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Joint distribution of waiting time and queue size for single server queues

WARSZAWA 1986

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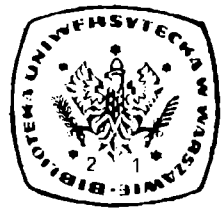

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CONTENTS

1. Introduction

1.1. Main problems under consideration. By a queue Q we mean the following queueing model. At the service station there is only one server. If at the moment of arrival of a unit (customer) the server is idle, the service of this unit is started immediately. A unit which, at its arrival, finds the server busy, joins the queue and waits for service. Whenever waiting units are present and the server has completed the service of a unit, he immediately starts the service of a waiting unit. The units are served in the order of arrival. Later on we say that a queue Q is generated by a sequence $\{(v_k, u_k), k \geqslant 0\}$ of pairs of nonnegative random variables defined on a common probability space, where v_k, $k \geqslant 1$, represents the service time of the kth unit, u_k, $k \geqslant 1$, represents the interarrival time between the kth and $(k+1)$th units, u_0 represents the time up to the first arrival into the system and v_0 represents the time a unit waits for service if it arrives at $t = 0$. We assume that $v_0 = 0$.

A queue Q is called GI/G/1 if the sequences $\{v_k, k \geqslant 0\}$ and $\{u_k, k \geqslant 0\}$ are independent and the random variables in each sequence $\{v_k, k \geqslant 0\}$ and $\{u_k, k \geqslant 0\}$ are independent and identically distributed for $k \geqslant 1$. In this case we introduce the notion of traffic intensity defined as $\varrho = Ev_1/Eu_1$.

For a queue Q denote by $D(t)$, $W(t)$, $L(t)$, $t \geqslant 0$, the number of departures from the system during the interval $[0, t]$, the virtual waiting time at time t (1), and the number of units in the system at time t (the queue size), respectively. If the queue Q is in the state of equilibrium we denote by w_∞, $W(\infty)$ and $L(\infty)$ the time between the arrival moment of a unit and the moment at which its service is started (the actual waiting time), the virtual waiting time and the queue size respectively. If it does not lead to ambiguity we will write w, W and L instead of w_∞, $W(\infty)$ and $L(\infty)$ respectively.

In the paper we investigate the following problems:

1. The joint asymptotic behaviour of the arrival process and the departure process.

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(1) The process $\{W(t), t \geqslant 0\}$ differs a little from the process $\{W^*(t), t \geqslant 0\}$ of the virtual waiting time defined in [5] (see p. 170) where $W^*(t)$, $t \geqslant 0$, is equal to the time a unit waits for service if it arrives at time t. The two processes satisfy the relation $W(t) = W^*(t)$ if t is not the arrival moment of a unit, and $W(t) = W^*(t) + v_k$ if t is the arrival moment of the kth unit. Both $W(t)$ and $W^*(t)$ have the same limit distribution as $t \to \infty$.

2. The distribution of $(W(t), L(t))$ for any queue and the distribution of (W, L) for a GI/G/1 queue.

3. The behaviour of the distributions of (W, L) and $(W(t), L(t))$ in a heavy traffic situation.

The first problem is important for a network of queues. In such a situation the departure process of a queue can be the arrival process for another one. The departure process is a renewal process only in a few cases. Thus it is important to be able to characterize that process.

One of the characteristics of the departure process is its intensity in the equilibrium state, i.e. the limit of $t^{-1}D(t)$, in some sense, as $t \to \infty$. The conditions for the existence and the form of the limit are given in Chapter 3, Theorem 3.1. It asserts that if $k^{-1}\sum_{i=1}^{k} v_i \to \mu^{-1}$ and $k^{-1}\sum_{i=1}^{k} u_i \to \lambda^{-1}$ in some sense as $k \to \infty$, where μ and λ are some finite positive numbers, then $t^{-1}D(t) \to \min(\mu, \lambda)$ in the same sense as $t \to \infty$. As far as I know, the result seems to be new. The last statement is formulated in Theorem 3.3 for a sequence of departure processes D_n from the queues Q_n, $n \geqslant 1$. The convergences mentioned above are called the laws of large numbers for departure processes. The rate of that convergence is given by the functional central limit theorem for departure processes in Section 3.2 (Theorem 3.4). As far as I know, the results seem to be stronger than their analogues known in the literature (see [9]). Theorems 3.3 and 3.4 are important from both the practical and the theoretical points of view. We exploit them later on in Chapter 5.

The departure process is completely described by the sequence $\{\Delta_k, k \geqslant 0\}$, where Δ_k is the interdeparture time of the kth unit and the $(k+1)$th unit, $k \geqslant 1$, and Δ_0 is the time to the departure of the first unit. It appears that if $\{\Delta_k, k \geqslant 0\}$ obeys some law (the functional law of large numbers or the functional central limit theorem) then so does the departure process. Precisely this is given in Corollaries 3.6 and 3.7.

The queues for which $\{(v_k, u_k), k \geqslant 1\}$ is a stationary sequence have been well investigated. Thus for studying networks of queues it is important to know when $\{\Delta_k, k \geqslant 0\}$ is a stationary or an asymptotically stationary sequence. Some results about the sequence $\{\Delta_k, k \geqslant 0\}$, which are consequences of the form (3.1.1) for the departure process and of Theorems 3.3 and 3.4, are given in the third part of Chapter 3.

The second problem, that of the joint distribution of the waiting time and the queue size, is important when the stochastic dependence of these random variables is taken into account. As far as I know, there exists no formula for the joint distribution of W and L whereas the separate forms for the distributions of W and L as well as relations between them can be found.

Chapter 4 deals with the distributions of random vectors $(W(t), L(t))$, $t \geqslant 0$, and (W, L). The main result of Chapter 4 is given in Theorem 4.1, where the form of the distribution of (W, L) for a GI/G/1 queue (see formula (4.1.10)) is established.

Various consequences of the distribution forms of (W, L) and $(W(t), L(t))$ are considered in Chapter 5. First of them is a relation between the distributions of L and $w+v$ (see formula (5.1.1)). Similar relations are known in queueing theory (see [2], [8], [22], [15], [16]) but here we obtain them immediately from the distribution of (W, L). We call them Little's formulas for the distributions of L and $w+v$ because they immediately yield Little's formula $EL = \lambda E(w+v)$.

The distribution of W for a GI/G/1 queue is given in the literature as the solution of an integral equation. Its form, however, is not suitable for effective numerical calculations. Thus we face an approximation problem for the distribution. One approach concerns the heavy traffic theory for queues. This theory deals with an asymptotic behaviour of queueing characteristics in a situation where the probability of the immediate service is small. For a GI/G/1 queue this means that $1-\varrho$ is a small positive number. The forms of the distributions of (W, L) and $(W(t), L(t))$ imply new results in the theory of heavy traffic of queues. They are formulated in Theorems. 5.2–5.5. Theorem 5.2 asserts, under appropriate conditions for GI/G/1 queues, that $(1-\varrho)(L-\lambda W) \xrightarrow{P} 0$ as $\varrho \nearrow 1$. Theorem 5.3 states, under appropriate conditions for GI/G/1 queues, that $E((1-\varrho)L)^m - E((1-\varrho)\lambda W)^m \to 0$ as $\varrho \nearrow 1$. The quantities L and W are defined for a queue Q with traffic intensity ϱ. Thus in the above convergences we ought to mark them by ϱ but for simplicity we do not do so. Theorem 5.4 describes a class of queues such that if all queues Q_n belong to it and the limit distribution of $k_n^{-1/2} W_n(k_n t)$ is nondegenerate, then

$$k_n^{-1/2}(L_n(k_n t) - \lambda W_n(k_n t)) \xrightarrow{P} 0 \quad \text{as } k_n \to \infty.$$

A functional analogue of the above statement is given in Theorem 5.5. The random variables $L_n(t)$ and $W_n(t)$ are analogues of $L(t)$ and $W(t)$ for Q_n and the constant λ has some queueing interpretation.

The paper is organized in the following way. Chapters 3, 4 and 5 deal with the first, second and third problem respectively. Chapter 2 contains auxiliary assertions, most of which are known in the literature. We give them here for two reasons. The first is that they are compiled from many papers, in which they are formulated in a form not suitable for us. The second reason is the wish to make the paper clearer and more complete.

I wish to express my thanks to Professor Bolesław Kopociński for his valuable comments. Our many discussions have been extremely stimulating and fruitful for me.

1.2. Notation and conventions. All notation used in the paper may be found in the cited literature.

Abbreviations:

r.v. – random variable or random vector,

r.e. – random element,

a.e. – almost everywhere,

i.i.d. – independent and identically distributed,

d.f. – distribution function.

Sets: $\boldsymbol{R} = (-\infty, \infty)$, $\boldsymbol{R}_+ = [0, \infty)$, $\boldsymbol{R}^l$ – Cartesian product of l copies of the set $\boldsymbol{R}$, $1 \leqslant l \leqslant \infty$.

Function spaces: the space D is defined as the family of all real-valued right-continuous functions on $\boldsymbol{R}_+$ with left-hand limits on $(0, \infty)$. This space considered with the metric d defined in [12] is a Polish space. The space D' is defined as the set of all x belonging to the space D which are non-negative and nondecreasing. This space considered with the metric d is a Polish space. The space C is defined as the family of all real-valued continuous functions on $\boldsymbol{R}_+$. By D^l we denote the Cartesian product of l copies of the space D, $1 \leqslant l \leqslant \infty$, and by d^l the product metric in D^l. We assume that subspaces of D are endowed with the relative topology and product spaces are endowed with the product topology.

Denote by $\mathscr{D}$ and $\mathscr{D}'$ the Borel σ-fields in (D, d) and (D', d) respectively. It may be shown that $D' \in \mathscr{D}$, $C \in \mathscr{D}$. Thus $\mathscr{D}' = \{A\colon A \subset D'$ and $A \in \mathscr{D}\}$. Furthermore, if $d(x_n, x) \to 0$ and $x \in C$ then the sequence $\{x_n, n \geqslant 1\}$ converges to x uniformly on compact sets in $\boldsymbol{R}_+$ (see [12]).

Elements: By e, $\mathbf{1}$, $\mathbf{0}$ and δ_a, where $a \geqslant 0$, we denote elements of the space D defined by $e(t) = t$, $\mathbf{1}(t) = 1$, $\mathbf{0}(t) = 0$ for $t \geqslant 0$ and $\delta_a(t) = 1$ for $t \geqslant a$ and 0 for $0 \leqslant t < a$, respectively. By $\tilde{\delta}_a$, where $a \in \boldsymbol{R}$, we denote the function on $\boldsymbol{R}$ defined by $\tilde{\delta}_a(t) = 1$ for $t \geqslant a$ and 0 for $t < a$, and $\underline{0}$ denotes the vector in $\boldsymbol{R}^l$ with all coordinates equal to zero.

Random elements: Let A be a topological space and $\mathscr{A}$ the Borel σ-field in A (the σ-field generated by the open sets in $A \equiv$ the σ-field generated by the topology). A mapping Y of a probability space $(\Omega, \mathscr{F}, P)$ into A such that $Y^{-1}\mathscr{A} \subset \mathscr{F}$ is called a *random element* of $(A, \mathscr{A})$. If the topology of A is fixed then instead of saying "Y is an r.e. of $(A, \mathscr{A})$" we say "Y is an r.e. of A".

The fact that two r.e.'s X and Y have the same distribution is denoted by $X \overset{\mathrm{D}}{=} Y$.

By $\mathfrak{W}$ we denote a Wiener process which is an r.e. of D (see [1]). By a two-dimensional Wiener process we mean an r.e. $(\mathfrak{W}_1, \mathfrak{W}_2)$ of D^2 such that $(\mathfrak{W}_1, \mathfrak{W}_2)$ is a Gaussian process with values in $\boldsymbol{R}^2$, $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are Wiener processes in D and, for all t, s belonging to $\boldsymbol{R}_+$, $E\mathfrak{W}_1(t)\mathfrak{W}_2(s) = E\mathfrak{W}_2(t)\mathfrak{W}_1(s) = \beta \min(s, t)$. The constant β is called the *covariance* of $\mathfrak{W}_1$ and $\mathfrak{W}_2$. If $\beta = 0$ then $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are independent Wiener processes. Notice

that for all $t_1, t_2 \in \boldsymbol{R}$ such that $t_1^2+2t_1t_2\beta+t_2^2>0$ the process $(t_1^2+2t_1t_2\beta+t_2^2)^{-1/2}(t_1\mathfrak{W}_1+t_2\mathfrak{W}_2)$ is a Wiener process in D.

Operations: For a set A, ∂A and χ_A denote the boundary of A and the indicator of A, respectively. For a real number x, $[x]$ stands for the greatest nonnegative number not greater than x and $x_+ = \max(0, x)$.

Sequences: Instead of $\{a_k, k \geqslant 1\}$ or $\{a_k, k \geqslant 0\}$ we write $\{a_k\}$ if it does not lead to ambiguity. Here a_k can be real numbers or r.v.'s or r.e.'s.

If $\{Z_k, k \geqslant 0\}$ is a sequence of r.v.'s (or real numbers) then $\bar{Z}_k$, $\hat{Z}_k$ and $\hat{\hat{Z}}_k$, $k \geqslant 1$, denote the r.e.'s of D (elements of D) defined by

$$\bar{Z}_k(t) = k^{-1}Z_{[kt]}, \quad \hat{Z}_k(t) = k^{-1/2}(Z_{[kt]}-\vartheta_k(t)), \quad \hat{\hat{Z}}_k(t) = k^{-1/2}Z_{[kt]}$$

$(t \geqslant 0)$ respectively, where $\vartheta_k(t) = EZ_{[kt]}$, $t \geqslant 0$.

If $\{Z_{n,k}, k \geqslant 0, n \geqslant 1\}$ is an array of r.v.'s (or real numbers) then $\bar{Z}_{n,k}$, $\hat{Z}_{n,k}$ and $\hat{\hat{Z}}_{n,k}$, $n, k \geqslant 1$, denote the r.e.'s of D (elements of D) defined by

$$\bar{Z}_{n,k}(t) = k^{-1}Z_{n,[kt]}, \quad \hat{Z}_{n,k}(t) = k^{-1/2}(Z_{n,[kt]}-\vartheta_{n,k}(t)), \quad \hat{\hat{Z}}_{n,k}(t) = k^{-1/2}Z_{n,[kt]}$$

$(t \geqslant 0)$ respectively, where $\vartheta_{n,k}(t) = EZ_{n,[kt]}$, $t \geqslant 0$.

Analogously, if $Y = \{Y(t), t \geqslant 0\}$ is an r.e. of D (or element of D) then $\bar{Y}_k$, $\hat{Y}_k$ and $\hat{\hat{Y}}_k$, $k \geqslant 1$, denote the r.e.'s of D (elements of D) defined by

$$\bar{Y}_k(t) = k^{-1}Y(kt), \quad \hat{Y}_k(t) = k^{-1/2}(Y(kt)-\vartheta'_k(t)), \quad \hat{\hat{Y}}_k(t) = k^{-1/2}Y(kt)$$

$(t \geqslant 0)$ respectively, where $\vartheta'_k(t) = EY(kt)$, $t \geqslant 0$.

If $\{Y_n, n \geqslant 1\}$ is a sequence of r.e.'s of D (elements of D) then $\bar{Y}_{n,k}$, $\hat{Y}_{n,k}$ and $\hat{\hat{Y}}_{n,k}$, $n, k \geqslant 1$, denote the r.e.'s of D (elements of D) defined by

$$\bar{Y}_{n,k}(t) = k^{-1}Y_n(kt), \quad \hat{Y}_{n,k}(t) = k^{-1/2}(Y_n(kt)-\vartheta'_{n,k}(t)),$$
$$\hat{\hat{Y}}_{n,k}(t) = k^{-1/2}Y_n(kt)$$

$(t \geqslant 0)$ respectively, where $\vartheta'_{n,k}(t) = EY_n(kt)$, $k \geqslant 1$.

If k is an r.v. assuming positive integer values then $\vartheta_k(t)$, $\vartheta_{n,k}(t)$, $\vartheta'_k(t)$, $\vartheta'_{n,k}(t)$ are defined as r.v.'s assuming $EZ_{[it]}$, $EZ_{n,[it]}$, $EY(it)$, $EY_n(it)$ on the set $\{k = i\}$.

In view of the above definitions the following relations hold:

$$\bar{Z}_k = k^{-1/2}(\hat{Z}_k+\vartheta_k) = k^{-1/2}\hat{\hat{Z}}_k, \quad \bar{Z}_{n,k} = k^{-1/2}(\hat{Z}_{n,k}+\vartheta_{n,k}) = k^{-1/2}\hat{\hat{Z}}_{n,k},$$
$$\bar{Y}_k = k^{-1/2}(\hat{Y}_k+\vartheta'_k) = k^{-1/2}\hat{\hat{Y}}_k, \quad \bar{Y}_{n,k} = k^{-1/2}(\hat{Y}_{n,k}+\vartheta'_{n,k}) = k^{-1/2}\hat{\hat{Y}}_{n,k}.$$

Convergences: By $\xrightarrow{d}$, $\xrightarrow{P}$ and $\xrightarrow{D}$ we denote convergence in D with respect to the metric d, convergence in probability and convergence in distribution, respectively. If ξ and ξ_k, $k \geqslant 1$, are r.e.'s of D (or $D \times D'$ or D^l), we assume the notation

$$\xi_k \to \xi \text{ a.e.}, \quad \xi_k \xrightarrow{P} \xi, \quad \xi_k \xrightarrow{D} \xi$$

for a.e. convergence, convergence in probability and convergence in distribution in (D, d) (in $(D \times D', d^2)$, or in (D^l, d^l)), respectively (see [1]); $\xi_k(t) \to \xi(t)$ a.e., $\xi_k(t) \xrightarrow{P} \xi(t)$ and $\xi_k(t) \xrightarrow{D} \xi(t)$ denote the convergence at the point t in an appropriate sense.

In the paper we also consider convergences with respect to pairs of indices (n, k), e.g. $a_{n,k} \to a$ as $n, k \to \infty$. In such cases one has to specify the manner in which the two indices tend to infinity. We use the following notation:

$$a_{n,k} \to a \quad \text{as } n, k \to \infty \text{ in the manner } (*), \tag{1}$$

having in mind a fixed nonspecified manner $(*)$. For example the notation $a_{n,k} \to a$ as $n, k \to \infty$ g.m. (general manner) means that for each sequence $\{k_n\}$ tending to infinity, $a_{n,k_n} \to a$ as $n \to \infty$ and $\lim_{n\to\infty} \lim_{k\to\infty} a_{n,k} = a$. Here $a_{n,k}$ and a can be real numbers or r.v.'s or r.e.'s and convergence in (1) can be of one of the above-mentioned types.

Mappings: We define the mappings π_t, h_a, ω_a, f_1 and f on the space D, where $a > 0$, by

$$\begin{aligned}
\pi_t(x) &= x(t),\\
h_a(x) &= \sup_{0\leqslant t\leqslant a} x(s),\\
\omega_a(x) &= \sup_{0\leqslant t\leqslant a} |x(t)-x(t-)|,\\
f_1(x)(t) &= \inf_{0\leqslant s\leqslant t} x(s), \qquad t \geqslant 0,\\
f(x)(t) &= x(t) - \inf_{0\leqslant s\leqslant t} x(s), \qquad t \geqslant 0,
\end{aligned}$$

the mapping τ on the space $D \times D'$ by

$$\tau(x, v)(t) = x(v(t)), \qquad t \geqslant 0, \qquad (x, v) \in D \times D',$$

the mapping $D' \ni x \mapsto x^{-1} \in D'$ (inverse function) by

$$x^{-1}(t) = \inf\{s\colon\ x(s) > t\}, \qquad t \geqslant 0,$$

and the mapping $D \ni x \mapsto x^+ \in D$ by $x^+(t) = (x(t))_+$, $t \geqslant 0$. Obviously, the mappings π_t, h_a, ω_a map D into $\boldsymbol{R}$, f_1 and f map D into D, τ maps $D \times D'$ into D, the mapping $x \mapsto x^{-1}$ maps D' into D' and the mapping $x \mapsto x^+$ maps D into D.

The mappings mentioned above have the following properties.

PROPERTY 1 (see [24], Appendix). *The mappings π_t, h_a, ω_a are measurable, i.e. the inverse image of the Borel σ-field in $\boldsymbol{R}$ under each of them lies in $\mathscr{D}$. Furthermore, they are continuous on the set of continuous functions.*

PROPERTY 2 (see [26], Theorem 6.1). *The mappings f_1 and f are continuous in (D, d) and have the following properties:*

$$f_1(ax) = af_1(x), \quad f(x)(t) \geqslant 0, \quad f(ax) = af(x) \quad \textit{for } a \geqslant 0.$$

PROPERTY 3 (see [24], Remark 5.1). *The mapping τ is measurable, i.e. $\tau^{-1}\mathcal{D} \subset \sigma(\mathcal{D} \times \mathcal{D}')$, and continuous on the set $\{(x, v)\colon x \in C,\ v \in C \cap D'\}$. Furthermore, for any $a \in \boldsymbol{R}$ and $(x, v) \in D \times D'$, $a\tau(x, v) = \tau(ax, v)$.*

PROPERTY 4. *The mapping $D \ni x \mapsto x^+ \in D$ is measurable and it is continuous on the set C.*

PROPERTY 5 (see [20], Lemma 3.4). *Let $x \in C$, $x_n \in D'$, $n \geqslant 1$, and $a_n \to \infty$. Then $a_n(x_n - e) \xrightarrow{d} x$ iff $a_n(x_n^{-1} - e) \xrightarrow{d} -x$.*

We frequently use the following property of addition in D.

PROPERTY 6 (see [26], Theorem 4.1). *Addition in D, i.e. the mapping $D \times D \ni (x, y) \mapsto x + y \in D$, is measurable, and it is continuous at those (x, y) for which x and y have no common point of discontinuity.*

Queueing systems: The symbols M/M/1, M/G/1 and GI/M/1 denote GI/G/1 queues in which the interarrival times and the service times have specified d.f.'s. Thus in the first case the interarrival times and the service times have some negative exponential d.f.'s while in the next cases the assumption about negative exponential d.f.'s concerns only the interarrival times (in a M/G/1 queue) and the service times (in a GI/M/1 queue), the other characteristics being arbitrary.

