POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE
(ROZPRAWY MATEMATYCZNE)

CXXII

ALEKSANDER STRASBURGER

Inducing spherical representations of semi-simple Lie groups

WARSZAWA 1975
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

Contents

Poświęcam mojej Matce

Introduction

The early achievements in the field of harmonic analysis on symmetric spaces were connected with the study of various algebras of functions canonically attached to the space, the simplest and perhaps the most important example being the algebra of continuous, compactly supported, K-invariant functions on the symmetric space G/K. Among the most spectacular results obtained that way we mention various equivalent characterizations of spherical functions, the establishment of the abstract Plancherel formula for G/K, the determination of the relation between spherical functions and spherical representations (cf. papers of Gelfand [3], Godement [6] and early papers of Harish-Chandra).

It had turned out pretty quickly, however, that those broad functional-theoretic methods are not capable to deal with more subtle problems, like e.g., the determination of the explicit form of the measure appearing in the Plancherel formula, determination of the dual space of G, etc., and that a lot more of the structure theory of the group G has to be brought in, in order to deal successfully with those problems.

Due mostly to the heroic labors of Harish-Chandra, whose work dominates the field (list of his publications, as given in Warner [28] comprises more than thirty papers, written over the period 1949—1973!), what has once been just one branch of harmonic analysis has now become a vast subject of its own interest, specific methods, aims, and last but not least, a source of inspiration for other parts of mathematics.

One of the first problems in the harmonic analysis on symmetric spaces which gained its full solution was the problem of determination of spherical functions, i.e., K-invariant eigenfunctions of G-invariant differential operators on a symmetric space G/K. It was done by Harish-Chandra who established now famous formula which gives parametrization of the set of spherical functions in terms of the structure of the group G. However, the corresponding problem for spherical representations has not yet been satisfactorily solved. Although we know enough representations for the Plancherel formula on G/K we still do not know all spherical representations of G, even unitary, or putting it another way, we do not know what λ, in the parametrization of spherical functions given by the formula (II.2) of Harish-Chandra, corresponds to positive definite spherical functions. We remark in passing that even for classical groups the so-called Gelfand–Naimark list of representations is incomplete, as the work of Kunze & Stein and Stein himself show (see a review paper of E. M. Stein, *Analytic continuation of group representations* [26]). In cases of groups of split rank 1 and also complex groups, this class of representations has been determined by Kostant [18], Parthasarathy, Ranga Rao & Varadarajan [22] who, however, used purely algebraic methods which rely on the determination of representations of the universal enveloping algebra.

In many of the earlier publications the induction procedure was used as a device to produce a sufficient supply of spherical (or even arbitrary) representations of G. In particular, Harish-Chandra subquotient theorem shows that inducing construction together with the usual formation of sub- and quotient representations is sufficient to obtain all, up to Naimark equivalence, representations of semi-simple Lie groups. This work, and especially Chapter V, shows that in the case of spherical (or as it is sometimes said, class one) representations situation is more favorable. Namely, we were able to prove, with mild restrictions on corresponding spherical functions (which are due to the weakness of the method rather than to some intrinsic complications of the theory), that spherical representations are Naimark equivalent to an irreducible subrepresentation of the representation induced (nonunitarily) by the character of the minimal parabolic subgroup MAN, with character determined uniquely by spherical function via the Harish-Chandra formula. This we consider to be a strengthening of the above-mentioned subquotient theorem of Harish-Chandra. For unitary representations is even better than that — we show that every such representation is unitary equivalent to the unitary extension of the representation induced (in the sense of Bruhat) by the mentioned character of MAN. We also obtain some information on the scalar product used to define the Hilbert space structure — it is shown to be given by conical distribution. This on the one hand shows yet another example of the role played by conical distributions in the representation theory and on the other, makes a link with the work of Knapp & Stein on the complementary series.

The organization of the paper is the following. In Chapter I we recall basic notions from various fields (representation theory, measure & distribution theory, structure theory of semi-simple Lie groups, etc.), we shall need later on. Chapter II is devoted to the brief formulation of the main properties of spherical functions and representations. It contains only one seemingly new result, namely Proposition II.5.

In Chapter III we introduce a certain series of reperesentations induced by characters of the subgroup MAN. What concerns that series of representations we essentially followed Helgason's paper [15], whereas in the treatment of Fourier and dual Radon transforms we prefered to give an alternate treatment emphasizing more the representation-theoretic side of matters. This resulted in establishing that the restriction of the dual Radon transform to an eigenspace of $\boldsymbol{D}(\Xi)$ is essentially the dual of the Fourier transformation, what enabled us to avoid an intermediate use of the Radon transform. Accordingly we preferred to discuss some properties of those eigenspaces (see the the notion of a simple $\lambda \in \mathfrak{a}_C^*$) in terms of the Fourier transform. This part owes much to discussions with A. Wawrzyńczyk.

In Chapter IV we recall some points of Helgason's theory of conical distributions and develop the theory of conical representations. It should be made clear, however, that we have found the notion of conical representation, as given by Helgason, unsatisfactory and so we have stuck to the modification thereof proposed in the joint work of A. Wawrzyńczyk and the author [27]. In comparison to that paper their study is pushed a little further on and some new results are added, especially an example showing that a conical representation need not be spherical contrary to the finite-dimensional case.

In that chapter we consider also the so-called intertwining operators for principal series, their study, however, limited to the case when integrals defining them are absolutely convergent. This will suffice for the study of spherical representations in Chapter V, where we first prove our version of subquotient theorem and then the unitary equivalence theorem, showing how spherical representations are constructed using conical distributions.

*

* *

It is my pleasant duty to acknowledge here the influence of Professor Krzysztof Maurin, from whom I learned all I know from mathematics, and whose interest and unconventional words of encouragement were always of great help for me.

The other kind of debt I owe to Doctor Antoni Wawrzyńczyk, whose share in this work accumulated over years of collaboration can hardly be overappreciated, and who through numerous discussions succeeded in casting more light on this difficult subject.

I am also much indebted to Professors Adam Piskorek and Ryszard Rączka for their cooperation and help during preparation of this work.

I would like to express my obligation to Professor Sigurdur Helgason, whose lectures at CIME (Montecatini) in 1970 introduced me to this fascinating subject and whose work, *Duality for symmetric spaces with applications to group representations*, served as a starting point for this research.

Chapter I

Preliminaries and notations

1. Manifolds — generalities. Let M be a differentiable manifold — we shall use Schwartz notation for spaces of functions attached to M and corresponding distribution spaces, i.e., $\mathscr{C}_0(M)$ for the space of compactly supported continuous functions, $\mathscr{E}(M)$ for the space of infinitely differentiable functions, $\mathscr{D}(M)$ for the space of compactly supported and infinitely differentiable functions — all spaces taken with their customary topologies. $\mathscr{D}'(M)$, $\mathscr{E}'(M)$ will stand for spaces of distributions and compactly supported distributions respectively.

If ψ is a distribution on M and τ a diffeomorphism of M onto itself, then we shall denote by ψ^τ the distribution defined by

$$\langle f, \psi^\tau \rangle = \langle f \circ \tau, \psi \rangle,$$

and for a differential operator D on M we denote by D^τ the operator defined by

$$D^\tau(f) = \big(D(f \circ \tau)\big) \circ \tau^{-1}.$$

We say that ψ resp. D is *invariant under* τ if ψ^τ, D^τ resp. is equal to ψ, D resp.

2. Representations. Let V be a locally convex, Hausdorff, complete, topological vector space and G a locally compact group. A homomorphism $G \ni g \to \pi(g)$ of G into $\mathrm{Aut}(V)$ is called *representation* (*continuous representation*) of G on V if the map $G \times V \ni (g, v) \to \pi(g)v \in V$ is jointly continuous. In the most important cases when V is barrelled (for example when V is Fréchet or Montel space), the joint continuity is implied by an apparently weaker condition of the separate continuity.

The requirement of joint continuity allows us to extend the given representation to a representation (i.e., an homomorphism) of the convolution algebra of measures with compact supports on G by means of the formula

$$M_c(G) \ni \mu \to \pi(\mu) : \pi(\mu)v = \int_G \pi(g) v \, d\mu.$$

Although the homomorphism $G \ni g \to \pi(g^{-1})^t$ does not, generally, fulfil the continuity requirement with respect to the strong dual topology on V', this is the case when V is semi-reflexive. Then $G \ni g \to \pi(g^{-1})^t$ is a continuous representation of G on V' equipped with its strong dual topology, which shall be denoted by π' and said to be the *contragredient representation.*

Two representations π, τ on V, W resp. are said to be *equivalent* if there exists a linear bicontinuous bijection $T\colon V \to W$ such that

$$T\pi(g) = \tau(g)T \quad \text{for all } g.$$

An operator satisfying the above equality (not necessarily continuous or bijective) is said to be an *intertwining operator.*

We shall also use several weaker forms of equivalences — so-called weak and Naimark equivalences. Two representations (π, V), (τ, W) are said to be *weakly equivalent* if their spaces contain dense invariant subspaces, say V_0 and W_0 resp., such that there exists a linear, bijective but not necessarily continuous operator $T\colon V_0 \to W_0$, intertwining for restrictions of π and τ to those subspaces.

The *Naimark equivalence* is obtained from the preceding definition by requiring additionally T to be closed.

A *unitary representation* is a representation on a Hilbert space by unitary operators.

A representation is called (*topologically*) *irreducible* if the representation space does not contain any proper invariant (and closed resp.) subspace.

Let F_0 be a prehilbert space with $h(\cdot\,,\cdot)$ as its prehilbert product, and suppose we are given a representation $G \ni g \to \pi_0(g)$ by operators preserving the form h. Let F be the Hilbert space completion of F_0; then $\pi_0(g)$ can be extended to unitary operators on F in the unique way. The representation obtained in such a way from π_0 is called the *unitary extension of* π_0.

Let a Lie group G act to the left on a manifold M; this action, carried over to various function or distribution spaces attached to M, e.g., $\mathscr{E}(M)$, $\mathscr{D}(M)$, etc., defines continuous representations on them which will be called, indifferently, *left regular representations of* G and will be denoted by $g \to L_g$, with L_g defined by

$$L_g f(m) = f(g^{-1}m), \quad f \in \mathscr{E}(M),$$

whereas for a distribution ψ,

$$\langle f, L_g\psi\rangle = \langle L_{g^{-1}}f, \psi\rangle.$$

We close this section with the discussion of differentiability properties of representations. Let (π, V) be a representation of a Lie group G. A vector $v \in V$ is said to be a *differentiable vector of* π if $G \ni g \to \pi(g)v \in V$ is a differentiable function (of C^∞ class). The set of differentiable vectors for π is π-stable and dense subspace of V denoted by V_∞. A restriction of π to V_∞ is denoted by π_∞ and is called a *differentiable representation induced by* π. The subspace spanned by vectors of the form $\pi(f)v$ with $f \in \mathscr{D}(G)$ is called the *Gårding space of* π and is readily shown to be dense, invariant, and contained in V_∞.

It is possible to define a representation of the convolution algebra $\mathscr{E}'(G)$ by closed operators on V, so that their domains will contain V_∞. In particular, the universal enveloping algebra $\mathfrak{A}(\mathfrak{g})$, viewed as the algebra of distributions supported at the origin $e \in G$, is represented by closed operators in such a way that, for $X \in \mathfrak{g}$, $\pi(X)$ is defined as

$$\pi(X)v = \lim_{t \to 0} t^{-1}\big(\pi(\exp(tX))v - v\big),$$

with domain $D_{\pi(X)}$ consisting of vectors for which the limit actually exists. For vectors of the form $\pi(f)v$ with $f \in \mathscr{D}(G)$,

$$\pi(X)\,\pi(f)v = \pi(Xf)v, \qquad X \in \mathfrak{A}(\mathfrak{g}).$$

Let now V be a Banach space. We shall equip V_∞ with such a topology that all the maps $\pi(X)$, for $X \in \mathfrak{A}(\mathfrak{g})$, will be continuous (the construction we owe to Goodman [7]). The topology is defined by the family of semi-norms ϱ_n: let $X_1, \ldots, X_d$ be a basis of $\mathfrak{g}$; then

$$\varrho_n(v) = \sum_{1 \leqslant i_k \leqslant d} \|\pi(X_{i_1}X_{i_2}\ldots X_{i_n})\,v\,\|, \qquad n \in N.$$

It can be shown that V_∞ is complete in that topology, hence a Fréchet space.

3. Induced representations. Let G be a locally compact group and Γ its closed subgroup. Given a representation $\Gamma \ni \gamma \to L(\gamma)$ of Γ on F we shall call the representation of G by left translations in the space $\mathscr{D}^L$, consisting of infinitely differentiable functions on G verifying

(a) $p(\operatorname{supp} f)$ is compact: $p: G \to G/\Gamma$ natural projection,

(b) $f(g\gamma) = \varrho^{1/2}(\gamma)\,L(\gamma^{-1})\,f(g)$ for all $\gamma \in \Gamma$, $g \in G$,

a *differentiable representation induced by* L.

Here ϱ is a C^∞ function satisfying

$$\varrho(g\gamma) = \frac{\Delta_\Gamma(\gamma)}{\Delta_G(\gamma)}\,\varrho(g), \qquad g \in G,\ \gamma \in \Gamma$$

and $\mathscr{D}^L$ is equipped with the topology of a strict inductive limit of $\mathscr{D}^L_\omega$, where $\omega \subset G$ is compact and $\mathscr{D}^L_\omega$ is the subspace of $\mathscr{D}^L$ consisting of functions with supports contained in $\omega\Gamma$ and with the relative topology inherited from $\mathscr{E}(G, F)$.

Bruhat [2] has shown that the map β: $\mathscr{D}(G, F) \to \mathscr{D}^L$, given by

$$(\beta f)(g) = \int_{\Gamma} \varrho^{-1/2}(\gamma)L(\gamma)f(g\gamma)\, d\gamma,$$

is a continuous open surjection, hence $\mathscr{D}^L$ is isomorphic to the quotient space of $\mathscr{D}(G, F)$.

In the case of a unitary inducing representation L there is a natural prehilbert structure on $\mathscr{D}^L$, defined via the form

$$\mathscr{D}^L \times \mathscr{D}^L \ni (f, h) \to \int_{G/\Gamma} \varrho^{-1}(x)\big(f(x)|\, h(x)\big)\, d\mu(x\Gamma),$$

where μ is a quasi-invariant measure on G/Γ determined by ϱ. The unitary extension of this representation is given by left translations, acting in the space $\mathscr{H}^L$ of Haar measurable functions on G, verifying (b) above, and with the integral $\int_{G/\Gamma} \varrho^{-1}(x)\big(f(x)\mid f(x)\big)d\mu(x\Gamma)$, which gives the norm, finite. This representation is said to be *unitarily induced by L.*

However, frequently enough another situation occurs — that of non-unitary inducing representation but with a postive definite, hermitian form on $\mathscr{D}^L$, left invariant by the representation. In this case the unitary extension of induced representation is called *unitary induced representation.* The Bruhat-Schwartz kernel theorem asserts that such a form is always defined by a vector distribution on G.

