv}}(B^e(a^{(1)}) \cup B^e(a^{(2)})) = B^e(a^{(1)} \wedge a^{(2)}), \tag{7.10}$$

$$B^e(a^{(3)} \vee a^{(4)}) \subset B^e(a^{(3)}) \cap B^e(a^{(4)}) \subset 2B^e(a^{(3)} \vee a^{(4)}), \tag{7.11}$$

where $a^{(1)} \wedge a^{(2)} = (\min\{a_{i1}, a_{i2}\})_{i\in I}$, $a^{(3)} \vee a^{(4)} = (\max\{a_{i3}, a_{i4}\})_{i\in I}$. Denote by $\mathcal{L}$ the left side of (7.4). Then by Proposition 7.1 and the relation (7.6) we get, applying (7.10), (7.11), the estimate (7.4) as follows:

$$\begin{aligned}\mathcal{L} &\ge \beta(B^e(a^{(1)} \wedge a^{(2)}), B^e(a^{(2)} \vee a^{(3)})) = \left|\left\{i : \frac{\max\{a_{i2}, a_{i3}\}}{\min\{a_{i1}, a_{i2}\}} \le 1\right\}\right| \\ &= \left|\left\{i : \frac{a_{i2}}{a_{i1}} \le 1;\ \frac{a_{i2}}{a_{i2}} \le 1;\ \frac{a_{i3}}{a_{i1}} \le 1;\ \frac{a_{i3}}{a_{i2}} \le 1\right\}\right| = \left|\left\{i : \frac{a_{i3}}{a_{i2}} \le 1;\ \frac{a_{i2}}{a_{i1}} \le 1\right\}\right|\end{aligned}$$

The last equality is true, because one of the omitted inequalities is trivial, and the other is a consequence of the two remaining ones.

Denoting by $\mathcal{L}'$ the left side of (7.5) we analogously get the estimate (7.5) as follows:

$$\begin{aligned}\mathcal{L}' &\le \beta(B^e(a^{(1)} \wedge a^{(2)}), 2B^e(a^{(3)} \vee a^{(4)})) \\ &= \left|\left\{i : \frac{2^{-1}\max\{a_{i3}, a_{i4}\}}{\min\{a_{i1}, a_{i2}\}} \le 1\right\}\right| \le \left|\left\{i : \frac{a_{i3}}{a_{i2}} \le 2;\ \frac{a_{i4}}{a_{i1}} \le 2\right\}\right|. \ \blacksquare\end{aligned}$$

7.3. For a pair of sets $U_0 = B^e(a^{(0)})$, $U_1 = B^e(a^{(1)})$ we define the one-parameter family of sets

$$U_\alpha = (U_0)^{1-\alpha}(U_1)^\alpha := B^e(a^{(\alpha)}), \quad -\infty < \alpha < \infty,$$

where $a^{(\alpha)} = (a_i^{(\alpha)})$, $a_i^{(\alpha)} = (a_i^{(0)})^{1-\alpha}(a_i^{(1)})^\alpha$, $i \in I$.

This construction can be used to compose invariants, due to the following simple interpolational statements.

Proposition 7.3. *Let T be a linear bounded operator from $l^1(a^{(0)})$ to $l^1(b^{(0)})$ and from $l^1(a^{(1)})$ to $l^1(b^{(1)})$, with both norms ≤ 1. Then T is a linear bounded operator from $l^1(a^{(\alpha)})$ to $l^1(b^{(\alpha)})$ with norm ≤ 1; here $a^\alpha = ((a_i^{(0)})^{1-\alpha}(a_i^{(1)})^\alpha)_{i\in I}$, $b^{(\alpha)} = ((b_i^{(0)})^{1-\alpha}(b_i^{(1)})^\alpha)_{i\in I}$, $0 \le \alpha \le 1$.*

Corollary 7.4. *Let $e = (e_i)$, $f = (f_j)$ be two absolute bases in a locally convex space X and $B^e(a^{(\alpha)}) \subset B^f(b^{(\alpha)})$, $\alpha = 0, 1$. Then $B^e(a^{(\alpha)}) \subset B^f(b^{(\alpha)})$, where $0 < \alpha < 1$.*

7.4. Let us consider two spaces $X = E_\infty(a) \widehat{\otimes} E'_\infty(b)$, $Y = E_\infty(\widetilde{a}) \widehat{\otimes} E'_\infty(\widetilde{b})$ and an isomorphism $T : Y \to X$.