Convention: We assume that a sum for which the lower bound is greater than the upper one is equal to zero. For example, if $Z_k = \sum_{j=0}^{k} \zeta_j$ then $Z_{-1} = 0$.

2. Preliminaries

2.1. Functional law of large numbers. Let $\{\zeta_k, k \geqslant 0\}$ be a sequence of nonnegative r.v.'s defined on a common probability space and let $\{\zeta_{n,k}, k \geqslant 0, n \geqslant 1\}$ be an array of nonnegative r.v.'s such that, for each $n \geqslant 1$, $\zeta_{n,k}$, $k \geqslant 0$, are defined on a common probability space.

Introduce the notation:

$$Z_k = \sum_{i=0}^{k} \zeta_i, \qquad Z_{n,k} = \sum_{i=0}^{k} \zeta_{n,i}, \qquad k \geqslant 0,\ n \geqslant 1.$$

Consider the processes $\bar{Z}_k$ and $\bar{Z}_{n,k}$, $k \geqslant 1$, $n \geqslant 1$ (for the definition see Section 1.2). Since all ζ_k and $\zeta_{n,k}$ are nonnegative r.v.'s it follows that all sample paths of $\bar{Z}_k$ and $\bar{Z}_{n,k}$ are in D'. This property allows us to determine relations between the convergence of the sequences $\{k^{-1}Z_k\}$, $\{k^{-1}Z_{n,k}\}$ and that of the sequences $\{\bar{Z}_k\}$, $\{\bar{Z}_{n,k}\}$ respectively. However, we start with the general facts.

PROPOSITION 2.1. *If $\{\eta_k\}$ is a sequence of r.e.'s of D' such that $\eta_k(t) \to \eta(t)$ a.e. (in probability) as $k \to \infty$, for each $t \geqslant 0$, where $\eta \in D$, then*

$$\eta_k \to \eta \quad \text{a.e. (in probability)} \quad \text{as } k \to \infty. \tag{1}$$

Proof. In view of Theorem 1 from [12] it is enough to show that for any nonnegative integer c

$$\sup_{0\leqslant t\leqslant c} |\eta_k(t)-\eta(t)| \to 0 \quad \text{a.e. (in probability)} \quad \text{as } k\to\infty. \tag{2}$$

By Lemma 1 in [1], p. 110, given $m \geqslant 1$, there exists a partition of $[0, c]$, $0 = t_0 < t_1 < \ldots < t_{k_m} = c$, such that

$$\sup_{t_i\leqslant t,s<t_{i+1}} |\eta(s)-\eta(t)| < 1/m \quad \text{for } i = 0, 1, \ldots, k_m-1.$$

Since all η_k have their sample paths in D' and $\eta_k(t)\to\eta(t)$ a.e. (in probability) for each t, it follows that η belongs to D'. Thus

$$\eta(t_{i+1}-)-\eta(t_i) < 1/m \quad \text{for } i = 0, 1, \ldots, k_m-1.$$

Let $t\in[0, c]$; then $t\in[t_{i-1}, t_i)$ for some $i = 1, 2, \ldots, k_m$ or $t = c$. In any case

$$\eta_k(t)-\eta(t) \leqslant \eta_k(t_i-)-\eta(t_{i-1}) \leqslant \eta_k(t_i-)-\eta(t_i-)+1/m,$$
$$\eta_k(t)-\eta(t) \geqslant \eta_k(t_{i-1})-\eta(t_i-) \geqslant \eta_k(t_{i-1})-\eta(t_{i-1})-1/m.$$

Hence

$$\sup_{0\leqslant t\leqslant c} |\eta_k(t)-\eta(t)| \leqslant 1/m+\varkappa_k, \tag{3}$$

where $\varkappa_k = \max\limits_{1\leqslant i\leqslant k_m} \max\{|\eta_k(t_i-)-\eta(t_i-)|, |\eta_k(t_{i-1})-\eta(t_{i-1})|\}$.

The convergence $\eta_k(t)\to\eta(t)$ a.e. (in probability) and the inequality $\eta(t_{i+1}-)-\eta(t_i) < 1/m$ imply that the upper limit of the right-hand side of inequality (3) does not exceed $2/m$. Since m is arbitrary we obtain the convergence (2) and thus (1). ∎

If the processes η_k are of the form $\bar{Z}_k$ or $\bar{Z}_{n,k}$, then by Proposition 2.1 and by Lemma 2.1 in [24] we obtain

PROPOSITION 2.2. *If $\bar{Z}_k(1)\to a$ a.e. (in probability) then $\bar{Z}_k\to ae$, a.e. (in probability) as $k\to\infty$. If, for each $t\geqslant 0$, $\bar{Z}_{n,k}(t)\xrightarrow{P} at$ as $n, k\to\infty$ in some manner (*), then $\bar{Z}_{n,k}\xrightarrow{P} ae$ as $n, k\to\infty$ in the same manner (*).* ∎

Now we give a well-known fact concerning the convergence of a composition of processes.

PROPOSITION 2.3. *For each fixed k, let ξ_k and θ_k be r.e.'s of D defined on a common probability space and let the sample paths of θ_k be unbounded and belong to D'. Furthermore, let $\theta_k\xrightarrow{P} ce$ and $\xi_k\xrightarrow{D}\xi$ as $k\to\infty$, where ξ is an r.e. of D with continuous sample paths. Then*

$$\tau(\xi_k, \theta_k)\xrightarrow{D}\tau(\xi, ce) \quad \text{as } k\to\infty, \tag{4}$$

$$\xi_k(t_k)\xrightarrow{D}\xi(s) \quad \text{as } k\to\infty \text{ and } t_k\to s. \tag{5}$$

Proof. Since the metric space $(D\times D', d^2)$ is separable, (ξ_k, θ_k) is an r.e. of $D\times D'$. By the assumptions and Theorem 4.1 from [1] we obtain the

convergence $(\xi_k, \theta_k) \xrightarrow{D} (\xi, ce)$ as $k \to \infty$. Since the mapping τ is measurable and continuous on the set $C \times (D' \cap C)$ the above convergence and Theorem 5.1 from [1] imply (4).

It is well known that if $x_k \xrightarrow{d} x$ and x is continuous then, for each sequence $\{t_k\}$ tending to $s < \infty$, the convergence $\pi_{t_k}(x_k) = x_k(t_k) \to x(s) = \pi_s(x)$ holds as $k \to \infty$. This fact, together with the measurability of π_t, its continuity on C and Theorem 5.5 from [1], yields (5). ∎

The weak law of large numbers for arrays of r.v.'s is the basic condition under which the assertions formulated in further sections hold. Thus, for completeness, we give here one version of the law. We rewrite it from [18] (Theorem 3, p. 315) in a simplified form.

PROPOSITION 2.4. *For each $n \geqslant 1$, let $\{\zeta_{n,k}, k \geqslant 1\}$ be a sequence of independent r.v.'s with a common mean such that*

(6) $$E\zeta_{n,1} \to a \quad \text{as } n \to \infty,$$

(7) *there exist finite positive numbers M and δ such that*

$$\sup_{n,k} E\zeta_{n,k}^{1+\delta} < M.$$

Then we have

(8) $$k^{-1} \sum_{i=1}^{k} \zeta_{n,i} \xrightarrow{P} a \quad \text{as } n, k \to \infty \text{ g.m.} \ ∎$$

2.2. Counting renewal process. The process N defined by a sequence $\{\zeta_k, k \geqslant 0\}$ as

(1) $$N(t) = \begin{cases} 0 & \text{if } Z_0 > t, \\ \max\{k\colon Z_{k-1} \leqslant t\} & \text{if } Z_0 \leqslant t, \end{cases}$$

is called a *counting renewal process* generated by $\{\zeta_k, k \geqslant 0\}$.

Sometimes it is convenient to consider the process N^0 defined by $N^0(t) = \inf\{k+1\colon k \geqslant 0, Z_k > t\}$. Obviously N and N^0 are r.e.'s of D' and $N^0(t) = N(t)+1$, $t \geqslant 0$.

It follows from formula (1) that the counting renewal process N depends only on the sequence $\{Z_k, k \geqslant 0\}$. Thus the behaviour of the sequence $\{Z_k, k \geqslant 0\}$ determines the behaviour of N. Below we show that N obeys a certain functional limit law (law of large numbers or functional central limit theorem) iff $\{Z_k, k \geqslant 0\}$ obeys that law. First we show that property for the law of large numbers.

For each $n \geqslant 1$, let N_n and N_n^0 be the processes generated by $\{\zeta_{n,k}, k \geqslant 0\}$.

PROPOSITION 2.5. *For $a \in (0, \infty)$, the following conditions are equivalent:*

(2) *for each $t \geqslant 0$, $\bar{Z}_{n,k}(t) \xrightarrow{P} at$,*

(3) *for each $t \geqslant 0$, $\bar{N}_{n,k}(t) \xrightarrow{P} t/a$,*

as $n, k \to \infty$ in the same manner $(*)$.

Proof. First we prove that (2) implies (3). Let t be a fixed positive number and let ε be a number such that $0 < \varepsilon < t/a$. Define $b = t/a+\varepsilon$ and $c = t/a-\varepsilon$. Notice that any counting renewal process N generated by $\{\zeta_k, k \geqslant 0\}$ has the properties

(4) $P\{N(t) > x\} = P\{N(t) > [x]\}, \quad P\{N(t) \geqslant x\} \geqslant P\{N(t) \geqslant [x+1]\},$

(5) $$P\{N(t) > m\} \leqslant P\{Z_{m-1} \leqslant t\} = P\{N(t) \geqslant m\},$$

where x is a nonnegative number and m is a positive integer. Hence we obtain

(6) $P\{ck \leqslant N_n(kt) \leqslant bk\} \geqslant P\{N_n(kt) \geqslant [ck+1]\} - P\{N_n(kt) > [bk]\},$

(7) $$P\{N_n(kt) \geqslant [ck+1]\} = P\{Z_{n,[ck+1]-1} \leqslant kt\},$$

(8) $$P\{N_n(kt) > [bk]\} \leqslant P\{Z_{n,[bk]-1} \leqslant kt\}.$$

In view of (2) and the definitions of b and c, the expression on the right-hand side of (7) tends to 1, and that on the right-hand side of (8) tends to 0. Since ε is an arbitrarily small number, by (6) and by the definitions of b and c we obtain the convergence (3) for every fixed t. Since t was arbitrary, we conclude that (2) implies (3).

The inverse implication follows directly from the inequality

$$P\{\bar{N}_{n,k}(t) > ([ku]+1)/k\} \leqslant P\{\bar{Z}_{n,k}(u) \leqslant t\} = P\{\bar{N}_{n,k}(t) \geqslant ([ku]+1)/k\}$$

for all u, $t \geqslant 0$. ∎

Now applying Propositions 2.1 and 2.5 we obtain

PROPOSITION 2.6. *For $a \in (0, \infty)$, the following convergences are equivalent:*

$$\bar{Z}_{n,k} \xrightarrow{P} ae,$$
$$\bar{N}_{n,k} \xrightarrow{P} a^{-1} e,$$

with n, $k \to \infty$ in the same manner (*). ∎

We now apply Proposition 2.5 to a particular class of counting renewal processes called stationary renewal processes. Let us first introduce the definitions.

If the r.v.'s in the sequence $\{\zeta_k, k \geqslant 0\}$ are independent with a common d.f. F for $k \geqslant 1$ (for $k \geqslant 0$), then the process N defined by (1) is called a *general renewal process* (a *renewal process*).

If a general renewal process N is such that the d.f. F of $\{\zeta_k, k \geqslant 1\}$, has the first moment μ finite and the r.v. ζ_0 has the d.f. $\tilde{F}$ of the form $\tilde{F}(x) = \mu^{-1} \int_0^x (1-F(s))\,ds$, $x \geqslant 0$, then the process N defined by (1) is called a *stationary renewal process* and we denote it by $\tilde{N}$.

PROPOSITION 2.7. *For each $n \geqslant 1$, let $\tilde{N}_n$ be a stationary renewal process generated by $\{\zeta_{n,k}, k \geqslant 0\}$ and let the array $\{\zeta_{n,k}, k \geqslant 1, n \geqslant 1\}$ satisfy*

conditions (2.1.6) *and* (2.1.7) *in which* $a > 0$. *Then* $c\tilde{N}_n(t/c) \xrightarrow{P} a^{-1}t$ *as* $n, 1/c \to \infty$ *g.m.* (*general manner*).

Proof. In the case in question, the r.v.'s in the sequence $\{\zeta_{n,k}, k \geqslant 0\}$ are independent and have a common d.f. F_n with mean μ_n for $k \geqslant 1$. Hence, in view of Proposition 2.4, (2.1.8) holds. Now using the form of the d.f. of $\zeta_{n,0}$, the Chebyshev inequality, and conditions (2.1.6) and (2.1.7) we obtain the following inequalities:

$$P\{c\zeta_{n,0} > \varepsilon\} = (c\mu_n)^{-1} \int_{\varepsilon}^{\infty} (1 - F_n(x/c))\,dx$$

$$\leqslant (c\mu_n)^{-1} E\zeta_{n,1}^{1+\delta} \int_{\varepsilon}^{\infty} (c/x)^{1+\delta}\,dx \leqslant c^{\delta} M(\mu_n \delta\varepsilon^{\delta})^{-1}.$$

Now applying (2.1.6) we obtain the convergence $c\zeta_{n,0} \xrightarrow{P} 0$ as n, $1/c \to \infty$ g.m. Hence (2.1.8) gives the convergence $\bar{Z}_{n,k}(1) \xrightarrow{P} a$ as n, $k \to \infty$ g.m. Thus, for each $t \geqslant 0$, $\bar{Z}_{n,k}(t) \xrightarrow{P} at$ as n, $k \to \infty$ g.m. In view of Proposition 2.5 we obtain the assertion. ∎

We now prove a fact which turns out to be useful for further considerations.

PROPOSITION 2.8. *If an array* $\{\zeta_{n,k}, k \geqslant 0, n \geqslant 1\}$ *satisfies the conditions of Proposition* 2.7, *then there exist* $\varepsilon > 0$ *and* $p > 0$ *such that for all* n, k

$$P\{\zeta_{n,k} > \varepsilon\} > p. \tag{9}$$

Proof. Let ε and p be such that (9) holds for $k = 1$. Then

$$P\{\zeta_{n,0} > \varepsilon/2\} \geqslant \varepsilon P\{\zeta_{n,1} > \varepsilon\}/(2\mu_n) \geqslant \varepsilon p/(2\mu_n).$$

Hence (2.1.6) shows that (9) holds for $k = 0$ and $n \geqslant 1$.

Now we shall show that there exist $\varepsilon > 0$ and $p > 0$ such that (9) holds for $n \geqslant 1$, $k = 1$. Assume, on the contrary, that this is false. Then for each $\varepsilon > 0$ and $p > 0$ there exists n such that $P\{\zeta_{n,1} > \varepsilon\} \leqslant p$. Thus for $\varepsilon_k \to 0$ and $p_k \to 0$ there exists a sequence $\{n_k\}$ tending to infinity such that $P\{\zeta_{n_k,1} > \varepsilon_k\} \leqslant p_k$. This means that $\zeta_{n_k,1} \xrightarrow{P} 0$. In view of the uniform integrability of $\{\zeta_{n,1}, n \geqslant 1\}$, which follows from (2.1.7), we obtain $E\zeta_{n_k,1} \to 0$, which contradicts (2.1.6) because of the assumption $a > 0$. ∎

It is well known in renewal theory that the stationary renewal process $\tilde{N}$ has stationary increments. This allows us to give an upper bound for its renewal function.

PROPOSITION 2.9. *For any stationary renewal process* $\tilde{N}$ *and any positive numbers* $\varepsilon < 1$, c *and* $m \geqslant 1$,

$$E(c\tilde{N}(t/c))^m \leqslant ((c+t)/\varepsilon)^m E\tilde{N}(\varepsilon)^m, \quad t \geqslant 0. \tag{10}$$

Proof. Since $\zeta_0 > 0$ a.e., we have $\tilde{N}(0) = 0$ a.e. Hence for each $k \geqslant 1$ we have the equality

$$\tilde{N}(kt) = \sum_{i=1}^{k} \left(\tilde{N}(it) - \tilde{N}((i-1)t)\right) \quad \text{a.e.}$$

The nonnegativity of $Y_k = \tilde{N}(kt) - \tilde{N}((k-1)t)$, $k \geqslant 1$, the convexity of the function x^m on $\boldsymbol{R}_+$, and the stationarity of the sequence $\{Y_k, k \geqslant 1\}$ now yield

$$E\left(k^{-1}\tilde{N}(kt)\right)^m = E\left(k^{-1}\sum_{i=1}^{k} Y_i\right)^m \leqslant E\left(k^{-1}\sum_{i=1}^{k} Y_i^m\right) = EY_1^m. \tag{11}$$

Since $t/c \leqslant [(c+t)/(c\varepsilon)]\varepsilon$ we have $\tilde{N}(t/c) \leqslant \tilde{N}(\varepsilon[(c+t)/(\varepsilon c)])$. Now from (11) we obtain the inequalities

$$E\left(c\tilde{N}(t/c)\right)^m \leqslant \left(c[(c+t)/(\varepsilon c)]\right)^m E\tilde{N}(\varepsilon)^m \leqslant \left((c+t)/\varepsilon\right)^m E\tilde{N}(\varepsilon)^m,$$

which finishes the proof. ∎

We now prove the elementary renewal theorem for series of stationary renewal processes. First, let us introduce some notation. Let $\{\zeta_k^*, k \geqslant 0\}$ be a sequence of independent r.v.'s with the common distribution $P\{\zeta_k^* = 0\} = 1-p$, $P\{\zeta_k^* = \varepsilon\} = p$, $k \geqslant 0$, where ε and p satisfy (9) and $\varepsilon < 1$. Furthermore, let N^* be the counting renewal process generated by $\{\zeta_k^*, k \geqslant 0\}$ (N^* is a renewal process).

PROPOSITION 2.10. *If an array $\{\zeta_{n,k}, k \geqslant 0, n \geqslant 1\}$ satisfies the conditions of Proposition 2.7 then for all $m \geqslant 1$ and $0 < \varepsilon \leqslant 1$ we have*

$$E\left(c\tilde{N}_n(t/c)\right)^m \leqslant \left((c+t)/\varepsilon\right)^m EN^*(\varepsilon)^m, \tag{12}$$

$$E\left(c\tilde{N}_n(t/c)\right)^m \to (t/a)^m \quad \textit{as } n, 1/c \to \infty \ \textit{g.m.} \tag{13}$$

Proof. The form of the distribution of ζ_k^* yields

$$P\{\zeta_{n,k} \leqslant x\} \leqslant P\{\zeta_0^* \leqslant x\}, \quad x \geqslant 0. \tag{14}$$

Hence we have $P\{\tilde{N}_n(t) \geqslant k\} \leqslant P\{N^*(t) \geqslant k\}$ for all $k \geqslant 0$. Thus for each $m \geqslant 1$ and for sufficiently large $1/c$ and n it follows from Proposition 2.9 that

$$E\left(c\tilde{N}_n(t/c)\right)^m \leqslant \left((c+t)/\varepsilon\right)^m E\tilde{N}_n(\varepsilon)^m \leqslant \left((1+t)/\varepsilon\right)^m EN^*(\varepsilon)^m.$$

But for each $m \geqslant 1$ we have $EN^*(\varepsilon)^m < \infty$. Hence for each $m \geqslant 1$ the family $\{(c\tilde{N}_n(t/c))^m, 0 < c \leqslant 1, n \geqslant 1\}$ is uniformly integrable. Now using Proposition 2.7 we obtain (13). ∎

We now give a relation established by R. Serfozo (see [20], Theorem 2.8) between the functional central limit theorem for $\{\hat{Z}_{n,k}\}$ and for $\{\hat{N}_{n,k}\}$. Here we formulate it in a modified form. Its proof is similar to that of Theorem 2.8 from [20].

Introduce the notation:

$$G_{n,k} = \sum_{i=0}^{k} E\zeta_{n,i}, \qquad k \geqslant 0,\ n \geqslant 1.$$

PROPOSITION 2.11. *Let $n, k \to \infty$ in a manner $(*)$ under which*

$$\bar{G}_{n,k} \xrightarrow{d} \lambda^{-1} e, \qquad \sqrt{k}(\bar{G}_{n,k} - \lambda_n^{-1} e) \xrightarrow{d} \mathbf{0}, \tag{15}$$

where λ and λ_n, $n \geqslant 1$, are some finite positive numbers. Furthermore, let

$$N_{n,k}^{(1)} = -\sqrt{k}(\lambda_n^{-1} \bar{N}_{n,k} - e) \quad \text{and} \quad N_{n,k}^{(2)} = -\sqrt{k}(\lambda_n^{-1} \bar{N}_{n,k}^{0} - e)$$

for $n, k \geqslant 1$. Then the following statements are equivalent as $n, k \to \infty$ in the manner $(*)$:

$$\sigma^{-1} \hat{Z}_{n,k} \xrightarrow{D} \mathfrak{W}, \tag{16}$$

$$(\sigma\sqrt{\lambda})^{-1} N_{n,k}^{(i)} \xrightarrow{D} \mathfrak{W} \quad \text{for } i = 1, 2, \tag{17}$$

where σ is a finite positive number.

Proof. Denote by $n, k \xrightarrow{(*)} \infty$ the convergence of n and k to infinity in the manner $(*)$.

Notice that $d(N_{n,k}^{(1)}, N_{n,k}^{(2)}) \leqslant k^{-1/2}\lambda_n^{-1} \to 0$ as $n, k \xrightarrow{(*)} \infty$. Thus for the proof of the proposition it is enough to show the equivalence of (16) and (17) for $i = 2$. Now notice that

$$\hat{Z}_{n,k} = \lambda_n^{-1}\sqrt{k}(\lambda_n \bar{Z}_{n,k} - \lambda_n(\bar{G}_{n,k} - \lambda_n^{-1} e) - e).$$

Hence under condition (15) we find that, as $n, k \xrightarrow{(*)} \infty$, the convergence (16) is equivalent to the convergence

$$(\sigma\lambda_n)^{-1}\sqrt{k}(\lambda_n \bar{Z}_{n,k} - e) \xrightarrow{D} \mathfrak{W} \quad \text{as } n, k \xrightarrow{(*)} \infty. \tag{18}$$

Thus it is enough to show the equivalence of the convergences (18) and (17) as $n, k \xrightarrow{(*)} \infty$.