4. Elements of structure theory of semi-simple Lie groups. In what follows G will denote a semi-simple connected Lie group with the finite center and $\mathfrak{g}$ its Lie algebra. We shall deal exclusively with the case of non-compact $\mathfrak{g}$. Denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$ and by θ the corresponding Cartan involution, i.e., an involutive automorphism of $\mathfrak{g}$ such that $\mathfrak{k}$ and $\mathfrak{p}$ are its 1 and -1 eigenspaces respectively. If $B(\cdot,\cdot)$ is the Killing form of $\mathfrak{g}$, then the form $(X, Y) \to -B(X, \theta Y)$ is positive definite and so determines a euclidean structure on $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$; then $\operatorname{ad}(H)$, $H \in \mathfrak{a}$, are self-adjoint with respect to the above scalar product and so they are simultaneously diagonalized. Then $\mathfrak{g} = \bigoplus_{\alpha\in\Sigma\cup\{0\}} \mathfrak{g}^\alpha$ where $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : \operatorname{ad}(H)X = \alpha(H)X\}$ and Σ is the set of non-vanishing identically simultaneous eigenvalues α called (*restricted*) *roots* of $\mathfrak{g}$ with respect to $\mathfrak{a}$. In general, we have $\dim \mathfrak{g}^\alpha = m^\alpha > 1$.

Hyperplanes $(\alpha, 0) = \{H \in \mathfrak{a}: \alpha(H) = 0\}$ partition $\mathfrak{a}$ into open connected and convex parts called *Weyl chambers*. If we denote by $\lambda \to H_\lambda$

the isomorphism of $\mathfrak{a}'$, the dual of $\mathfrak{a}$, with $\mathfrak{a}$ itself given by the euclidean structure on $\mathfrak{a}$ by the formula $\lambda(H) = B(H_\lambda, H)$ for all $H \in \mathfrak{a}$, then there is a dual partitioning of $\mathfrak{a}'$ into Weyl chambers. Let $\mathfrak{a}^+ \subset \mathfrak{a}$ be any Weyl chamber — call a root *positive* if it is positive on $\mathfrak{a}^+$. This introduces an order into the set of roots and if Σ_+ is the set of positive roots, then $\Sigma = \Sigma_+ \cup (-\Sigma_+)$. We put $\varrho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} m^\alpha \alpha$.

Let $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma_+} \mathfrak{g}^\alpha$; then $\mathfrak{n}$ is nilpotent subalgebra of $\mathfrak{g}$ and the Iwasawa theorem states a direct sum decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$$

and a diffeomorphism

$$K \times A \times N \ni (k, a, n) \to kan \in G,$$

with K, A, N, — analytic subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, respectively. Also K is a maximal compact subgroup of G. For $g \in G$ we shall put $g = k(g) \exp H(g) n(g)$ with $k(g) \in K$, $H(g) \in \mathfrak{a}$, $n(g) \in N$ determined uniquely in virtue of the Iwasawa decomposition.

The dimension of $\mathfrak{a}$ is called the *split rank of G* and is equal to the rank of the associated symmetric space G/K.

Let M and M' be the centralizer and normalizer, respectively, of A in K. Since both have the same Lie algebra — the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ — and are compact, their quotient M'/M, called the *Weyl group*, is finite. The Weyl group W acts on $\mathfrak{a}$ via the adjoint representation and on $\mathfrak{a}'$ by duality, i.e., if $w = m_w M$, then

$$w(H) = \mathrm{Ad}_G(m_w)H, \qquad w\lambda(H) = \lambda\big(w^{-1}(H)\big).$$

It can be shown that the action of W on the set of Weyl chambers in $\mathfrak{a}$ is simply transitive and also that it sends roots into roots.

5. Homogeneous spaces of semi-simple Lie groups. Together with a symmetric space attached to a semi-simple G which has a representation as the quotient G/K with K maximal compact subgroup of G, we shall deal with its dual space, namely, the space of horocycles G/MN. (We shall leave aside the geometrical interpretation of the space of horocycles since we shall not need it.)

In view of the Iwasawa decomposition it is not surprising that we have a diffeomorphism of $K/M \times A$ with G/MN given by $(kM, a) \to kaMN$, and indeed this was proved by Helgason in [13]. This allows us to regard G/MN as a trivial fiber bundle over K/M with a fiber A and with the action of G given by

$$g(kM, a) = \big(k(gk)M,\ \exp\big(H(gk)\big)a\big).$$

This point of wiew will be convenient in determination of G-invariant differential operators on G/MN.

There is yet another space of constant use in the sequel — the maximal boundary space of G/K which is equal to G/MAN. From the Iwasawa decomposition it follows that G/MAN is naturally diffeomorphic with K/M and the action of G, carried over to K/M, is given by $g(kM) = k(gk)M$.

6. Measures on G and its homogeneous spaces. The bi-invariant measure on G is denoted by dg. Using the Iwasawa decomposition one finds that with suitable normalization of measures

$$\int_G f(g)dg = \int_{K\times A\times N} f(kan)e^{2\varrho(\log a)}dk\, da\, dn,$$

here dk is normalized so that the total measure of K is 1.

With M as above, $\Gamma = MAN$ is a closed subgroup of G and its left Haar measure and modular function are given as

$$\int_\Gamma f(\gamma)\, d\gamma = \int_{M\times A\times N} f(man)\, dm\, da\, dn,$$

$$\int_\Gamma f(\gamma x^{-1})d\gamma = \int_\Gamma f(\gamma)d(\gamma x) = e^{-2\varrho\log a}\int_\Gamma f(\gamma)d\gamma, \quad \text{for } x = man.$$

Let U, H be locally compact groups, $H \subset U$ and assume they both are unimodular; then U/H has a U-invariant measure $d(xH)$, given by

$$\int_U f(x)\, dx = \int_{U/H}\Big(\int_H f(xh)dh\Big)\, d(xH). \tag{I.1}$$

If H is compact and of total measure 1, then one can identify $\mathscr{C}_0(U/H)$ with the subspace of $\mathscr{C}_0(U)$ consisting of right H-invariant functions, and with this identification we have

$$\int_U f(x)\, dx = \int_{U/H} f(xH)\, d(xH), \quad f\in\mathscr{C}_0(U/H). \tag{I.2}$$

In general, there is no G-invariant measure on the coset space U/H, there are present, however (provided U is countable at infinity), quasi-invariant measures.

A measure μ on U/H is said to be *quasi-invariant* if for all $x\in U$ μ and its x-translate are mutually equivalent, i.e., have the same null sets.

Quasi-invariant measures are most easily constructed by means of so-called ϱ-functions.

A Borel positive function ϱ on U, bounded below and above on compact subsets and verifying for all $h\in H$

$$\varrho(xh) = \frac{\Delta_H(h)}{\Delta_U(h)}\varrho(x); \quad x\in U,$$

is called *ϱ-function*. Such function gives rise to a quasi-invariant measure $d\mu_\varrho$ as follows

$$\text{(I.3)} \qquad \int_U f(x)\, \varrho(x)\, dx = \int_{U/H} d\mu_\varrho(xH) \int_H f(xh) dh; \quad f \in \mathscr{C}_0(U).$$

Such ϱ-functions always exist — when U is a Lie group, then ϱ may be chosen of C^∞ class.

Returning to a semi-simple G we see that there exist G-invariant measures on G/K and G/MN, given by formula (I.1). On G/MAN, however, there is no G-invariant measure — a quasi-invariant we shall be using constantly is given, in the identification of G/MAN with K/M, by the K-invariant measure $d(kM)$, and the corresponding ϱ-function is $e^{-2\varrho H(g)}$.

7. Invariant differential operators on homogeneous spaces. It is well known that the algebra of G-invariant differential operators on G is canonically isomorphic to the universal enveloping algebra $\mathfrak{A}(\mathfrak{g})$ of $\mathfrak{g}$. The algebras of G-invariant operators on G/K and G/MN admit a simple and complete description in terms of invariant operators on A — the abelian part of the Iwasawa decomposition of G.

The proof of the following result may be found in Helgason's treatise [12] (part a) and [13] (part b).

THEOREM I.1.

(a) (Harish-Chandra) *The algebra $\boldsymbol{D}(G/K)$ of all G-invariant differential operators on G/K is canonically isomorphic with $I(A)$ — the algebra of Weyl group invariant differential operators on A with constant coefficients.*

(b) (Helgason). *The algebra $\boldsymbol{D}(G/MN)$ of G-invariant differential operators on G/MN is canonically isomorphic with $\boldsymbol{D}(A)$ — the algebra of differential operators on A with constant coefficients.*

In order to get some idea of the above isomorphisms note that $\boldsymbol{D}(G/K)$ may be viewed as an algebra of left G, right K-invariant operators on G, and hence identified with the centralizer $\mathfrak{A}^{\mathfrak{k}}(\mathfrak{g})$ of $\mathfrak{k}$ in $\mathfrak{A}(\mathfrak{g})$. For every $p \in \mathfrak{A}(\mathfrak{g})$ there exists a unique $p^a \in \mathfrak{A}(\mathfrak{a})$ such that $p - p^a \in \mathfrak{n}\,\mathfrak{A}(\mathfrak{g}) + \mathfrak{A}(\mathfrak{g})\mathfrak{k}$ and $\mathfrak{A}^{\mathfrak{k}}(\mathfrak{g}) \ni p \to p^a$ is a homomorphism of algebras. This is essentially, up to an automorphism of $\mathfrak{A}(\mathfrak{a})$ which renders p^a to be W-invariant, the isomorphism of (a).

As for (b), recall that the action of G on $G/MN \simeq K/M \times A$ is fiber preserving and induces translations on each fiber. So we define for every $U \in \boldsymbol{D}(A)$, $D_U \in \boldsymbol{D}(G/MN)$ by means of

$$(D_U f)(kM, a) = (Uf|_{kM})(a); \qquad (f|_{kM})(a) = f(kM, a).$$

This is the isomorphism of (b).

Chapter II

Spherical representations and spherical functions
General results

In many respects the behaviour of the representations of G depends on the behaviour of its restriction to the maximal compact subgroup K. In particular, the problem of multiplicities of irreducible representations of K in restrictions to K of irreducible representations of G is of decisive importance in many cases. The following theorem is basic in that direction.

THEOREM II.1 (Harish-Chandra). *Let G be a connected semi-simple Lie group with the finite center and K its maximal compact subgroup. Let further (π, V) be a topologically irreducible representation of G in V such that the center of $\mathfrak{A}(\mathfrak{g})$ — $\mathfrak{Z}(\mathfrak{g})$ — acts on V_∞ by multiplication by scalars (such representations are called quasi-simple). Then there exists a positive integer N such that*

$$m([\delta]:\pi|_K) \leqslant N \dim([\delta])^2$$

for every $\delta \in \hat{K}$. (Here $m([\delta]:\pi|_K)$ is the multiplicity of $[\delta]$ in $\pi|_K$.)

In fact, an integer N can be taken 1 — this was shown by Harish-Chandra in the case of representations in Banach space, and recently more generally by J. Lepovsky. Therefore the space of K-fixed vectors is at most 1-dimensional. We are thus led to the

DEFINITION II.1. An irreducible representation (π, V) is called *spherical* (or *of class one*) if the restriction of π to K contains a trivial representation of K.

Let $\hat{K}$ be the set of equivalence classes of irreducible representations of K. For any $[\delta] \in \hat{K}$ consider the maximal subspace $V(\delta)$ of V such that $\pi|_K$ is a multiple of $[\delta]$ on $V(\delta)$. In view of the preceding theorem, $V(\delta)$ is finite-dimensional and uniquely determined. Also if χ_δ is the character of δ and $d(\delta)$ is the dimension of δ, then the projection of V onto $V(\delta)$ is given as

$$P_\delta = \pi\big(d(\delta)\,\overline{\chi}_\delta\big) = d(\delta)\int_K \overline{\chi}_\delta(k)\pi(k)\,dk.$$

Let now V be semi-reflexive and V' be the topological dual of V; then the transpose of P_δ is a projection of V' onto $V'(\delta)$, as may be easily seen.

Let us note that K-finite vectors in V, that is, vectors which K-translates lie in a finite-dimensional subspace of V, are infinitely differentiable. Hence a K-fixed vector is a common eigenvector for $\mathfrak{A}^{\mathfrak{k}}(\mathfrak{g})$ — the centralizer of $\mathfrak{k}$ in $\mathfrak{A}(\mathfrak{g})$, or what is the same, the subalgebra of $\mathfrak{A}(\mathfrak{g})$ stable under $\mathrm{Ad}(K)$.

Let for an $X \in \mathfrak{A}^{\mathfrak{k}}(\mathfrak{g})$, $p(X)$ be the corresponding eigenvalue; then $\mathfrak{A}^{\mathfrak{k}}(\mathfrak{g}) \ni X \to p(X) \in C$ is a character of $\mathfrak{A}^{\mathfrak{k}}(\mathfrak{g})$ (i.e., homomorphism of $\mathfrak{A}^{\mathfrak{k}}(\mathfrak{g})$ into C).

Consider now the function

$$G \ni g \to \langle \pi(g^{-1})\, v_0,\ v^0 \rangle = \varphi(g), \tag{II.1}$$

where v_0, v^0 are K-fixed vectors in V, V' resp., such that $\langle v_0, v^0\rangle = 1$. (Such vectors always exist.) If we let act $\mathfrak{A}(\mathfrak{g})$ on $\mathscr{E}(G)$ by left invariant differential operators, then it is easy to see that for every $X \in \mathfrak{A}^{\mathfrak{k}}(\mathfrak{g})$ we have

$$(X\varphi)(g) = \langle \pi(g^{-1})\pi\big(\mathrm{Ad}(g)X\big)v_0,\ v^0\rangle = p\big(\mathrm{Ad}(g)X\big)\varphi(g),$$

hence φ is an eigenfunction for an algebra $\boldsymbol{D}_0(G)$ of left invariant differential operators on G, commuting with right translations by elements of K. This motivates the following

DEFINITION II.2. A function $\varphi \in \mathscr{E}(G)$, bi-invariant under K, satisfying $\varphi(e) = 1$ and a common eigenfunction for an algebra $\boldsymbol{D}_0(G)$, is said to be a *(zonal) spherical function.*

There are several equivalent characterizations of zonal spherical functions — we collect them in one theorem.

