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THE PICARD GROUP OF THE CANONICAL RESOLUTION OF A 3-DIMENSIONAL SIMPLE SINGULARITY

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Let X be the canonical resolution of one of the 3-dimensional singularities A_n, D_n, E_n and E its exceptional locus. We show how to find the Picard group of the 3-dimensional formal scheme $\hat{X_E}$ in terms of the "resolution diagram". More precisely, let $E_1, \ldots, E_m$ be the irreducible components of E, $\mu \in Z^m$, J_i the ideal of E_i in $\mathcal{O}_X$ and $J^\mu = J_1^{\mu_1} \cdot \ldots \cdot J_m^{\mu_m}$ the ideal of the divisor μE on the smooth 3-fold X.

0.1. THEOREM. *If we choose μ such that $\mu E \hookrightarrow X$ is a (symmetric) negative embedding, then*

$$\operatorname{Pic} \mu E \simeq \operatorname{Pic} \dot{E}$$

(here $\dot{E} = (1, \ldots, 1)\, E$ denotes the reduced exceptional locus) via the canonical homomorphism.

The term on the right hand may be expressed as a subgroup of $\operatorname{Pic} E_1 \times \ldots \times \operatorname{Pic} E_m$, given by conditions arising from [1], pp. 306, 312–315.

1. Proof of the theorem

For $\mu_i \geqslant 1$ ($i \in \{1, \ldots, m\}$) consider the exact sequence

$$1 \to 1 + J^\mu/J^{\mu+e_i} \to (\mathcal{O}_X/J^{\mu+e_i})^* \to (\mathcal{O}_X/J^\mu)^* \to 1$$

of sheaves of abelian groups on E. Obviously, $1 + J^\mu/J^{\mu+e_i}$ is isomorphic $J^\mu/J^{\mu+e_i}$.

1.1. *Remark.* $H^1(J^\mu \,|\, E_i) = H^2(J^\mu \,|\, E_i) = 0$ implies

$$\operatorname{Pic}(\mu + e_i)\, E = \operatorname{Pic} \mu E.$$

Therefore, if μ is given, it is sufficient to find a sequence

$$S = [i_1, \dots, i_s], \quad i_j \in \{1, \dots, m\},$$

such that

$$H^1(J^{\mu^{(t)}} | E_{i_t}) = H^2(J^{\mu^{(t)}} | E_{i_t}) = 0$$

$(\mu^{(t)} = \mu - e_{i_1} - \dots - e_{i_t})$ for all $t = 1, \dots, s$ and $\mu^{(s)} = (1, \dots, 1)$.

1.2. Definition. A sequence S with the above properties is said to be *admissible* for μ. For simplicity, write

$$[i_1, \dots, i_r] + [i_{r+1}, \dots, i_s] = [i_1, \dots, i_s]$$

and

$$n \cdot [i_1, \dots, i_s] = [i_1, \dots, i_s] + \dots + [i_1, \dots, i_s] \ (n \text{ terms}).$$

Now consider e.g. the case of the canonical resolution for A_n, n odd, $n > 1$. We have $m = (n+1)/2$, $E_1 \simeq \dots \simeq E_{m-1} \simeq F_2$ (the ruled surface $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(-2))$ with the Picard group generated by a fibre f and the section C_0) $E_m \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$, and

$$J^\mu | E_1 \simeq \mathcal{O}_{E_1}((2\mu_1 - \mu_2) C_0 + 2\mu_1 f),$$
$$J^\mu | E_i \simeq \mathcal{O}_{E_i}((2\mu_i - \mu_{i-1} - \mu_{i+1}) C_0 + 2(\mu_i - \mu_{i-1}) f), \quad i = 2, \dots, m-1,$$
$$J^\mu | E_m \simeq \mathcal{O}_{E_m}(\mu_m - \mu_{m-1}, \mu_m - \mu_{m-1}),$$