Assume that (18) holds. The Skorokhod–Dudley a.e. representation (see [20]) yields the existence of $\xi_{n,k}$ and $\mathfrak{W}'$, r.e.'s of D defined on a common probability space such that $\xi_{n,k} \overset{D}{=} \lambda_n \bar{Z}_{n,k}$, $\mathfrak{W}' \overset{D}{=} \mathfrak{W}$, and

$$(\sigma\lambda_n)^{-1}\sqrt{k}(\xi_{n,k} - e) \to \mathfrak{W}' \quad \text{a.e.} \quad \text{as } n, k \xrightarrow{(*)} \infty. \tag{19}$$

But in view of Property 5, (19) is equivalent to

$$(\sigma\lambda_n)^{-1}\sqrt{k}(\xi_{n,k}^{-1} - e) \to -\mathfrak{W}' \quad \text{a.e.} \quad \text{as } n, k \xrightarrow{(*)} \infty. \tag{20}$$

Applying twice the mapping τ we deduce that (20) is equivalent to

$$\sigma^{-1}\sqrt{k}(\lambda_n^{-1}\tau(\xi_{n,k}^{-1}, \lambda_n e) - e) \to -\tau(\mathfrak{W}', \lambda e) \quad \text{a.e.} \quad \text{as } n, k \xrightarrow{(*)} \infty. \tag{21}$$

Applying the mapping $D' \ni x \mapsto x^{-1} \in D'$ to the process $\lambda_n \bar{Z}_{n,k}$ we obtain

$(\lambda_n \bar{Z}_{n,k})^{-1}(t) = \bar{N}^0_{n,k}(t\lambda_n^{-1})$, $t \geqslant 0$. Hence in view of $\lambda_n \bar{Z}_{n,k} \overset{D}{=} \xi_{n,k}$ we have $\xi_{n,k}^{-1} \overset{D}{=} \tau(\bar{N}^0_{n,k}, \lambda_n^{-1} e)$, which together with (21) implies (17) as $n, k \overset{(*)}{\to} \infty$.

Now assume that (17) holds as $n, k \overset{(*)}{\to} \infty$. Then the Skorokhod–Dudley a.e. representation gives the existence of $\eta_{n,k}$ and $\mathfrak{W}'$, r.e.'s of D defined on a common probability space such that $\eta_{n,k} \overset{D}{=} \lambda_n^{-1} \bar{N}^0_{n,k}$, $\mathfrak{W}' \overset{D}{=} \mathfrak{W}$ and

$$(22) \qquad (\sigma\sqrt{\lambda})^{-1}\sqrt{k}(\eta_{n,k}^{-1} - e) \to -\mathfrak{W}' \quad \text{a.e.} \quad \text{as } n, k \overset{(*)}{\to} \infty.$$

Applying the mapping $D' \ni x \mapsto x^{-1} \in D'$ to the process $\lambda_n^{-1} \bar{N}^0_{n,k}$ we obtain $(\lambda_n^{-1} \bar{N}^0_{n,k})^{-1}(t) = \bar{Z}_{n,k}(t\lambda_n)$, $t \geqslant 0$. Hence in view of $\eta_{n,k} \overset{D}{=} \lambda_n^{-1} \bar{N}^0_{n,k}$ we have $\eta_{n,k}^{-1} \overset{D}{=} \tau(\bar{Z}_{n,k}, \lambda_n e)$, which together with (22) implies the convergence

$$(23) \qquad (\sigma\sqrt{\lambda})^{-1}\sqrt{k}(\tau(\bar{Z}_{n,k}, \lambda_n e) - e) \to -\mathfrak{W}' \quad \text{a.e.} \quad \text{as } n, k \overset{(*)}{\to} \infty.$$

Now applying twice the mapping τ we deduce that (23) is equivalent to

$$\sqrt{k}(\sigma\sqrt{\lambda})^{-1}(\bar{Z}_{n,k} - \lambda_n^{-1} e) \to -\tau(\mathfrak{W}', \lambda^{-1} e) \quad \text{a.e.} \quad \text{as } n, k \overset{(*)}{\to} \infty,$$

which in turn implies the convergence (18) as $n, k \overset{(*)}{\to} \infty$. ∎

As an immediate consequence of Proposition 2.11 and of the definition of $\hat{N}_{n,k}$ we have

PROPOSITION 2.12. *If the convergences in* (15) *and* (16) *of Proposition* 2.11 *hold as* $n, k \to \infty$ *in a manner* $(*)$ *and for each* $a \geqslant 0$

$$k^{-1/2} \sup_{0 \leqslant t \leqslant a} |EN_n(kt) - \lambda_n kt| \to 0,$$

as $n, k \to \infty$ *in the same manner* $(*)$, *then* $(\sigma\sqrt{\lambda^3})^{-1} \hat{N}_{n,k} \overset{D}{\to} \mathfrak{W}$ *as* $n, k \to \infty$ *in the manner* $(*)$. ∎

As an immediate consequence of the definition of general renewal processes (and stationary renewal processes) and of Proposition 2.11 we have

PROPOSITION 2.13. *If* $\{N_n\}$ *is a sequence of general renewal processes or stationary renewal processes such that*

$$\lambda_n^{-1} \overset{\mathrm{df}}{=} E\zeta_{n,1} \to \lambda^{-1}, \qquad \operatorname{Var}\zeta_{n,1} \to \sigma^2, \qquad \sup_n E\zeta_{n,0} < \infty,$$

$$E|\zeta_{n,1} - E\zeta_{n,1}|^{2+\delta} < M,$$

where σ, λ, δ, *and* M *are some finite positive numbers, then the following convergences hold as* $n, k \to \infty$ *g.m.*:

$$\sigma^{-1}\hat{Z}_{n,k} \overset{D}{\to} \mathfrak{W}, \quad (\sigma\sqrt{\lambda})^{-1} N^{(1)}_{n,k} \overset{D}{\to} \mathfrak{W}, \quad (\sigma\sqrt{\lambda^3})^{-1} \hat{N}_{n,k} \overset{D}{\to} \mathfrak{W}.$$

3. Departure process

3.1. Law of large numbers. Let a queue Q be generated by a sequence $\{(v_k, u_k), k \geqslant 0\}$. Introduce the following notation ($k \geqslant 0$):

$$V_k = \sum_{i=1}^{k} v_i, \qquad U_k = \sum_{i=0}^{k} u_i,$$

$$X_k = v_k - u_k, \qquad S_k = \sum_{i=1}^{k} X_i,$$

$$w = \sup_{0 \leqslant k < \infty} S_k.$$

Denote by N the arrival process of the queue Q. It is of the form (2.2.1) with $Z_k = U_k$, $k = 0$. We say that N is generated by $\{u_k, k \geqslant 0\}$.

Let I^0 denote the idle time process, i.e. the process for which $I^0(t)$ represents the total idle time during $[0, t]$. Furthermore, let I_k, $k \geqslant 0$, denote the total idle time during $[0, U_k]$ (total idle time up to the arrival moment of the $(k+1)$th unit). It is well known that those quantities have the following form:

$$I^0(t) = -\inf_{0 \leqslant s \leqslant t} (V_{N(s)} - s), \qquad t \geqslant 0,$$

$$I_k = u_0 - \min_{0 \leqslant j \leqslant k} S_j, \qquad k \geqslant 0.$$

In view of this the departure process D has the form

$$D(t) = \begin{cases} 0 & \text{if } u_0 + v_1 > t, \\ \sup\{k\colon V_k + I_{k-1} \leqslant t\} & \text{if } u_0 + v_1 \leqslant t, \end{cases} \tag{1}$$

and the virtual waiting time process W has the form

$$W(t) = V_{N(t)} - t + I^0(t), \qquad t \geqslant 0.$$

In this chapter and also later on a sequence $\{Q_n\}$ of queues will be the subject of our considerations. All quantities pertaining to the queue Q_n are then marked by the subscript n: $v_{n,k}$, $u_{n,k}$, etc. in place of v_k, u_k, etc.

We now give the relation between the law of large numbers for the sequences $\{v_k\}$, $\{u_k\}$ and the law of large numbers for the sequence of the total idle time.

LEMMA 3.1. *If the following conditions hold*:

$$k^{-1} V_k \to \mu^{-1} \quad \textit{a.e. (in probability) as } k \to \infty, \tag{2}$$

$$k^{-1} U_k \to \lambda^{-1} \quad \textit{a.e. (in probability) as } k \to \infty, \tag{3}$$

where μ and λ are some finite positive numbers, then

(4) $$\bar{I}_k \to \max(0, \lambda^{-1} - \mu^{-1})e \quad \textit{a.e. (in probability) as } k \to \infty.$$

Proof. Consider the processes $\bar{S}_k$, $k \geqslant 1$. Obviously $\bar{S}_k = \bar{V}_k - \bar{U}_k + k^{-1} u_0 \mathbf{1}$, $k \geqslant 1$. Using (2), (3), Proposition 2.2 and the fact that the mapping $D \times D \ni (x, y) \mapsto x - y \in D$ is continuous in the metric d as x or y is continuous, we obtain

$$\bar{S}_k \to (\mu^{-1} - \lambda^{-1})e \quad \text{a.e. (in probability) as } k \to \infty.$$

In view of the definition of I_k we have

$$\bar{I}_k(t) = k^{-1} u_0 - \inf_{0 \leqslant s \leqslant t} \bar{S}_k(s) = k^{-1} u_0 \mathbf{1}(t) - f_1(\bar{S}_k)(t), \qquad t \geqslant 0.$$

But the mapping f_1 is continuous in (D, d). Hence we obtain (4). ∎

LEMMA 3.2. *If the conditions of Lemma 3.1 are satisfied, then*

(5) $$\bar{I}_k^0 \to \max(0, 1 - \lambda/\mu)e \quad \textit{a.e. (in probability) as } k \to \infty.$$

Proof. First we show that the convergences a.e. in (2) and (3) imply the following ones: $\bar{U}_k \to \lambda^{-1} e$, $\bar{S}_k \to (\mu^{-1} - \lambda^{-1})e$ and $\bar{N}_k \to \lambda e$ a.e. as $k \to \infty$. The first two of them are consequences of (2) and (3) (in the a.e. case) and of Proposition 2.2 and the continuity of addition on the set C. To show the third convergence let us note that $U_k \to \infty$ a.e. as $k \to \infty$, which in turn gives $N(t) \to \infty$ a.e. as $t \to \infty$. This together with the inequalities

$$\frac{1}{1+N(t)} U_{N(t)} \leqslant \frac{t}{1+N(t)} \leqslant \frac{1}{1+N(t)} U_{N(t)+1} \quad \text{for } t \geqslant 0$$

yields

$$\frac{1}{N(t)} U_{N(t)} \to \lambda^{-1} \quad \text{a.e.}, \qquad \frac{1}{N(t)} U_{N(t)+1} \to \lambda^{-1} \quad \text{a.e.}$$

as $t \to \infty$, where $0 < \lambda < \infty$. Hence $t^{-1} N(t) \to \lambda$ a.e. as $t \to \infty$, which in turn for each $t \geqslant 0$ implies $\bar{N}_k(t) \to \lambda t$ a.e. as $k \to \infty$. Now using Proposition 2.1 we have $\bar{N}_k \to \lambda e$ a.e. as $k \to \infty$.

If (2) and (3) hold in probability then the convergences $\bar{U}_k \xrightarrow{P} \lambda^{-1} e$, $\bar{S}_k \xrightarrow{P} (\mu^{-1} - \lambda^{-1})e$ and $\bar{N}_k \xrightarrow{P} \lambda e$ immediately follow from Propositions 2.2 and 2.5 and the continuity of addition on C.

Thus if conditions (2) and (3) hold then by the continuity of the mapping τ on $C \times (C \cap D')$ and of the mapping ω_a on C we have the convergences

(6) $$\begin{gathered} \tau(\bar{U}_k, \bar{N}_k) \to e, \qquad \tau(\bar{S}_k, \bar{N}_k) \to (\lambda/\mu - 1)e, \\ \omega_a(\tau(\bar{U}_k, \bar{N}_k)) \to 0 \quad \text{a.e. (in probability) as } k \to \infty. \end{gathered}$$

Let us define the process $\gamma'(t) = t + u_0 - U_{N(t)}$, $t \geqslant 0$. From the definition

of the counting renewal process N we obtain $\gamma'(t) \leqslant u_0 + u_{N(t)}$ for $t \geqslant 0$. But for each $a > 0$

$$k^{-1} \sup_{0 \leqslant t \leqslant a} u_{N(kt)} = \omega_a(\tau(\bar{U}_k, \bar{N}_k)).$$

Hence in view of the inequality

$$\sup_{0 \leqslant t \leqslant a} \bar{\gamma}_k'(t) \leqslant k^{-1} u_0 + \omega_a(\tau(\bar{U}_k, \bar{N}_k))$$

we obtain $\bar{\gamma}_k' \to \mathbf{0}$ a.e. (in probability) as $k \to \infty$. Now the definition of $\bar{I}_k^0$ gives

$$\bar{I}_k^0(t) = -\inf_{0 \leqslant s \leqslant t} (\tau(\bar{S}_k, \bar{N}_k)(s) - \bar{\gamma}_k'(s)). \tag{7}$$

To finish the proof let us note that the assertion follows from the above convergences and the continuity of addition on C and of the mapping f_1. ∎

Using the relations

$$W(t) = S_{N(t)} - \gamma'(t) - \inf_{0 \leqslant s \leqslant t} (S_{N(s)} - \gamma'(s)), \quad t \geqslant 0,$$

$$\bar{W}_k = f(\tau(\bar{S}_k, \bar{N}_k) - \bar{\gamma}_k'), \quad k \geqslant 1,$$

where $\gamma'(t) = t + u_0 - U_{N(t)}$, $t \geqslant 0$, and the continuity of the mapping f, we may prove in the same way as in Lemma 3.2 the following fact.

LEMMA 3.3. *If the conditions of Lemma* 3.1 *are satisfied, then*

$$\bar{W}_k \to \max(0, \lambda/\mu - 1)e \quad \textit{a.e. (in probability)} \quad \textit{as } k \to \infty. \ ∎$$

By considerations similar to those in the proof of Lemma 3.1 and Lemma 3.2 we get

LEMMA 3.4. *If the following condition holds*:

$$\textit{for each } t \geqslant 0, \quad \bar{V}_{n,k}(t) \xrightarrow{P} \mu^{-1} t, \quad \bar{U}_{n,k}(t) \xrightarrow{P} \lambda^{-1} t \tag{8}$$

as $n, k \to \infty$ *in some manner* $(*)$, *where* μ *and* λ *are some finite positive numbers, then*

$$\bar{I}_{n,k} \xrightarrow{P} \max(0, \lambda^{-1} - \mu^{-1})e, \tag{9}$$

$$\bar{I}_{n,k}^0 \xrightarrow{P} \max(0, 1 - \lambda/\mu)e \tag{10}$$

as $n, k \to \infty$ *in the same manner* $(*)$. ∎

Formula (1) for the departure process is similar to (2.2.1), which defines the counting renewal process. Thus one can expect that the two processes obey similar laws. The main result of this section gives the relation between the law of large numbers for the sequences $\{v_k\}$, $\{u_k\}$ and the law of large numbers for the departure process.

THEOREM 3.1. *If the conditions of Lemma* 3.1 *are satisfied, then*

$$t^{-1} D(t) \to \min(\lambda, \mu) \quad \textit{a.e. (in probability)} \quad \textit{as } t \to \infty. \tag{11}$$

Proof. We prove the a.e. convergence in (11) assuming the same type of convergence in (2) and (3). Convergence in probability in (11) under the same assumption in (2) and (3) will be an immediate consequence of Corollary 3.1 proved below.

Under our assumptions, $V_k \to \infty$ and $U_k \to \infty$ a.e. as $k \to \infty$. Thus $V_k + I_{k-1} \to \infty$ a.e. as $k \to \infty$, which together with (1) implies the convergence $D(t) \to \infty$ a.e. as $t \to \infty$. Hence (2), (3) and Lemma 3.1 yield

$$\frac{1}{D(t)} V_{D(t)} \to \mu^{-1} \quad \text{a.e.},$$

$$\frac{1}{D(t)} I_{D(t)-1} \to -\min(0, \mu^{-1} - \lambda^{-1}) \quad \text{a.e.}$$

as $t \to \infty$. By the form of the departure process (see (1)) in the case $D(t) \geqslant 1$, we have the inequalities

$$\frac{1}{D(t)}(V_{D(t)} + I_{D(t)-1}) \leqslant \frac{t}{D(t)} < \frac{1}{D(t)}(V_{D(t)+1} + I_{D(t)}).$$

Thus the left-hand side and the right-hand side of the above inequalities converge a.e., as $t \to \infty$, to

$$\mu^{-1} - \min(0, \mu^{-1} - \lambda^{-1}) = \max(\mu^{-1}, \lambda^{-1}).$$

Hence we obtain the a.e. convergence in (11). ∎

As an immediate consequence of Proposition 2.1 and Theorem 3.1 we have

THEOREM 3.2. *If the conditions of Lemma* 3.1 *are satisfied, then* $\bar{D}_k \to \min(\lambda, \mu)e$ *a.e. (in probability) as* $k \to \infty$. ∎

THEOREM 3.3. *If the conditions of Lemma* 3.4 *are satisfied, then*

$$\bar{D}_{n,k} \xrightarrow{P} \min(\lambda, \mu)e \tag{12}$$

as $n, k \to \infty$ *in the manner* (*).

Proof. Let t be a fixed positive number and let ε satisfy $0 < \varepsilon < ta$, where $a = \min(\lambda, \mu)$. Define $b = ta + \varepsilon$, $c = ta - \varepsilon$. From (1), for any $k, m \geqslant 1$, we have

$$P\{D_n(kt) > m\} \leqslant P\{V_{n,m} + I_{n,m-1} \leqslant tk\} = P\{D_n(kt) \geqslant m\}. \tag{13}$$

Now from the fact that the departure process has nondecreasing sample paths and they take only nonnegative integer values we have

$$\begin{aligned} P\{ck \leqslant D_n(kt) \leqslant bk\} &= P\{D_n(kt) \geqslant ck\} - P\{D_n(kt) > bk\} \\ &\geqslant P\{D_n(kt) \geqslant [ck+1]\} - P\{D_n(kt) > [bk]\}. \end{aligned} \tag{14}$$

But from (13) we obtain

$$(15)\qquad P\{D_n(kt) \geqslant [ck+1]\} = P\{V_{n,[ck+1]} + I_{n,[ck+1]-1} \leqslant tk\}$$
$$= P\{\bar{V}_{n,k}(c+1/k) + \bar{I}_{n,k}(([kc+1]-1)/k) \leqslant t\},$$

$$(16)\qquad P\{D_n(kt) > [bk]\} \leqslant P\{V_{n,[bk]} + I_{n,[bk]-1} \leqslant kt\}$$
$$= P\{\bar{V}_{n,k}(b) + \bar{I}_{n,k}(([bk]-1)/k) \leqslant t\}.$$

Thus in view of (8), the definitions of b, c and Lemma 3.4, the expression on the right-hand side of (15) tends to 1 and the expression on the right-hand side of (16) tends to 0 as $n, k \to \infty$ in the manner $(*)$. Since t was arbitrary, we conclude that for each $t \geqslant 0$, $\bar{D}_{n,k}(t) \xrightarrow{P} at$ as n, $k \to \infty$ in the manner $(*)$. Now using Lemma 2.1 from [24] we obtain the assertion (12). ∎

Notice that the assertion of Theorem 3.2 concerning convergence in probability is an immediate consequence of Theorem 3.3. Furthermore, by Theorem 3.3 we get

COROLLARY 3.1. *If the conditions*

$$k^{-1} V_k \xrightarrow{P} \mu^{-1}, \quad k^{-1} U_k \xrightarrow{P} \lambda^{-1} \quad \text{as } k \to \infty,$$

are fulfilled, where μ and λ are some finite positive numbers, then $t^{-1} D(t) \xrightarrow{P} \min(\lambda, \mu)$ as $t \to \infty$.

Proof. Let a sequence $\{t_k\}$ tend to infinity and $a_k = [t_k]$. We have

$$\frac{1}{t_k} D(t_k) = \frac{a_k}{t_k} \bar{D}_{a_k}(t_k/a_k) \quad \text{for } t_k \geqslant 1.$$

Hence Theorem 3.3 with $\bar{D}_{n,k} = \bar{D}_k$ and Proposition 2.3 give the assertion. ∎

As an immediate consequence of the relation $\bar{I}_k = \bar{N}_k - \bar{D}_k$, together with Theorem 3.2 and Proposition 2.6, we have

COROLLARY 3.2. *If the conditions od Lemma 3.1 are satisfied, then $\bar{I}_k \to \max(0, \lambda - \mu)e$ a.e. (in probability) as $k \to \infty$.*

3.2. Functional central limit theorem. Introduce the following notation (for $n \geqslant 1$, $k \geqslant 0$):

$$B_{n,k} = \sum_{i=1}^{k} Ev_{n,i}, \quad C_{n,k} = \sum_{i=0}^{k} Eu_{n,i}, \quad A_{n,k} = \sum_{i=1}^{k} EX_{n,i}.$$

THEOREM 3.4. *Let $n, k \to \infty$ in some manner $(*)$ under which*

$$(1)\qquad \bar{B}_{n,k} \xrightarrow{d} \mu^{-1} e, \quad \sqrt{k}(\bar{B}_{n,k} - \mu_n^{-1} e) \xrightarrow{d} \mathbf{0},$$

$$(2)\qquad \bar{C}_{n,k} \xrightarrow{d} \lambda^{-1} e, \quad \sqrt{k}(\bar{C}_{n,k} - \lambda_n^{-1} e) \xrightarrow{d} \mathbf{0},$$

$$(3)\qquad (\sigma_1^{-1} \hat{V}_{n,k}, \sigma_2^{-1} \hat{U}_{n,k}) \xrightarrow{D} (\mathfrak{W}_1, \mathfrak{W}_2),$$

where $\sigma_1, \sigma_2, \mu, \lambda, \mu_n, \lambda_n, n \geqslant 1$, *are some finite positive numbers and* $(\mathfrak{W}_1, \mathfrak{W}_2)$ *is a two-dimensional Wiener process. Furthermore, let*

$$N_{n,k}^{(1)} = \sqrt{k}(e - \lambda_n^{-1} \bar{N}_{n,k}),$$

$$D_{n,k}^{(1)} = \sqrt{k}(e - \lambda_n^{-1} \bar{D}_{n,k}), \qquad D_{n,k}^{(2)} = \sqrt{k}(e - \mu_n^{-1} \bar{D}_{n,k}),$$

$n, k \geqslant 1$. *Then the following convergences hold as* $n, k \to \infty$ *in the manner* $(*)$:

(4) $$(N_{n,k}^{(1)}, D_{n,k}^{(1)}) \xrightarrow{\mathrm{D}} (\sigma_2 \sqrt{\lambda}\, \mathfrak{W}_2, \sqrt{\lambda} f(\sigma_1 \mathfrak{W}_1 - \sigma_2 \mathfrak{W}_2 + \sqrt{\lambda} qe) + \sqrt{\lambda} \sigma_2 \mathfrak{W}_2)$$

if $\lambda = \mu$ *and* $\sqrt{k}(\mu_n^{-1} - \lambda_n^{-1}) \to q$, $|q| < \infty$;

(5) $$(N_{n,k}^{(1)}, D_{n,k}^{(1)}) \xrightarrow{\mathrm{D}} (\sigma_2 \sqrt{\lambda}\, \mathfrak{W}_2, \sigma_2 \sqrt{\lambda}\, \mathfrak{W}_2)$$

if $\lambda < \mu$;

(6) $$(N_{n,k}^{(1)}, D_{n,k}^{(2)}) \xrightarrow{\mathrm{D}} (\sigma_2 \sqrt{\lambda}\, \mathfrak{W}_2, \sigma_1 \sqrt{\mu}\, \mathfrak{W}_1)$$

if $\lambda > \mu$.