THEOREM II.2. *The following conditions are equivalent:*

(i) *φ is a zonal spherical function;*

(ii) *φ is continuous function on G not identically equal to* 0, *and satisfying*

$$\int_K \varphi(xky)\,dk = \varphi(x)\varphi(y), \quad \text{for all } x,\ y \in G;$$

(iii) *φ is a continuous function, bi-invariant under K, such that*

$$f \to \int_G f(x)\varphi(x)dx$$

is a homomorphism of the convolution algebra of all continuous compactly supported and K-bi-invariant functions on G;

(iv) *φ is a continuous function, bi-invariant under K, such that*

$$f * \varphi = \lambda(f)\varphi; \quad \lambda(f) \in C \text{ for all } f \in \mathscr{C}_0(K\backslash G/K).$$

The following theorem gives a description of zonal spherical functions in terms of the dual space to $\mathfrak{a}$ — the abelian part in the Iwasawa decomposition of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$.

THEOREM II.3 (Harish-Chandra). *Every zonal spherical function is given by the formula*

$$\varphi(g) = \int_K e^{(i\lambda-\varrho)H(gk)}dk \tag{II.2}$$

with $\lambda \in \mathfrak{a}_0'$ determined uniquely modulo the Weyl group. Conversely, for every λ formula (II.2) *gives a spherical function.*

We have seen above how to construct spherical functions, given a spherical representation. An obvious question arises whether there is only one spherical function corresponding that way to a representation, and also whether the representation can be recovered from the function.

As to the first question, the answer is essentially positive, assuming only the representation space semi-reflexive. In fact, let (π, V) be spherical and V semi-reflexive; then the contragredient representation is continuous and irreducible. Then Theorem II.1 gives essential unicity of a K-fixed vector in V' and this implies in turn the uniqueness of the spherical function.

As to the second question, there is a satisfactory answer in the case of unitary representations, whereas in the general case due, partly at least, to the lack of an appropriate notion of the equivalence of representations, one cannot hope to obtain the same representation from the spherical function.

We quote here theorem, pertinent to the unitary case.

THEOREM II.4. *Let (U, H) be a unitary spherical representation, $e \in H$ a K-fixed vector of unit norm, and*

$$G \ni g \to \varphi(g) = \big(U(g^{-1})\, e, e\big)$$

a corresponding spherical function. Then φ is positive definite and the correspondence $(U, H) \to \varphi$ gives rise to a bijection of the set of equivalence classes of spherical representations upon the set of positive definite spherical functions.

To provide a better insight into the content of the theorem, we recall here the construction of a representation from a given positive definite function.

Let $\mathscr{C}_0(G)$ be equipped with the prehilbert structure, given by the positive hermitian form

$$\mathscr{C}_0(G) \times \mathscr{C}_0(G) \ni (f, h) \to \int_G\int_G \varphi(x^{-1}y) f(x)\bar{h}(y)\, dx\, dy = (f, h)_\varphi,$$

with φ a positive definite spherical function. This form is left invariant, hence a unitary extension of the left regular representation in $\mathscr{C}_0(G)$ may be formed, giving a spherical representation (cf. for example, K. Maurin, *General eigenfunction expansions and unitary representations of topological groups*, pp. 233 ff.).

Remark. The same, within equivalence of course, representation can be obtained by taking the space $\mathscr{C}_0(G/K)$, or even $\mathscr{D}(G/K)$, and defining the prehilbert structure by the formula

$$(f, h) \to (f, h)_\varphi = \int_G f * \varphi(x) \bar{h}(x)\, dx$$

(with the usual identification of functions on G/K with K-invariant functions on G).

Remark. There is a theorem, due to Helgason, giving within the weak equivalence, all spherical representations of G in terms of spherical functions. A representation is constructed from a spherical function φ by taking the smallest closed subspace in $\mathscr{E}(G/K)$, containing all G-translates of φ. It is shown that this representation is spherical, quasi-simple and φ is related to it by formula (II.1) (cf. [15]).

Recall now that a vector $e \in V$ is said to be *cyclic* for the representation (π, V), if the set of G-translates of e—$\{\pi(g)e \colon g \in G\}$ is total in V.

We shall now prove

PROPOSITION II.5. *Let (π, V) be a representation of G in a semi-reflexive space V and let the subspace of K-fixed vectors in V be one-dimensional. (We do not assume irreducibility of π.) If there is a K-fixed vector in V', cyclic for the contragredient representation, then G acts irreducibly in the closure of $\{\pi(g)v_0 \colon v_0$ K-fixed, $g \in G\}$.*

Proof. Let $v^0 \in V'$ be cyclic and invariant under K. By semi-reflexivity that means that for every $v \in V$ the function $G \ni g \to \langle \pi(g)v, v^0 \rangle$ does not vanish identically.

Assume on the contrary that there is a closed invariant subspace $W \subset V_1$, $V_1 = \{\pi(g)v_0 \colon g \in G\}$ and let $0 \neq w \in W$. Then there exists $g \in G$ such that $\langle \pi(g)w, v^0 \rangle \neq 0$. If P_0 is the projection onto the space of K-fixed vectors in V, then $\langle P_0 \pi(g)w, v^0 \rangle = \langle \pi(g)w, P_0' v^0 \rangle = \langle \pi(g)w, v^0 \rangle \neq 0$. So we see that $P_0 \pi(g)w \neq 0$ and is K-invariant, hence proportional to v_0. Therefore $W = V_1$.

Chapter III

Some representations in function spaces

In this section we collect some information on function theory on certain homogeneous spaces of G. Some results of this chapter are, explicitly or implicitly, contained in Helgason's paper [15], where, however, a function-theoretic point of view prevailed the representation-theoretic content. When put in their natural context of various realizations of the same representation of G many of isomorphisms established by Helgason become more natural. In particular, the Fourier transform appears either as a multiplier picture of the Bruhat map β^λ or as a result of the action of $\mathscr{D}(G/K)$ at the K-fixed vector in the multiplier realization of the induced representation π^λ (cf. Proposition III.7 and Lemma III.8). We have omitted the proofs of those facts due to Helgason, where our point of view does not lead to any simplification of an argument.

We begin by defining a certain series of representations called, in [15], a *principal spherical series* (however, in view of Definition II.1, not all representations of that series are spherical since some fail to be irreducible).

DEFINITION III.1. Let $(\pi^\lambda, \mathscr{H}^\lambda)$ be the Banach space representation induced by the character $\Gamma \ni man \to e^{-i\lambda \log a}$ of $\Gamma = MAN$. The space $\mathscr{H}^\lambda$ consists of measurable on G functions which verify

$$f(gman) = e^{(i\lambda - \varrho)\log a} f(g) \tag{III.1}$$

almost everywhere (with respect to the Haar measure) for all $man \in \Gamma$, and such that

$$\int_K |f(k)|^2 \, dk < \infty .$$

The last expression serves as a norm in $\mathscr{H}^\lambda$, and π^λ acts in $\mathscr{H}^\lambda$ via left translations, non-unitarily however, except when $\lambda \in \mathfrak{a}_C^*$ is real (i.e., the inducing representation is unitary).

It is known that the space of differentiable vectors in $\mathscr{H}^\lambda$ is exactly the space of differentiable induced representation $\mathscr{D}^\lambda$ consisting of smooth functions in $\mathscr{H}^\lambda$ (cf. [27]). The Fréchet topology of the space of differentiable vectors is exactly the usual topology of uniform convergence of all orders of restrictions to KA.

For the time being we let $(\mathscr{D}^\lambda)'$ denote the topological dual to $\mathscr{D}^\lambda$. In this case the Bruhat map β^λ: $\mathscr{D}(G)\to\mathscr{D}^\lambda$ takes the form

$$(\beta^\lambda f)(g) = \int_{MAN} f(gman)e^{(-i\lambda+\varrho)\log a}dm\,da\,dn.$$

Functions from $\mathscr{H}^\lambda$ ($\mathscr{D}^\lambda$ resp.) may be regarded as functions on G/MN, due to their right MN-invariance. On the other hand, they are unambiguously determined by their restrictions to K/M.

Lemma III.1. *Let τ_λ be the multiplier representation on $B = K/M$ defined by*

$$\big(\tau_\lambda(g)f\big)(kM) = e^{(i\lambda-\varrho)H(g^{-1}k)}f\big(g^{-1}(kM)\big)$$

for $f\in\mathscr{L}^2(K/M)$. Then the restriction of functions is an isometric isomorphism of $\mathscr{H}^\lambda$ onto $\mathscr{L}^2(K/M)$, intertwining for π^λ and τ_λ and mapping $\mathscr{D}^\lambda$ onto $\mathscr{E}(K/M)$.

Recall that the algebra $\boldsymbol{D}(G/MN)$ of left invariant differential operators on G/MN can be identified with the algebra of translationally invariant differential operators on A, hence every complex linear form μ on $\mathfrak{a}$ determines a homomorphism, say χ_μ, of $\boldsymbol{D}(G/MN)$ into $\boldsymbol{C}$, and conversely, every such homomorphism can be obtained that way from the unique $\mu\in\mathfrak{a}_C'$.

Fairly simple computations show that $\mathscr{D}^\lambda$ consists of common eigenfunctions of the algebra $\boldsymbol{D}(G/MN)$, that is,

$$Df = \chi_{(i\lambda-\varrho)}(D)f, \quad \text{for all } D\in\boldsymbol{D}(G/MN). \tag{III.2}$$

We may then define $\mathscr{D}_\lambda'$ as the space of all distributions, which are solutions to (III. 2), and check that we have $\mathscr{D}^\lambda\subset\mathscr{H}^\lambda\subset\mathscr{D}_\lambda'$.

The following characterization of $\mathscr{D}_\lambda'$ was given in [15].

Proposition III. 2 (Helgason). *Let for each $a\in A$, $\sigma(a)\colon G/MN\to G/MN$ be a diffeomorphism defined by $\sigma(a)(gMN) = gaMN$. (This is a correct definition since A normalizes MN.) Then a distribution $\Psi\in\mathscr{D}'(G/MN)$ belongs to $\mathscr{D}_\lambda'$ if and only if*

$$\Psi^{\sigma(a)} = e^{-(i\lambda+\varrho)\log a}\Psi. \tag{III.3}$$

Identifying G/MN with $K/M\times A$ by $(kM, a)\to kaMN$, we see that if $\Psi\in\mathscr{D}_\lambda'$, then $e^{(i\lambda+\varrho)\log a}\Psi$ is a distribution on $K/M\times A$ invariant under the action of A given by $a(kM, b) = (kM, ab)$.

By a theorem of Bruhat every such distribution Φ is given by a distribution $S\in\mathscr{E}'(K/M)$ as follows

$$\Phi(f) = \int_{K/M}\Big(\int_A f(kM, a)da\Big)dS(kM),$$

hence

$$\Psi(f) = \int_{K/M} \Big(\int_A f(kaMN) e^{(i\lambda+\varrho)\log a} da \Big) \, dS(kM), \tag{III.4}$$

and the correspondence $\Psi \to S$ is unique.

Conversely, for every distribution $S \in \mathscr{E}'(K/M)$, (III.4) defines a distribution from $\mathscr{D}'_\lambda$, as one easily computes using (III.3).

So we have arrived at

PROPOSITION III. 3 (Helgason).

1. *The mapping* $S \to \Psi$, *given by the formula* (III.4) *above, is a linear bijection of* $\mathscr{E}'(K/M)$ *onto* $\mathscr{D}'_\lambda$, *intertwining for the contragredient representation to* $\tau_{-\lambda}$ *and the natural action of* G *on* $\mathscr{D}'_\lambda$ *by translations.*

2. *The pairing between* $\mathscr{D}^\lambda$ *and* $\mathscr{D}'_{-\lambda}$, *given by*

$$\langle \varphi, \Psi \rangle = \langle \varphi|_{K/M}, S \rangle = \int_{K/M} \varphi(kMN) \, dS(kM),$$

is G*-invariant and gives an isomorphism of* $\mathscr{D}'_{-\lambda}$ *onto* $(\mathscr{D}^\lambda)'$.

Proof. To complete 1, we are left with the intertwining property. Compute

$$\begin{aligned} (L_g\Psi)(\varphi) &= \Psi(L_{g^{-1}}\varphi) \\ &= \int_{K/M} \Big(\int_A \varphi\big(k(gk) \exp H(gk) a\big) e^{(i\lambda+\varrho)\log a} \, da \Big) \, dS(kM) \\ &= \int_{K/M} e^{-(i\lambda+\varrho)H(gk)} \Big(\int_A \varphi\big(k(gk) a\big) e^{(i\lambda+\varrho)\log a} da \Big) \, dS(kM). \end{aligned}$$

This shows that the distribution corresponding to $L_g\Psi$ is $\big(\tau_{-\lambda}(g)\big)' S$, as asserted.

For 2, it suffices to prove the invariance of $\langle \cdot, \cdot \rangle$, the rest following from the non-degeneracy of the pairing between $\mathscr{E}(K/M)$ and $\mathscr{E}'(K/M)$. However, it is enough to show $\langle L_g\varphi, L_g\Psi \rangle = \langle \varphi, \Psi \rangle$ for Ψ represented by a function on K/M, since for other Ψ it will follow by an approximation. In this case we compute

$$\langle L_g\varphi, L_g\Psi \rangle = \int_{K/M} \varphi\big(g^{-1}(kM)\big) e^{(i\lambda-\varrho)H(g^{-1}k)} \big(\tau_{-\lambda}(g)\big)' S(kM) d(kM).$$

But $\big(\tau_{-\lambda}(g)\big)' S = \tau_{-\lambda}(g) S$ for every function S on K/M, so the right-hand side equals to

$$\begin{aligned} &\int_{K/M} \varphi\big(g^{-1}(kM)\big) e^{(i\lambda-\varrho)H(g^{-1}k)} e^{-(i\lambda+\varrho)H(g^{-1}k)} S\big(g^{-1}(kM)\big) \, d(kM) \\ &\qquad = \int_{K/M} \varphi\big(g^{-1}(kM)\big) S\big(g^{-1}(kM)\big) e^{-2\varrho H(g^{-1}k)} d(kM) \\ &\qquad = \int_{K/M} \varphi(kM) S(kM) \, d(kM) = \langle \varphi, \Psi \rangle, \end{aligned}$$

since $e^{-2\varrho H(g^{-1}k)}$ is just the Radon–Nikodym derivative of the g-translated measure $d(kM)$ on K/M (cf. (I. 3)).

In all what follows we shall identify, via the pairing above, $\mathscr{D}'_\lambda$ with the (topological) dual of $\mathscr{D}^{-\lambda}$. With these spaces identified we have the natural description of $\mathscr{D}'_\lambda$ (we owe this fact to Dr. Wawrzyńczyk).

PROPOSITION III.4. *There exists a natural isomorphism of $\mathscr{D}'_\lambda$ with the closed subspace of distributions on G which satisfy*

$$\text{(III. 5)} \qquad \Phi(R_{man}f) = e^{(-i\lambda+\varrho)\log a}\Phi(f).$$

The isomorphism is realized by the transpose to the Bruhat map $\beta^{-\lambda}$: $\mathscr{D}(G)\to\mathscr{D}^{-\lambda}$.

