SEQUENTIAL METHODS IN STATISTICS
BANACH CENTER PUBLICATIONS, VOLUME 16
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

ON CALCULATING THE LAPLACE TRANSFORM OF A SPECIAL QUADRATIC FUNCTIONAL OF THE ORNSTEIN–UHLENBECK VELOCITY PROCESS

L. PARTZSCH

Section of Mathematics, Technical University of Dresden, Dresden, G.D.R.

1

Let $X(t)$, $t \geqslant 0$, be a stationary Ornstein–Uhlenbeck velocity process, that is, a real separable Gaussian process with mean 0 and the covariance function

$$K(s, t) = EX(s)X(t) = \frac{\sigma^2}{2\alpha} \cdot e^{-\alpha|s-t|}, \qquad s, t \geqslant 0,$$

($\sigma > 0$, $\alpha > 0$ parameters).

The aim of this note consists in pointing out two different possibilities of calculating the Laplace transform $F_t(\lambda)$, $\lambda \geqslant 0$, of the quadratic functional

$$Y(t) := \int_0^t X^2(s)\,ds \qquad (t > 0).$$

This problem arose in connection with sequential estimations of density parameters of stationary Gaussian processes (see [4]), where the Markov times

$$\tau_a := \min\{t \geqslant 0:\ Y(t) = a\}, \qquad a > 0,$$

are considered and the existence of the moments $E\tau_a^n$ ($n = 1, 2, \ldots$) has to be proved. Unfortunately, we have not succeeded in finding a simple proof of this fact. From

$$\{\tau_a > t\} = \{Y(t) < a\}$$

and the Chebyshev inequality we obtain

$$P(\tau_a > t) \leqslant e^a \cdot Ee^{-Y(t)},$$

and in this way the problem in question could be handled by finding the asymptotics of $F_t(1)$ as $t \to \infty$. We shall explicitly calculate $F_t(\lambda)$. We have the relation $E_t(1) \sim e^{-\gamma_0 \cdot t}$ as $t \to \infty$ for some $\gamma_0 > 0$, and from this the proof of existence of all moments of τ_a easily follows.

2

The first possibility of calculating $F_t(\lambda)$ consists in reducing the Ornstein–Uhlenbeck velocity process to the Wiener process and in applying a Cameron–Martin type formula. Without loss of generality we can assume

$$X(s) = \frac{\sigma}{\sqrt{2\alpha}} \cdot e^{-\alpha s} \cdot W(e^{2\alpha s}), \qquad s \geqslant 0,$$

where $W(t)$, $t \geqslant 0$, is a Wiener standard process. We have

$$E_t(\lambda) = E\exp\Big(-\lambda \cdot \int_0^t X^2(s)\,ds\Big) = E\exp\Big(-\int_1^{t_1} q_1(u)\,W^2(u)\,du\Big)$$

$$\text{with} \qquad t_1 := e^{2\alpha t},\ q_1(u) = \frac{\beta}{u^2},\ \beta = \frac{\sigma^2 \cdot \lambda}{4\alpha^2}. \tag{1}$$

A Cameron–Martin type formula (see, e.g., [3]) leads to

$$F_t(\lambda) = E\exp\Big(-\int_0^{t_1} q(u)\,W^2(u)\,du\Big) = \exp\left(\frac{1}{2}\int_0^{t_1} \gamma(u)\,du\right),$$

$$\text{with} \qquad q(u) = \begin{cases} 0 & \text{for } 0 \leqslant u < 1, \\ q_1(u) & \text{for } 1 \leqslant u \leqslant t_1, \end{cases} \tag{2}$$

where $\gamma(u)$, $0 \leqslant u \leqslant t_1$, is the unique continuous solution of the Riccati equation

$$\begin{aligned} &\gamma'(u) = 2q(u) - \gamma^2(u), \qquad 0 \leqslant u < t_1, \quad u \neq 1, \\ &\gamma(t_1) = 0. \end{aligned} \tag{3}$$

Equation (3) can be solved explicitly. From the continuity of $\gamma(u)$ and the relation $q(u) = 0$ for $0 \leqslant u < 1$ we calculate

$$F_t(\lambda) = \exp\Big(\tfrac{1}{2}\int_1^{t_1} \gamma(u)\,du\Big) \cdot (1-\gamma(1))^{-1/2}. \tag{4}$$

In the interval $[1, t_1]$ we have to solve the Riccati equation

$$\gamma'(u) = \frac{2\beta}{u^2} - \gamma^2(u). \tag{5}$$

A special solution of (5) is

$$\gamma_{\text{spec}}(u) = \frac{c}{u},$$

where

$$c^2 - c - 2\beta = 0.$$

Let us choose

$$c = c_2 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+8\beta} \qquad (c_2 < 0). \tag{6}$$

With the aid of $\gamma_{\text{spec}}(u)$ the general solution of (5) can be determined, and we get – under the boundary condition – the result

$$\gamma(u) = \frac{c_2}{u} + \frac{1}{u}\cdot\left[\frac{1}{1-2c_2} + \left(\frac{u}{t_1}\right)^{2c_2-1}\cdot\left(\frac{1}{2c_2-1} - \frac{1}{c_2}\right)\right]^{-1}, \qquad 1 \leqslant u \leqslant t_1. \tag{7}$$

Finally, a direct calculation yields by (4), (6), (7) the following

THEOREM.