Proof. By $n, k \overset{(*)}{\to} \infty$ we denote the convergence of pairs (n, k) to (∞, ∞) in the manner $(*)$.

Define, for $t \geqslant 0$, the processes

$$\theta_{n,k}^{(i)}(t) = (\bar{N}_{n,k}(t) - i/k)_+, \qquad \varphi_{n,k}^{(i)}(t) = (\bar{D}_{n,k}(t) - (1-i)/k)_+,$$

$$\phi_{n,k}^{(i)}(t) = k^{-1/2} U_{n, N_n(kt) - i},$$

$$\xi_{n,k}^{(i)}(t) = k^{-1/2}(V_{n, D_n(kt)+i} + I_{n, D_n(kt) - (1-i)}),$$

$$\psi_{n,k}^{(i)}(t) = k^{-1/2} v_{n, D_n(kt)+i},$$

$$H_{n,k}^{(i)}(t) = k^{-1/2}(1-i)\delta_a(t), \qquad K_{n,k}^{(i)}(t) = k^{-1/2} i \delta_b(t),$$

where $a = k^{-1}(v_{n,1} + u_{n,0})$, $b = k^{-1} u_{n,0}$; define also the functions

$$\eta_{n,k} = \sqrt{k}(\bar{A}_{n,k} - (\mu_n^{-1} - \lambda_n^{-1})e)$$

and the numbers

$$\alpha = \min(\mu, \lambda), \qquad a_{n,k} = \sqrt{k}(\mu_n^{-1} - \lambda_n^{-1}),$$

where $n, k \geqslant 1$, and $i = 0, 1, 2$.

Notice that $\theta_{n,k}^{(i)}$ and $\varphi_{n,k}^{(i)}$ are r.e.'s of D', $\phi_{n,k}^{(i)}$, $\xi_{n,k}^{(i)}$, $\psi_{n,k}^{(i)}$, $H_{n,k}^{(i)}$ and $K_{n,k}^{(i)}$ are r.e.'s of D, $\eta_{n,k}$ belong to D and

$$H_{n,k}^{(i)} \xrightarrow{\mathrm{P}} \mathbf{0}, \quad K_{n,k}^{(i)} \xrightarrow{\mathrm{P}} \mathbf{0}, \quad \eta_{n,k} \xrightarrow{d} \mathbf{0} \quad \text{as } n, k \overset{(*)}{\to} \infty.$$

Notice also that for $i = 0, 1$

$$\xi_{n,k}^{(i)}(t) = k^{-1/2}(S_{n, D_n(kt) - (1-i)} + I_{n, D_n(kt) - (1-i)} + U_{n, D_n(kt) - (1-i)})$$
$$+ \psi_{n,k}^{(i)}(t) - u_{n,0}(H_{n,k}^{(i)}(t) + ik^{-1/2}\mathbf{1}(t)), \quad t \geqslant 0.$$

In this notation, we have the following relations for $i = 0, 1$:

$$\phi_{n,k}^{(i)} = \tau(\hat{U}_{n,k}, \theta_{n,k}^{(i)}) + \tau(\hat{\bar{C}}_{n,k}, \theta_{n,k}^{(i)}) + u_{n,0}(K_{n,k}^{(i)} - ik^{-1/2}\mathbf{1}), \tag{7}$$

$$\xi_{n,k}^{(i)} = \tau(f(\hat{\bar{S}}_{n,k}), \varphi_{n,k}^{(i)}) + \tau(\hat{U}_{n,k}, \varphi_{n,k}^{(i)}) + \tau(\hat{\bar{C}}_{n,k}, \varphi_{n,k}^{(i)}) + \psi_{n,k}^{(i)} - u_{n,0} H_{n,k}^{(i)} - ik^{-1/2} u_{n,0}\mathbf{1}. \tag{8}$$

Assumptions (1), (2) and (3) give the convergences $\bar{U}_{n,k} \xrightarrow{\mathrm{P}} \lambda^{-1} e$, $\bar{V}_{n,k} \xrightarrow{\mathrm{P}} \mu^{-1} e$ as $n, k \xrightarrow{(*)} \infty$, which in turn by Theorem 3.3 and Proposition 2.6 give $\bar{N}_{n,k} \xrightarrow{\mathrm{P}} \lambda e$ and $\bar{D}_{n,k} \xrightarrow{\mathrm{P}} \alpha e$ as $n, k \xrightarrow{(*)} \infty$. Thus for $i = 0, 1, 2$

$$\theta_{n,k}^{(i)} \xrightarrow{\mathrm{P}} \lambda e, \quad \varphi_{n,k}^{(i)} \xrightarrow{\mathrm{P}} \alpha e \quad \text{as } n, k \xrightarrow{(*)} \infty. \tag{9}$$

Now the convergences (9) and assumptions (1)–(3) lead to the following convergences as $n, k \xrightarrow{(*)} \infty$:

$$\tau(\sqrt{k}(\bar{B}_{n,k} - \mu_n^{-1} e), \varphi_{n,k}^{(i)}) \xrightarrow{\mathrm{P}} \tau(\mathbf{0}, \alpha e) = \mathbf{0}, \tag{10}$$

$$\tau(\sqrt{k}(\bar{C}_{n,k} - \lambda_n^{-1} e), \theta_{n,k}^{(i)}) \xrightarrow{\mathrm{P}} \mathbf{0}, \tag{11}$$

$$\tau(\sqrt{k}(\bar{C}_{n,k} - \lambda_n^{-1} e), \varphi_{n,k}^{(i)}) \xrightarrow{\mathrm{P}} \mathbf{0}. \tag{12}$$

Furthermore, if

$$M_{n,k}^{(i)} = \tau(\hat{U}_{n,k}, \theta_{n,k}^{(i)}), \qquad R_{n,k}^{(i)} = \tau(\hat{U}_{n,k}, \varphi_{n,k}^{(i)}), \qquad T_{n,k}^{(i)} = \tau(\hat{S}_{n,k}, \varphi_{n,k}^{(i)})$$

for $i = 0, 1$, $n, k \geqslant 1$, and

$$M_1 = \tau(\sigma_2 \mathfrak{W}_2, \lambda e), \qquad M_2 = \tau(\sigma_2 \mathfrak{W}_2, \alpha e), \qquad M_3 = \tau(\sigma_1 \mathfrak{W}_1 - \sigma_2 \mathfrak{W}_2, \alpha e),$$

then

$$(M_{n,k}^{(0)}, M_{n,k}^{(1)}, R_{n,k}^{(0)}, R_{n,k}^{(1)}, T_{n,k}^{(0)}, T_{n,k}^{(1)}) \xrightarrow{\mathrm{D}} (M_1, M_1, M_2, M_2, M_3, M_3). \tag{13}$$

In view of the relation $\hat{\bar{V}}_{n,k} = \hat{V}_{n,k} + \sqrt{k}\,\bar{B}_{n,k}$ we have for $i = 0, 1$ and for each $a > 0$

$$\sup_{0 \leqslant t \leqslant a} \psi_{n,k}^{(i)}(t) = \omega_a(\tau(\hat{\bar{V}}_{n,k}, \varphi_{n,k}^{(i+1)})) \leqslant \omega_a(\tau(\hat{V}_{n,k}, \varphi_{n,k}^{(i+1)})) + \omega_a(\tau(\sqrt{k}(\bar{B}_{n,k} - \mu_n^{-1} e), \varphi_{n,k}^{(i+1)})) + \omega_a(\tau(\sqrt{k}\,\mu_n^{-1} e, \varphi_{n,k}^{(i+1)})).$$

Hence (10)–(13) and the obvious inequality

$$\omega_a(\tau(\sqrt{k}\,\mu_n^{-1} e, \varphi_{n,k}^{(i)})) \leqslant k^{-1/2} \mu_n^{-1}$$

yield for $i = 0, 1$

$$\psi_{n,k}^{(i)} \xrightarrow{\mathrm{P}} \mathbf{0} \quad \text{as } n, k \xrightarrow{(*)} \infty. \tag{14}$$

Notice that for each $a > 0$ and $i = 0, 1$

$$a\sqrt{k}\,\bar{D}_{n,k} = \tau(a\sqrt{k}\,e, \varphi_{n,k}^{(i)}) + aH_{n,k}^{(i)},$$

$$\lambda_n^{-1}\sqrt{k}\,\bar{N}_{n,k} = \tau(\lambda_n^{-1}\sqrt{k}\,e, \theta_{n,k}^{(i)}) + \lambda_n^{-1} K_{n,k}^{(i)}.$$

Thus for $i = 0, 1$ we have

$$\tau(\hat{\bar{C}}_{n,k}, \varphi_{n,k}^{(i)}) - \lambda_n^{-1}\sqrt{k}\,\bar{D}_{n,k} = \tau\big(\sqrt{k}(\bar{C}_{n,k} - \lambda_n^{-1} e), \varphi_{n,k}^{(i)}\big) - \lambda_n^{-1} H_{n,k}^{(i)}, \tag{15}$$

$$\tau(\hat{\bar{C}}_{n,k}, \theta_{n,k}^{(i)}) - \lambda_n^{-1}\sqrt{k}\,\bar{N}_{n,k} = \tau\big(\sqrt{k}(\bar{C}_{n,k} - \lambda_n^{-1} e), \theta_{n,k}^{(i)}\big) - \lambda_n^{-1} K_{n,k}^{(i)}, \tag{16}$$

$$\begin{aligned}\tau(\hat{\bar{C}}_{n,k}, \varphi_{n,k}^{(i)}) + a_{n,k}\tau(e, \varphi_{n,k}^{(i)}) - \mu_n^{-1}\sqrt{k}\,\bar{D}_{n,k} \\ = \tau\big(\sqrt{k}(\bar{C}_{n,k} - \lambda_n^{-1} e), \varphi_{n,k}^{(i)}\big) - \mu_n^{-1} H_{n,k}^{(i)}.\end{aligned} \tag{17}$$

Hence assumptions (1), (2), the convergences in (9) and the convergences $H_{n,k}^{(i)} \xrightarrow{P} \mathbf{0}$, $K_{n,k}^{(i)} \xrightarrow{P} \mathbf{0}$, yield, for $i = 0, 1$ as $n, k \xrightarrow{(\omega)} \infty$,

$$\tau(\hat{\bar{C}}_{n,k}, \varphi_{n,k}^{(i)}) - \lambda_n^{-1}\sqrt{k}\,\bar{D}_{n,k} \xrightarrow{P} \mathbf{0}, \tag{18}$$

$$\tau(\hat{\bar{C}}_{n,k}, \varphi_{n,k}^{(i)}) + a_{n,k}\tau(e, \varphi_{n,k}^{(i)}) - \mu_n^{-1}\sqrt{k}\,\bar{D}_{n,k} \xrightarrow{P} \mathbf{0}, \tag{19}$$

$$\tau(\hat{\bar{C}}_{n,k}, \theta_{n,k}^{(i)}) - \lambda_n^{-1}\sqrt{k}\,\bar{N}_{n,k} \xrightarrow{P} \mathbf{0}. \tag{20}$$

Now observe that $\hat{S}_{n,k} = \hat{\bar{S}}_{n,k} - \eta_{n,k} - a_{n,k} e$. Since $\eta_{n,k} \xrightarrow{d} \mathbf{0}$ as $n, k \xrightarrow{(\omega)} \infty$, assumption (3) together with the continuity of addition on C and Theorem 6.4 from [26] give, as $n, k \xrightarrow{(\omega)} \infty$,

$$(\hat{U}_{n,k}, f(\hat{\bar{S}}_{n,k})) \xrightarrow{D} (\sigma_2 \mathfrak{W}_1, f(\sigma_1 \mathfrak{W}_1 - \sigma_2 \mathfrak{W}_2 + qe)) \tag{21}$$

if $a_{n,k} \to q$, $|q| < \infty$;

$$(\hat{U}_{n,k}, f(\hat{\bar{S}}_{n,k})) \xrightarrow{D} (\sigma_2 \mathfrak{W}_2, \mathbf{0}) \tag{22}$$

if $a_{n,k} \to -\infty$;

$$(\hat{U}_{n,k}, f(\hat{\bar{S}}_{n,k}) - a_{n,k} e) \xrightarrow{D} (\sigma_2 \mathfrak{W}_2, \sigma_1 \mathfrak{W}_1 - \sigma_2 \mathfrak{W}_2) \tag{23}$$

if $a_{n,k} \to \infty$.

Now using the relation

$$\tau(f(\hat{\bar{S}}_{n,k}), \varphi_{n,k}^{(i)}) = \tau(f(\hat{\bar{S}}_{n,k}) - a_{n,k} e, \varphi_{n,k}^{(i)}) + a_{n,k}\tau(e, \varphi_{n,k}^{(i)})$$

and then relations (7), (8), convergences (9), (14), (18)–(23), the continuity of addition on the set C and Theorem 5.1 from [1], we obtain the following convergences in distribution in (D^4, d^4) as $n, k \xrightarrow{(\omega)} \infty$:

$$(M_{n,k}^{(2)}, M_{n,k}^{(3)}, R_{n,k}^{(2)}, R_{n,k}^{(3)}) \xrightarrow{D} (M_1, M_1, M_4 + M_1, M_4 + M_1) \tag{24}$$

if $\lambda = \mu$ and $a_{n,k} \to q$, $|q| < \infty$;

$$(M_{n,k}^{(2)}, M_{n,k}^{(3)}, R_{n,k}^{(2)}, R_{n,k}^{(3)}) \xrightarrow{D} (M_1, M_1, M_1, M_1) \tag{25}$$

if $\lambda < \mu$;

$$(26)\qquad (M_{n,k}^{(2)}, M_{n,k}^{(3)}, R_{n,k}^{(4)}, R_{n,k}^{(5)}) \xrightarrow{\mathrm{D}} (M_1, M_1, M_5+M_6, M_5+M_6)$$

if $\lambda > \mu$, where

$$M_{n,k}^{(2+i)} = \phi_{n,k}^{(i)} - \lambda_n^{-1}\sqrt{k}\,\bar{N}_{n,k}, \qquad R_{n,k}^{(2+i)} = \xi_{n,k}^{(i)} - \lambda_n^{-1}\sqrt{k}\,\bar{D}_{n,k},$$

$$R_{n,k}^{(4+i)} = \xi_{n,k}^{(i)} - \mu_n^{-1}\sqrt{k}\,\bar{D}_{n,k}, \qquad i = 0, 1,\ n, k \geqslant 1,$$

$$M_4 = \tau\big(f(\sigma_1 \mathfrak{W}_1 - \sigma_2 \mathfrak{W}_2 + qe), \lambda e\big),$$

$$M_5 = \tau(\sigma_1 \mathfrak{W}_1 - \sigma_2 \mathfrak{W}_2, \mu e),$$

$$M_6 = \tau(\sigma_2 \mathfrak{W}_2, \mu e).$$

Using the relation $(\tau(\mathfrak{W}_1, ae), \tau(\mathfrak{W}_2, be)) \overset{\mathrm{D}}{=} (\sqrt{a}\,\mathfrak{W}_1, \sqrt{b}\,\mathfrak{W}_2)$, where a and b are any positive numbers, we obtain

$$\big(\tau(\sigma_2 \mathfrak{W}_2, \lambda e), \tau\big(f(\sigma_1 \mathfrak{W}_1 - \sigma_2 \mathfrak{W}_2 + qe), \lambda e\big) + \tau(\sigma_2 \mathfrak{W}_2, \lambda e)\big)$$
$$\overset{\mathrm{D}}{=} \big(\sigma_2\sqrt{\lambda}\,\mathfrak{W}_2, \sqrt{\lambda} f(\sigma_1 \mathfrak{W}_1 - \sigma_2 \mathfrak{W}_2 + \sqrt{\lambda}\,qe) + \sqrt{\lambda}\,\sigma_2 \mathfrak{W}_2\big),$$
$$\big(\tau(\sigma_2 \mathfrak{W}_2, \lambda e), \tau(\sigma_1 \mathfrak{W}_1 - \sigma_2 \mathfrak{W}_2, \mu e) + \tau(\sigma_2 \mathfrak{W}_2, \mu e)\big)$$
$$\overset{\mathrm{D}}{=} (\sigma_2\sqrt{\lambda}\,\mathfrak{W}_2, \sigma_1\sqrt{\mu}\,\mathfrak{W}_1).$$

Hence the inequalities

$$\phi_{n,k}^{(1)}(t) \leqslant \sqrt{kt} < \phi_{n,k}^{(0)}(t) \quad \text{and} \quad \xi_{n,k}^{(0)}(t) \leqslant \sqrt{kt} < \xi_{n,k}^{(1)}(t)$$

and (24)–(26) imply assertions (4)–(6). ∎

Now we give the functional central limit theorem jointly for the arrival process and the departure process for one single server queue Q. We get the result as a consequence of Theorem 3.4, taking $Q_n = Q$ and $n, k \to \infty$ in such a manner that the limit is first taken for $k \to \infty$ and then for $n \to \infty$.

Let B_k, C_k and A_k, $k \geqslant 0$, be the numbers defined for the queue Q in the same way as are $B_{n,k}$, $C_{n,k}$ and $A_{n,k}$ for the queues Q_n. Then by Theorem 3.4 we get

COROLLARY 3.3. *Let the following conditions hold*:

$$(27)\qquad \sqrt{k}(\bar{B}_k - \mu^{-1} e) \xrightarrow{d} \mathbf{0},$$

$$(28)\qquad \sqrt{k}(\bar{C}_k - \lambda^{-1} e) \xrightarrow{d} \mathbf{0},$$

$$(29)\qquad (\sigma_1^{-1}\hat{V}_k, \sigma_2^{-1}\hat{U}_k) \xrightarrow{\mathrm{D}} (\mathfrak{W}_1, \mathfrak{W}_2),$$

where σ_1, σ_2, μ, λ are some finite positive numbers and $(\mathfrak{W}_1, \mathfrak{W}_2)$ is a two-dimensional Wiener process. Furthermore, let

$$N_k^{(1)} = \sqrt{k}(e - \lambda^{-1}\bar{N}_k), \qquad D_k^{(1)} = \sqrt{k}(e - \lambda^{-1}\bar{D}_k), \qquad D_k^{(2)} = \sqrt{k}(e - \mu^{-1}\bar{D}_k).$$

Then, as $k \to \infty$, *we have*

$$(N_k^{(1)}, D_k^{(1)}) \xrightarrow{D} (\sigma_2 \sqrt{\lambda}\, \mathfrak{W}_2,\ \sqrt{\lambda} f(\sigma_1 \mathfrak{W}_1 - \sigma_2 \mathfrak{W}_2) + \sqrt{\lambda}\, \sigma_2 \mathfrak{W}_2)$$

if $\lambda = \mu$;

$$(N_k^{(1)}, D_k^{(1)}) \xrightarrow{D} (\sigma_2 \sqrt{\lambda}\, \mathfrak{W}_2,\ \sigma_2 \sqrt{\lambda}\, \mathfrak{W}_2)$$

if $\lambda < \mu$;

$$(N_k^{(1)}, D_k^{(2)}) \xrightarrow{D} (\sigma_2 \sqrt{\lambda}\, \mathfrak{W}_2, \sigma_1 \sqrt{\mu}\, \mathfrak{W}_1)$$

if $\lambda > \mu$. ∎

As a consequence of Theorem 3.4 we give the functional central limit theorem for queue size in a sequence of queues.

Let $L_{n,k}^{(1)} = \sqrt{k}(\bar{L}_{n,k} - (\lambda_n - \mu_n) e)$, where the numbers λ_n and μ_n are the same as in the definitions of the processes $N_{n,k}^{(1)}$ and $D_{n,k}^{(2)}$, $n, k \geqslant 1$. Hence the following relation is obvious:

$$L_{n,k}^{(1)} = -\lambda_n N_{n,k}^{(1)} + \lambda_n D_{n,k}^{(2)} - (\lambda_n - \mu_n) D_{n,k}^{(2)}, \qquad n, k \geqslant 1. \tag{30}$$

Thus by Theorem 3.4 we get

COROLLARY 3.4. *Let the conditions of Theorem* 3.4 *be fulfilled, let* β *be the covariance of the coordinates of the two-dimensional Wiener process* $(\mathfrak{W}_1, \mathfrak{W}_2)$ *given in condition* (3) *and let*

$$c^2 = \sigma_1^2 \mu^3 + 2\beta\sigma_1 \sigma_2 \sqrt{\mu^3 \lambda^3} + \sigma_2^2 \lambda^3.$$

Then the following convergences hold as $n, k \to \infty$ *in the same manner* (*) *as in Theorem* 3.4:

$$L_{n,k}^{(1)} \xrightarrow{D} c\mathfrak{W}$$

if $\lambda > \mu$, *and*

$$L_{n,k}^{(1)} \xrightarrow{D} f(c\mathfrak{W} + \lambda^2 qe)$$

if $\lambda = \mu$ *and* $\sqrt{k}(\mu_n - \lambda_n) \to q$, $|q| < \infty$. ∎

Notice that relation (30) and Corollary 3.3 allow us to obtain the convergence of $(N_k^{(1)}, D_k^{(2)}, L_k^{(1)})$ as $k \to \infty$.