Proof. Since $\beta^{-\lambda}$ is surjective, continuous and open, hence its transpose is injective and continuous for the strong dual topology. It is easy to see that if $\Psi\in\mathscr{D}'_\lambda$, then $\Phi = (\beta^{-\lambda})^t\Psi$ satisfies property (III.5). In fact,

$$\Phi(R_{man}f) = \langle\beta^{-\lambda}(R_{man}f), \Psi\rangle = \langle e^{(-i\lambda+\varrho)\log a}\beta^{-\lambda}f, \Psi\rangle$$
$$= e^{(-i\lambda+\varrho)\log a}\Phi(f).$$

To prove the converse it suffices to show that if $\Phi\in\mathscr{D}'(G)$ verifies (III.5), then it vanishes on $\ker\beta^{-\lambda}$, hereby defining a distribution from $\mathscr{D}'_\lambda$ (i.e., a functional on $\mathscr{D}^{-\lambda} = \mathscr{D}(G)/\ker\beta^{-\lambda}$). We reason by approximation as above. Assuming Φ represented by a locally integrable function φ, we have

$$\Phi(f) = \int_G f(g)\varphi(g)\,dg = \int_{KAN} f(kan)\varphi(kan)e^{2\varrho\log a}\,dk\,da\,dn.$$

But for this function, (III.5) means that

$$\varphi(gman) = e^{(i\lambda-\varrho)\log a}\varphi(g),$$

hence

$$\Phi(f) = \int_{KAN} f(kan)\varphi(k)e^{(i\lambda+\varrho)\log a}\,dk\,da\,dn,$$

and by the factorization of the measure on K (cf. Chapter I), we have

$$\Phi(f) = \int_{K/M} \beta^{-\lambda}f(kM)\varphi(kM)\,d(kM).$$

Since this depends only on the restriction of $\beta^{-\lambda}f$ to K/M, the assertion follows. Now, as these two operations are inverse to each other, the proof is complete.

We also note for future use the following two corollaries.

COROLLARY III. 5. *$(\pi^{-\lambda}, \mathscr{H}^{-\lambda})$ is contragredient to $(\pi^{\lambda}, \mathscr{H}^{\lambda})$.*

COROLLARY III. 6. *There is a canonical, G-invariant, sesquilinear form between $\mathscr{H}^{\lambda}$ and $\mathscr{H}^{\bar\lambda}$, given as*

$$\mathscr{H}^{\lambda}\times\mathscr{H}^{\bar\lambda}\ni(f, h)\to\langle f, h\rangle = \int_K f(k)\overline{h(k)}\,dk.$$

Establishing Plancherel formula for the left regular representation of G on the symmetric space G/K Helgason introduced the notion of Fourier transform which generalizes that of spherical Fourier transform of Harish-Chandra.

DEFINITION III.2. For a $\lambda \in \mathfrak{a}'_C$ define the Fourier transform as a map $\mathcal{F}^\lambda$: $\mathcal{D}(G/K) \to \mathcal{E}(K/M)$ by the formula

$$(\mathrm{III.6}) \qquad (F^\lambda f)(kM) = \tilde{f}(\lambda, kM) = \int_{G/K} f(gK) e^{(i\lambda-\varrho)H(g^{-1}k)} d(gK).$$

As discussed at length in [14], functions $gK \to e^{(i\lambda-\varrho)H(g^{-1}k)}$ are analogs for symmetric spaces of exponential functions in R^n, and hence serve as building bricks for introducing the generalized Fourier transform. From our point of view, it is important to note that these functions are common eigenfunctions of all left invariant differential operators on G/K.

There is an intimate connection between the Bruhat map β^λ and the Fourier transform F^λ. We owe the following observation to Dr. Wawrzyńczyk.

PROPOSITION III.7. *Consider as usual $\mathcal{D}(G/K)$ as the subspace of $\mathcal{D}(G)$, consisting of right invariant, with respect to K functions, and let β^λ be the Bruhat map β^λ: $\mathcal{D}(G) \to \mathcal{D}^\lambda$. Then for all $f \in \mathcal{D}(G/K)$ we have*

$$(\beta^\lambda f)(g) = (F^\lambda f)\big(k(g)M\big) e^{(i\lambda-\varrho)H(g)}.$$

(As in Chapter I, we have put $g = k(g) \exp H(g) n$ according to the Iwasawa decomposition.)

Proof. We shall prove $(\beta^\lambda f)(k) = (F^\lambda f)(kM)$, the rest being clear. First note that by virtue of $H(g^{-1}k) = H\big((k^{-1}g)^{-1}\big)$, and the invariance of the measure $d(gK)$, we have

$$F^\lambda(kM) = \int_{G/K} f(gK) e^{(i\lambda-\varrho)H(g^{-1}k)} d(gK)$$

$$= \int_{G/K} f(kgK) e^{(i\lambda-\varrho)H(g^{-1})} d(gK) = F^\lambda(L_k^{-1} f)(eM),$$

so it suffices to compute $(F^\lambda f)(eM)$.

To do this, we recall that G/K is diffeomorphic with AN and the integral over G/K goes over to the integral on AN as follows

$$\int_{G/K} f(gK)\, dgK = \int_{AN} f(anK)\, dn\, da$$

(with an appropriate normalization of measures).

Further, since $H(g^{-1}) = -H(g)$ for $g \in AN \subset G$, we have

$$(F^\lambda f)(eM) = \int_{G/K} f(gK)\, e^{(i\lambda-\varrho)H(g^{-1})} d(gK)$$
$$= \int_{AN} f(anK)\, e^{(-i\lambda+\varrho)\log a} dn\, da$$
$$= (\beta^\lambda f)(e),$$

what was to be proved.

Now we note the following observation.

LEMMA III.8. *Let τ_λ be the multiplier representation in $\mathscr{E}(K/M)$ with multiplier* (cf. Lemma III.1)

$$\sigma(g, kM) = e^{(i\lambda-\varrho)H(g^{-1}k)},$$

that is,

$$\big(\tau_\lambda(g)h\big)(kM) = \sigma(g, kM)\, h\big(g^{-1}(kM)\big), \quad h \in \mathscr{E}(K/M),$$

and let $\mathscr{D}(G) \ni f \to \tau_\lambda(f)$ denote the extension of τ_λ to the representation of the convolution algebra $\mathscr{D}(G)$ in $\mathscr{E}(K/M)$. Then with Ψ_0 denoting the function equal to 1 on all of K/M,

$$\tau_\lambda(f)\Psi_0 = F^\lambda f, \quad \textit{for all } f \in \mathscr{D}(G/K).$$

Proof. A straightforward comparison of formulae yields the desired result:

$$\big(\tau_\lambda(f)\Psi_0\big)(kM) = \int_G f(g)\, e^{(i\lambda-\varrho)H(g^{-1}k)} dg$$
$$= \int_{G/K} f(gK)\, e^{(i\lambda-\varrho)H(g^{-1}k)} d(gK).$$

This lemma will allow us to define the concept of the restricted dual Radon transform without invoking the concept of Radon transformation.

DEFINITION III.3. The map $\mathscr{E}'(K/M) \to \mathscr{E}(G/K)$, dual to Fourier transform F^λ: $\mathscr{D}(G/K) \to \mathscr{E}(K/M)$, will be called the *restricted dual Radon transform,* and will be denoted by $R_{-\lambda}$. (The use of "$-$" will become clear later on.)

That $R_{-\lambda}$ maps $\mathscr{E}'(K/M)$ into $\mathscr{E}(G/K)$ and not into $\mathscr{D}'(G/K)$, as was expected a priori, is the consequence of an explicit formula for $R_{-\lambda}$ which we are now going to derive. Since it is the same formula Helgason gets from his function-theoretic definition of dual Radon transform, the identity of the two notions will be shown at the same time.

Let $f \in \mathscr{D}(G/K)$ and $S \in \mathscr{E}'(K/M)$. Then

$$\langle F^\lambda f, S\rangle = \langle \tau_\lambda(f)\psi_0, S\rangle = \int_G f(g)\langle \tau_\lambda(g)\psi_0, S\rangle\, dg.$$

Now $g \to \langle \tau_\lambda(g)\psi_0, S\rangle$ is infinitely differentiable function, since the multiplier representation is differentiable representation of G, and actually a right K-invariant function, so it belongs to $\mathscr{E}(G/K)$. At the same time we have

$$(\text{III.7}) \qquad (R_{-\lambda}S)(gK) = \langle \tau_\lambda(g)\Psi_0, S\rangle = \int_{K/M} e^{(i\lambda-\varrho)H(g^{-1}k)} dS(kM),$$

which is exactly the formula of Proposition 4.6, p. 93 of [15], applied to S representing $\Psi \in \mathscr{D}'_{-\lambda}$.

Corollary III.9. *$R_{-\lambda}$ intertwines contragredient to $\tau_{-\lambda}$ acting on $\mathscr{E}'(K/M)$ with the left regular representation of G acting on $\mathscr{E}(G/K)$.*

This is seen by duality, since F^λ intertwines the left regular representation with τ_λ.

Corollary III.10. *$R_{-\lambda}$ is injective if and only if the set $F^\lambda\mathscr{D}(G/K)$ is dense in $\mathscr{E}(K/M)$.*

In [15], Helgason introduced the following definition.

Definition III.4. $\lambda \in \mathfrak{a}'_C$ is said to be *simple* if R_λ is injective.

Corollary III.11. *λ is simple if and only if $\{F^{-\lambda}f\colon f \in \mathscr{D}(G/K)\}$ is dense in $\mathscr{E}(K/M)$.*

From the formula for R_λ above, we have

Corollary III.12 (Helgason). *λ is simple if and only if Ψ_0 is cyclic for $\tau_{-\lambda}$ acting on $\mathscr{E}(K/M)$.*

The notion of simplicity of elements of $\mathfrak{a}'_C$ turns out to be crucial for Helgason's study of conical distributions. Since those are of great importance for the following, we quote here his result giving a sufficient condition for λ to be simple.

Proposition III.13 (Helgason). *$\lambda \in \mathfrak{a}'_C$ is simple if for all positive roots α*

$$(\text{III.8}) \qquad \mathrm{Re}\big(B(i\lambda, \alpha)\big) > 0.$$

The set of λ satisfying condition (III.8) will be denoted by C.

We shall omit the proof since it is rather involved.

We close this chapter with a discussion of one more subject of a purely technical nature, namely, the convolution of distributions on K/M. To define a suitable notion of convolution we shall make an essential use of the compactness of K.

As usually, for any two distributions $S_1, S_2 \in \mathscr{E}'(K)$, we define their convolution via

$$(\text{III.9}) \qquad S_1 * S_2(f) = \int_K \int_K f(k_1k_2)\, dS_1(k_1)\, dS_2(k_2)$$

for all $f \in \mathscr{E}'(K)$.

It is a classical fact that $\mathscr{E}'(K/M)$ is naturally isomorphic to the closed subspace of $\mathscr{E}'(K)$, consisting of right M-invariant distributions. Let $\tilde{S}$ be the image of S under that isomorphism. For regular (i.e., defined by functions) distributions, $\tilde{S} = S\circ\pi$, where $\pi\colon K\to K/M$ is the canonical projection.

It is easy to see from (III. 9) that if S_2 is right M-invariant, then $S_1 * S_2$ is such for any S_1, so given any S_1, $S_2 \in \mathscr{E}'(K/M)$, $\tilde{S}_1 * \tilde{S}_2$ is determined by a unique distribution from $\mathscr{E}'(K/M)$, which we denote by $S_1\times S_2$.

More formally, let us put for an $f\in\mathscr{E}(K)$, $f_M(kM) = \int_M f(km)\,dm$, then $\tilde{S}$ is defined by $\tilde{S}(f) = S(f_M)$ for $S\in\mathscr{E}'(K/M)$, whereas $S_1\times S_2$ is determined by

$$S_1\times S_2(f) = \tilde{S}_1 * \tilde{S}_2(f\circ\pi), \qquad f\in\mathscr{E}(K/M).$$

It will come out at hands later to have an explicit formula for $S_1\times S_2$, where S_2 is a left M-invariant.

LEMMA III. 14 (Helgason). *Let S_1, $S_2\in\mathscr{E}'(K/M)$ and S_2 be left M-invariant. Then*

$$S_1\times S_2(f) = \int_{K/M}\int_{K/M} f(k_1k_2\,M)\,dS_1(k_1\,M)\,dS_2(k_2M).$$

Proof.

$$S_1\times S_2(f) = \tilde{S}_1 * \tilde{S}_2(f\circ\pi) = \int_K\int_K f\circ\pi(k_1k_2)\,d\tilde{S}_1(k_1)\,d\tilde{S}_2(k_2).$$

Let

$$g(k) = \int_K f\circ\pi(kk_2)\,d\tilde{S}_2(k_2) = \int_{K/M} f(kk_2\,M)\,dS_2(k_2\,M).$$

Then owing to M-invariance of S_2, g is right M-invariant, what implies $g_M(kM) = g(k)$ and so we have

$$\begin{aligned} S_1\times S_2(f) &= \int_K g(k)\,d\tilde{S}_1(k_1) = \int_{K/M} g_M(kM)\,dS_1(kM) \\ &= \int_{K/M}\int_{K/M} f(kk_2\,M)\,dS_2(k_2\,M)\,dS_1(kM) \\ &= \int_{K/M}\int_{K/M} f(k_1k_2\,M)\,dS_1(k_1\,M)\,dS_2(k_2\,M), \end{aligned}$$

the last step by Fubini's theorem for distributions.

Chapter IV

Conical representations and conical distributions

Conical representations were introduced by Helgason in the course of a study of harmonic analysis on horocycle spaces. They are supposed to be counterparts of spherical representations, their theory, however, contrary to the abundant theory of the latter, is hardly begun. There is one definitive result, due to Helgason, stating that a finite-dimensional representation is conical if and only if it is spherical. This phenomenon is not, however, pertinent to the infinite-dimensional case — an example of a conical and not spherical representation will be given at the end of the chapter.

Helgason has defined conical representations as one possessing an MN-invariant vector — this is unsatisfactory because, as the work of Sherman [25] shows, such definition excludes unitary representations.

In the joint work of A. Wawrzyńczyk and the author [27] another definition was proposed and some properties of so defined representations were investigated. We shall now recall main points of that work and push the theory a little further on.

Let, as always, G be a connected semi-simple Lie group with finite center.

DEFINITION IV.1. Let (π, V) be an irreducible representation of G on the Banach space V. If there exist a form $\omega \in (V_\infty)'$ and a character χ of $\Gamma = MAN$, trivial on M, such that for $x \in V_\infty$

$$\langle \pi(man)x, \omega\rangle = \chi(man)\langle x, \omega\rangle, \qquad \text{all } man \in \Gamma,$$

then (π, V) is said to be *conical*, ω *— a conical form.*

Remarks. 1) Every such character, which we shall sometimes call conical character is determined by a linear complex form μ on $\mathfrak{a}$ by the formula $\chi(man) = e^{\mu(\log a)} = \chi_\mu(man)$.