Then we can consider two absolute bases in X: the canonical one $e_i \otimes e_j$, $(i,j) \in \mathbb{N}^2$, and the image of the canonical basis of Y: $f_{kl} = T(e_k \otimes e_l)$, $(k,l) \in \mathbb{N}^2$. Hence each $x \in X$ has two basis expansions:

$$x = \sum \xi_{ij} e_i \otimes e_j = \sum \eta_{kl} f_{kl}.$$

Consider two systems of sets, defined respectively by those expansions ($p, q \in \mathbb{N}$):

$$A_{p,q} = \Big\{x \in X : \sum |\xi_{ij}| \exp(pa_i - qb_j) \le 1\Big\}, \tag{7.12}$$

$$B_{p,q} = \Big\{x \in X : \sum |\eta_{kl}| \exp(p\widetilde{a}_k - q\widetilde{b}_l) \le 1\Big\}, \tag{7.13}$$

By [14], II, p. 113, we have $X = \lim\operatorname{proj}_p \lim\operatorname{ind}_q l^1(\exp(pa_i - qb_j))$ and $X \simeq (\widetilde{X})^*$, where $\widetilde{X} = \lim\operatorname{ind}_p \lim\operatorname{proj}_q c_0(\exp(-pa_j + qb_j))$, and the analogous representations for Y hold. Hence by Grothendieck's factorization theorem ([14], I, p. 16) we derive that the systems (7.12), (7.13) are equivalent in the following sense:

$$\forall r\ \exists p\ \forall q\ \exists s\ \exists c : B_{p,q} \subset cA_{r,s}, A_{p,q} \subset cB_{r,s}.$$

Therefore we can consider some chains of indices (as long as we need, but finite),

$$r_1 < p_1 < \ldots < r_m < p_m < q_m < s_m < \ldots < q_1 < s_1,$$

such that the following imbeddings are valid:

$$A_{p_\nu,q_\nu} \subset c_\nu B_{r_\nu,s_\nu}, \qquad B_{r_{\nu+1},s_{\nu+1}} \subset c_\nu A_{p_\nu,q_\nu}, \tag{7.14}$$

where the constant c_ν does not depend on the parameters q_μ, s_μ with $\mu < \nu$, $\nu = 2, \ldots, m$. So we will assume that those indices are taken sufficiently far apart:

$$\min\left\{\frac{p_1}{r_1}, \frac{r_2}{p_1}, \ldots, \frac{p_m}{r_m}, \frac{q_m}{p_m}, \frac{s_m}{q_m}, \ldots, \frac{s_1}{q_1}\right\} \ge 2. \tag{7.15}$$

Those systems of sets are good raw material to construct some new linear topological invariants, which are natural for the class of spaces considered here. Namely, we apply the functions (7.3) to the following artificial "synthetic" absolutely convex sets, constructed with the sets taken from the two fixed collections (7.12), (7.13):

$$\begin{aligned} W_1 &= \overline{\operatorname{conv}}((A_{p_2,q_2})^{1/2}(A_{p_7,q_7})^{1/2} \cup (\exp\tau)(A_{p_7,q_7})),\\ W_2 &= W_4 = A_{p_4,q_4},\\ W_3 &= (A_{p_1,q_1})^{1/2}(A_{p_6,q_6})^{1/2} \cap (\exp t)(A_{p_6,q_6}),\end{aligned}$$

$$
\begin{aligned}
V_1 &= \overline{\mathrm{conv}}\biggl(\frac{1}{\sqrt{c_2c_7}}(B_{r_3s_3})^{1/2}(B_{r_8s_8})^{1/2} \cup \Bigl(\frac{1}{c_7}\exp\tau\Bigr)B_{r_8s_8}\biggr),\\
V_2 &= \frac{1}{2c_4}B_{r_5s_5},\\
V_3 &= \sqrt{c_1c_6}(B_{r_1s_1})^{1/2}(B_{r_6s_6})^{1/2} \cap (c_6\exp t)B_{r_6s_6},\\
V_4 &= 2c_4B_{r_4s_4},
\end{aligned}
$$

therewith we assume that (7.14) and (7.15) hold.