Using the well-known invariance principle for arrays of r.v.'s we obtain

COROLLARY 3.5. *Let* $\{Q_n\}$ *be a sequence of* GI/G/1 *queues such that*

$$Ev_{n,1} \to \mu^{-1}, \quad Eu_{n,1} \to \lambda^{-1}, \quad \operatorname{Var} v_{n,1} \to \sigma_1^2, \quad \operatorname{Var} u_{n,1} \to \sigma_2^2,$$

$$E|v_{n,1} - Ev_{n,1}|^{2+\delta} < M, \quad E|u_{n,1} - Eu_{n,1}|^{2+\delta} < M,$$

where μ, λ, σ_1, σ_2, δ, *and* M *are some finite positive numbers. Then, with* $\mu_n^{-1} = Ev_{n,1}$, $\lambda_n^{-1} = Eu_{n,1}$, *conditions* (1)–(3) *of Theorem* 3.4 *are satisfied as* $n, k \to \infty$ *g.m.* (*general manner*) *and the Wiener processes* $\mathfrak{W}_1$ *and* $\mathfrak{W}_2$ *are independent*. ∎

By Corollary 3.3 we obtain separately the convergence in distribution of $N_k^{(1)}$ and $D_k^{(i)}$ to appropriate r.e.'s. The convergence of $N_k^{(1)}$ to an appropriate r.e. was obtained in [1] (see Theorem 17.3) and the convergences of $D_k^{(i)}$ and $L_k^{(1)}$ separately were obtained in [9]. In [9] it is assumed that the sequences $\{v_k, k \geqslant 0\}$ and $\{u_k, k \geqslant 0\}$ are independent and the counting renewal processes generated by them satisfy the functional central limit theorem. Let us notice that these assumptions together with conditions (27) and (28) imply, in view of Proposition 2.11, that condition (29) holds, i.e.

$$(\sigma_1^{-1}\hat{U}_k, \sigma_2^{-1}\hat{U}_k) \xrightarrow{D} (\mathfrak{W}_1, \mathfrak{W}_2) \quad \text{as } k \to \infty.$$

Furthermore, we note that in view of the independence of $\{v_k, k \geqslant 0\}$ and $\{u_k, k \geqslant 0\}$ the Wiener processes $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are independent. Notice that in Theorem 3.4 and in Corollary 3.3 we do not assume that $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are independent. We assume only that $(\mathfrak{W}_1, \mathfrak{W}_2)$ is a two-dimensional Wiener process. Such processes are obtained as the limit in distribution of $(\hat{V}_k, \hat{U}_k)$ when $\{(v_k, u_k), k \geqslant 1\}$ is a stationary sequence of pairs of r.v.'s.

3.3. Interdeparture times. The intensity of the departure process does not give too much information about it. Below we give some other properties of the departure process in more specified cases. Some of them are well known but we recall them here as an illustration of the usefulness of formula (3.1.1).

In view of formula (3.1.1) the departure moment of the kth unit is equal to $t_k = V_k + u_0 - \min_{0 \leqslant j \leqslant k-1} S_j$, $k \geqslant 1$. Hence the interdeparture time of the kth unit and the $(k+1)$th unit is equal to

$$\begin{aligned}\Delta_k &= v_{k+1} - \min\big(\min_{0 \leqslant j \leqslant k-1} S_j, S_k\big) + \min_{0 \leqslant j \leqslant k-1} S_j \\ &= v_{k+1} - \min\big(0, X_k + S_{k-1} - \min_{0 \leqslant j \leqslant k-1} S_j\big).\end{aligned}$$

The r.v. $\Delta_0 = u_0 + v_1$ represents the time to the departure of the first unit.

It is easily seen that the departure process is the counting renewal process generated by the sequence $\{\Delta_k, k \geqslant 0\}$. From Propositions 2.5 and 2.11 we now immediately obtain

COROLLARY 3.6. *For* $0 < a < \infty$, *the following convergences are equivalent*:

(1) $$k^{-1} \sum_{i=0}^{k} \Delta_i \xrightarrow{P} a \quad \text{as } k \to \infty,$$

(2) $$\bar{D}_k \xrightarrow{P} a^{-1} e \quad \text{as } k \to \infty. \ \blacksquare$$

COROLLARY 3.7. *Let the sequence of functions*

$$\Big\{\sqrt{k}\Big(k^{-1} \sum_{i=1}^{[kt]} E\Delta_i - at\Big), t \geqslant 0\Big\}$$

converge in (D, d) *to* $\mathbf{0}$. *Then the following conditions are equivalent*:

(3) *the sequence of processes* $\{\sigma^{-1}\sqrt{k}(k^{-1}\sum_{i=1}^{[kt]}\Delta_i - at),\ t \geqslant 0\}$ *converges in distribution in* (D, d) *to a Wiener process*;

(4) *the sequence of processes* $\sigma^{-1}\sqrt{ka}(a\bar{D}_k - e)$ *converges in distribution to a Wiener process.* ∎

We now show that under some conditions the sequence $\{\Delta_k, k \geqslant 0\}$ is asymptotically stationary.

Let $\{(v_k, u_k), k \geqslant 1\}$ be a stationary sequence and let $\{(v_k^*, u_k^*), -\infty < k < \infty\}$ be a double-ended stationary sequence such that $\{(v_k^*, u_k^*), k \geqslant 1\}$ and $\{(v_k, u_k), k \geqslant 1\}$ have the same distribution.

Introduce the notation:

$$X_k^* = v_k^* - u_k^*, \qquad -\infty < k < \infty,$$

$$\Delta_{m,k} = v_{k+1}^* - \min\Big(0, X_k^* + \max_{-m \leqslant j \leqslant k-1} \sum_{i=j+1}^{k-1} X_i^*\Big), \qquad k \geqslant 0,\ m \geqslant 1.$$

COROLLARY 3.8. *If* $\{(v_k, u_k), k \geqslant 1\}$ *is a stationary sequence then for all* $m \geqslant 1$ *and* $l \geqslant 1$ *the r.v.* $(\Delta_{m+k}, \Delta_{m+k+1}, \ldots, \Delta_{m+k+l})$ *has the same distribution as the r.v.* $(\Delta_{m,k}, \Delta_{m,k+1}, \ldots, \Delta_{m,k+l})$. *Furthermore, if* $\{(v_k, u_k), k \geqslant 1\}$ *is stationary and ergodic and if* $EX_1 \neq 0$, *then for each* $l \geqslant 1$ *we have the convergence*

$$(\Delta_{m,k}, \Delta_{m,k+1}, \ldots, \Delta_{m,k+l}) \xrightarrow{\mathrm{D}} (\Delta_k^*, \Delta_{k+1}^*, \ldots, \Delta_{k+l}^*) \quad \text{as } m \to \infty,$$

where in the case $EX_1 > 0$ *we have* $\Delta_k^* = v_{k+1}$, $k \geqslant 0$, *and in the case* $EX_1 < 0$,

$$\Delta_k^* = v_{k+1}^* - \min\Big(0, X_k^* + \sup_{-\infty < j \leqslant k-1} \sum_{i=j+1}^{k-1} X_i^*\Big), \qquad k \geqslant 0.$$

Proof. The first assertion is an immediate consequence of the stationarity of $\{(v_k, u_k), k \geqslant 1\}$ and the definition of $\Delta_{m,k}$. For the second assertion observe that from the ergodicity of $\{(v_k, u_k), k \geqslant 1\}$ we have the convergences $\bar{S}_k(1) \to EX_1$ a.e. as $k \to \infty$, and $k^{-1}\sum_{j=-k}^{r} X_j^* \to EX_1$ a.e. as $k \to \infty$, for each r. Hence for each $k \geqslant 1$ the r.v.'s

$$X_k^* + \max_{-m \leqslant j \leqslant k-1} \sum_{i=j+1}^{k-1} X_i^*, \qquad m \geqslant 1, \tag{5}$$

converge a.e. as $m \to \infty$, to infinity if $EX_1 > 0$ and to

$$X_k^* + \sup_{-\infty < j \leqslant k-1} \sum_{i=j+1}^{k-1} X_i^* \quad \text{if } EX_1 < 0.$$

Now from the definition of $\Delta_{m,k}$ and from the first part of the corollary we obtain the second assertion. ∎

The above corollary has its analogue in [19], where it has been proved that if $\{(v_k, u_k), k \geqslant 1\}$ is ergodically stable then so is $\{\Delta_k, k \geqslant 0\}$.

By the obvious relation

$$(6) \quad P\{D(s_i) = r_i,\ i = 1, \ldots, k\} = P\{\sum_{j=0}^{r_i - 1} \Delta_j \leqslant s_i < \sum_{j=0}^{r_i} \Delta_j,\ i = 1, \ldots, k\}$$

and Corollary 3.8 we obtain

COROLLARY 3.9. *If $\{(v_k, u_k), k \geqslant 1\}$ is stationary and ergodic then in the case $EX_1 \neq 0$ the finite-dimensional distributions of the processes $\{D_k(t) = D(t_k + t), t \geqslant 0\}$, where t_k is the departure moment of the k-th unit, converge to those of the counting renewal process N generated by $\{\Delta_k^*, k \geqslant 0\}$.* ∎

Observe that for a GI/G/1 queue the r.v.'s defined by (5) converge a.e. as $m \to \infty$, to infinity if $EX_1 \geqslant 0$. Hence we obtain

COROLLARY 3.10. *For a* GI/G/1 *queue, provided $EX_1 \geqslant 0$, we have the following convergence for all $k \geqslant 0$, $l \geqslant 1$:*

$$(\Delta_{m,k}, \Delta_{m,k+1}, \ldots, \Delta_{m,k+l}) \xrightarrow{D} (v_1, v_2, \ldots, v_{l+1}) \quad \text{as } m \to \infty. \ ∎$$

The above fact and relation (6) imply that for a GI/G/1 queue with $EX_1 \geqslant 0$ the finite-dimensional distributions of the processes $\{D_k(t) = D(t_k + t), t \geqslant 0\}$, $k \geqslant 1$, converge to those of the renewal process N generated by $\{v_{k+1}, k \geqslant 0\}$. Here t_k is the departure moment of the kth unit. For a GI/G/1 queue this fact was proved in [5] (see p. 233), where t_k is the arrival moment of the kth unit.

From Corollary 3.8 by some calculations we obtain the following known fact:

COROLLARY 3.11. *For an* M/M/1 *queue in the case $EX_1 < 0$ we have $\Delta_1^* \overset{D}{=} u_1$.*

Proof. Write $w^* = \sup_{0 \leqslant k < \infty} \sum_{j=1}^{k} X_{-j}^*$. If a queue is M/M/1 then the r.v.'s v_k^*, u_k^*, $-\infty < k < \infty$, are independent and negative exponentially distributed. Therefore the r.v.'s v_1^*, X_0^*, w^* are also independent and $w \overset{D}{=} w^*$. In this case we have (see [5])

$$P\{w \geqslant x\} = \lambda\mu^{-1} \exp(-(\mu - \lambda)x), \qquad x \geqslant 0,$$

where $\lambda^{-1} = Eu_1$, $\mu^{-1} = Ev_1$. It is easy to calculate that for $x \leqslant 0$

$$P\{X_0^* \leqslant x\} = \frac{\mu}{\lambda + \mu} \exp(\lambda x).$$

Thus for $x \leqslant 0$ we have

$$P\{w^* + X_0^* \leqslant x\} = \frac{\mu}{\lambda+\mu}\left(1-\left(\frac{\lambda}{\mu}\right)^2\right)\exp(\lambda x).$$

Now it is easy to calculate that for $x \geqslant 0$

$$P\{v_1^* - \min(0, w^* + X_0^*) \geqslant x\} = \exp(-\lambda x). \ \blacksquare$$

Now using Corollary 3.8 we prove by simple calculations the following known fact:

COROLLARY 3.12. *If $\{(v_k, u_k), k \geqslant 1\}$ is a stationary and ergodic sequence and $EX_1 < 0$ and $Ew^* < \infty$ then $E\Delta_0^* = Eu_1$.*

Proof. $E\Delta_0^* = Ev_1^* - E\min(0, X_0^* + w^*) = Ev_1 - EX_0^* - Ew^* + E\max(0, X_0^* + w^*) = Ev_1 - Ev_0^* + Eu_0^* - Ew^* + Ew^* = Eu_1$. $\blacksquare$

4. Joint distribution of waiting time and queue size

Introduce the following definitions:

$$\gamma(t) = t - U_{N(t)-1}, \qquad t \geqslant 0,$$

$$\gamma \ - \ \text{r.v. with the d.f. } P\{\gamma \leqslant x\} = \frac{1}{Eu_1}\int_0^x P\{u_1 > s\}\,ds,$$

$$w_0 = 0, \qquad w_{k+1} = S_k - \min_{0 \leqslant j \leqslant k} S_j, \qquad k \geqslant 0,$$

$$w^{(m)} = \sup_{0 \leqslant j < \infty}\Big(\sum_{i=1}^{j} X_{m+i}\Big), \ m \geqslant 0 \text{ (if it exists)}.$$

From the definitions of w (see Section 3.1) and $w^{(m)}$ it is obvious that $w = w^{(0)}$. Recall that a sum for which the lower bound is greater than the upper one is equal to zero. For instance, $U_{-1} = 0$.

The following relation between the virtual waiting time and the actual waiting time is obvious:

$$W(t) = \big(w_{N(t)} + v_{N(t)} - \gamma(t)\big)_+, \qquad t \geqslant 0. \tag{1}$$

LEMMA 4.1. *For any Borel set A in the real line and for any nonnegative integer m we have*

$$P\{W(t) \in A,\ L(t) > m\} \tag{2}$$

$$= P\Big\{W(t) \in A,\ v_{N(t)-m} - \gamma(t) + w_{N(t)-m} - \sum_{j=N(t)-m}^{N(t)-1} u_j > 0,\ N(t) > m\Big\}.$$

Proof. Notice that $L(t) = N(t) - D(t)$. Hence in view of formula (3.1.1) we obtain

$$P\{W(t)\in A,\ L(t) > m\} = P\{W(t)\in A,\ D(t) < N(t)-m\}$$

$$= \sum_{k=m+1}^{\infty} P\{W(t)\in A,\ V_{k-m}+I_{k-m-1} > t,\ N(t) = k\}$$

$$= P\{W(t)\in A,\ V_{N(t)-m}-t+I_{N(t)-m-1} > 0,\ N(t) > m\}$$

$$= P\{W(t)\in A,\ v_{N(t)-m}-\gamma(t)+w_{N(t)-m}-\sum_{j=N(t)-m}^{N(t)-1} u_j > 0,\ N(t) > m\}. \ \blacksquare$$

Applying Lemma 4.1 for a GI/G/1 queue we obtain

LEMMA 4.2. *For a* GI/G/1 *queue we have*

$$(3)\qquad P\{W(t)\in A,\ L(t) > m\} = P\{(\sup_{0\leqslant j\leqslant N(t)-1} S_j+v_{N(t)}-\gamma(t))_+ \in A,$$

$$\Delta(t, m)-\gamma(t)-\sum_{i=1}^{m} u_i+\sup_{0\leqslant j\leqslant N(t)-m-1}\sum_{i=1}^{j} X_{m+i} > 0,\ N(t) > m\},$$

where A is any Borel set in the real line, m is a nonnegative integer and $\Delta(t, m) = v_{N(t)}\tilde{\delta}_0(-m)+v_m\tilde{\delta}_0(m-1)$.

Proof. From formula (1) and Lemma 4.1 we obtain

$$(4)\qquad P\{W(t)\in A,\ L(t) > m\}$$

$$= \sum_{k=m+1}^{\infty} P\{(\sup_{0\leqslant j\leqslant k-1} (S_{k-1}-S_j)+v_k-(t-U_{k-1}))_+ \in A,$$

$$v_{k-m}-(t-U_{k-1})-\sum_{i=k-m}^{k-1} u_i+\sup_{0\leqslant j\leqslant k-m-1} (S_{k-1-m}-S_j) > 0,\ U_{k-1}\leqslant t < U_k\}.$$

Notice that the vector $(v_1, u_1, v_2, u_2, \ldots, v_{k-1}, u_{k-1})$ has the same distribution as the vector $(v_{k-1}, u_{k-1}, \ldots, v_1, u_1)$. Thus the right-hand side of (4) is equal to

$$\sum_{k=m+1}^{\infty} P\{(\sup_{0\leqslant j\leqslant k-1} S_j+v_k-(t-U_{k-1}))_+ \in A,\ v_k\tilde{\delta}_0(-m)+v_m\tilde{\delta}_0(m-1)$$

$$-t+U_{k-1}-\sum_{i=1}^{m} u_i+\sup_{0\leqslant j\leqslant k-m-1}\sum_{i=1}^{j} X_{m+i} > 0,\ U_{k-1}\leqslant t < U_k\}.$$

Hence we obtain (3). $\blacksquare$

We now derive the limiting distribution of $(W(t), L(t))$ as $t\to\infty$. First we prove Lemma 4.3, which is a generalization of (5.100) from [5] (see p. 293). (5.100) states that $w_{N(t)}$, the waiting time of the unit which arrives as the last one before t, is asymptotically independent, as $t\to\infty$, of $\gamma(t)$, the length

of the interval from the arrival moment of that unit to the moment t, i.e. $\gamma(t) = t - U_{N(t)-1}$. We generalize this by proving that an r.v. $Y_{N(t)}$, which depends on $X_1, X_2, \ldots, X_{N(t)-1}$ and converges in distribution to Y with $P\{Y = \underline{0}\} > 0$, is assymptotically independent of $\gamma(t)$ as $t \to \infty$. The proof of Lemma 4.3 is similar to that of (5.100) from [5], although the basic formula (7) for getting the convergence in Lemma 4.3 follows from other considerations than its analogue (5.96) in [5].

LEMMA 4.3. *Let the sequence $\{(v_k, u_k), k \geqslant 0\}$ generating a* GI/G/1 *queue be such that u_1 has a nonlattice distribution and $P\{u_1 > 0\} > 0$. Define the sequence $\{Y_k, k \geqslant 0\}$ of r.v.'s in $\boldsymbol{R}^l$ by $Y_0 = Y_1 = \underline{0}$, $Y_k = f_k(X_1, X_2, \ldots, X_{k-1})$, $k \geqslant 2$, where, for some $l \geqslant 1$, f_k is a measurable function from $\boldsymbol{R}^{k-1}$ into $\boldsymbol{R}^l$. If $Y_k \to Y$ a.e. with $P\{Y = \underline{0}\} > 0$ and for each Borel set B in $\boldsymbol{R}^l$ the function $P\{Y_{N(t)} \in B\}$ as a function of t on $\boldsymbol{R}_+$ has a bounded variation then*

$$P\{Y_{N(t)} \in B,\ \gamma(t) \leqslant y\} \to P\{Y \in B\}\, P\{\gamma \leqslant y\} \quad \text{as } t \to \infty, \tag{5}$$

where y is any nonnegative real number and B is any Borel set in $\boldsymbol{R}^l$ such that $P\{Y \in \partial B\} = 0$.

Proof. The r.v.'s $u_0, u_1, u_2, \ldots$ are independent and $u_1, u_2, \ldots$ are identically distributed. Hence by the condition $P\{u_1 > 0\} > 0$ we have the convergence $P\{U_k = 0\} \to 0$ as $k \to \infty$. Further, let ε be any sufficiently small number. Then there exists n such that, for $k \geqslant n$, $P\{U_k = 0\} \leqslant \varepsilon$. Now introduce the following definitions. Let g be a function on $\boldsymbol{R}$ defined as

$$g(t) = P\{u_1 > t\}\big(1 - \delta_0(t-y)\big) \quad \text{for } t \geqslant 0$$

and zero otherwise, where y is a fixed nonnegative number. Further, for $t \geqslant 0$ and a Borel set B let

$$H(t) = 1 + \sum_{k=1}^{\infty} P\Big\{\sum_{i=1}^{k} u_i \leqslant t\Big\},$$

$$K(B, t) = P\{Y_{N(t)} \in B,\ N(t) \geqslant n\},$$

$$L(B, t) = \sum_{k=1}^{\infty} P\{Y_k \in B,\ U_{k-1} \leqslant t\}.$$

For $y < t$ we have

$$P\{Y_{N(t)} \in B,\ \gamma(t) \leqslant y\}$$
$$= P\{Y_{N(t)} \in B,\ \gamma(t) \leqslant y,\ N(t) < n\} + P\{Y_{N(t)} \in B,\ \gamma(t) \leqslant y,\ N(t) \geqslant n\},$$

$$P\{Y_{N(t)} \in B,\ \gamma(t) \leqslant y,\ N(t) \geqslant n\} = \sum_{k=n}^{\infty} P\{Y_k \in B,\ t - U_{k-1} \leqslant y,\ U_{k-1} \leqslant t < U_k\}$$

$$= \sum_{k=n}^{\infty} \int_{t-y}^{t} P\{Y_k \in B,\ U_{k-1} \in ds\}\, P\{u_k > t-s\} = \int_{t-y}^{t} P\{u_1 > t-s\}\, L(B, ds).$$

Thus for $y < t$ we have

$$P\{Y_{N(t)} \in B,\ \gamma(t) \leqslant y,\ N(t) \geqslant n\} = (g * L(B, \cdot))(t) \tag{6}$$

where $*$ denotes the convolution operation.