and the embedding $\mu E \hookrightarrow X$ is negative iff $J^\mu | E_i$ is ample for all i, i.e. $\mu_1 < \mu_2 < \dots < \mu_m$,

$$\mu_{i-1} + \mu_{i+1} < 2\mu_i, \quad i = 2, \dots, m-1,$$
$$\mu_2 < 2\mu_1.$$

Using this, it is possible to verify

1.3. Admissible sequences. Let μ be such that μE is negatively embedded. Then

(i) $S = (\mu_m - \mu_{m-1}) \cdot [m] + \dots + (\mu_i - \mu_{i-1}) \cdot [i, \dots, m] +$

$$\dots + (\mu_3 - \mu_2) \cdot [3, \dots, m] + (\mu_2 - \mu_1) \cdot [2, \dots, m] + (\mu_1 - 1) \cdot [1, \dots, m]$$

is admissible for μ in the case of A_n, $n > 1$, n odd. For the remaining cases, we adopt the notations of [1], pp. 341–345, and complete our list (in (iii), (iv) we assume μ symmetric).

(ii) In the case of A_n, n even, $n > 1$, the following sequence is admissible:

$$S = (\mu_m - 2\mu_{m-1})\cdot[m] + (\mu_{m-1} - \mu_{m-2})\cdot([m] + [m-1, m]) + \dots$$
$$\dots + (\mu_i - \mu_{i-1})\cdot([m] + [i, i+1, \dots, m-1, m]) + \dots$$
$$\dots + (\mu_2 - \mu_1)\cdot([m] + [2, \dots, m]) + (\mu_1 - 1)\cdot([m] + [1, \dots, m]) + [m].$$

(iii) For D_4

$$S = \sum_{i=1}^{3} (\mu_i - \mu'_1)[i] + (\mu'_1 - 1)\cdot[1, 2, 3, 1', 1''],$$

and for D_5

$$S = \delta_2\cdot[2] + \delta_1\cdot[1, 2] + \delta_3\cdot[3] + (\alpha - 1)\cdot[2, 1', 1, 2, 3, 1''] + [2]$$

are admissible.

D_{2k+1}, $k \geqslant 3$:

$$S = \delta_2\cdot[2] + \delta_1\cdot[1, 2] + \sum_{j=1}^{k-2} ((\varepsilon_{j+1} - \delta_{j+2} - 1)\cdot S_j + (\varepsilon_{j+1} - 1)\cdot([j+2] + S_j) + S_j)$$
$$+ (\varepsilon_k - \delta_{k+1} - 1)\cdot S_{k-1} + (\varepsilon_k - 1)\cdot([k+1] + S_{k-1}) + [2, \dots, k];$$

here S_j is defined by

$$S_1 = [2, 1', 1, 2, 3, 1''], \qquad S_j = [j+1, j', j+2, j''] + S_{j-1}.$$

D_{2k}, $k \geqslant 3$:

$$S = \delta_1\cdot[1] + \delta_2\cdot[2] + \sum_{j=1}^{k-2} ((\varepsilon_{j+1} - \delta_{j+2} - 1)\cdot S_j + (\varepsilon_{j+1} - 1)\cdot([j+2] + S_j) + S_j)$$
$$+ (\varepsilon_k - \delta_{k+1} - 1)\cdot S_{k-1} + (\varepsilon_k - 1)\cdot([k+1] + S_{k-1}) + [3, \dots, k];$$

S_j is defined by

$$S_1 = [1', 1, 2, 3, 1''], \qquad S_j = [j+1, j', j+2, j''] + S_{j-1}.$$

(iv) The case of E_n:

E_6

$$S = S_5(\lambda) + (\alpha - 1)\cdot[1', 3, 2, 1, 3, 2, 3, 1''] + [3, 2, 3];$$

E_7

$$S = S_6(\lambda, \beta) + (\alpha_3 - 1)\cdot S' + [1, 2, 3, 1', 1'', 1, 2];$$

E_8

$$S = S_6(\lambda, \beta) + (\alpha_3 - \alpha_4)\cdot S' + (\alpha_4 - 1)\cdot(S'' + S') + S'$$
$$+ [2', 1'', 3, 3, 4, 2'', 1, 2, 1', 1', 1, 2, 3, 1'', 1, 3, 4].$$