So, we get two families of functions:

$$
\begin{aligned}
\beta_P(t,\tau) &:= \beta(W_1,W_2,W_3,W_2), \qquad P=(p_1,q_1;\dots;p_7,q_7),\\
\widetilde{\beta}_R(t,\tau) &:= \beta(V_1,V_2,V_3,V_4), \qquad R=(r_1,s_1;\dots;r_8,r_8).
\end{aligned}
$$

The first carries some information on the space X, as does the second about the space Y. By Proposition 7.1 we can compare these data:

$$
\beta_P(t,\tau) \le \widetilde{\beta}_R(t,\tau). \tag{7.16}
$$

In view of symmetry we can arrange the corresponding estimates in the opposite direction:

$$
\widetilde{\beta}'_{R'}(t,\tau) \le \beta'_{P'}(t,\tau), \tag{7.17}
$$

after some analogous preparation of appropriate “synthetic” sets V_1', $V_2'=V_4'$, V_3' and W_1', W_2', W_3', W_4', constructed from the same material (7.10) and (7.11).

After some calculations, based on Proposition 7.2, we can give necessary conditions for the isomorphism $X \simeq Y$ in terms of the sequences a, b, $\widetilde{a}$, $\widetilde{b}$. These conditions will be given in the following section.

8. Necessary conditions

Here we give the proof of Theorem 3.7. We need to estimate the functions β_P, $\widetilde{\beta}_R$ by some characteristics of the sequences a, b, $\widetilde{a}$, $\widetilde{b}$. Denote by $a^{(\nu)}=(a_{ij}^{(\nu)})$, $b^{(\nu)}=(b_{kl}^{(\nu)})$ the weight sequences, corresponding to the sets A_{p_v,q_v} and B_{r_v,s_v}:

$$
\begin{aligned}
a_{i,j}^{(\nu)} &:= \exp(p_\nu a_i - q_\nu b_j), \qquad (i,j)\in\mathbb{N}^2,\ \nu=1,\dots,7,\\
b_{k,l}^{(\nu)} &:= \exp(r_\nu \widetilde{a}_k - s_\nu \widetilde{b}_l), \qquad (k,l)\in\mathbb{N}^2,\ \nu=1,\dots,8.
\end{aligned}
$$

Then by Proposition 7.2 the following estimates hold:

$$
\beta_P(t,\tau) \ge \biggl|\biggl\{(i,j) : \frac{\sqrt{a_{ij}^{(1)}a_{ij}^{(6)}}}{a_{ij}^{(4)}} \le 1;\ \frac{\sqrt{a_{ij}^{(2)}a_{ij}^{(7)}}}{a_{ij}^{(4)}} \ge 1;\ \frac{a_{ij}^{(6)}}{a_{ij}^{(4)}} \le \exp t;\ \frac{a_{ij}^{(7)}}{a_{ij}^{(4)}} \ge \exp\tau\biggr\}\biggr|,
$$

$$
\widetilde{\beta}_R(t,\tau) \le
$$

$$
\biggl|\biggl\{(k,l) : \frac{\sqrt{b_{kl}^{(1)}b_{kl}^{(6)}}}{b_{kl}^{(5)}} \le c_4\sqrt{c_1c_6};\ \frac{\sqrt{b_{kl}^{(3)}b_{kl}^{(8)}}}{b_{kl}^{(4)}} \ge \frac{1}{c_4\sqrt{c_2c_7}};\ \frac{b_{kl}^{(6)}}{b_{kl}^{(5)}} \le c_4c_6\exp t;\ \frac{b_{kl}^{(8)}}{b_{kl}^{(4)}} \ge \frac{\exp\tau}{c_4c_7}\biggr\}\biggr|.
$$