Now for $k \geqslant n$ we have

$$P\{Y_k \in B,\ U_{k-1} \leqslant t\} = \sum_{j=k}^{\infty} P\{Y_k \in B,\ U_{j-1} \leqslant t < U_j\}.$$

For a fixed $k \geqslant n$, introduce the notation $Z_r = \sum_{i=k}^{r} u_i$, $r \geqslant k-1$. Hence for $j \geqslant k$ we have

$$\begin{aligned} P\{Y_k \in B,\ U_{j-1} \leqslant t < U_j\} &= P\{Y_k \in B,\ U_{k-1} + Z_{j-1} \leqslant t < U_{k-1} + Z_{j-1} + u_j\} \\ &= \int_0^t P\{Y_k \in B,\ U_{k-1} + s \leqslant t < U_{k-1} + s + u_j \mid Z_{j-1} = s\}\, P\{Z_{j-1} \in ds\} \\ &= \int_0^t P\{Y_k \in B,\ U_{k-1} \leqslant t - s < U_{k-1} + u_j \mid Z_{j-1} = s\}\, P\{Z_{j-1} \in ds\}. \end{aligned}$$

The r.v.'s (Y_k, U_{k-1}) and $(u_k, u_{k+1}, \ldots, u_j)$ are independent. Thus the right-hand side of the above equality can be rewritten as

$$\int_0^t P\{Y_k \in B,\ U_{k-1} \leqslant t - s < U_{k-1} + u_k\}\, P\{Z_{j-1} \in ds\}.$$

As a consequence we obtain

$$\begin{aligned} P\{Y_k \in B,\ U_{k-1} \leqslant t\} &= \sum_{j=k}^{\infty} \int_0^t P\{Y_k \in B,\ N(t-s) = k\}\, P\{Z_{j-1} \in ds\} \\ &= \int_0^t P\{Y_{N(t-s)} \in B,\ N(t-s) = k\}\, H(ds). \end{aligned}$$

From the definition of $L(B, t)$ we thus have

$$L(B, t) = \int_0^t P\{Y_{N(t-s)} \in B,\ N(t-s) \geqslant n\}\, H(ds) = (K(B, \cdot) * H)(t). \tag{7}$$

Now using formulas (6) and (7) we get

$$P\{Y_{N(t)} \in B,\ \gamma(t) \leqslant y,\ N(t) \geqslant n\} = (g * K(B, \cdot) * H)(t). \tag{8}$$

Notice that in the proof of formulas (7) and (8) we did not use the fact that u_1 has a nonlattice distribution. Thus formulas (7) and (8) are valid if $t = na + \sigma$, $0 < \sigma < a$, whenever u_1 has a lattice distribution with period a.

The function $g * K(B, \cdot)$ is of bounded variation. Thus using the key

renewal theorem of Smith (see [5], Theorem 6.2) we deduce that the right-hand side of (8) tends, as $t \to \infty$, to

$$\frac{1}{Eu_1} \int_0^\infty \int_0^\infty g(t-s) K(B, ds) dt. \tag{9}$$

By Fubini's theorem we have

$$\int_0^\infty \int_0^t g(t-s) H(B, ds) dt = \int_0^\infty \int_0^\infty g(t-s) dt K(B, ds)$$

$$= \int_0^\infty \int_s^{s+y} P\{u_1 > t-s\} dt K(B, ds) = \int_0^y P\{u_1 > t\} dt \int_0^\infty K(B, ds).$$

Since $u_1, u_2, \ldots$ are i.i.d. we have $N(t) \to \infty$ a.e. as $t \to \infty$. Hence in view of the convergence $Y_k \to Y$ a.e. we have $Y_{N(t)} \to Y$ a.e. as $t \to \infty$. Thus for any Borel set B in $\boldsymbol{R}^l$ such that $P\{Y \in \partial B\} = 0$ we have the convergence $K(B, t) \to P\{Y \in B\}$ and

$$\limsup_{t \to 0} K(B, t) \leqslant \lim_{t \to 0} P\{N(t) > n\} = P\{U_n = 0\}.$$

In view of the above facts we have the inequality

$$P\{\gamma \leqslant y\}(P\{Y \in B\} - \varepsilon) \leqslant \lim_{t \to \infty} P\{Y_{N(t)} \in B, \gamma(t) \leqslant y\}$$
$$\leqslant P\{\gamma \leqslant y\} P\{Y \in B\}$$

for any $y \geqslant 0$ and any Borel set $B \subset \boldsymbol{R}^l$ such that $P\{Y \in \partial B\} = 0$. Since ε was arbitrary we obtain assertion (5). ∎

Applying a similar argument to that used in Section I.6.3 of [5] (see the derivation of I.6.24) we come to

Remark 4.1. If, in Lemma 4.3, u_1 has a lattice distribution with period a then the convergence in (5) is valid as $n \to \infty$ if $t = na + \sigma$ with $0 \leqslant \sigma < a$.

The main result of Chapter 4 is

THEOREM 4.1. *For a* GI/G/1 *queue such that* u_1 *has a nonlattice distribution and* $\varrho < 1$, *we have the convergence*

$$P\{W(t) \in A,\ L(t) > m\} \tag{10}$$
$$\to P\{(w+v-\gamma)_+ \in A,\ v\delta_0(-m) + v_m \delta_0(m-1) - \sum_{i=1}^{m} u_i - \gamma + w^{(m)} > 0\}$$

as $t \to \infty$, *for any Borel set* A *in* $\boldsymbol{R}_+$ *which is a continuity set of* $(w+v-\gamma)_+$ *and any nonnegative integer* m, *where* w, v, γ *are independent,* $v \overset{D}{=} v_1$, *and* $w^{(m)}$, v_m, γ, u_1, $u_2, \ldots, u_m$ *are independent for any* m.

Proof. We prove the convergence (10) separately in two cases: $m \geqslant 1$ and $m = 0$.

For a fixed $m \geqslant 1$, define Y and Y_k, $k \geqslant 1$, to be the pairs $Y = (Y', Y'')$, $Y_k = (Y'_k, Y''_k)$, where $Y' = w$, $Y'' = w^{(m)}$, $Y'_0 = Y'_1 = Y''_0 = Y''_1 = 0$,

$$Y'_k = \sup_{0 \leqslant j \leqslant k-1} S_j, \quad Y''_k = \sup_{0 \leqslant j \leqslant k-m-1} \sum_{i=1}^{j} X_{m+i}, \quad k \geqslant 2.$$

For $m = 0$ define Y and Y_k, $k \geqslant 0$, by $Y = w$, $Y_0 = Y_1 = 0$,

$$Y_k = \sup_{0 \leqslant j \leqslant k-1} S_j, \; k \geqslant 2.$$

Since the r.v.'s $X_1, X_2, \ldots$, are i.i.d. and $EX_1 < 0$ we have in both cases the convergence $Y_k \to Y$ a.e. as $k \to \infty$, and, in view of the convergence $N(t) \to \infty$ a.e. as $t \to \infty$, also the convergence $Y_{N(t)} \to Y$ a.e. as $t \to \infty$.

Notice that the sample paths of the processes $\{Y'_{N(t)}, t \geqslant 0\}$ and $\{Y''_{N(t)}, t \geqslant 0\}$ belong to D'. Hence in both cases the function $P\{Y_{N(t)} \in B\}$, as a function of t, is of bounded variation; here B is a Borel set in $\boldsymbol{R}^2$ or in $\boldsymbol{R}$ according as $m \geqslant 1$ or $m = 0$. Thus from Lemma 4.3 we have in both cases the convergence

$$P\{Y_{N(t)} \in B, \gamma(t) \leqslant y\} \to P\{Y \in B\} P\{\gamma \leqslant y\} \tag{11}$$

as $t \to \infty$, where B is such that $P\{Y \in \partial B\} = 0$.

Now it is easy to see that in both cases the r.v. $(Y_{N(t)}, \gamma(t))$ and the r.v. $v_{N(t)}$ are independent for each $t \geqslant 0$. From (11) we thus have in both cases the convergence

$$P\{Y_{N(t)} \in B, \gamma(t) \leqslant y, v_{N(t)} \leqslant x\} \to P\{Y \in B\} P\{\gamma \leqslant y\} P\{v \leqslant x\} \tag{12}$$

for B such that $P\{Y \in \partial B\} = 0$.

The mappings $(x, y) \mapsto x + y$ and $x \mapsto (x)_+$, $x, y \in \boldsymbol{R}$, are continuous. Thus for $m \geqslant 1$, Lemma 4.2 and (12), together with Theorem 5.1 from [1], give the convergence

$$P\{W(t) \in A, L(t) > m\} \to P\{(w + v - \gamma)_+ \in A, v_m + w^{(m)} - \gamma - \sum_{i=1}^{m} u_i > 0\} \tag{13}$$

as $t \to \infty$, for a Borel set A such that $P\{(w + v - \gamma)_+ \in \partial A\} = 0$. For $m = 0$ we have

$$P\{W(t) \in A, L(t) > 0\} = P\{(Y_{N(t)} + v_{N(t)} - \gamma(t))_+ \in A, \\ v_{N(t)} - \gamma(t) + Y'_{N(t)} > 0, N(t) > 0\}.$$

From (12) and Theorem 5.1 of [1], in the same way as before we obtain the convergence

$$P\{W(t) \in A, L(t) > 0\} \to P\{(w + v - \gamma)_+ \in A, w + v - \gamma > 0\} \tag{14}$$

as $t \to \infty$, for a Borel set A such that $P\{(w + v - \gamma)_+ \in \partial A\} = 0$, which completes the proof. ∎

In the same way as in Theorem 4.1, except that we pass to infinity in the manner $t = na+\sigma \to \infty$ as $n \to \infty$ (where $0 \leqslant \sigma < a$) and we use Remark 4.1 instead of Lemma 4.3, we obtain

Remark 4.2. If in Theorem 4.1 the r.v. u_1 has a lattice distribution with period a then the convergence (10) is valid as $n \to \infty$ if $t = na+\sigma$, $0 \leqslant \sigma < a$.

5. New forms of Little's formula

In 1961 Little observed that for a GI/G/1 queue the following relation holds:

$$EL = \lambda E(w+v), \tag{1}$$

where $\lambda^{-1} = Eu_1$. Many papers connected with the relation between L and w have appeared so far (see [3], [4], [6], [8], [10], [14], [21]). The authors either give simpler proofs of (1), or weaken the assumptions (see [3], [4], [21]), or modify (1) formulating a relation between factorial moments (see [4], [11]), or put it in a new form in queueing theory ([4], [21]) or in the theory of point processes (see [7]).

As consequences of Lemma 4.1 and Theorem 4.1 we obtain in this chapter: the relation between the distributions of L and $w+v$ for a GI/G/1 queue, an analogue of formula (1) for the mth moment of L for a GI/G/1 queue, the convergence $(1-\varrho)(L-\lambda W) \xrightarrow{P} 0$ and $(1-\varrho)^m(EL^m-\lambda^m EW^m) \to 0$ as $\varrho \nearrow 1$ for GI/G/1 queues satisfying some conditions, and the convergence $k_n^{-1/2}(L_n(k_n t)-\lambda W_n(k_n t)) \xrightarrow{P} 0$ for all queues Q_n belonging to some class of queues such that $k_n^{-1/2} W_n(k_n t)$ has nondegenerate limit distribution as $k_n \to \infty$. Here the quantities L and W are defined for a queue Q with traffic intensity ϱ.

5.1. Little's formula for distributions. Directly from Theorem 4.1 and Remark 4.2 we obtain the following relation between the d.f.'s of L and $w+v$.

THEOREM 5.1. *For any* GI/G/1 *queue with* $\varrho < 1$,

$$P\{L \leqslant k\} = P\{w+v \leqslant \sum_{i=0}^{k} u_i'\}, \quad k \geqslant 0, \tag{1}$$

where $w, v, u_0', u_1', \ldots$ *are independent r.v.'s such that* $v \overset{D}{=} v_1$, $u_0' \overset{D}{=} \gamma$, $u_i' \overset{D}{=} u_i$, $i \geqslant 1$. ∎

The problem of the relation between the distributions of L and w is known in the literature (see [2], [9], [11], [17], [22]). Borovkov (see [2], p. 208) gives the following relation:

$$P\{L > k+1\} = P\{w > \sum_{i=0}^{k} u_i'\}, \quad k \geqslant 0. \tag{2}$$

Remark 5.1. *Formulas* (1) *and* (2) *are equivalent.*

Proof. By (1) for $k \geqslant 0$ we obtain

$$P\{L > k\} = P\{w+v-u'_k > \sum_{i=0}^{k-1} u'_i\}. \tag{3}$$

Since the u_k are nonnegative and $(w+v-u_k)_+ \overset{D}{=} w$ for all k, it follows that the right-hand side of (3) is equal to the right-hand side of (2). Hence we obtain the assertion. ∎

We call formula (1) *Little's formula* for the distributions of L and $w+v$. By formula (1) we obtain a relation between the moments of L and $w+v$.

Immediately from the formula $EL^m = m \sum_{k=1}^{\infty} k^{m-1} P\{L \geqslant k\}$, $m > 0$, and from formula (1) we get the following result.

COROLLARY 5.1. *For a* GI/G/1 *queue with* $EL^m < \infty$,

$$EL^m = \int_0^{\infty} P\{w+v > x\}\, dE(\tilde{N}(x))^m, \tag{4}$$

where m is any positive real number and $\tilde{N}$ is the stationary renewal process generated by $\{u'_k, k \geqslant 0\}$. ∎

Since $E\tilde{N}(x) = \lambda x$, formula (4) in the case $m = 1$ is of the form (5.1). Now integrating (4) term by term and using Proposition 2.9 we obtain

COROLLARY 5.2. *For a* GI/G/1 *queue with* $EL^m < \infty$,

$$EL^m = \int_0^{\infty} E(\tilde{N}(x))^m\, dP\{w+v \leqslant x\}. \tag{5}$$ ∎

The corollary has its analogue for GI/G/k, e.g. Theorem 5 in [4], where the following relation is given:

$$EL^{(m)} = \lambda \int_0^{\infty} M^{(m-1)}(x)\, dP\{w \leqslant x\},$$

where $EL^{(m)}$ is the mth factorial moment of L and

$$M^{(m)}(x) = \sum_{i=m}^{\infty} \binom{i-1}{m-1} P\{\sum_{j=1}^{i} u_j \leqslant x\}.$$

Formula (5) and Jensen's inequality directly imply

COROLLARY 5.3. *For a* GI/G/1 *queue with* $EL^m < \infty$,

$$EL^m \geqslant \lambda^m E(w+v)^m \quad \textit{for } m \geqslant 1 \quad (\leqslant \textit{ for } 0 < m \leqslant 1).$$ ∎

5.2. Asymptotic equality of L and λW in heavy traffic. For each $n \geqslant 1$, let Q_n be a GI/G/1 queue generated by a sequence $\{(v_{n,k}, u_{n,k}), k \geqslant 0\}$. Here $v_{n,k}$ and $u_{n,k}$ have the same interpretation as v_k and u_k for Q (see Chapter 1). All the characteristics of Q_n occurring here are defined in Chapters 1, 3 and

4. They ought to be written with the index n, but we drop it here for simplicity. Thus, instead of $v_{n,k}$, $u_{n,k}$, $X_{n,k}$, $S_{n,k}$, ϱ_n, $w_{n,k}$, $w_n^{(m)}$, $w_{n,\infty}$, $W_n(\infty)$, $L_n(\infty)$, we write v_k, u_k, X_k, S_k, ϱ, w_k, $w^{(m)}$, w, W, L. In this section we consider only sequences $\{Q_n\}$ of GI/G/1 queues for which $\varrho_n \nearrow 1$. Thus, instead of writing $n \to \infty$, we write $\varrho \nearrow 1$.

THEOREM 5.2. *Let $\{Q_n\}$ be a sequence of* GI/G/1 *queues such that*

(1) *there exist positive numbers M and δ such that for all ϱ*

$$Ev_1^{1+\delta} < M, \quad Eu_1^{1+\delta} < M,$$

(2) $$Ev_1 \to \lambda^{-1}, \quad Eu_1 \to \lambda^{-1} \quad \text{as } \varrho \nearrow 1,$$

where λ is some finite positive number,

(3) $$(1-\varrho)w \xrightarrow{D} Z \quad \text{as } \varrho \nearrow 1.$$

Then

(4) *for each $m \geqslant 0$,* $$(1-\varrho)(w - w^{([m/(1-\varrho)])}) \xrightarrow{P} 0 \quad \text{as } \varrho \nearrow 1,$$

(5) $$(1-\varrho)(\lambda W - L) \xrightarrow{P} 0, \quad (1-\varrho)(\lambda w - L) \xrightarrow{P} 0 \quad \text{as } \varrho \nearrow 1.$$

Proof. In the proof we write $1-\varrho = a$. By assumptions (1), (2) and by Proposition 2.4 we obtain the following convergences as $a \searrow 0$:

(6) $$a \sum_{k=1}^{[1/a]} v_k \xrightarrow{P} \lambda^{-1}, \quad a \sum_{k=1}^{[1/a]} u_k \xrightarrow{P} \lambda^{-1}.$$

We now show the convergence in (4). For this purpose notice that

$$w = \max\Big(\sup_{0 \leqslant k \leqslant m} S_k, \sup_{m \leqslant k < \infty} S_k\Big)$$
$$= \max\Big(\sup_{0 \leqslant k \leqslant m} S_k, S_m + \sup_{0 \leqslant k < \infty} \sum_{i=m+1}^{k+m} X_i\Big).$$

Hence by the definition of $w^{(m)}$ (see Chapter 4) we have

$$w - w^{(m)} = \max\Big(\sup_{0 \leqslant k \leqslant m} S_k - w^{(m)}, S_m\Big).$$

Thus

(7) $$a(w - w^{([m/a])}) = \max\Big(\sup_{0 \leqslant t \leqslant m} aS_{[t/a]} - aw^{([m/a])}, aS_{[m/a]}\Big).$$

Using (6), the nonnegativity of v_k and u_k and Proposition 2.2 we deduce that the sequence of processes $\{aS_{[t/a]}, t \geqslant 0\}$ converges in probability in (D, d), as $a \searrow 0$, to the function equal to zero. Thus (3), the relation $w \overset{D}{=} w^{(m)}$, the continuity of the mapping $D \ni x \mapsto \sup_{0 \leqslant t \leqslant m} x(t)$ on C and (7) give the convergence (4).

For (5), observe that the following convergences hold:

$$av_1 \xrightarrow{P} 0, \quad a\gamma \xrightarrow{P} 0 \quad \text{as } a \searrow 0. \tag{8}$$

Indeed, the first follows from (1) and from Chebyshev's inequality and the second from (2) and

$$P\{a\gamma > \varepsilon\} \leqslant M(a\varepsilon)^{\delta}(\delta Eu_1)^{-1}$$

(see the proof of Proposition 2.7).

Now (8) and (3) lead to the convergence

$$\big(aw + a(v-\gamma)\big)_+ - aw \xrightarrow{D} (Z)_+ - Z = 0 \quad \text{as } a \searrow 0. \tag{9}$$

Using formula (4.1.10) we have

$$\begin{aligned} P\{aW \in A,\ aL \leqslant y\} = P\{&aw + a\big((w+v-\gamma)_+ - w\big) \in A, \\ &av_{[y/a]}\tilde{\delta}_0([y/a]-1) + av\tilde{\delta}_0(-[y/a]) - a\gamma \\ &+ a(w^{([y/a])} - w) + aw - a\sum_{k=1}^{[y/a]} u_k \leqslant 0\}. \end{aligned} \tag{10}$$

Using simultaneously (3), (8), (9) and (6) we deduce that the right-hand side of (10) tends to $P\{Z \in A,\ Z \leqslant y/\lambda\}$ if A and $(-\infty, y/\lambda]$ are continuity sets of Z. This means that $(aW, aL) \xrightarrow{D} (Z, \lambda Z)$, which leads to the convergence $a(\lambda W - L) \xrightarrow{P} 0$ as $a \searrow 0$.

The convergence $(1-\varrho)(\lambda w - L) \xrightarrow{P} 0$ as $a \searrow 0$ is obtained from the relation $W = (w+v-\gamma)_+$, by (8) and the first convergence in (5). ∎

The following result of Kingman (see [23] or [24]) indicates a case for which condition (3) holds.

KINGMAN'S RESULT. *Let conditions* (1) *and* (2) *hold with* $\delta > 1$ *in* (1). *Furthermore, let*

$$\operatorname{Var} v_1 + \operatorname{Var} u_1 \to \sigma^2 \quad \textit{as } \varrho \nearrow 1, \textit{ where } 0 < \sigma^2 < \infty. \tag{11}$$

Then $(1-\varrho)w \xrightarrow{D} \Lambda$ *as* $\varrho \nearrow 1$, *where* Λ *has a negative exponential d.f. with mean* $\lambda\sigma^2/2$. ∎

COROLLARY 5.4. *Let conditions* (1) *and* (2) *hold with* $\delta > 1$ *in* (1). *If, furthermore, condition* (11) *holds then* $\lambda^{-1}(1-\varrho)L \xrightarrow{D} \Lambda$ *as* $\varrho \nearrow 1$. ∎

Corollary 5.4 was obtained by Borovkov (see [2], p. 213) as a consequence of the relation between the distributions of L and w. Here it follows from a stronger result, namely from the convergence (5) in Theorem 4, where Z has a negative exponential distribution with mean $\lambda\sigma^2/2$.

The assertions of Theorem 5.2 concerning convergence in probability imply, under some obvious conditions, convergence in the mean.

THEOREM 5.3. *Let* $\{Q_n\}$ *be a sequence of* GI/G/1 *queues such that conditions* (2) *and* (11) *hold and for some real number* $m \geqslant 1$

(13) *there exist finite positive numbers M and δ such that for all ϱ*

$$Eu_1^{2+\delta} < M, \quad Ev_1^{m+1+\delta} < M.$$

If, furthermore, the sequence $\{((1-\varrho)w)^m, 0 < \varrho < 1\}$ is uniformly integrable then

$$E((1-\varrho)W)^m \to E\Lambda^m, \quad E((1-\varrho)w)^m \to E\Lambda^m \quad \text{as } \varrho \nearrow 1, \tag{14}$$

$$E((1-\varrho)L)^m - E((1-\varrho)\lambda W)^m \to 0 \quad \text{as } \varrho \nearrow 1. \tag{15}$$

Proof. In the proof we write $a = 1-\varrho$. Conditions (13) and (2) imply the convergences $av \xrightarrow{P} 0$ and $a\gamma \xrightarrow{P} 0$ as $a \searrow 0$ (see the proof of Theorem 5.3). Hence $a(v-\gamma) \xrightarrow{P} 0$ as $a \searrow 0$.