2) For finite-dimensional representations Helgason's definition and our are equivalent.

We shall now prove an improvement of a result from [27].

THEOREM IV.1. *Let* (π, V) *be a conical representation and* χ *its conical character. Define* $\lambda \in \mathfrak{a}'_C$ *by*

$$\chi(man) = e^{(-i\lambda+\varrho)\log a} = \chi_{(-i\lambda+\varrho)}(man).$$

Then the formula

$$V_\infty \ni x \to \big(G \ni g \to \langle \pi(g^{-1})x, \omega\rangle\big)$$

defines an operator

$$T_0\colon V_\infty \to \mathscr{H}^\lambda,$$

which gives a Naimark equivalence of (π, V) *to a subrepresentation of* $(\pi^\lambda, \mathscr{H}^\lambda)$.

Proof. It is clear that the function $G \ni g \to \langle \pi(g^{-1})x, \omega\rangle$ is an element of $\mathscr{D}^\lambda$ for $x \in V_\infty$. To prove that $T_0\colon V_\infty \to \mathscr{D}^\lambda$ has a closed extension T observe that, if $V_\infty \supset \{x_n\}$ with $\|x_n\| \xrightarrow[n\to\infty]{} 0$ and $T_0 x_n \xrightarrow[n\to\infty]{} h$ in $\mathscr{H}^\lambda$, then for all $f \in \mathscr{D}(G)$ from the continuity of maps $\pi(f)\colon V \to V_\infty$ follows that $T_0\pi(f)x_n \xrightarrow[n\to\infty]{} \pi^\lambda(f)h$ simultaneously with $T_0\pi(f)x_n \xrightarrow[n\to\infty]{} 0$. So $\pi^\lambda(f)h = 0$ and by arbitrariness of $f \in \mathscr{D}(G)$ we see $h = 0$.

Intertwining property of T_0 is clear, so it remains to prove injectivity of T. But this is equivalent to injectivity of T_0 (to see this, let $x \in \ker T$; then $\pi(f)x \in V_\infty \cap \ker T$ for all $f \in \mathscr{D}(G)$, so T_0 is not injective — the other way it is clear). As T_0 maps continuously V_∞ into $\mathscr{D}^\lambda$ and (π_∞, V_∞) is irreducible T_0 has to be injective (cf. [2], [23]).

This theorem enables us to give a sufficient condition for the sphericity of a conical representation.

PROPOSITION IV.2. *Let* (π, V) *be a conical representation with* $\chi_{(-i\lambda+\varrho)}$ *as its conical character. Then if* λ *is simple,* (π, V) *is spherical.*

Proof. Recall that λ is simple if and only if the K-fixed vector ψ_0 in $\mathscr{D}'_{-\lambda}$ is cyclic. Since $\mathscr{D}'_{-\lambda}$ is the dual to $\mathscr{D}^\lambda$, the transpose to T_0 maps $\mathscr{D}'_{-\lambda}$ into $(V_\infty)'$. We claim that $T_0^t\psi_0 \neq 0$. In fact, if $T_0^t\psi_0 = 0$, then for all $x \in V_\infty$ $\langle T_0 x, \psi_0\rangle = 0$, in particular, $T_0^t\pi^{-\lambda}(g)'\psi_0 = 0$ for all $g \in G$. But T_0^t is continuous, hence it vanishes on a dense set only if it vanishes everywhere, and this is impossible since $T_0 \neq 0$.

So take $V_\infty \ni x$ for which $\langle T_0 x, \psi_0\rangle \neq 0$. By K-invariance of ψ_0, if P_0 denotes a projection on the set of K-invariant vectors in V, we see that $P_0 x \neq 0$, what was to be shown.

Now we turn to the resume of the theory of conical distributions, as given in Helgason's [15].

DEFINITION IV.2 (Helgason). A distribution $\Psi \in \mathscr{D}'(G/MN)$ is called *conical* if it is a common eigendistribution of $\boldsymbol{D}(G/MN)$ and is (left) MN-invariant.

Every common eigendistribution Ψ of $\boldsymbol{D}(G/MN)$ determines a character of $\boldsymbol{D}(G/MN)$, namely $\boldsymbol{D}(G/MN) \ni D \to \chi(D) \in \boldsymbol{C}$, where $\chi(D)$ is the eigenvalue of D corresponding to Ψ, i.e.,

$$D\Psi = \chi(D)\Psi.$$

We know that those characters are determined uniquely by linear forms $\lambda \in \mathfrak{a}_C'$, and in Chapter III we have introduced spaces $\mathscr{D}_\lambda'$ of eigendistributions of $D(G/MN)$ with the same character $\chi_{(i\lambda-\varrho)}$ as an eigenvalue. So every conical distribution lies in one of the spaces $\mathscr{D}_\lambda'$, and to determine all conical distributions it suffices to find all MN-invariants in $\mathscr{D}_\lambda'$. Their importance for the theory of conical representations resides in the fact that they are also A-eigenvectors, thus are conical forms for $\pi^{-\lambda}$.

Following Helgason, we shall now give a construction of a certain family of conical measures and discuss some properties of those. However, we shall omit entirely an analytic continuation procedure as we shall never need it.

First recall some structure of the connected semi-simple G with finite center. As in Chapter I, $G = KAN$ is the Iwasawa decomposition of G corresponding to a chosen order of roots, and let Σ_+ denote the set of positive roots in this order. There is a famous result, known as the Bruhat lemma, asserting that the set of double cosets $MAN \backslash G/MAN$ is in a bijective correspondence with the Weyl group, or what amounts to the same fact, $G = \bigcup_{w \in W} MANm_wMAN$, where the union is disjoint and m_w are representatives of $w \in W = M'/M$ in M'.

The elements of the double coset $MANm_wMAN$ are not uniquely determined in the form $\gamma m_w \gamma'$ with $\gamma, \gamma' \in \Gamma = MAN$; so we shall have to modify this a little in order to obtain uniqueness.

Since M' normalizes M and A, we can obviously write the Bruhat decomposition as $G = \bigcup_{w \in W} Nm_wMAN$.

$\mathfrak{n}$ — the Lie algebra of N is the sum of root spaces corresponding to positive roots, $\mathfrak{n} = \bigoplus_{a \in \Sigma_+} \mathfrak{g}^a$, and $N = \exp \mathfrak{n}$. W permutes root spaces as it permutes roots, namely

$$\mathrm{Ad}_G(m_w)\mathfrak{g}^a \subset \mathfrak{g}^{wa}.$$

If we define N_w' as the closed, analytic subgroup with the Lie algebra $\mathfrak{n}_w' = \bigoplus_{a \in \Sigma_+ \cap w\Sigma_+} \mathfrak{g}^a$, then it turns out that N decomposes as a product of N_w', and N_w, i.e. the closed analytic subgroup with the Lie algebra $\mathfrak{n}_w = \bigoplus_{a \in \Sigma_+ \cap w(-\Sigma_+)} \mathfrak{g}^a$. We have $N = N_w' \cdot N_w = N_w \cdot N_w'$ uniquely for every $n \in N$ and the Bruhat decomposition can be written $G = \bigcup_{w \in W} N_w m_w MAN$, what gives unique decomposition of g except for the choice of m_w in its class.

We come now to the construction of conical distributions. However, using the isomorphism given in Proposition III.4, we shall construct rather distributions on G than distributions on G/MN. Heuristically, in order to assure the homogenity property (III.5), which takes the form:

$$\Phi(R_{man}f) = e^{(-i\lambda+\varrho)\log a}\Phi(f),$$

it is necessary to integrate on the right over MAN with the kernel $e^{(i\lambda+\varrho)\log a}$. To get MN-invariance on the left, we have to integrate over MN. The question of convergence of such integrals is a priori unclear, however.

Proceeding more rigorously, consider the following integrals

$$\Phi_{\lambda,w}(f) = \int\limits_{N_w} d(n_w) \int\limits_{MAN} f(n_w m_w man) e^{(i\lambda+\varrho)\log a} dm\, da\, dn,$$

with $f \in \mathscr{D}(G)$.

It is evident at first that, for the neutral element e of the Weyl group, $N_e = \{\mathrm{id}\}$, and the corresponding double coset equals MAN, so it is closed. Then $\Phi_{\lambda,e}$ is absolutely convergent and is seen to be simply $\delta_e(\beta^{-\lambda}f)$. We summarize this in the form of

Corollary IV.3. *For all $\lambda \in \mathfrak{a}'_C$ the space $\mathscr{D}'_\lambda$ contains a measure δ concentrated on MAN and verifying*

$$\delta(L_{man}R_{m_1a_1n_1}f) = e^{(i\lambda+\varrho)\log a}e^{(-i\lambda+\varrho)\log a_1}\delta(f).$$

For other cosets the question is not as simple as that, and we shall give requirements for λ in order $\Phi_{\lambda,w}$ to be absolutely convergent.

Let $V_w = m_w^{-1}N_w m_w$; then since the Haar measure on N_w is carried over upon the Haar measure on V_w by the map $N_w \ni n \to m_w^{-1} n m_w \in V_w$, the integral defining $\Phi_{\lambda,w}$ can be rewritten as

$$\Phi_{\lambda,w}(f) = \int\limits_{V_w} d(v_w) \int\limits_{MAN} f(m_w v_w man) e^{(i\lambda+\varrho)\log a} dm\, da\, dn.$$

Decomposing v_w into Iwasawa decomposition factors, $v_w = k(v_w)\exp H(v_w) n(v_w)$ and using the invariance of measures, we transform this expression into

$$\begin{aligned}\Phi_{\lambda,w}(f) &= \int\limits_{V_w} d(v_w) \int\limits_{MAN} f\big(m_w k(v_w) m \exp H(v_w)\, an\big) e^{(i\lambda+\varrho)\log a} dm\, da\, dn\\ &= \int\limits_{V_w} e^{-(i\lambda+\varrho)H(v_w)} d(v_w) \int\limits_{MAN} f\big(m_w k(v_w) man\big) e^{(i\lambda+\varrho)\log a} dm\, da\, dn.\end{aligned}$$

The integral over MAN is always convergent (giving the Bruhat map), so we are left with

$$\Phi_{\lambda,w}(f) = \int\limits_{V_w} e^{-(i\lambda+\varrho)H(v_w)} (\beta^{-\lambda}f)\big(m_w k(v_w)\big)\, d(v_w).$$

Because $\beta^{-\lambda}f$ restricted to K is continuous, this is absolutely convergent if and only if

$$\int_{V_w} e^{-(i\lambda+\varrho)H(v_w)}d(v_w)$$

is absolutely convergent. But this is known to be the case if and only if λ satisfies $\mathrm{Re}(B(i\lambda, a)) > 0$ for all $a \in \Sigma_+ \cap w^{-1}\Sigma_-$ (compare [5], [15]). We sum up.

PROPOSITION IV.4 (Helgason). *Assume $\lambda \in \mathfrak{a}'_C$ satisfies*

$$\text{(IV.1)} \qquad \mathrm{Re}(B(i\lambda, a)) > 0 \quad \textit{for all} \ a \in \Sigma_+ \cap w^{-1}\Sigma_-.$$

Then the formula

$$\text{(IV.2)} \qquad \Phi_{\lambda,w}(f) = \int_{V_w} (\beta^{-\lambda}f)\big(m_w k(v_w)\big) e^{-(i\lambda+\varrho)H(v_w)}d(v_w)$$

defines a conical distribution (in fact, even a measure), belonging to $\mathcal{D}'_\lambda$ (with the identification of $\mathcal{D}'_\lambda$ with the space of distributions on G verifying (III.5)).

A measure $S_{\lambda,w}$ on K/M corresponding to it according to Proposition III.3, *is defined by*

$$S_{\lambda,w}(h) = \int_{V_w} e^{-(i\lambda+\varrho)H(v_w)} h\big(m_w k(v_w)M\big)\, d(v_w).$$

Let us determine homogenity properties of thus defined distributions. The left MN-invariance is most easily seen by observing that the measure dn on N is factored into product of measures $d(n_w)d(n'_w)$, wherefrom we obtain the invariance of $d(n_w)$ under translations by $n \in N$. Since the module of M-translation is 1, we infer MN-invariance.

Now consider A-translations of $\Phi_{\lambda,w}$,

$$\begin{aligned}\Phi_{\lambda,w}(L_a f) &= \int_{N_w} (\beta^{-\lambda}f)(a^{-1}n_w m_w)\, d(n_w)\\ &= \int_{N_w} (\beta^{-\lambda}f)(a^{-1}n_w a a^{-1} m_w)\, d(n_w)\\ &= \int_{N_w} (\beta^{-\lambda}f)(a^{-1}n_w a m_w m_w{}^{-1} a^{-1} m_w)\, d(n_w).\end{aligned}$$

Since A normalizes N_w; $a^{-1}N_w a \subset N_w$ and similarly for M', $m_w^{-1} a m_w \in A$ for $a \in A$. To finish computations we need a formula for the module of transformation $N_w \ni n_w \to a^{-1}n_w a$. It is computed using $N_w = \exp \mathfrak{n}_w$ and the formula for $d(n_w)$ in terms of the Lie algebra and found to be $e^{2\varrho_w \log a}$, where $\varrho_w = 1/2 \sum_{a>0, w^{-1}a<0} m^a a$. Since we have also $\log(m_w^{-1}a^{-1}m_w)$

$= -w^{-1}\log a$, we finally obtain

$$\Phi_{\lambda,w}(L_a f) = e^{2\varrho_w \log a} e^{(-i\lambda-\varrho)(-w^{-1}\log a)} \Phi_{\lambda,w}(f) = e^{(iw\lambda+\varrho)\log a} \Phi_{\lambda,w}(f).$$

Summarizing, we have

$$\Phi_{\lambda,w}(L_{man} f) = e^{(iw\lambda+\varrho)\log a} \Phi_{\lambda,w}(f). \tag{IV.3}$$

This formula will allow us to define intertwining operators for principal spherical series — that is operators mapping $\mathscr{D}^{\lambda}$ into $\mathscr{D}^{w\lambda}$ and intertwining for π^{λ} and $\pi^{w\lambda}$.

Those operators are of great importance for the study of representations of semi-simple Lie groups (we mention here their connection with problems of irreducibility of principal series, of analytic continuation, of existence and irreducibility of complementary series, etc.), and were investigated by several authors in different contextes. Their systematic study was begun by Kunze & Stein in [19], although special cases of such operators were already considered by Gelfand, Naimark and others in cases of classical groups, then it was undertaken by Schiffmann and also by Helgason in context of spherical representations. The last named author observed their close connection with conical distributions. Applications to the problem of existence of complementary series were given, among other results, in the work of Knapp & Stein [16]. From this vast theory we shall give here the simplest facts which will be needed for our purposes.