The simple estimates below, following from assumptions (7.15), are useful now, if we try to work with the concrete form of $a^{(\nu)}$, $b^{(\nu)}$:

$$\frac{p_1+p_6}{2}-p_4\le p_6;\quad \frac{q_1+q_6}{2}-q_4\ge\frac{q_1}{4};\quad \frac{p_2+p_7}{2}-p_4\ge\frac{p_7}{4};\quad \frac{q_2+q_7}{2}-q_4\le q_2;$$
$$p_6-p_4\le p_6\le q_4;\quad q_4-q_6\le q_4;\quad p_7-p_4\ge\frac{p_7}{2};\quad q_4-q_7\ge\frac{q_4}{2}\ge\frac{p_7}{2};$$
$$\frac{r_6}{4}\le\frac{r_1+r_6}{2}-r_5\le r_6;\quad \frac{s_1}{4}\le\frac{s_1+s_6}{2}-s_5\le s_1;$$
$$\frac{r_8}{4}\le\frac{r_3+r_8}{2}-r_4\le r_8;\quad \frac{s_3}{4}\le\frac{s_3+s_8}{2}-s_4\le s_3;$$
$$\frac{r_6}{2}\le r_6-r_5\le r_6;\quad \frac{r_6}{2}\le\frac{s_5}{2}\le s_5-s_6\le s_5;$$
$$\frac{r_8}{2}\le r_8-r_4\le r_8\le s_4;\quad \frac{s_4}{2}\le s_4-s_8\le s_4.$$

With these relations the next estimates will be obtained:

$$\begin{aligned}\beta_P(t,\tau)\ge\Big|\Big\{(i,j):&\Big(\frac{p_1+p_6}{2}-p_4\Big)a_i-\Big(\frac{q_1+q_6}{2}-q_4\Big)b_j\le 0;\\ &\Big(\frac{p_2+p_7}{2}-p_4\Big)a_i-\Big(\frac{q_2+q_7}{2}-q_4\Big)b_j\ge 0;\\ &(p_6-p_4)a_i+(q_4-q_6)b_j\le t;\ (p_7-p_4)a_i+(q_4-q_7)b_j\ge\tau\Big\}\Big|\\ \ge\Big|\Big\{(i,j):&\frac{4p_6}{q_1}\le\frac{b_j}{a_i+b_j}\le\frac{p_7}{8q_2};\ \frac{2\tau}{p_7}\le a_i+b_j\le\frac{t}{q_4}\Big\}\Big|;\\ \widetilde{\beta}_R(t,\tau)\le\Big|\Big\{(k,l):&\Big(\frac{r_1+r_6}{2}-r_5\Big)\widetilde{a}_k-\Big(\frac{s_1+s_6}{2}-s_5\Big)\widetilde{b}_l\le\ln c_4\sqrt{c_1c_6};\\ &\Big(\frac{r_3+r_8}{2}-r_4\Big)\widetilde{a}_k-\Big(\frac{s_3+s_8}{2}-s_4\Big)\widetilde{b}_l\ge-\ln c_4\sqrt{c_2c_7};\\ &(r_6-r_5)\widetilde{a}_k+(s_5-s_6)\widetilde{b}_l\le t+\ln c_4c_6;\\ &(r_8-r_4)\widetilde{a}_k+(s_4-s_8)\widetilde{b}_l\ge\tau-\ln c_4c_7\Big\}\Big|\\ \le\Big|\Big\{(k,l):&\frac{r_6}{8s_1}\le\frac{\widetilde{b}_l}{\widetilde{a}_k+\widetilde{b}_l}\le\frac{8r_8}{s_3};\ \frac{\tau}{4s_4}\le\widetilde{a}_k+\widetilde{b}_l\le\frac{4t}{r_6}\Big\}\Big|,\end{aligned}$$

where the last inequality is true for $\tau\ge\tau_0$, and τ_0 depends on all the parameters r_ν, s_ν. Thus we have, after replacing $2\tau/p_8$ by τ and t/q_4 by t,

$$\begin{aligned}&\Big|\Big\{(i,j):\frac{4p_6}{q_1}\le\frac{b_j}{a_i+b_j}\le\frac{p_7}{8q_2};\ \tau\le a_i+b_j\le t\Big\}\Big|\\ &\qquad\le\Big|\Big\{(k,l):\frac{r_6}{8s_1}\le\frac{\widetilde{b}_l}{\widetilde{a}_k+\widetilde{b}_l}\le\frac{8r_8}{s_3};\ \frac{p_7\tau}{8s_4}\le\widetilde{a}_k+\widetilde{b}_l\le\frac{4q_4}{r_6}t\Big\}\Big|.\end{aligned}$$

Fixing the parameters p_6, p_7, p_8, r_6, r_8, q_4, s_4 and leaving the rest of them free, we get assertion (3.2) of Theorem 3.7 with

$$\varDelta=\max\{4q_4/r_6, 8s_4/p_7\},$$

but asymptotically, i.e., for $\tau\ge\tau_0$, where τ_0 depends on all the parameters. The relations (3.3)–(3.5) can be proved similarly, but in a considerably simpler fashion.