2. Calculation of $\operatorname{Pic} E$ in the case of A_n

For $\{i_1, \dots, i_t\} \subseteq \{1, \dots, m\}$, put $\bar{J} = J_{i_1} \cdot \ldots \cdot J_{i_t}$, $\bar{E}$ the divisor of X corresponding to $\bar{J}$. Consider the exact sequence

$$1 \to \mathcal{O}^*_{\bar{E}+E_i} \to \mathcal{O}^*_{\bar{E}} \times \mathcal{O}^*_{E_i} \to \mathcal{O}^*_{\bar{E}\cap E_i} \to 1. \tag{2.1}$$

We obtain

$$H^0(\mathcal{O}^*_{\bar{E}}) \times H^0(\mathcal{O}^*_{E_i}) \xrightarrow{\beta} H^0(\mathcal{O}^*_{\bar{E}\cap E_i}) \to \operatorname{Pic}(\bar{E}+E_i)$$
$$\to \operatorname{Pic} \bar{E} \times \operatorname{Pic} E_i \to \operatorname{Pic}(\bar{E} \cap E_i).$$

If the condition

$$H^0(\mathcal{O}_{\bar{E}\cap E_i}) = k \tag{2.2}$$

is satisfied, then β is surjective. Therefore,

$$\operatorname{Pic}(\bar{E}+E_i) = \ker\big(\operatorname{Pic}(\bar{E}) \times \operatorname{Pic} E_i \to \operatorname{Pic}(\bar{E} \cap E_i)\big).$$

Applying this stepwise for the A_n-resolutions, we get

$$\operatorname{Pic} E = \operatorname{Pic} E_1 \times_{\operatorname{Pic} E_1 \cap E_2} \operatorname{Pic} E_2 \times \ldots \times \operatorname{Pic} E_{m-1} \times_{\operatorname{Pic} E_{m-1} \cap E_m} \operatorname{Pic} E_m.$$

For $(L_1, \dots, L_m) \in \operatorname{Pic} E$ put

$$L_i = \mathcal{O}_{F_2}(a_i C_0 + b_i f), \quad i = 1, \dots, m-1,$$
$$L_m = \begin{cases} \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a_m, b_m) & \text{for } n \text{ odd}, \\ \mathcal{O}_{\mathbf{P}^2}(a_m) & \text{for } n \text{ even} \end{cases}$$

and obtain

2.1. Picard groups ($m = \operatorname{int}((n+2)/2)$).
n odd:

$$\operatorname{Pic} E \cong \{(a_1, b_1, \dots, a_m, b_m) \in \mathbf{Z}^{2m},\ -2a_1 + b_1 = b_2,\ -2a_2 + b_2 = b_3, \dots$$
$$\dots, -2a_{m-2} + b_{m-2} = b_{m-1},\ -2a_{m-1} + b_{m-1} = a_m + b_m\};$$

n even:

$$\operatorname{Pic} E \cong \{(a_1, b_1, \dots, a_{m-1}, b_{m-1}, a_m) \in \mathbf{Z}^{2m-1},\ -2a_1 + b_1 = b_2,$$
$$-2a_2 + b_2 = b_3, \dots$$
$$\dots, -2a_{m-2} + b_{m-2} = b_{m-1},\ -2a_{m-1} + b_{m-1} = 2a_m\}.$$

Reference

[1] M. Roczen, *Some properties of the canonical resolutions of the 3-dimensional singularitites* A_n, D_n, E_n *over a field of characteristic* $\neq 2$, in *Algebraic Geometry, Bucharest 1982*, Lecture Notes in Math. 1056, Springer-Verlag, Berlin–New York 1984, 297–365.

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