Let us observe that, by virtue of (IV. 3), the function h given by $G \ni g \to \Phi_{\lambda,w}(L_{g^{-1}} f)$ verifies (III.1), with λ replaced by $-w\lambda$. Since this function is infinitely differentiable, it belongs to $\mathscr{D}^{-w\lambda}$, and as the function h depends only on $\beta^{-\lambda} f$ and not f itself (by virtue of (IV. 2)), the map $f \to h$ defines a map $\mathscr{D}^{-\lambda}$ upon $\mathscr{D}^{-w\lambda}$. Because of regularity of $\Phi_{\lambda,w}$ this map has a nice continuity property.

Proposition IV.5 (Schiffmann). *Let the map $A(\lambda, w)\colon \mathscr{D}^{-\lambda} \to \mathscr{D}^{-w\lambda}$ be defined by*

$$\big(A(\lambda, w)f\big)(g) = \int_{N_w} f(g n_w m_w)\, d(n_w)$$

for λ satisfying $\operatorname{Re}\big(B(i\lambda, a)\big) > 0$ for all $a \in \Sigma_+ \cap w_{-1}\Sigma_-$. Then it can be extended to the bounded map $\mathscr{H}^{-\lambda} \to \mathscr{H}^{-w\lambda}$ of the norm less or equal to $\int_{V_w} |e^{-(i\lambda+\varrho)H(v_w)}|\, d(v_w)$.

Proof. It suffices to show the boundedness of $A(\lambda, w)$. We have

$$\begin{aligned} \|A(\lambda, w)f\|^2 &= \int_K \big|\big(A(\lambda, w)f\big)(k)\big|^2\, dk \\ &= \int_K \Big| \int_{N_w} f(k n_w m_w)\, d(n_w) \Big|^2 dk \\ &= \int_K \Big| \int_{V_w} f\big(k m_w k(v_w)\big)\, e^{-(i\lambda+\varrho)H(v_w)} d(v_w) \Big|^2 dk. \end{aligned} \tag{IV.4}$$

Applying Schwarz inequality to the inner integral we get

$$\Big|\int_{V_w} f(km_w k(v_w)\, e^{-(i\lambda+\varrho)H(v_w)}\, d(v_w)\Big|$$
$$\leqslant \Big(\int_{V_w} |e^{-(i\lambda+\varrho)H(v_w)}|\, d(v_w)\Big)^{1/2} \Big(\int_{V_w} |f(km_w k(v_w))|^2 |e^{-(i\lambda+\varrho)H(v_w)}|\, d(v_w)\Big)^{1/2}$$

Inserting this into (IV.4) and changing the order of integration, we obtain

$$\|A(\lambda, w)f\|^2 \leqslant \Big(\int_{V_w} |e^{-(i\lambda+\varrho)H(v_w)}|\, d(v_w)\Big)^2 \|f\|^2,$$

what finishes the proof.

Remark. We have defined conical distributions and intertwining operators for λ lying in a certain tube in $\mathfrak{a}'_C$, namely for λ satisfying $\mathrm{Re}\big(B(i\lambda, a)\big) > 0$ for all $a \in \Sigma_+ \cap w^{-1}\Sigma_-$. In particular, our construction applies for λ belonging to the tube over the fundamental Weyl chamber, that is the set $C = \{\lambda\colon \mathrm{Re}\big(B(i\lambda, a)\big) > 0$ for all $a \in \Sigma_+\}$.

However, mentioned authors elaborated a method of analytic (with respect to λ) continuation of integrals of the kind which allows us to define intertwining operators for all $\lambda \in \mathfrak{a}'_C$. It rests on the observation that with an appropriate factor $\gamma(\lambda, w)$, accounting for possible poles, the function

$$\mathfrak{a}'_C \ni \lambda \to \gamma(\lambda, w) A(\lambda, w) f(g)$$

is analytically prolongable to an entire function on $\mathfrak{a}'_C$. For details see [15], [16], [17], [24].

A problem arises whether so defined distributions $\Phi_{\lambda,w}$ are all conical distributions in $\mathscr{D}'_\lambda$. That this is so was shown by Helgason.

We shall state here his theorem only for $\lambda \in C$, as this is the case of our main concern. It is readily seen that for such λ condition (IV.1) is satisfied, hence $\Phi_{\lambda,w}$ are given by formula (IV.2).

Recall that $c(\cdot)$ function of Harish-Chandra is given as

$$c(\lambda) = \int_V e^{-(i\lambda+\varrho)H(v)} dv, \qquad V = V_e.$$

We now state

THEOREM IV.6 (Helgason). *Assume*

(i) λ *regular* (*i.e.*, $w\lambda \neq \lambda$ *for* $w \neq e$),

(ii) $\lambda \in C = \{\lambda \in \mathfrak{a}'_C\colon \mathrm{Re}\big(B(i\lambda, a)\big) > 0$ *for all* $a \in \Sigma_+\}$,

(iii) $c(\lambda) \neq 0$.

Then every conical distribution in $\mathscr{D}'_\lambda$ *is a linear combination of* $\Phi_{\lambda,w}$.

For the proof see [15], pp. 96 ff.

Let us now return to the study of conical representations, and consider the case of representations in spaces with an indefinite metric. We recall that an indefinite metric on a Banach space V is a hermitian continuous form on V, setting up an isomorphism of V with its topological dual, and representation in such space is a representation which acts by operators preserving this form. In such case, the above isomorphism of V with V' gives an identification of contragredient representation with the given one. Of course, this class of representations includes unitary representations as well as representations in Pontryagin spaces. It is natural and perhaps even compulsory to investigate such representations, because they arise naturally in the process of an analytic continuation of representations of principal series (cf. [17]).

In the joint work of A. Wawrzyńczyk and the author it was shown that every conical representation in the space with an indefinite metric is equivalent (with preservation of norm and metric) to a representation induced by an appropriate character of $\Gamma = MAN$ with the indefinite metric given by conical distribution.

We shall now summarize those results and pursue this line a little further on. Let, as at the beginning of this chapter, (π, V) be a conical representation with $\chi_{(-i\lambda+\varrho)}$ as its conical character and ω its conical form. Let also $T\colon V\to\mathscr{H}^{\lambda}$ be the operator constructed in Theorem IV.1. Using the canonical sesquilinear and G-invariant pairing between $\mathscr{H}^{\lambda}$ and $\mathscr{H}^{\bar{\lambda}}$

$$\mathscr{H}^{\lambda}\times\mathscr{H}^{\bar{\lambda}}\ni(f, h)\to\int\limits_{K} f(k)\overline{h(k)}dk = \langle f|h\rangle,$$

we can define a sesquilinear G-invariant and partially continuous form between V_{∞} and $\mathscr{D}^{\bar{\lambda}}$, $\xi(v, h) = \langle Tv|h\rangle$. Then, according to a theorem of N. Skovhus Poulsen, there exists an operator, say S, mapping continuously $\mathscr{D}^{\bar{\lambda}}$ into V'_{∞}, intertwining for the action of G in respective spaces and connected with ξ by the formula $\xi(v, h) = \langle v, Sh\rangle$. Since π preserves an indefinite metric $[\cdot, \cdot]$, it is naturally equivalent to its contragredient, and so we can regard S as the map into V, using $\langle v, Sh\rangle = [v, Sh]$. Carrying over the indefinite metric to $\mathscr{D}^{\bar{\lambda}}$ via S, and then lifting resulting sesquilinear form to $\mathscr{D}(G)$ by means of $\beta^{\bar{\lambda}}$, we obtain existence of a distribution Ψ such that the following holds:

$$[S\beta^{\bar{\lambda}}f, S\beta^{\bar{\lambda}}h] = \Psi(h^{*}*f).$$

It is a trivial matter to verify that Ψ satisfies the following:

(IV. 5) $$\Psi(L_{man}f) = e^{(-i\lambda+\varrho)\log a}\Psi(f),$$

(IV. 6) $$\Psi(R_{man}f) = e^{(i\bar{\lambda}+\varrho)\log a}\Psi(f).$$

Comparing with Proposition III.4, we see that Ψ belongs to $\mathcal{D}'_{-\bar{\lambda}}$ and is a conical distribution.

These are ideas behind the proof of the following theorem from [27].

THEOREM IV.7. *Let* (π, V) *be a conical representation in the space with an indefinite metric* $[\cdot, \cdot]$; *and* $\chi_{(-i\lambda+\varrho)}$, ω *its character and conical form respectively. Then there exist a continuous semi-norm* q *on* $\mathcal{D}^{\bar{\lambda}}$ *and conical distribution* $\Psi \in \mathcal{D}'_{-\bar{\lambda}}$, *such that* (π, V) *is isometrically equivalent to the Banach space representation, obtained from* $(\mathcal{D}^{\bar{\lambda}}, q)$ *by completion.*

Moreover, Ψ *defines an indefinite metric on the above extension of* $(\mathcal{D}^{\bar{\lambda}}, q)$ *by means of the formula*

$$(\beta^{\bar{\lambda}} f, \beta^{\bar{\lambda}} h) \to \Psi(h^* * f), \tag{IV.7}$$

which corresponds to $[\cdot, \cdot]$ *under this isometry.*

COROLLARY IV. 8. *Every unitary conical representation with a conical character* $\chi_{(-i\lambda+\varrho)}$ *is unitary equivalent to the unitary extension of the left regular representation in* $\mathcal{D}^{\bar{\lambda}}$, *with a scalar product defined by a positive definite conical distribution from* $\mathcal{D}'_{-\bar{\lambda}}$.

Let us now look at the situation in the multiplier realization of this representation.

PROPOSITION IV. 9. *Let* $\Psi \in \mathcal{D}(G)$ *satisfy* (IV.5) *and* (IV.6), *and let* S *correspond to it via Proposition* III.3. *Denote by* $f \to f_\lambda$ *the map* $\mathcal{D}(G)$ *onto* $\mathcal{E}(K/M)$, *given by* $f_\lambda(kM) = \beta^\lambda f(k)$. *Then in the notation of Lemma* III.14,

$$\Psi(h^* * f) = (\overline{h_{\bar{\lambda}}}) \times S(f_{\bar{\lambda}}) = (\bar{h})_\lambda \times S(f_{\bar{\lambda}}).$$

Proof. First note that

$$(h^* * f)_\lambda(kM) = \int_G \bar{h}(g^{-1}) \tau_\lambda(g) f_\lambda(kM)\, dg.$$

Now, by the definition of S (see Proposition III. 3),

$$\begin{aligned} \Psi(h^* * f) = \langle (h^* * f)_\lambda, S\rangle &= \int_G \overline{h(g)} \langle \tau_{\bar{\lambda}}(g^{-1}) f_{\bar{\lambda}}, S\rangle \\ &= \int_{KAN} \overline{h(kan)} \langle \tau_{\bar{\lambda}}((kan)^{-1}) f_{\bar{\lambda}}, S\rangle e^{2\varrho \log a}\, dk\, da\, dn \\ &= \int_{KAN} \overline{h(kan)} e^{(i\lambda+\varrho)\log a} \langle \tau_{\bar{\lambda}}(k^{-1}) f_{\bar{\lambda}}, S\rangle\, dk\, da\, dn \\ &= \int_{K/M} \overline{h_{\bar{\lambda}}(kM)} \langle f_{\bar{\lambda}}(kk_1M), S_{k_1}\rangle\, d(kM). \end{aligned}$$

The last equality, by virtue of Lemma III.14, gives the desired result.

The above proposition reduces, at least to a certain extent, the study of conical representations to the problem of harmonic analysis on the compact space K/M.

We shall get more information about conical representations assuming that λ determining conical character, is simple. Note that then $-\bar{\lambda}$ is simple as well.

COROLLARY IV. 10. *Let the hypotheses be as in Theorem* IV. 6, *and let* (π, V) *be a conical representation in the space with an indefinite metric, with* $\chi_{(-i\lambda+\varrho)}$ *as its conical character. Then there exist a* $w \in W$ *such that* $w\lambda = \bar{\lambda}$ *and a unique, up to a scalar multiple, conical distribution* Φ *satisfying* (IV. 5) *and* (IV. 6) *and defining indefinite metric via* (IV. 7).

Proof. By (IV. 6), Φ is equal to $\sum_{w \in W} c_w \Phi_{\lambda,w}$, but by virtue of (IV. 3), all $c_w = 0$, except for such $w \in W$ for which $w\lambda = \bar{\lambda}$.

Remark. This together with Proposition IV. 2 shows that (π, V) is both spherical and conical, and the indefinite metric is given by a conical distribution. We shall see later on that a kind of converse is true, namely that certain unitary spherical representations are conical and the scalar product is given by conical distribution. However, not all conical representations are spherical as the following example will show.

EXAMPLE. Let $G = SU(1,1)$ be the group of conformal transformations of the unit disc in the complex plane. G consists of matrices $\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$ with $a, b \in C$ and $|a|^2 - |b|^2 = 1$. The subgroups of the Iwasawa decomposition are

$$K = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}, \theta \in \boldsymbol{R}\right\}, \quad A = \left\{\begin{bmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{bmatrix}, t \in \boldsymbol{R}\right\},$$

$$N = \left\{\begin{bmatrix} 1+in & -in \\ in & 1-in \end{bmatrix}, n \in \boldsymbol{R}\right\}$$

with their Lie algebras spanned over $\boldsymbol{R}$ by

$$k = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad n = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} \quad \text{resp..}$$

G acts in the complex plane by linear fractional transformations

$$z \to g(z) = \frac{az+b}{\bar{b}z+\bar{a}} \quad \text{where} \quad g = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}.$$

If we identify K/M with the unit circle S^1 via the map $K \ni g \to g(1) \in S^1$, then the representations τ_λ of Lemma III.1 are

$$\mathscr{L}^2(S^1) \ni f \to \big(\tau_\lambda(g)f\big)(\omega) = |\bar{a}\omega - b|^{2s} f\left(\frac{\bar{a}\omega - b}{-\bar{b}\omega + a}\right)$$

where $g = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$, and we have put $2s = (i\lambda - \varrho)(h) \in C$.

The space of smooth vectors for the representation τ_λ consists of precisely the smooth functions on S^1 and the Dirac's δ supported at the origin $1 \in S^1$ corresponds to the conical distribution $\Phi_{\lambda,c}$ of Proposition IV.3.