9. Spaces $s \widehat{\otimes} E'_\infty(b)$

P r o o f o f T h e o r e m 3.1. We shall apply Theorem 3.7 and Proposition 6.1 to the spaces $X = E_\infty(a) \widehat{\otimes} E'_\infty(b)$ and $Y = E_\infty(\widetilde{a}) \widehat{\otimes} E'_\infty(\widetilde{b})$ with

$$a_i = \widetilde{a}_i = \max\{1, \ln i\}, \quad i \in \mathbb{N}$$

(note that $s \simeq E_\infty(a)$). Without loss of generality, we can assume the following conditions:

$$a_i \le b_i, \quad \widetilde{a}_i \le \widetilde{b}_i, \quad i \in \mathbb{N},$$

or, what is the same,

$$m_b(t) \le m_a(t), \quad m_{\tilde{b}}(t) \le m_{\tilde{a}}(t), \qquad t \ge 1.$$

Note that $m_a(t) = m_{\tilde{a}}(t)$ and

$$e^{t/2} < e^t - 1 < m_a(t) \le e^t, \quad t \ge 1. \tag{9.1}$$

Suppose that $X \simeq Y$. Then, by Theorem 3.7, $\exists \varDelta\ \forall \widetilde{\varepsilon}\ \exists \varepsilon\ \forall \delta\ \exists \widetilde{\delta}$ such that the inequality (3.2) is valid. We denote by I and $\widetilde{I}$, respectively, the left and right sides of (3.2) with $t = \tau\varDelta$. It can be assumed that all the parameters are chosen in such a way that $\varDelta \ge 5$, $\delta\varDelta < \varepsilon < \widetilde{\varepsilon}$. Then, applying (9.1), the following estimates are true:

$$\begin{aligned} I &\ge [m_a(\tau(\varDelta - \varepsilon)) - m_a(\tau)][m_b(\varepsilon\tau) - m_b(\delta\varDelta\tau)] \\ &\ge \left[\exp\frac{\tau(\varDelta-\varepsilon)}{2} - \exp\tau\right] n_b(\delta\varDelta\tau, \varepsilon\tau) \ge (\exp\tau) n_b(\delta\varDelta\tau, \varepsilon\tau); \\ \widetilde{I} &\le m_{\tilde{a}}(\varDelta^2\tau)[m_{\tilde{b}}(\widetilde{\varepsilon}\varDelta^2\tau) - m_{\tilde{b}}(\widetilde{\delta}\tau/\varDelta)] \le (\exp 2\varDelta^2\tau) n_{\tilde{b}}(\widetilde{\delta}\tau/\varDelta, \widetilde{\varepsilon}\varDelta^2\tau). \end{aligned}$$

Therefore,

$$(\exp\tau) n_b(\delta\varDelta\tau, \varepsilon\tau) \le (\exp(2\varDelta^2\tau)) n_{\tilde{b}}(\widetilde{\delta}\tau/\varDelta, \widetilde{\varepsilon}\varDelta^2\tau). \tag{9.2}$$

From this it follows that

$$n_b(\delta\varDelta\tau, \varepsilon\tau) \le n_{\tilde{b}}(\widetilde{\delta}\tau/\varDelta, \widetilde{\varepsilon}\varDelta^2\tau), \quad \tau \ge 1; \tag{9.3}$$

otherwise, for some τ_0, the left side of (9.2) would be positive, but the right side would be equal to zero. Putting $t = \sqrt{\varepsilon\delta\varDelta}\tau$, from (9.3) we get

$$n_b(t/A, At) \le n_{\tilde{b}}(t/B, Bt), \quad t \ge 1, \tag{9.4}$$

where

$$A = \sqrt{\frac{\varepsilon}{\delta\varDelta}} > 1, \quad B = \max\left\{\widetilde{\varepsilon}\sqrt{\frac{\varDelta^3}{\varepsilon\delta}}, \frac{\sqrt{\varepsilon\delta\varDelta^3}}{\widetilde{\delta}}\right\}.$$

Because of symmetry we also have the inequality obtained from (3.4) by exchanging b and $\widetilde{b}$. By Proposition 1.1 this means that b and $\widetilde{b}$ are identical in lacunarity.