By Kingman's result we have the convergences $aw \xrightarrow{D} \Lambda$, $aW = (aw + a(v-\gamma))_+ \xrightarrow{D} \Lambda$ as $a \searrow 0$, where Λ is a r.v. having a negative exponential d.f. with mean $\lambda\sigma^2/2$. Thus we have

$$(aw)^m \xrightarrow{D} \Lambda^m, \quad (aW)^m \xrightarrow{D} \Lambda^m, \quad E(aw)^m \to E\Lambda^m \quad \text{as } a \searrow 0. \tag{16}$$

It is well known that the sum of two uniformly integrable sequences is also uniformly integrable. Using (13), the uniform integrability of $\{(aw)^m, 0 < a < 1\}$ and the inequality $(aw+av)^m \leqslant 2^{m-1}((aw)^m + (av)^m)$, we obtain the uniform integrability of $\{(aw+av)^m, 0 < a < 1\}$. Hence the inequality $P\{W > x\} \leqslant P\{w+v > x\}$ shows that $\{(aW)^m, 0 < a < 1\}$ is uniformly integrable. Thus from (16) we obtain (14).

Now in view of Corollary 5.2 we have

$$E(aL)^m = \int_0^\infty E(a\tilde{N}(x/a))^m \, dP\{aw+av \leqslant x\} = Ef_a(aw+av),$$

where $f_a(x) = E(a\tilde{N}(x/a))^m$, $x \geqslant 0$. By Proposition 2.10 we have $f_a(x) \to f(x) = (\lambda x)^m$ as $a \searrow 0$, for each $x \geqslant 0$. Let A be the set of all x from the real line such that there exists a sequence $\{x_a\}$ tending to x as $a \searrow 0$, and $f_a(x_a) \not\to f(x)$. Since f_a is increasing for each a and f is continuous, it follows that A is empty. Hence Theorem 5.5 in [1] gives the convergence $f_a(aw+av) \xrightarrow{D} \Lambda^m \lambda^m$ as $a \searrow 0$. By formula (2.2.12) for all $x \geqslant 0$ and sufficiently small a we have the inequality $f_a(x) \leqslant (1+x/\varepsilon)^m EN^*(\varepsilon)^m$, where $EN^*(\varepsilon)^m < \infty$. Thus for sufficiently small a

$$f_a(aw+av) \leqslant ((aw+av)/\varepsilon + 1)^m EN^*(\varepsilon)^m.$$

In view of the uniform integrability of $\{(aw+av)^m, 0 < a < 1\}$ we obtain the uniform integrability of $\{f_a(aw+av), 0 < a < 1\}$. Thus $Ef_a(aw+av) \to \lambda^m E\Lambda^m$ as $a \searrow 0$. Hence (16) implies (15). ∎

5.3. Further asymptotic results in heavy traffic. For each $n \geqslant 1$, let Q_n be a queue generated by $\{(v_{n,k}, u_{n,k}), k \geqslant 0\}$. In this section all characteristics of Q_n are affixed by the index n. We use the symbols $X_{n,k}$, $S_{n,k}$, $w_{n,k}$, $V_{n,k}$, $U_{n,k}$, N_n, W_n, L_n, $B_{n,k}$, $C_{n,k}$, $A_{n,k}$, $k \geqslant 0$, $n \geqslant 1$, defined earlier. In addition, we

introduce the processes $w_{n,k}^{(1)}$ and $w_{n,k}^{(2)}$ given by

$$w_{n,k}^{(1)}(t) = k^{-1/2} w_{n,N_n(kt)},$$

$$w_{n,k}^{(2)}(t) = k^{-1/2}(w_{n,N_n(kt)} - w_{n,N_n(kt)-m_n})$$

for $t \geqslant 0$, and the function $C_{n,k}^{(2)}$ defined by

$$C_{n,k}^{(2)}(t) = k^{-1/2}(C_{n,[kt]} - C_{n,[kt]-m_n}), \qquad t \geqslant 0,$$

where $\{m_n\}$ is some sequence of nonnegative integers. Notice that $w_{n,k}^{(1)}$ and $w_{n,k}^{(2)}$ are r.e.'s of D and $C_{n,k}^{(2)}$ belongs to D.

Below we prove that under appropriate conditions

$$k_n^{-1/2}(\lambda W_n(k_n t) - L_n(k_n t)) \xrightarrow{P} 0 \quad \text{for each } t \geqslant 0.$$

THEOREM 5.4. *Let a sequence $\{k_n\}$, tending to infinity, be such that*

(1) $\bar{B}_{n,k_n} \xrightarrow{d} \mu^{-1} e, \qquad \sqrt{k_n}(\bar{B}_{n,k_n} - \mu_n^{-1} e) \xrightarrow{d} \mathbf{0},$

(2) $\bar{C}_{n,k_n} \xrightarrow{d} \lambda^{-1} e, \qquad \sqrt{k_n}(\bar{C}_{n,k_n} - \lambda_n^{-1} e) \xrightarrow{d} \mathbf{0},$

(3) *the convergence $m_n/\sqrt{k_n} \to y$, where y is a finite number, implies the convergence $C_{n,k_n}^{(2)} \xrightarrow{d} \mathbf{1} y \lambda^{-1}$,*

(4) $$(\sigma_1^{-1} \hat{V}_{n,k_n}, \sigma_2^{-1} \hat{U}_{n,k_n}) \xrightarrow{D} (\mathfrak{W}_1, \mathfrak{W}_2)$$

as $n \to \infty$, where μ, λ, μ_n, λ_n, σ_1, σ_2 are finite positive numbers and $(\mathfrak{W}_1, \mathfrak{W}_2)$ is a two-dimensional Wiener process with the covariance of its coordinates equal to β. Furthermore, denote $\sigma^2 = \sigma_1^2 - 2\sigma_1\sigma_2\beta + \sigma_2^2$ and let $(\mu_n^{-1} - \lambda_n^{-1})\sqrt{k_n} \to q$, $|q| < \infty$. Then

(5) $w_{n,k_n}^{(1)} \xrightarrow{D} Z, \qquad w_{n,k_n}^{(2)} \xrightarrow{P} \mathbf{0},$

(6) $\hat{\hat{W}}_{n,k_n} \xrightarrow{D} Z,$

(7) *for each $t \geqslant 0$,* $\lambda \hat{\hat{W}}_{n,k_n}(t) - \hat{\hat{L}}_{n,k_n}(t) \xrightarrow{P} 0,$

as $n \to \infty$, where $Z = \mathbf{0}$ if $\sigma = 0$, $q < 0$, $Z = \lambda q e$ if $\sigma = 0$, $q > 0$, and $Z = \sqrt{\lambda}\sigma f(\mathfrak{W} + \sigma^{-1}\sqrt{\lambda} q e)$ if $\sigma > 0$.

Proof. Introduce the processes ($t \geqslant 0$):

$$\gamma_n(t) = k_n t - U_{n,N_n(k_n t)-1},$$

$$\theta_{n,i}(t) = \left(\bar{N}_{n,k_n}(t) - \frac{i}{k_n}\right)_+, \qquad \varphi_{n,i}(t) = \left(\bar{N}_{n,k_n}(t) - \frac{i+m_n}{k_n}\right)_+,$$

$$U_{n,i}^{(1)} = \hat{U}_{n,\bar{N}_{n,k_n}-i}, \qquad U_{n,i}^{(2)} = \hat{U}_{n,\bar{N}_{n,k_n}-i-m_n},$$

$$U_{n,i}^{(3)}(t) = k_n^{-1/2}(U_{n,N_n(k_n t)-i} - U_{n,N_n(k_n t)-i-m_n}),$$

$$A_n(t) = k_n^{-1/2} A_{n,N_n(k_n t)},$$

$$C_{n,i}^{(3)}(t) = k_n^{-1/2}(C_{n,N_n(k_n t)-i} - C_{n,N_n(k_n t)-i-m_n}),$$

where $i = 0, 1$, $n \geqslant 1$ and the sequence $\{m_n\}$ is such that $m_n/\sqrt{k_n} \to y$ and y is a finite number. The processes $\theta_{n,i}$ and $\varphi_{n,i}$ are r.e.'s of D' and $U_{n,i}^{(1)}$, $U_{n,i}^{(2)}$, $U_{n,i}^{(3)}$, A_n, $C_{n,i}^{(3)}$ are r.e.'s of D.

Using the obvious relation $\bar{U}_{n,k} = k^{-1/2}\hat{U}_{n,k} + \bar{C}_{n,k}$ and assumptions (2) and (4) we obtain $\bar{U}_{n,k_n} \xrightarrow{P} \lambda^{-1} e$, which together with Proposition 2.6 gives $\bar{N}_{n,k_n} \xrightarrow{P} \lambda e$ as $n \to \infty$. Hence the definitions of $\theta_{n,i}$, $\varphi_{n,i}$ and the continuity of the mapping $x \mapsto x^+$ on the set C give

$$\theta_{n,i} \xrightarrow{P} \lambda e, \quad \varphi_{n,i} \xrightarrow{P} \lambda e \quad \text{as } n \to \infty, \quad \text{for } i = 0, 1. \tag{8}$$

Now applying the definitions introduced above and those of the mappings f and τ we obtain the following relations:

$$U_{n,i}^{(3)} = U_{n,i}^{(1)} - U_{n,i}^{(2)} + C_{n,i}^{(3)}, \tag{9}$$

$$U_{n,i}^{(1)} = Y_{n,i}^{(1)} + \tau(\hat{U}_{n,k_n}, \theta_{n,i}), \qquad U_{n,i}^{(2)} = Y_{n,i}^{(2)} + \tau(\hat{U}_{n,k_n}, \varphi_{n,i}), \tag{10}$$

$$C_{n,i}^{(3)} = Y_{n,i}^{(3)} + \tau(C_{n,k_n}^{(2)}, \theta_{n,i}), \qquad A_n = \tau(\hat{\bar{A}}_{n,k_n}, \theta_{n,0}), \tag{11}$$

$$w_{n,k_n}^{(1)} = \tau\big(f(\hat{S}_{n,k_n} + \hat{\bar{A}}_{n,k_n}), \theta_{n,0}\big), \tag{12}$$

$$w_{n,k_n}^{(2)} = w_{n,k_n}^{(1)} - \tau\big(f(\hat{S}_{n,k_n} + \hat{\bar{A}}_{n,k_n}), \varphi_{n,0}\big), \tag{13}$$

where $Y_{n,i}^{(j)}$, $j = 1, 2, 3$, $i = 0, 1$, $n \geqslant 1$, are processes defined by

$$Y_{n,i}^{(1)}(t) = k_n^{-1/2}(Eu_{n,0} - u_{n,0})\big(1 - \delta_{b_{n,i}}(t)\big),$$
$$Y_{n,i}^{(2)}(t) = k_n^{-1/2}(Eu_{n,0} - u_{n,0})\big(1 - \delta_{c_{n,i}}(t)\big),$$
$$Y_{n,i}^{(3)}(t) = k_n^{-1/2} Eu_{n,0}\big(\delta_{b_{n,i}}(t) - \delta_{c_{n,i}}(t)\big), \qquad t \geqslant 0,$$

and $b_{n,i}$, $c_{n,i}$ are r.v.'s defined by $b_{n,i} = k_n^{-1} i u_{n,0}$, $c_{n,i} = k_n^{-1} U_{n,i+m_n-1}$. Notice that $Y_{n,i}^{(j)}$ are r.e.'s of D and $Y_{n,i}^{(j)} \xrightarrow{P} \mathbf{0}$ as $n \to \infty$, for all i, j.

Applying (8)–(13), assumptions (1)–(4), the continuity of the mapping τ on the set $C \times (C \cap D')$, the continuity of f on D, the continuity of addition on C and Theorem 5.1 from [1] we obtain for $i = 0, 1$, as $n \to \infty$, the convergences

$$U_{n,i}^{(1)} - U_{n,i}^{(2)} \xrightarrow{P} \mathbf{0}, \tag{14}$$

$$C_{n,i}^{(3)} \xrightarrow{P} y\lambda^{-1}\mathbf{1}, \tag{15}$$

$$A_n \xrightarrow{P} \lambda q e, \tag{16}$$

$$U_{n,i}^{(3)} \xrightarrow{P} \lambda^{-1} y\mathbf{1}, \tag{17}$$

and the assertions in (5) in both cases: $\sigma = 0$ and $\sigma > 0$.

Now notice that for any $a > 0$ the following inequalities hold:

$$\sup_{0 \leqslant t \leqslant a} k_n^{-1/2} u_{n,N_n(k_n t)} \leqslant \omega_a(U_{n,0}^{(3)}), \tag{18}$$

$$(19)\qquad k_n^{-1/2}\sup_{0\le t\le a} v_{n,N_n(k_nt)-m_n} \le k_n^{-1/2}\sup_{0\le t\le a} v_{n,N_n(k_nt)}$$
$$\le \omega_a\big(\tau(\hat{V}_{n,k_n},\theta_{n,0})\big)+\omega_a\big(\tau\big(\sqrt{k_n}(\bar{B}_{n,k_n}-\mu_n^{-1}e),\theta_{n,0}\big)\big)$$
$$+\omega_a\big(\tau(\sqrt{k_n}\,\mu_n^{-1}e,\theta_{n,0})\big).$$

But $\omega_a\big(\tau(\sqrt{k_n}\,\mu_n^{-1}e,\theta_{n,0})\big)\le(k_n\mu_n)^{-1}$. Thus using (17), (8), (4), (1) and the continuity of the mapping ω_a on the set C we deduce that the right-hand sides of (18) and (19) tend in probability to zero as $n\to\infty$. Hence in view of the inequality $\gamma_n(t)\le u_{n,N_n(k_nt)}$, $t\ge 0$, we have

$$(20)\qquad k_n^{-1/2}\gamma_n \xrightarrow{P} \mathbf{0},$$

$$(21)\qquad k_n^{-1/2}\sup_{0\le t\le a} v_{n,N_n(k_nt)-m_n} \xrightarrow{P} 0,$$

$$(22)\qquad k_n^{-1/2}\sup_{0\le t\le a} v_{n,N_n(k_nt)} \xrightarrow{P} 0.$$

Now applying the relation between the virtual waiting time and the actual waiting time (see formula (4.1.1)):

$$\hat{\hat{W}}_{n,k_n}(t)=\big(w^{(1)}_{n,k_n}(t)+k_n^{-1/2}\big(v_{n,N_n(k_nt)}-\gamma_n(t)\big)\big)_+,\qquad t\ge 0,$$

assertion (5), the convergences (20)–(22), Theorem 4.4 from [1], the continuity of the mapping $x\mapsto x^+$ on the set C and Theorem 5.1 from [1] we obtain assertion (6) in both cases: $\sigma=0$ and $\sigma>0$.

For the proof of (7) notice that from Lemma 4.1 we have

$$P\{k_n^{-1/2}W_n(k_nt)\in A,\ L_n(k_nt)>y\sqrt{k_n}\}$$
$$=P\{(w^{(1)}_{n,k_n}(t)+\xi_n(t))_+\in A,\ w^{(1)}_{n,k_n}(t)+\eta_n(t)>0,\ \bar{N}_{n,k_n}(t)>[y\sqrt{k_n}]/k_n\},$$

where

$$\xi_n(t)=k_n^{-1/2}\big(v_{n,N_n(k_nt)}-\gamma_n(t)\big),$$
$$\eta_n(t)=k_n^{-1/2}\big(v_{n,N_n(k_nt)-m_n}-\gamma_n(t)\big)-w^{(2)}_{n,k_n}(t)-U^{(3)}_{n,1}(t)$$

for $t\ge 0$ and $m_n=[y\sqrt{k_n}]$. Using (20)–(22), assertion (5) and (17) we obtain

$$\xi_n\xrightarrow{P}\mathbf{0},\quad \eta_n\xrightarrow{P}-y\lambda^{-1}\mathbf{1}\quad \text{as } n\to\infty.$$

Hence (8) and assertion (5) give

$$P\{\hat{\hat{W}}_{n,k_n}(t)\in A,\ \hat{\hat{L}}_{n,k_n}(t)>y\}\to P\{Z(t)\in A,\ Z(t)>y/\lambda\}$$

for any Borel set A such that $P\{Z(t)\in\partial A\}=0$, which completes the proof. ∎

Now we give a functional analogue of the convergence (7).

THEOREM 5.5. *If the conditions of Theorem 5.4 are fulfilled then*

$$(23)\qquad \lambda\hat{\hat{W}}_{n,k_n}-\hat{\hat{L}}_{n,k_n}\xrightarrow{P}\mathbf{0}\quad \text{as } n\to\infty.$$

Proof. Observe that assertion (7) of Theorem 5.4 yields the convergence of the finite-dimensional distributions of $\lambda\hat{\hat{W}}_{n,k_n}-\hat{\hat{L}}_{n,k_n}$ to that of the process $\mathbf{0}$. Thus for the proof of (23) it is enough to show that the sequence $\{\lambda\hat{\hat{W}}_{n,k_n}-\hat{\hat{L}}_{n,k_n}\}$ is tight in (D, d). But assertion (6) of Theorem 5.4 gives the convergence in distribution in (D, d) of $\{\lambda\hat{\hat{W}}_{n,k_n}\}$ and thus $\{\lambda\hat{\hat{W}}_{n,k_n}\}$ is tight in (D, d). Hence it is enough to show that $\hat{\hat{L}}_{n,k_n}$ is tight in (D, d), which follows from Corollary 3.4. ∎

As an immediate consequence of Theorem 5.5 and of the convergence $\pi_{t_n}(x_n)\to\pi_t(x)$ as $t_n\to t$, $x_n\xrightarrow{d}x$, $x\in C$, we obtain

COROLLARY 5.5. *If the sequence $\{k_n\}$, tending to infinity, is such that the conditions of Theorem 5.5 are fulfilled then $k_n^{-1/2}(\lambda W_n(t)-L_n(t))\xrightarrow{P}0$ as n and t tend to infinity in such a manner that $\sqrt{t}(\mu_n^{-1}-\lambda_n^{-1})$ tends to a finite number.* ∎

Let us turn our attention to the manner of the passage to the limit in Theorems 5.4 and 5.2. In Theorem 5.2 we take the limit of $(1-\varrho_n)(\lambda W_n(\infty)-L_n(\infty))$ as $\varrho_n\nearrow 1$, where the pair $(W_n(\infty), L_n(\infty))$ has the same distribution as the limiting distribution of $(W_n(t), L_n(t))$ as $t\to\infty$. The scheme of this passage is

$$(1-\varrho_n)(\lambda W_n(t)-L_n(t))\underset{t\to\infty}{\overset{D}{\to}}(1-\varrho_n)(\lambda W_n(\infty)-L_n(\infty))\xrightarrow{P}0 \quad \text{as } \varrho_n\nearrow 1.$$

If we restrict ourselves to a GI/G/1 queue, the scheme of the passage to the limit in Theorem 5.4 is the following:

$$(1-\varrho_n)(\lambda W_n(t)-L_n(t))\xrightarrow{P}0$$

as $(\varrho_n, t)\to(1, \infty)$ in such a manner that $(1-\varrho_n)\sqrt{t}$ tends to a finite number.

5.4. Examples. In this section we give a few examples of queues for which the assertions of Theorem 5.4 hold. Here we assume that if a queue belonging to a given class has index n then the same index marks all the quantities which define the system and all its characteristics.

5.4.1. *Queues generated by independent r.v.'s v_k, u_k, $k\geqslant 0$.* Consider the class $Q^{(1)}$ of queues Q generated by $\{(v_k, u_k), k\geqslant 0\}$ such that all r.v.'s in the sequence $\{v_k, u_k, k\geqslant 0\}$ are independent. Furthermore, let the following conditions hold:

(i$_1$) $\quad Ev_k=b, \quad \operatorname{Var} v_k=\sigma_1^2 \quad$ for $k\geqslant 1, \quad 0<b, \sigma_1^2<\infty,$

(ii$_1$) $\quad Eu_k=c, \quad \operatorname{Var} u_k=\sigma_2^2 \quad$ for $k\geqslant 1, \quad 0<c, \sigma_2^2<\infty,$

(iii$_1$) there exist finite positive numbers δ and M such that

$$E|v_k-b|^{2+\delta}<M, \quad E|u_k-c|^{2+\delta}<M \quad \text{for } k\geqslant 0.$$

LEMMA 5.1. *Let $\{Q_n, n\geqslant 1\}$ be a sequence of queues belonging to the class*

$Q^{(1)}$. *Furthermore, assume that the sequence* $\{Q_n\}$ *satisfies* (iii$_1$) *uniformly in* n,

$$b_n \to \mu^{-1}, \quad c_n \to \lambda^{-1}, \quad \sigma_{n,1}^2 \to \sigma_1^2, \quad \sigma_{n,2}^2 \to \sigma_2^2,$$

and μ, λ, σ_1, σ_2 *are finite positive numbers.*

Then conditions (5.3.1)–(5.3.4) *of Theorem* 5.4 *are fulfilled with* $\mu_n^{-1} = b_n$, $\lambda_n^{-1} = c_n$, *for each sequence* $\{k_n\}$ *tending to infinity, and the Wiener processes* $\mathfrak{W}_1$ *and* $\mathfrak{W}_2$ *in* (5.3.4) *are independent.*

Proof. The fact that conditions (5.3.1)–(5.3.3) are satisfied for any sequence $\{k_n\}$ tending to infinity follows immediately from the definitions of $B_{n,k}$, $C_{n,k}$, $C_{n,k}^{(2)}$ and the choice of μ_n and λ_n. Since for each $n \geqslant 1$ the sequences $\{v_{n,k}, k \geqslant 0\}$, $\{u_{n,k}, k \geqslant 0\}$ are independent and the r.v.'s in each sequence are independent, by using well-known theorems concerning the convergence to a Wiener process (see e.g. [24], Theorem 2.3) we obtain

$$(\sigma_1^{-1} \hat{V}_{n,k}, \sigma_2^{-1} \hat{U}_{n,k}) \xrightarrow{\mathrm{D}} (\mathfrak{W}_1, \mathfrak{W}_2) \quad \text{as } n, k \to \infty \text{ g.m.},$$

and the Wiener processes $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are independent. This completes the proof. ∎

As an immediate consequence of Lemma 5.1 and Theorem 5.5 we obtain

COROLLARY 5.6. *If the conditions of Lemma* 5.1 *are satisfied then for each sequence* $\{k_n\}$ *tending to infinity in such a manner that* $\sqrt{k_n}(b_n - c_n) \to q$, $|q| < \infty$ *as* $n \to \infty$, *we have*

$$(\sigma\sqrt{\lambda})^{-1} \hat{\hat{W}}_{n,k_n} \xrightarrow{\mathrm{D}} f(\mathfrak{W} + \sigma^{-1}\sqrt{\lambda}\, qe),$$
$$\hat{\hat{W}}_{n,k_n} - \hat{\hat{L}}_{n,k_n} \xrightarrow{\mathrm{P}} \mathbf{0}$$

as $n \to \infty$, *where* $\sigma^2 = \sigma_1^2 + \sigma_2^2$. ∎

5.4.2. *Queues generated by the stationary sequence* $\{(v_k, u_k), k \geqslant 0\}$. For the second example let us introduce the following notion.