Consider the action of the $\mathfrak{A}(\mathfrak{g})$ induced by τ_λ on the space of smooth vectors. The elements $k = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $x_+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $x_- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ together with the unity generate $\mathfrak{A}(\mathfrak{g})$ and their action on $\mathscr{E}(S^1)$ is given by

$$\tau_\lambda(k) = -2\frac{d}{d\theta},$$

$$\tau_\lambda(x_+) = -se^{i\theta} - e^{i\theta}\frac{d}{d\theta},$$

$$\tau_\lambda(x_-) = -se^{-i\theta} + e^{-i\theta}\frac{d}{d\theta}.$$

By looking at the action of $\mathfrak{A}(\mathfrak{g})$ on the basis consisting of exponentials $f_n(\theta) = e^{in\theta}$ we see that

$$\tau_\lambda(k)f_n = -2nf_n,$$
$$\tau_\lambda(x_+)f_n = (-s+n)f_{n+1},$$
$$\tau_\lambda(x_-)f_n = (-s-n)f_{n-1}.$$

Hence in the case where s is a negative integer $\mathscr{L}^2(S^1)$ contains closed invariant subspaces $\mathscr{F}_+$ and $\mathscr{F}_-$, spanned by combinations of exponentials f_n with $n \geqslant -s$ and $n \leqslant s$ resp. One can show that these subspaces are irreducible under corresponding τ_λ but the resulting representations are not unitary with respect to the scalar product inherited from $\mathscr{L}^2(S^1)$. Nevertheless they provide us with the desired example — they are conical, the conical form being δ_1 and obviously not spherical as they do not contain the K-invariant function f_0.

Chapter V

Induced spherical representations

This chapter centers around two theorems, one is "equivalence theorem for spherical representations" (Theorem V.1) which, in the case of spherical representations, strengthens Harish-Chandra's theorem which is in the literature called "subquotient theorem" (cf. Harish-Chandra [8], [9]; Warner [28]). The other is Theorem V.11 which gives realization of unitary spherical representations as representations of the complementary series in the sense of Knapp & Stein (cf. [16], [17]).

A word about Kostant results from [18] and their relation to ours, especially Theorem V.1 is in order. Kostant introduces a class of what he calls "admissible modules" of the universal enveloping algebra $\mathfrak{A}(\mathfrak{g})$ — these are algebraically irreducible representations of $\mathfrak{A}(\mathfrak{g})$ which in addition contain 1-dimensional subspace anihilating $\mathfrak{k}$ – the Lie algebra of K. This is modeled after the case of representations of $\mathfrak{A}(\mathfrak{g})$ on the space of K-finite vectors from the space of spherical representation. Then he proves that the $\mathfrak{A}(\mathfrak{g})$-modules, generated from the K-invariant vector in $\mathscr{H}^{\lambda}$, are algebraically irreducible if λ verifies $\mathrm{Re}(B(i\lambda, a)) \geqslant 0$ for all positive roots a, and that one obtains in this way all admissible $\mathfrak{A}(\mathfrak{g})$-modules (up to an algebraic equivalence, of course).

We prove here a similar result but for the action of G rather than that of $\mathfrak{A}(\mathfrak{g})$ and for λ verifying $\mathrm{Re}(B(i\lambda, a)) > 0$ only. The reason is that we rely on Proposition III.11, and it is not known whether its conclusions persist in the case when λ is orthogonal to some of the roots.

Observe now that if $\lambda \in \mathfrak{a}'_C$ and $\mathrm{Im}\,\lambda$ is regular, i.e., none of the roots is orthogonal to $\mathrm{Im}\,\lambda$, then there exists unique $w \in W$ such that $w\,\mathrm{Im}\,\lambda$ belongs to the (open) fundamental Weyl chamber defined as $\{\lambda \in \mathfrak{a}' : B(\lambda, a) > 0$ for all $a \in \Sigma_+\}$. With this in mind we state

THEOREM V.1. *As above, let* $C = \{\lambda \in \mathfrak{a}'_C \colon \mathrm{Re}(B(i\lambda, a)) > 0$ *for all* $a \in \Sigma_+\}$.

(i) *If* $\lambda \in C$, *then* $\mathscr{D}^{\lambda}_K$ *— the closed subspace of* $\mathscr{D}^{\lambda}$ *spanned by* G*-translates of the* K*-invariant vector* ψ_0 *in* $\mathscr{D}^{\lambda}$ *is irreducible under the action of* G.

(ii) *If* (π, V) *is spherical quasi-simple representation on a complete semi-reflexive locally convex space* V *with the spherical function* φ_{μ} *with*

$\operatorname{Im}\mu$ regular, then (π, V) is Naimark equivalent to $\mathscr{D}_K^\lambda$ for a unique $\lambda \in C$ which is congruent modulo the Weyl group to μ.

Remark. We recall here that, according to the discussion in Chapter II, there is one and only one spherical function associated with such representation and μ is determined modulo the Weyl group uniquely.

Proof. To prove (i), it suffices to observe that, on the basis of Proposition III.13, λ is simple so ψ_0 is cyclic for $(\pi_{-\lambda}, \mathscr{D}'_{-\lambda})$, and then apply Proposition II.5.

(ii) Let μ and λ be as in the hypotheses. Consider the map

$$\eta\colon \mathscr{D}(G/K) \to V; \quad \eta(f) = \pi(f)v_0,$$

where $v_0 \in V$ is K-fixed. Evidently η intertwines the left regular representation in $\mathscr{D}(G/K)$ with π. We shall show that if $F^\lambda f = 0$, F^λ is the Fourier transform, then $\eta(f) = 0$. In fact, take an arbitrary $h \in \mathscr{D}(G/K)$, $v^0 \in V'$ a K-invariant vector. Then

$$\langle \eta(f), \pi'(h)v^0 \rangle = \int_{G/K} h(gK) f * \varphi_\lambda(gK)\, d(gK), \tag{V.1}$$

so we have to compute convolution with a spherical function.

LEMMA V.2 (Helgason). *For each $f \in \mathscr{D}(G/K)$,*

$$f * \varphi_\lambda(g) = \int_K e^{-(i\lambda+\varrho)H(g^{-1}k)} \tilde{f}(\lambda, kM)\, dk. \tag{V.2}$$

Proof. By virtue of Lemma III.1 and Harish-Chandra's formula (II.1), we have

$$\varphi_\lambda(g) = \big(\tau_\lambda(g^{-1})\psi_0, \psi_0\big) = \int_K e^{(i\lambda-\varrho)H(gk)}\, dk,$$

so

$$f * \varphi_\lambda(g) = \int_G f(y)\big(\tau^\lambda(y^{-1}g)\psi_0, \psi_0\big)\, dy = \big(\tau_\lambda(g^{-1})\tau_\lambda(f)\psi_0, \psi_0\big)$$
$$= \int_K e^{(i\lambda-\varrho)H(gk)} \tilde{f}\big(\lambda, g(kM)\big)\, dk,$$

the last equality resulting from Lemma III.8.

Now, the rest is a simple change of variables, using $H(gk) = -H\big(g^{-1}k(gk)\big)$ and the formula

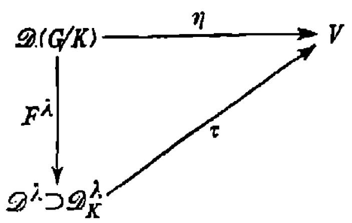
$$\int_{K/M} h\big(g(kM)\big) e^{-2\varrho H(gk)}\, d(kM) = \int_{K/M} h(kM)\, d(kM).$$

Now, going back to the proof of the theorem and using the lemma above, we turn (V.1) into

$$\langle \eta(f), \pi'(h)v^0 \rangle = \int\limits_{G/K} \int\limits_{K} h(gK) e^{-(i\lambda+\varrho)H(g^{-1}k)} \tilde{f}(\lambda, kM)\, dk\, d(gK),$$

what is seen to be equal 0 when $F^\lambda f = 0$. Remembering that the set $\{\pi'(h)v^0\colon h \in \mathscr{D}(G/K)\}$ is dense in V', we infer that $\eta(f)$ must be equal 0.

This allows us to define $\tau\colon F^\lambda(\mathscr{D}(G/K)) \to V$ by means of the identity $\tau \circ F^\lambda = \eta$, and so we have a commutative diagram of mappings



In fact, τ is injective on the set $F^\lambda(\mathscr{D}(G/K))$, since if $\tilde{f}(\lambda, \cdot) \in \ker \tau$, then

$$\begin{aligned} 0 &= \langle \tau\tilde{f}, \pi'(h)v^0 \rangle = \langle \eta(f), \pi'(h)v^0 \rangle \\ &= \int\limits_{G/K} \int\limits_{K} h(gK) e^{-(i\lambda+\varrho)H(g^{-1}k)} \tilde{f}(\lambda, kM)\, dk\, d(gK) \\ &= \int\limits_{K/M} \tilde{h}(-\lambda, kM) \tilde{f}(\lambda, kM)\, d(kM). \end{aligned}$$

Now λ is simple so the set $\{F^{-\lambda}h\colon h \in \mathscr{D}(G/K)\}$ is dense in $\mathscr{L}^2(K/M)$, what implies $\tilde{f}(\lambda, \cdot) = 0$.

Injectivity of τ results from the irreducibility of $\mathscr{D}^\lambda_K$ and V, so it remains to show that τ has closed extension.

Assume then $\|\tilde{f}_n(\lambda, \cdot)\|^2 \xrightarrow[n\to\infty]{} 0$ and $\tau\tilde{f}_n \to v \in V$. From

$$\langle \tau\tilde{f}_n, \pi'(h)v^0 \rangle = \int\limits_{K/M} \tilde{h}(-\lambda, KM) \tilde{f}(\lambda, kM)\, d(kM)$$

one can easily see that the right-hand side tends to 0 and so must the left, which, due to density of $\{\pi'(h)v^0\}$, means $v = 0$. This shows that τ is a Naimark equivalence, as asserted.

Remark. Comparing Theorem V.1 with Harish-Chandra's subquotient theorem we see that at the cost of supposing λ sufficiently regular, we obtain strengthening of conclusions — namely equivalence to an irreducible subrepresentation rather than to an irreducible quotient representation of some subspaces of $\mathscr{D}^\lambda$.

Using this theorem it is easy to obtain two dual classes of models for spherical representations. Similarly to the case of conical representations, we shall consider representations in spaces with an indefinite metric rather than unitary representations for reasons indicated above.

First, by combining Kostant's Proposition 3 from [18] with some Harish-Chandra theorem, we obtain the following

Proposition V.3. *Let (π, V) be a spherical representation leaving invariant an indefinite metric $[\cdot, \cdot]$ on V and let φ_λ be its spherical function. Then there exists an element w of the Weyl group W such that $w\lambda = \bar{\lambda}$.*

Proof. Throughout the proof we shall use Kostant's notations from [18]. Consider the derived action of $\mathfrak{A}(\mathfrak{g})$ on the space of K-finite vectors from V. By the Harish-Chandra theorem from [8] (see also Warner, Theorem 4.5.5.4), we see that the resulting $\mathfrak{A}(\mathfrak{g})$-module is admissible and from the preceding Theorem V.1 we immediately obtain that this module belongs to the class $C_{(-i\lambda+\varrho)}$. Now indefinite metric induces an $\mathfrak{A}(\mathfrak{g})$-invariant form on the module and thus Proposition 3 of [18] applies giving the existence of such $w \in W$.

We pass now to the construction of these dual models of spherical representations. Let as above (π, V) be a spherical representation and $[\cdot, \cdot]$ an invariant indefinite metric. Choose λ such that it belongs to C and $\varphi_\mu = \varphi_\lambda$. (We assume, as in the theorem above, that $\operatorname{Im}\mu$ is regular.) Then, by means of the η, τ constructed in Theorem V.1, we can carry over the structure of $(\pi, V, [\cdot, \cdot])$ to $\mathcal{D}(G/K)$ on the one hand, and to $F^\lambda(\mathcal{D}(G/K))$, on the other. By the usual procedure of extending operators of representation and the indefinite metric to the completion of the spaces in question, we obtain two representations $(\pi^\varphi, V^\varphi, [\cdot, \cdot]^\varphi)$ and $(\pi^\lambda, V^\lambda, [\cdot, \cdot]^\lambda)$ isometrically equivalent to the given representation $(\pi, V, [\cdot, \cdot])$.

Corollary V.4 (compare Helgason [15], Wawrzyńczyk [29]). *Let (π, V) be a spherical representation by operators leaving invariant the form $[\cdot, \cdot]$, and $\varphi = \varphi_\lambda$ its spherical function with $\lambda \in C$. Equip $\mathcal{D}(G/K)$ with the norm and the metric (indefinite) transported by η of Theorem* V.1 *and $F^\lambda(\mathcal{D}(G/K))$ with the norm and indefinite metric transported by τ. Denote their completions by V^φ and V^λ respectively, extended representations by π^φ, π^λ and indefinite metrics by $[\cdot, \cdot]^\varphi$, $[\cdot, \cdot]^\lambda$ resp. Then the Fourier transform F^λ extends to an isometry of V^φ onto V^λ, intertwining for π^φ and π^λ and carrying over $[\cdot, \cdot]^\varphi$ upon $[\cdot, \cdot]^\lambda$.*

It is a simple exercise to derive a formula for indefinite metrics $[\cdot, \cdot]^\varphi$ and $[\cdot, \cdot]^\lambda$. The former case is classic, the latter was obtained by Helgason in the case of unitary representations and λ real.

Proposition V. 5. *Let $(\pi, V, [\cdot, \cdot])$ be a spherical representation with a K-fixed vector v^0 and the spherical function φ_μ. With $\eta\colon \mathcal{D}(G/K) \to V$ defined as $\eta(f) = \pi(f)v^0 \in V$, we have*

$$[f, h]^\varphi = [\eta(f), \eta(h)] = \int_G \overline{h(x)} f * \varphi_\mu(x)\, dx$$

$$= \int_{K/M} \tilde{f}(\mu, kM)\overline{\tilde{h}(\bar{\mu}, kM)}\, d(kM).$$

Proof. The spherical function $\varphi_\mu(g)$ is given as $\varphi_\mu(g) = [\pi(g^{-1})v^0, v^0]$. Then $[\eta(f), \eta(h)] = [\pi(f)v^0, \pi(h)v^0] = \int_G \overline{h(x)} f*\varphi_\mu(x)\, dx$. To prove the second equality we shall make an appeal to Lemma V.2. Inserting (V.2) into the last expression above, we have exactly as in the proof of Theorem V.1

$$[\eta(f), \eta(h)] = \int_G \overline{h(x)} \int_K e^{-(i\mu+\varrho)H(x^{-1}k)} \tilde{f}(\mu, kM)\, dk\, dx$$

$$= \int_K \tilde{f}(\mu, kM) \int_G \overline{h(x)} e^{-(i\mu+\varrho)H(x^{-1}k)}\, dx\, dk$$

$$= \int_{K/M} \tilde{f}(\mu, kM) \overline{\tilde{h}(\bar{\mu}, kM)}\, d(kM).$$

We have then the following extension of Helgason's Proposition 5.4, p. 116 from [15].