On the other hand, with b and $\widetilde{b}$ identical in lacunarity, this means that for arbitrary $A > 1$ and some $B = B(A)$ the condition (9.4) holds together with the above-mentioned symmetric inequality. We choose a constant $\varDelta$ and a sequence ε_m, $m \in \mathbb{Z}_+$, in such a way that

$$\varDelta \ge 8, \quad \varDelta^3\varepsilon_{m+1} \le \varepsilon_m, \quad m \in \mathbb{Z}_+, \tag{9.5}$$

and

$$\Delta\varepsilon_m \le A^2\varepsilon_{m+1}, \quad B \le \min\left\{\frac{\varepsilon_{m-1}^2}{\Delta\varepsilon_m\varepsilon_{m+1}},\ \frac{\varepsilon_m\varepsilon_{m+1}}{\Delta^3\varepsilon_{m+2}^2}\right\}, \qquad m \in \mathbb{Z}_+. \tag{9.6}$$

Let us now show that, after choosing all the parameters, condition (iii) of Proposition 6.1 is valid. Because of symmetry, only (6.1) needs to be proved.

By $I(m,s)$ and $\widetilde{I}(m,s)$, respectively, we denote the left and right sides of (6.1). The assumption taken at the beginning of this proof, together with (9.5), ensures the following inequalities:

$$\begin{aligned} I(m,s) &\le m_a(\Delta^{s+1})[m_b(\varepsilon_m\Delta^{s+1}) - m_b(\varepsilon_{m+1}\Delta^s)] \\ &\le (\exp(2\Delta^{s+1}))n_b(\varepsilon_{m+1}\Delta^s,\, \varepsilon_m\Delta^{s+1}), \\ \widetilde{I}(m,s) &\ge [m_{\tilde a}(\Delta^{s+2} - \varepsilon_{m+1}\Delta^{s-1}) - m_a(\Delta^{s-1})]n_{\tilde b}(\varepsilon_{m+2}\Delta^{s+2},\, \varepsilon_{m-1}\Delta^s) \\ &\ge (\exp(2\Delta^{s+1}))n_{\tilde b}(\varepsilon_{m+2}\Delta^{s+2}, \varepsilon_{m-1}\Delta^s). \end{aligned}$$

Taking into account (9.6), from this we get

$$I(m,s) \le \widetilde{I}(m,s), \qquad m,s \in \mathbb{Z}_+.$$

So, by Proposition 6.1, $X \overset{\mathrm{qd}}{\simeq} Y$ (the more so as $X \simeq Y$).

10. Spaces $s' \widehat{\otimes} E_\infty(a)$

10.1. Proof of Theorem 3.11. Since (i)⇒(ii) and (iii)⇒(iv) are obvious, we need to prove (ii)⇒(iii) and (iv)⇒(i). First we show (ii)⇒(iii). For this purpose we use Theorem 3.7. Denote by I and $\widetilde{I}$ the left and right sides of (3.2), respectively, and assume that

$$\varepsilon \ge \frac{4\delta}{1-\delta}, \qquad \varepsilon = \varepsilon(\widetilde{\varepsilon}) < \widetilde{\varepsilon} < \frac{1}{2}.$$

Then

$$I \ge \sum_{\tau < a_i \le t(1-\varepsilon)} \left[m_b(\varepsilon a_i) - m_b\left(\frac{\delta}{1-\delta}a_i\right)\right] \ge m_a(t(1-\varepsilon)) - m_a(\tau), \qquad \tau \ge \tau_0 := \frac{4}{\varepsilon},$$

since

$$m_b(\varepsilon\tau) - m_b\left(\frac{\delta}{1-\delta}\tau\right) \ge \exp\frac{\varepsilon\tau}{2} - \exp\frac{\varepsilon\tau}{4} \ge 1,$$

if $\tau \ge \tau_0$. Further,

$$\widetilde{I} \le (\exp\widetilde{\varepsilon}\Delta t)\left[m_a(\Delta t) - m_a\left(\frac{1-\widetilde{\varepsilon}}{\Delta}\tau\right)\right].$$

Therefore, from (3.2) it follows that (3.8) is satisfied with $\gamma = \widetilde{\varepsilon}\Delta/(1-\varepsilon)$, $A = 2\Delta$ and τ_0 defined as above. Because of symmetry (3.9) can be obtained in the same way.