A sequence $\{(v_k, u_k), k \geqslant 1\}$ defined on the same probability space $(\Omega, \mathscr{F}, P)$ is called *φ-mixing* with the function $\varphi = \{\varphi_k\}$, $\varphi_k \to 0$, if for the σ-fields $\mathscr{F}_k$ and $\mathscr{F}^k$ generated by the families of r.v.'s $\{v_i, u_i, i \leqslant k\}$ and $\{v_i, u_i, i \geqslant k\}$, respectively, and for any $E_1 \in \mathscr{F}_i$ and $E_2 \in \mathscr{F}^{k+i}$ such that $P(E_1) > 0$ we have

$$|P\{E_2 | E_1\} - P\{E_2\}| < \varphi_k.$$

Consider the class $Q^{(2)}$ of queues Q generated by $\{(v_k, u_k), k \geqslant 0\}$ such that

(i$_2$) the sequence $\{(v_k, u_k), k \geqslant 1\}$ is stationary and φ-mixing with $\varphi = \{\varphi_k\}$ satisfying the condition $\sum_{k=1}^{\infty} \sqrt{\varphi_k} < \infty$,

(ii$_2$) $$Ev_1^2 < \infty, \quad Eu_1^2 < \infty,$$

By [1] (see Lemma 3, p. 172), under conditions (i_2) and (ii_2) the sequences $\{\operatorname{Var}\hat{V}_k\}$ and $\{\operatorname{Var}\hat{U}_k\}$ have finite limits equal to

$$\sigma_1^2 = \operatorname{Var} v_1 + 2\sum_{k=2}^{\infty} \operatorname{cov}(v_1, v_k),$$

$$\sigma_2^2 = \operatorname{Var} u_1 + 2\sum_{k=2}^{\infty} \operatorname{cov}(u_1, u_k),$$

respectively. Assume that

(iii_2) $$\sigma_1^2 > 0, \quad \sigma_2^2 > 0.$$

By [1] (see Lemma 1, p. 170) we have

(1) $$|E(v_i - Ev_1)(u_j - Eu_1)| < 2\sqrt{\varphi_{|i-j|}}\,(\operatorname{Var} v_1 \operatorname{Var} u_1)^{1/2}.$$

Thus the numbers

$$\varkappa_1 = \tfrac{1}{2}\operatorname{cov}(v_1, u_1) + \sum_{k=2}^{\infty} \operatorname{cov}(v_1, u_k),$$

$$\varkappa_2 = \tfrac{1}{2}\operatorname{cov}(v_1, u_1) + \sum_{k=2}^{\infty} \operatorname{cov}(u_1, v_k)$$

are finite. Assume that

(iv_2) the matrix $A = \begin{bmatrix} \sigma_1^2 & \varkappa_1 + \varkappa_2 \\ \varkappa_1 + \varkappa_2 & \sigma_2^2 \end{bmatrix}$ is positive definite.

LEMMA 5.2. *If a sequence* $\{(v_k, u_k), k \geqslant 1\}$ *satisfies* (i_2)–(iv_2) *then*

$$(\sigma_1^{-1}\hat{V}_k, \sigma_2^{-1}\hat{U}_k) \xrightarrow{D} (\mathfrak{W}_1, \mathfrak{W}_2),$$

where $(\mathfrak{W}_1, \mathfrak{W}_2)$ *is a two-dimensional Wiener process with the covariance of its coordinates equal to* $\beta = (\varkappa_1 + \varkappa_2)/(\sigma_1\sigma_2)$.

Proof. By Theorem 20.1 from [1] we have $\sigma_1^{-1}\hat{V}_k \xrightarrow{D} \mathfrak{W}$ and $\sigma_2^{-1}\hat{U}_k \xrightarrow{D} \mathfrak{W}$, which implies the tightness of $\{\sigma_1^{-1}\hat{V}_k\}$ and $\{\sigma_2^{-1}\hat{U}_k\}$ in (D, d). This in turn gives the tightness of $\{(\sigma_1^{-1}\hat{V}_k, \sigma_2^{-1}\hat{U}_k)\}$ in (D^2, d^2).

Now in the same way as in [1], p. 174, we find that $\{(\sigma_1^{-1}\hat{V}_k, \sigma_2^{-1}\hat{U}_k)\}$ has asymptotically independent increments in the sense that for $0 \leqslant s_1 \leqslant t_1 < s_2 \leqslant t_2 < \ldots < s_r \leqslant t_r$ and for all Borel sets $H_1, H_2, \ldots, H_r$ in $\boldsymbol{R}^2$ the difference

$$P\{(\sigma_1^{-1}(\hat{V}_k(t_i) - \hat{V}_k(s_i)), \sigma_2^{-1}(\hat{U}_k(t_i) - \hat{U}_k(s_i))) \in H_i,\ i = 1, 2, \ldots, r\}$$
$$- \prod_{i=1}^{r} P\{(\sigma_1^{-1}(\hat{V}_k(t_i) - \hat{V}_k(s_i)), \sigma_2^{-1}(\hat{U}_k(t_i) - \hat{U}_k(s_i))) \in H_i\}$$

converges to zero as $k \to \infty$. Thus it is enough to show that for any $t \geqslant 0$ and $t_1, t_2 \in \boldsymbol{R}$

$$t_1 \sigma_1^{-1} \hat{V}_k(t) + t_2 \sigma_2^{-1} \hat{U}_k(t) \xrightarrow{D} t_1 \mathfrak{W}_1(t) + t_2 \mathfrak{W}_2(t) \quad \text{as } k \to \infty.$$

But for $t \geqslant 0$ we have

$$t_1 \sigma_1^{-1} \hat{V}_k(t) + t_2 \sigma_2^{-1} \hat{U}_k(t) = k^{-1/2} \sum_{i=1}^{[kt]} \tilde{X}_i,$$

where the sequence $\{\tilde{X}_k = \sigma_1^{-1} t_1 (v_k - Ev_1) + \sigma_2^{-1} t_2 (u_k - Eu_1), \quad k \geqslant 1\}$ is stationary and φ-mixing. Hence in view of Theorem 20.1 from [1] and the equality $(t_1^2 + 2t_2 t_1 (\varkappa_1 + \varkappa_2)/(\sigma_1 \sigma_2) + t_2^2)^{1/2} \mathfrak{W} \overset{D}{=} t_1 \mathfrak{W}_1 + t_2 \mathfrak{W}_2$ it is enough to show that

$$k^{-1} \operatorname{Var} \sum_{i=1}^{k} \tilde{X}_i \to t_1^2 + 2t_1 t_2 (\varkappa_1 + \varkappa_2)/(\sigma_1 \sigma_2) + t_2^2 > 0. \tag{2}$$

Writing $\xi_k = v_k - Ev_1$, $\eta_k = u_k - Eu_1$, we have

$$\operatorname{Var} \sum_{i=1}^{k} \tilde{X}_i = \sigma_1^{-2} t_1^2 E\Big(\sum_{i=1}^{k} \xi_i\Big)^2 + \sigma_2^{-2} t_2^2 E\Big(\sum_{i=1}^{k} \eta_i\Big)^2$$
$$+ 2t_1 t_2 (\sigma_1 \sigma_2)^{-1} E\Big(\sum_{i=1}^{k} \xi_i\Big)\Big(\sum_{i=1}^{k} \eta_i\Big).$$

But

$$E \sum_{i=1}^{k} \xi_i \sum_{i=1}^{k} \eta_i = \sum_{i=2}^{k} \sum_{j=1}^{i-1} E\xi_j \eta_{i-j}$$
$$= [\tfrac{1}{2}(k-1)] E\xi_i \eta_i + \sum_{i=2}^{k-2} [\tfrac{1}{2}(k-i)](E\xi_1 \eta_i + E\xi_i \eta_1).$$

Now using inequality (1), the finiteness of $\sum_{k=1}^{\infty} \sqrt{\varphi_k}$ and Kronecker's Lemma we get

$$k^{-1} E \sum_{i=1}^{k} \xi_i \sum_{i=1}^{k} \eta_i \to \varkappa_1 + \varkappa_2 \quad \text{as } k \to \infty.$$

Hence Lemma 3 in [1], p. 172 gives the convergence in (2). Using the fact that the matrix A is positive definite we obtain the positivity of $t_1^2 + 2t_1 t_2 (\varkappa_1 + \varkappa_2)/(\sigma_1 \sigma_2) + t_2^2$ for all $t_1 > 0$ and $t_2 > 0$, which completes the proof. ∎

COROLLARY 5.7 *Let a queue Q belong to the class $Q^{(2)}$ and $Ev_1 - Eu_1 < 0$. Consider a sequence $\{Q_n\}$ of queues such that $v_{n,k} = v_k + b_n$ for $k \geqslant 1$, where $b_n \geqslant 0$ and $b_n \to -E(v_1 - u_1)$, $u_{n,k} = u_k$, and $u_{n,0}$ are r.v.'s such that $\sup_n Eu_{n,0} < \infty$.*

Then for each sequence $\{k_n\}$ *tending to infinity in such a manner that* $\{\sqrt{k_n}(Ev_1 - Eu_1 + b_n)\}$ *has a finite limit we have*

$$\lambda \hat{\hat{W}}_{n,k_n} - \hat{\hat{L}}_{n,k_n} \xrightarrow{P} 0 \quad \text{as } n \to \infty$$

where $\lambda^{-1} = Eu_1$.

Proof. Conditions (5.3.1)–(5.3.3) of Theorem 5.4, with $\mu_n^{-1} = Ev_1 + b_n$, $\lambda_n^{-1} = Eu_1$ and any sequence $\{k_n\}$ tending to infinity, are obviously satisfied. For the proof of (5.3.4) observe that $\hat{V}_{n,k_n} = \hat{V}_{k_n}$, $\hat{U}_{n,k_n} = \hat{U}_{k_n}$. Hence Lemma 5.2 shows that (5.3.4) holds for each sequence $\{k_n\}$ tending to infinity. This, in view of Theorem 5.5, completes the proof. ∎

5.4.3. *Regenerative queues.* For each $i \geqslant 1$, let $\{v_k^{(i)}, k \geqslant 1\}$ and $\{u_k^{(i)}, k \geqslant 1\}$ be independent sequences of independent nonnegative r.v.'s and let $T_i = \{(v_k^{(i)}, u_k^{(i)}), k \geqslant 1\}$, $i \geqslant 1$, r.e.'s of $(\boldsymbol{R}^2)^\infty$, be i.i.d. Furthermore, for all $k, i \geqslant 1$, let

(i$_3$) $$Ev_k^{(i)} = b, \quad E(v_k^{(i)})^2 = \varkappa_1, \quad 0 < b, \varkappa_1 < \infty,$$

(ii$_3$) $$Eu_k^{(i)} = c, \quad E(u_k^{(i)})^2 = \varkappa_2, \quad 0 < c, \varkappa_2 < \infty,$$

(iii$_3$) there exist finite positive numbers δ and M such that

$$E|v_k^{(i)} - b|^{2+\delta} < M, \quad E|u_k^{(i)} - c|^{2+\delta} < M.$$

Define sequences of r.v.'s by

$$X_k^{(i)} = v_k^{(i)} - u_k^{(i)}, \quad i, k \geqslant 1,$$

$$S_k^{(i)} = \sum_{j=1}^{k} X_j^{(i)}, \quad i \geqslant 1,\ k \geqslant 0,$$

$$r_i = \min\{k > 0\colon S_k^{(i)} \leqslant 0\}, \quad i \geqslant 1,$$

$$R_i = \sum_{j=1}^{i} r_j, \quad i \geqslant 0,$$

$$\eta_k = \max\{i\colon R_i < k\}, \quad k \geqslant 1,$$

$$\gamma_k = k - R_{\eta_k}, \quad v_k = v_{\gamma_k}^{(\eta_k+1)}, \quad u_k = u_{\gamma_k}^{(\eta_k+1)},$$

$$X_k = v_k - u_k, \quad k \geqslant 1, \quad v_0 = 0.$$

Consider the class $Q^{(3)}$ of queues Q generated by $\{(v_k, u_k), k \geqslant 0\}$ formed as above where u_0 is a nonnegative r.v. independent of $\{(v_k, u_k), k \geqslant 1\}$ and $Eu_0 < \infty$. Queues of this type were considered in [2], [23], and [24]. Notice that $v_k^{(i)}$ is the service time of the kth unit in the ith busy period and $u_k^{(i)}$ is the interarrival time between the arrival times of the kth and $(k+1)$th units in the ith busy period. The queues described in this way were called in [24] regenerative queueing systems. (Incidentally, [24] contains a misprint in the definitions of η_k and γ_k).

LEMMA 5.3. *Let $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and let $\mathscr{F}_k$ be the σ-field generated by the r.v.'s $v_1, u_1, v_2, u_2, \ldots, v_k, u_k$, $k \geqslant 1$, defined above. Then for $k \geqslant 1$*

$$E(v_k \mid \mathscr{F}_{k-1}) = b, \quad E(u_k \mid \mathscr{F}_{k-1}) = c, \tag{3}$$

$$E((v_k)^2 \mid \mathscr{F}_{k-1}) = \varkappa_1, \quad E((u_k)^2 \mid \mathscr{F}_{k-1}) = \varkappa_2, \tag{4}$$

$$E(v_k u_k \mid \mathscr{F}_{k-1}) = bc, \tag{5}$$

$$E|v_k - b|^{2+\delta} < M, \quad E|u_k - c|^{2+\delta} < M. \tag{6}$$

Proof. To establish (3)–(5) it suffices to show that for any set A belonging to $\mathscr{F}_{k-1}$, $k \geqslant 1$, we have

$$Ev_k \chi_A = bP\{A\}, \quad Eu_k \chi_A = cP\{A\}, \tag{7}$$

$$E(v_k)^2 \chi_A = \varkappa_1 P\{A\}, \quad E(u_k)^2 \chi_A = \varkappa_2 P\{A\}, \tag{8}$$

$$Ev_k u_k \chi_A = bcP\{A\}. \tag{9}$$

If $k = 1$, the equalities are obvious. For $k > 1$, observe that it suffices to establish (7)–(9) for any set A of the form $A = \{v_i \in H_i, u_i \in H_i', 1 \leqslant i \leqslant k-1\}$, where H_i, H_i', $i \geqslant 1$, are Borel sets in $\boldsymbol{R}$.

Let i_l and j_l, $l \geqslant 1$, be nonnegative integers which do not exceed l. Assume that $X_j^{(0)} = 0$ for $j \geqslant 1$. Let $\{X \sim 0\}$ denote any subset of Ω on which the r.v. X is either everywhere positive or everywhere nonpositive. Let us consider a maximal set $B \in \mathscr{F}_{k-1}$ with the property $X_k = X_{j_k}^{(i_k)}$ on B, where i_k and j_k are not random.

Each set B having this property can be expressed as a finite intersection of sets of the form

$$\{X_{j_1}^{(i_1)} + X_{j_2}^{(i_2)} + \ldots + X_{j_{k-1}}^{(i_{k-1})} \sim 0\},$$

which in turn leads to the fact that for each $k \geqslant 1$ the space Ω may be expressed as a finite union of disjoint sets B. Since all r.v.'s $v_j^{(i)}$, $u_j^{(i)}$, $i, j \geqslant 1$, are independent we obtain

$$Ev_{j_k}^{(i_k)} \chi_{A \cap B} = bP\{A \cap B\}, \qquad Eu_{j_k}^{(i_k)} \chi_{A \cap B} = cP\{A \cap B\},$$
$$E(v_{j_k}^{(i_k)})^2 \chi_{A \cap B} = \varkappa_1 P\{A \cap B\}, \quad E(u_{j_k}^{(i_k)})^2 \chi_{A \cap B} = \varkappa_2 P\{A \cap B\},$$
$$Ev_{j_k}^{(i_k)} u_{j_k}^{(i_k)} \chi_{A \cap B} = bcP\{A \cap B\}.$$

The above relations and the fact that Ω may be expressed as a finite union of disjoint sets B lead to (7)–(9). This in turn implies (3), (4) and (5). Relation (6) may be proved in a similar way. ∎

LEMMA 5.4. *Let $\{Q_n, n \geqslant 1\}$ be a sequence of queues belonging to $Q^{(3)}$ with $\sup_n Eu_{n,0} < \infty$. Furthermore, assume that the sequence $\{Q_n\}$ satisfies* (iii$_3$) *uniformly in n and*

$$b_n \to \mu^{-1}, \quad c_n \to \lambda^{-1}, \quad \sigma_{n,1} \to \sigma_1, \quad \sigma_{n,2} \to \sigma_2,$$

where μ, λ, σ_1, σ_2 *are finite positive numbers and* $\sigma_{n,1}^2 = \varkappa_{n,1} - b_n^2$, $\sigma_{n,2}^2 = \varkappa_{n,2} - c_n^2$.

Then conditions (5.3.1)–(5.3.4) *of Theorem* 5.4 *are fulfilled for each sequence* $\{k_n\}$ *tending to infinity with* $\mu_n^{-1} = b_n$, $\lambda_n^{-1} = c_n$, *and the Wiener processes* $\mathfrak{W}_1$ *and* $\mathfrak{W}_2$ *in* (5.3.4) *are independent.*

Proof. Conditions (5.3.1)–(5.3.4) follow from the definitions of $B_{n,k}$, $C_{n,k}$, $C_{n,k}^{(2)}$, Lemma 5.3, and from the choice of μ_n and λ_n. Now we show that (5.3.4) holds and the processes $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are independent. For this purpose notice that for each $n \geqslant 1$,

$$\{(\sum_{i=1}^{k} (v_{n,i} - b_n), \mathscr{F}_{n,k}),\ k \geqslant 0\} \quad \text{and} \quad \{(\sum_{i=1}^{k} (u_{n,i} - c_n), \mathscr{F}_{n,k}),\ k \geqslant 0\}$$

are martingales, where $\mathscr{F}_{n,0} = \{\emptyset, \Omega\}$ and $\mathscr{F}_{n,k}$ is the σ-field generated by r.v.'s from the set $\{v_{n,i}, u_{n,i}, 1 \leqslant i \leqslant k\}$. Thus by Remark 2.4 from [24] and by Theorem 2.3 from [24] we obtain $\sigma_1^{-1} \hat{V}_{n,k_n} \xrightarrow{\mathrm{D}} \mathfrak{W}_1$ and $\sigma_2^{-1} \hat{U}_{n,k_n} \xrightarrow{\mathrm{D}} \mathfrak{W}_2$ as $n \to \infty$, for any sequence $\{k_n\}$ tending to infinity. Thus for the proof of (5.3.4) it is enough to show that for any $0 \leqslant t_1 \leqslant t_2$, $0 \leqslant s_1 \leqslant s_2$, the r.v.'s $\hat{V}_{n,k_n}(t_2) - \hat{V}_{n,k_n}(t_1)$ and $\hat{U}_{n,k_n}(s_2) - \hat{U}_{n,k_n}(s_1)$ are asymptotically independent. But notice that for any z_1 and z_2 in $\boldsymbol{R}$

$$\{\sum_{i=1}^{k} (z_1 (v_{n,i} - b_n) + z_2 (u_{n,i} - c_n)), \mathscr{F}_{n,k},\ k \geqslant 0\}$$

is a martingale. Thus by Remark 2.4 from [24] and Theorem 2.3 from [24] we obtain

$$z_1 \sigma_1^{-1} (\hat{V}_{n,k_n}(t_2) - \hat{V}_{n,k_n}(t_1)) + z_2 \sigma_2^{-1} (\hat{U}_{n,k_n}(s_2) - \hat{V}_{n,k_n}(s_1)) \xrightarrow{\mathrm{D}} (z_1^2 (t_2 - t_1) + z_2^2 (s_2 - s_1))^{1/2} \mathfrak{W}(1).$$

Together with the convergences $\sigma_1^{-1} \hat{V}_{n,k_n} \xrightarrow{\mathrm{D}} \mathfrak{W}$ and $\sigma_2^{-1} \hat{U}_{n,k_n} \xrightarrow{\mathrm{D}} \mathfrak{W}$ as $n \to \infty$, this implies the asymptotic independence of $\{\sigma_1^{-1} \hat{V}_{n,k_n}\}$ and $\{\sigma_2^{-1} \hat{U}_{n,k_n}\}$, which in turn implies the convergence

$$(\sigma_1^{-1} \hat{V}_{n,k_n}, \sigma_2^{-1} \hat{U}_{n,k_n}) \xrightarrow{\mathrm{D}} (\mathfrak{W}_1, \mathfrak{W}_2),$$

where the Wiener processes $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are independent. This completes the proof. ∎

As an immediate consequence of Lemma 5.4 and Theorems 5.4 and 5.5 we have the following

COROLLARY 5.8. *If the conditions of Lemma* 5.4 *are satisfied then for each sequence* $\{k_n\}$ *tending to infinity in such a manner that* $\sqrt{k_n}(b_n - c_n) \to q$ *and* $|q| < \infty$ *as* $n \to \infty$, *we have*

$$(\sigma \sqrt{\lambda})^{-1} \hat{\hat{W}}_{n,k_n} \xrightarrow{\mathrm{D}} f(\mathfrak{W} + \sigma^{-1} \sqrt{\lambda}\, qe), \qquad \hat{\hat{W}}_{n,k_n} - \hat{\hat{L}}_{n,k_n} \xrightarrow{\mathrm{P}} \mathbf{0}$$

as $n \to \infty$, *where* $\sigma = (\sigma_1^2 + \sigma_2^2)^{1/2}$.

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