Corollary V. 6. *In addition to the assumptions of Proposition* V.5, *let* $\mu \in C$. *Then for* $\tilde{f}, \tilde{h} \in F^\mu(\mathscr{D}(G/K))$,

$$[\tilde{f}, \tilde{h}]^\mu = [\tau(\tilde{f}), \tau(\tilde{h})] = \int_{K/M} \tilde{f}(\mu, kM) \overline{\tilde{h}(\bar{\mu}, kM)}\, d(kM).$$

In particular, if (π, V) *is a unitary spherical representation, then the form*

$$\text{(V.3)} \qquad F^\mu(\mathscr{D}(G/K)) \times F^\mu(\mathscr{D}(G/K)) \ni (\tilde{f}, \tilde{h}) \to \int_{K/M} \tilde{f}(\mu, kM) \overline{\tilde{h}(\bar{\mu}, kM)}\, d(kM)$$

is a positive definite hermitian form on $F^\mu(\mathscr{D}(G/K))$, *invariant under* τ_μ, *and hence defines a unitary spherical representation equivalent with* (π, V)

In the rest of the chapter we shall confine ourselves to unitary spherical representations and implications for those of Theorem V.1. Particularly we shall be interested whether the form (V.3) can be extended to the whole of $\mathscr{D}^\mu$, hence defining a complementary series representation in the sense of the following definition of Knapp & Stein.

Definition V.1. Let $(\pi_\lambda, \mathscr{D}^\lambda)$ be a differentiable representation induced by a *nonunitary* character of MAN, $man \to e^{-i\lambda \log a}$ (so λ is not real). If there exists a continuous hermitian form on $\mathscr{D}^\lambda$ invariant under π_λ, then this representation is said to belong to:

(a) *complementary series* if the form is positive definite,

(b) *quasi-complementary series* if the form is positive semi-definite.

Remark. The definition makes sense and was in fact formulated for not necessarily 1-dimensional representation of MAN.

On the basis of Proposition IV. 8, we infer that the following holds.

COROLLARY V.8. *Every unitary conical representation belongs either to principal or to quasi-complementary series.*

Turning now to unitary spherical representations, observe first that if $\lambda \in C = \{\lambda \in \mathfrak{a}'_C \colon \operatorname{Re}(B(i\lambda, \alpha)) > 0$ for all $\alpha \in \Sigma_+\}$, then $-\bar{\lambda} \in C$ also, hence for $w \in W$, for which $w\lambda = \bar{\lambda}$ $\Phi_{-\bar{\lambda},w}$ is defined by formula (IV.2) and is a distribution — even the measure — from $\mathcal{D}'_{-\bar{\lambda}}$ which satisfies

$$\Phi_{-\bar{\lambda},w}(R_{man}f) = e^{(i\bar{\lambda}+\varrho)\log a}\Phi_{-\bar{\lambda},w}(f),$$

$$\Phi_{-\bar{\lambda},w}(L_{man}f) = e^{(-i\lambda+\varrho)\log a}\Phi_{-\bar{\lambda},w}(f).$$

We have also the following

LEMMA V.9 (Helgason). *The dual Radon transform $R_{-\bar{\lambda}}(\Phi_{-\bar{\lambda},w})(\cdot)$ is given by the formula*

$$R_{-\bar{\lambda}}(\Phi_{-\bar{\lambda},w})(g) = ce^{(i\lambda-\varrho)H(g^{-1})},$$

with a constant c different from 0 if $c(\lambda) \neq 0$.

For the proof of the lemma we refer to Helgason's [15], p. 95.

Using this lemma we shall prove

PROPOSITION V. 10. *Let f, $h \in \mathcal{D}(G/K)$. Then*

$$\Phi_{-\bar{\lambda},w}(h^* * f) = c\int_{K/M} \tilde{f}(\lambda, KM)\overline{\tilde{h}(\bar{\lambda}, kM)}\, d(kM), \tag{V.4}$$

where c is the constant above.

Proof. We first transport the computations to K/M by means of Proposition III.3 and Proposition IV.9,

$$\Phi_{-\bar{\lambda},w}(h^* * f) = \int_{K/M} (\overline{h_{\bar{\lambda}}}) \times S(kM) f_{\bar{\lambda}}(kM)\, d(kM)$$

$$= \int_G f(g) \int_{K/M} e^{(i\bar{\lambda}-\varrho)H(g^{-1}k)}\, dS_1(kM)\, dg, \tag{V.5}$$

where we have put $S_1(kM) = (\overline{h_{\bar{\lambda}}}) \times S(kM)$.

Compute now $R_{-\bar{\lambda}}(S_1)(g) = \int_{K/M} e^{(i\bar{\lambda}-\varrho)H(g^{-1}k)}\, dS_1(kM)$. By virtue of Lemma III.14 we transform this into

$$\big(R_{-\bar{\lambda}}(S_1)\big)(g) = \int_{K/M}\int_{K/M} e^{(i\bar{\lambda}-\varrho)H(g^{-1}k_1k_2)}(\overline{h_{\bar{\lambda}}})(k_1M)\, d(k_1M)\, dS(k_2M)$$

$$= \int_{K/M} (\overline{h_{\bar{\lambda}}})(k_1 M) \int_{K/M} e^{(i\bar{\lambda}-\varrho)H(g^{-1}k_1k_2)}\, dS(k_2 M)\, d(k_1M).$$

By making an appeal to Lemma V.9, with g replaced by $k_1^{-1}g$, we see that the inner integral is equal to

$$\int_{K/M} e^{(i\bar{\lambda}-\varrho)H(g^{-1}k_1k_2)}\,dS(k_2M) = R_{-\bar{\lambda},w}(\Phi_{-\bar{\lambda},w})(k_1^{-1}g) = ce^{(i\lambda-\varrho)H(g^{-1}k_1)},$$

hence

$$R_{-\bar{\lambda}}(S_1)(g) = c\int_{K/M} (\overline{h_{\bar{\lambda}}})(k_1M)\,e^{(i\lambda-\varrho)H(g^{-1}k_1)}\,d(k_1M).$$

Inserting this into (V.5), we obtain

$$\Phi_{-\bar{\lambda},w}(h^* * f) = c\int_G f(g)\int_{K/M} (\overline{h_{\bar{\lambda}}})(k_1M)\,e^{(i\lambda-\varrho)H(g^{-1}k_1)}\,d(k_1M)\,dg$$
$$= c\int_{K/M} \tilde{f}(\lambda, kM)\overline{\tilde{h}(\bar{\lambda}, kM)}\,d(kM),$$

as asserted.

Let us note the following corollary of the proof

COROLLARY V.11 (Helgason). *Let $\tilde{A}(\lambda, w)$ be the intertwining operator for τ_λ and $\tau_{w\lambda}$ obtained from $A(\lambda, w)$ by passage to the multiplier representations. Then*

$$\tilde{A}(\bar{\lambda}, w)\circ F^{\bar{\lambda}} = F^{\lambda},$$

where $w\in W$ is such that $w\lambda = \bar{\lambda}$.

We are now in the position to give our main result on unitary spherical representations.

THEOREM V.12. *Let (π, V) be a unitary spherical representation with φ as its corresponding spherical function. Let $\mu\in\mathfrak{a}_C'$ be determined by φ, according to the formula of Harish-Chandra $\varphi = \varphi_\mu$. Assume $\operatorname{Im}\mu$ regular and let $\lambda\in C$ be congruent to μ (mod. the Weyl group). Assume also $c(\lambda) \neq 0$.*

Then there exists a conical distribution $\Psi\in\mathscr{D}'_{-\bar{\lambda}}$, positive definite and such that the formula

$$(f, h)\to\Psi(h^* * f)$$

defines an invariant semi-definite hermitian form on $\mathscr{D}^{\bar{\lambda}}$, and (π, V) is unitarily equivalent to the unitary extension of $(\pi^{\bar{\lambda}}, \mathscr{D}^{\bar{\lambda}}, \Psi)$.

Proof. We have observed in Corollary V.6 that π is unitary equivalent to the unitary extension of τ_λ acting on $F^\lambda(\mathscr{D}(G/K))$, equipped with the scalar product $(\cdot,\cdot)^\lambda$ given by $(\tilde{f}, \tilde{h})^\lambda = \int_{K/M} \tilde{f}(\lambda, kM)\overline{\tilde{h}(\bar{\lambda}, kM)}\,d(kM)$. Our task now will be to transport this representation to the space $\mathscr{D}^{\bar{\lambda}}$.

Let $f, h\in\mathscr{D}(G/K)$; then

$$\left(\tilde{A}(\bar{\lambda}, w)\circ F^{\bar{\lambda}}f,\ \tilde{A}(\bar{\lambda}, w)\circ F^{\bar{\lambda}}h\right)^\lambda = (F^\lambda f, F^\lambda h)^\lambda = c\Phi_{-\bar{\lambda},w}(h^* * f).$$

It is seen from the above that $(\cdot,\cdot)^{\lambda}$ transported by $\tilde{A}(\bar{\lambda}, w)$ to $F^{\bar{\lambda}}(\mathscr{D}(G/K))$ is given by the form $\Phi_{-\bar{\lambda},w}(h^* * f)$ for $f, h \in \mathscr{D}(G/K)$. But since $-\bar{\lambda} \in C$, $\bar{\lambda}$ is cyclic, what means that $F^{\bar{\lambda}}(\mathscr{D}(G/K))$ is dense in $\mathscr{E}(K/M)$. Applying now Proposition IV.5, we infer that

$$|\Phi_{-\bar{\lambda},w}(h^* * f)| \leqslant \text{const} \, \|h_{\bar{\lambda}}\| \, \|f_{\bar{\lambda}}\|, \tag{V.6}$$

so this form extends by continuity to the form on $\mathscr{D}^{\bar{\lambda}} \times \mathscr{D}^{\bar{\lambda}}$ or even $\mathscr{K}^{\bar{\lambda}} \times \mathscr{K}^{\bar{\lambda}}$. Evidently this extension is still given as $\Phi_{-\bar{\lambda},w}(h^* * f)$, but this time for any $f, h \in \mathscr{D}(G)$.

(V.6) shows also that $\tau \circ \tilde{A}(\bar{\lambda}, w)\colon F^{\bar{\lambda}}(\mathscr{D}(G/K)) \to V$ is a continuous, densely defined operator, thus extends to the whole of $\mathscr{D}^{\bar{\lambda}}$.

Further, an appropriate multiple of $\tau \circ \tilde{A}(\bar{\lambda}, w)$ gives the required unitary equivalence of V to the extension of $(\pi^{\bar{\lambda}}, \mathscr{D}^{\bar{\lambda}}, \Phi_{-\bar{\lambda},w})$.

Remark. Results of Bruhat on bilinear intertwining forms for differentiable induced representations show that the representation defined by a conical distribution $\Phi_{-\bar{\lambda},w}$ is actually conical, hence completing our already established relations between spherical and conical representations.

This paper contains, with several modifications and improvements, author's thesis for Doctor of Mathematics degree, written under the supervision of Professor K. Maurin at the Division of Mathematical Methods of Physics (University of Warsaw) and presented to the Department of Mathematics and Mechanics of the same University.

References

[1] A. Borel, *Representations de groupes localement compacts*, Lecture Notes in Mathematics 276, Springer-Verlag 1972.

[2] F. Bruhat, *Sur les représentations induites des groupes de Lie*, Bull. Soc. Math. France 84 (1956), pp. 97–205.

[3] I. M. Gelfand, *Spherical functions on symmetric spaces* (in Russian), Dokl. Akad. Nauk SSSR 70 (1950), pp. 5–8.

[4] I. M. Gelfand and M. A. Naimark, *Unitary representations of classical groups* (in Russian), Tr. Mat. Inst. Steklova AN SSSR 36 (1950).

[5] S. G. Gindikin and F. I. Karpelevic, *Plancherel measure of Riemannian symmetric spaces of non-positive curvature* (in Russian), Dokl. Akad. Nauk SSSR 145 (1962), pp. 252–255.

[6] R. Godement, *A theory of spherical functions I*, Trans. Amer. Math. Soc. 73 (1952), pp. 496–556.

[7] R. Goodman, *Analytic and entire vectors for representations of Lie groups*, Trans. Amer. Math. Soc. 143 (1969), pp. 55–76.

[8] Harish-Chandra, *Representations of semi-simple Lie groups I*, Trans. Amer. Math. Soc. 75 (1953), pp. 185–243.

[9] — *Representations of semi-simple Lie groups II*, ibidem 76 (1954), pp. 26–65.

[10] — *Spherical functions on a semi-simple Lie group I*, Amer. J. Math. 80 (1958), pp. 241–310.

[11] — *Spherical functions on a semi-simple Lie group II*, ibidem 803 (1958), pp. 553–613.

[12] S. Helgason, *Differential geometry and symmetric spaces*, New York 1962.

[13] — *Duality and Radon transform for symmetric spaces*, Amer. J. Math. 85 (1963), pp. 667–692.

[14] — *Lie groups and symmetric spaces*, in Battelle Rencontres ed. by C. M. De Witt and J. A. Wheeler, pp. 1–71, Princeton 1968.

[15] — *A duality for symmetric spaces with applications to group representations*, Adv. in Math. 5 (1970), pp. 1–154.

[16] A. W. Knapp and E. M. Stein, *Existence of complementary series*, Problems in Analysis, Symposium in Honor of Salomon Bochner, Princeton 1970.

[17] —, — *Intertwining operators for semi-simple Lie groups*, Ann. of Math. 93 (1971), pp. 489–578.

[18] B. Kostant, *On the existence and irreducibility of certain series of representations*, Bull. Amer. Soc. 75 (1969), pp. 627–649.

[19] R. A. Kunze and E. M. Stein, *Uniformly bounded representations III*, Amer. J. Math. 89 (1967), pp. 385–442.

[20] G. W. Mackey, *Infinite-dimensional group representations*, Bull. Amer. Math. Soc. 69 (1963), pp. 628–686.

[21] K. Maurin, *General eigenfunction expansions and unitary representations of topological groups*, Warszawa 1968.

[22] W. Parthasarathy, W. Ranga Rao, and W. Varadarajan, *Representations of complex semi-simple Lie groups and Lie algebras*, Ann. of Math. 86 (1967), pp. 383–429.
[23] N. Skovhus Poulsen, *On C^∞-vectors and intertwining bilinear forms for representations of Lie groups*, J. Funct. Anal. 9 (1972), pp. 87–120.
[24] G. Schiffman, *Sur les integrales d'entrelacement de R. A. Kunze et E. M. Stein* (thèse sc. math.), Bull. Soc. Math. France 99 (1971), pp. 3–72.
[25] T. Sherman, *A weight theory for unitary representations*, Canad. J. Math. 18 (1966), pp. 159–168.
[26] E. M. Stein, *Analytic continuation of group representations*, Adv. in Math. 4, pp. 172–207.
[27] A. Strasburger and A. Wawrzyńczyk, *Automorphic forms for conical representations*, Bull. Acad. Pol. Sci., Sér. sci. math., astr., phys. 20 (1972), pp. 921–927.
[28] G. Warner, *Harmonic analysis on semi-simple Lie groups I*, Springer-Verlag 1972.
[29] A. Wawrzyńczyk, *A duality in the theory of group representations*, Studia Math. 36 (1970), pp. 227–257.