(iv)⇒(i). By Theorem 3.9, and for reasons of symmetry, it is enough to prove the inequalities (3.2) and (3.3) with the corresponding quantifiers and $\tau \ge t_0$ for some constant t_0, depending on all parameters.

First, we deal with (3.2). Let (iv) hold. Take some $D > 2$. Without loss of generality it can be assumed that

$$\widetilde{\delta} < \delta < \varepsilon < \frac{\widetilde{\varepsilon}}{16D^{4(r+1)}}, \tag{10.1}$$

where r will be fixed later.

Let us introduce some notation:

$$\begin{aligned}
&b_j = \max(1, \ln j), \quad \widetilde{b}_j = b_j, \quad j \in \mathbb{N};\\
&\mathcal{N}(s) = \left\{(i,j) \in \mathbb{N}^2 : \delta < \frac{b_j}{a_i + b_j} \le \varepsilon;\ D^{2s-1} \le a_i + b_j < D^{2s+1}\right\},\\
&\mathcal{M}(s) = \left\{(k,l) \in \mathbb{N}^2 : \widetilde{\delta} < \frac{\widetilde{b}_l}{\widetilde{a}_k + \widetilde{b}_l} \le \widetilde{\varepsilon};\ D^{2s-1} \le \widetilde{a}_k + \widetilde{b}_l < D^{2s+1}\right\},\\
&I(s) = |\mathcal{N}(s)|, \quad J(s) = |\mathcal{M}(s)|, \quad s \in \mathbb{Z}_+,\\
&I(s_0, s_1) = \Big|\bigcup_{s=s_0}^{s_1} \mathcal{N}(s)\Big| = \sum_{s=s_0}^{s_1} I(s), \quad J(s_0, s_1) = \Big|\bigcup_{s=s_0}^{s_1} \mathcal{M}(s)\Big| = \sum_{s=s_0}^{s_1} J(s), \quad s_0 \le s_1.
\end{aligned}$$

Put $A = D^2$ and choose $r \in \mathbb{N}$ in such a way that the constant $B = B(A)$ in (iv) satisfies the condition $B \le D^{2r}$. Then with the assumption (10.1) and the condition (iv),

$$\begin{aligned}
I(s) &\le [m_b(\varepsilon D^{2s+1}) - m_b(\delta D^{2s-1})]\left[m_a(D^{2s+1}) - m_a\left(\frac{D^{2s-1}}{2}\right)\right]\\
&\le \exp \varepsilon D^{2s+1}[m_a(D^{2(s+1)}) - m_a(D^{2(s-1)})]\\
&\le \exp\, 2\varepsilon D^{2s+1}[m_{\tilde{a}}(D^{2(s+r)}) - m_a(D^{2(s-r)})]
\end{aligned}$$

for $D^{2s+1} \ge \tau_0(\varepsilon)$. On the other hand,

$$\begin{aligned}
\widetilde{J}(s) &:= J(s-r-1,\ s+r+1)\\
&\ge [m_{\tilde{b}}(\widetilde{\varepsilon} D^{2(s-r-1)}) - m_{\tilde{b}}(\widetilde{\delta} D^{2(s+r+1)})]\left[m_{\tilde{a}}\left(\frac{D^{2(s+r+1)}}{2}\right) - m_{\tilde{a}}(D^{2(s-r-1)})\right]\\
&\ge \left[\exp\frac{\widetilde{\varepsilon} D^{2(s-r-1)}}{2} - \exp(\widetilde{\delta} D^{2(s+r+1)})\right][m_{\tilde{a}}(D^{2(s+r)}) - m_{\tilde{a}}(D^{2(s-r)})]\\
&\ge \exp\frac{\widetilde{\varepsilon} D^{2(s-r-1)}}{4}[m_{\tilde{a}}(D^{2(s+r)}) - m_{\tilde{a}}(D^{2(s-r)})]
\end{aligned}$$

for $D^{2s} \ge 1/\widetilde{\delta}$. Therefore the inequality

$$I(s) \le \frac{1}{2(r+1)}\widetilde{J}(s), \quad s \ge S, \tag{10.2}$$

would be true with an appropriate constant S if the following inequality held:

$$\exp 2\varepsilon D^{2s+1} \le \frac{1}{2(r+1)} \exp\frac{\widetilde{\varepsilon} D^{2(s-r-1)}}{4}, \quad s \ge R,$$

with some constant R. But the last inequality follows from the assumption (10.1) if $D^{2R+1} \ge (\ln 2(r+1))/(2\varepsilon)$, so (10.2) is proved. Hence

$$I(s_0, s_1) \le \frac{1}{2(r+1)} \sum_{s=s_0}^{s_1} \widetilde{J}(s) = \frac{1}{2(r+1)} \sum_{s=s_0}^{s_1} \sum_{\alpha=0}^{2(r+1)} J(s-r-1+\alpha)$$

$$\leq \frac{1}{2(r+1)} \sum_{s=s_0}^{s_1} J(s_0 - r - 1, s_0 + r + 1) = J(s_0 - r - 1, s_0 + r + 1).$$

Putting $t_0 = D^{2S-1}$, we choose for $t_0 \leq \tau < t < \infty$ two natural numbers s_0 and s_1 such that

$$D^{2s_0-1} \leq \tau < D^{2s_0}, \qquad D^{2s_1} \leq t < D^{2s_1+1}.$$

Then we get (3.2) with $\Delta = D^{2(r+1)+1}$ and $\tau \geq t_0$. To prove (3.3) we use the conditions (3.12), (3.13). Denote by I and J the left and right sides of (3.3), respectively. Without loss of generality, we can assume $\widetilde{\delta} < \delta < 1$. Put $\Delta = 16E$. Then

$$\begin{aligned}
&I \leq m_b(t)m_a(t) \leq (\exp t)m_a(t) \leq \exp((E+1)t)m_{\tilde a}(Et) \leq (\exp 2Et)m_{\tilde a}(Et),\\
&J \geq [m_{\tilde b}(\Delta t/2) - m_{\tilde b}(\Delta t/8)]m_{\tilde a}(\Delta t/2)\\
&\quad \geq (\exp \Delta t/8)m_{\tilde a}(\Delta t/2) \geq (\exp 2Et)m_{\tilde a}(Et), \qquad t \geq 1.
\end{aligned}$$

Thus we get (3.3). ∎

10.2. Proof of Theorem 3.3. Let a and $\widetilde{a}$ be identical in lacunarity, i.e., $\exists A > 1$ such that

$$\text{(10.3)} \qquad n_a(t,\tau) \leq n_{\tilde a}(\tau/A, At), \quad n_{\tilde a}(t,\tau) \leq n_a(\tau/A, At), \qquad 1 \leq \tau < t < \infty.$$

Then, due to (3.1), we have for each $\gamma > 0$,

$$m_a(\tau, t) \leq (\exp \gamma t)n_a(\tau, t), \qquad \tau \geq \tau_0 = \tau_0(\gamma).$$

Using (10.3), from this we get

$$m_a(\tau, t) \leq (\exp \gamma t)n_{\tilde a}(\tau/A, At) \leq (\exp \gamma t)m_{\tilde a}(\tau/A, At), \qquad \tau \geq \tau_0,$$

i.e., (3.10); by symmetry we also have (3.11). Therefore, by Theorem 3.11 we get $X \simeq Y$.

On the other hand, let $X \simeq Y$. Then condition (iii) of Theorem 3.11 holds. Suppose that the left side of (3.10) is not equal to zero; then the right side of (3.10) is not equal to zero either, which means that

$$n_a(t,\tau) \leq n_{\tilde a}(\tau/A, At).$$

Similarly we get $n_{\tilde a}(t,\tau) \leq n_a(\tau/A, At)$. So the sequences a and $\widetilde{a}$ are identical in lacunarity. ∎

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