$$F_t(\lambda) = E\exp\left(-\lambda\int_0^t X^2(s)\,ds\right)$$
$$= \left(4\alpha\sqrt{\alpha^2+2\sigma^2\lambda}\cdot e^{\alpha t}\right)^{1/2}\left[(\alpha+\sqrt{\alpha^2+2\sigma^2\lambda})^2\exp(t\sqrt{\alpha^2+2\sigma^2\lambda}) - (\alpha-\sqrt{\alpha^2+2\sigma^2\lambda})^2\exp(-t\sqrt{\alpha^2+2\sigma^2\lambda})\right]^{-1/2}$$
$$= e^{\alpha t/2}\cdot\left[\frac{\alpha^2+\sigma^2\lambda}{\alpha\cdot\sqrt{\alpha^2+2\sigma^2\lambda}}\cdot\sinh(t\cdot\sqrt{\alpha^2+2\sigma^2\lambda}) + \cosh(t\cdot\sqrt{\alpha^2+2\sigma^2\lambda})\right]^{-1/2}. \tag{8}$$

COROLLARY.

$$\lim_{t\to\infty} E\exp\left(-\int_0^t X^2(s)\,ds\right)\exp\left(\frac{\alpha}{2}\cdot t\left(\sqrt{1+\frac{2\sigma^2}{\alpha^2}}-1\right)\right) = \frac{2\sqrt{\alpha}\cdot(\alpha^2+2\sigma^2)^{1/4}}{\alpha+\sqrt{\alpha^2+2\sigma^2}}.$$

3

As a second possibility of calculating $F_t(\lambda)$ we use the Karhunen representation and an identification theorem of Hadamard in complex analysis. This method of deriving the Laplace transform is applicable to other Gaussian processes too.

Now let us consider the Karhunen representation of the Ornstein–Uhlenbeck velocity process:

$$X(s) = \sum_{n=1}^{\infty} \varphi_n(s) \cdot \sqrt{\lambda_n} \cdot X_n, \qquad 0 \leqslant s \leqslant t. \tag{9}$$

Here, the X_n, $n = 1, 2, \ldots$, are independent identically $N(0, 1)$ distributed random variables, and λ_n and φ_n ($n = 1, 2, \ldots$) are the eigenvalues and eigenfunctions respectively of the nuclear integral operator K corresponding to the kernel $K(u, v)$, $0 \leqslant u, v \leqslant t$, in the space $L^2[0, t]$ of square-integrable functions on the interval $[0, t]$.

The representation (9) yields

$$F_t(\lambda) = E\exp\Big(-\lambda \int_0^t X^2(s)\,ds\Big) = E\exp\Big(-\sum_n \lambda\lambda_n \cdot X_n^2\Big)$$
$$= \prod_n (1+2\lambda\lambda_n)^{-1/2} = D(-2\lambda)^{-1/2}, \tag{10}$$

where

$$D(\lambda) := \prod_n (1-\lambda\lambda_n)$$

denotes the Fredholm determinant of K (compare, e.g., [6], [2]).

We consider the equation of eigenfunctions for determination of $D(\lambda)$:

$$K\varphi(s) = \int_0^t K(s, u)\,\varphi(u)\,du = \lambda\varphi(s), \qquad 0 \leqslant s \leqslant t. \tag{11}$$

By differentiating twice we can see that (11) is equivalent to

$$\varphi''(s) - \left(\alpha^2 - \frac{\sigma^2}{\lambda}\right) \cdot \varphi(s) = 0, \qquad 0 < s < t, \tag{12}$$

with the boundary conditions

$$\varphi'(0) - \alpha\varphi(0) = 0, \qquad \varphi'(t) + \alpha\varphi(t) = 0. \tag{13}$$

With the general solution of (12)

$$\varphi(s) = C_1 \cdot e^{\eta s} + C_2 \cdot e^{-\eta s}, \qquad \eta = \left(\alpha^2 - \frac{\sigma^2}{\lambda}\right)^{1/2},$$

the boundary conditions (13)

$$C_1 \cdot (\eta - \alpha) - C_2 \cdot (\eta + \alpha) = 0,$$
$$C_1 \cdot (\eta + \alpha) \cdot e^{\eta t} - C_2 (\eta - \alpha) \cdot e^{-\eta t} = 0$$

yield an equation for the determination of the eigenvalues:

A number λ is an eigenvalue of K iff $G(\lambda) = 0$ with

$$G(\lambda) = -e^{-\eta t}(\eta-\alpha)^2 + e^{\eta t}(\eta+\alpha)^2$$

$$= -\left(\alpha - \sqrt{\alpha^2 - \frac{\sigma^2}{\lambda}}\right)^2 \cdot \exp\left(-\sqrt{\alpha - \frac{\sigma^2}{\lambda}\cdot t}\right) + \left(\alpha + \sqrt{\alpha^2 - \frac{\sigma^2}{\lambda}}\right)^2 \cdot \exp\left(\sqrt{\alpha^2 - \frac{\sigma^2}{\lambda}\cdot t}\right)$$

The function $\tilde{G}(\lambda) := G\left(\frac{1}{\lambda}\right)\cdot\frac{1}{\sqrt{\alpha^2 - \sigma^2\cdot\lambda}}$ is an analytic one with the zeros at $\lambda = \frac{1}{\lambda_k}$ and with an exponential rate of increase less than 1. As the Fredholm determinant $D(\lambda)$ has the same properties, the Hadamard identification theorem (see, e.g., [5]) yields

$$D(\lambda) = \text{const}\cdot\tilde{G}(\lambda).$$

For the determination of the constant we set $\lambda = 0$ and finally get

$$D(\lambda) = (4\alpha e^{\alpha t})^{-1}\cdot\tilde{G}(\lambda) = (4\alpha e^{\alpha t}\cdot\sqrt{\alpha^2 - \sigma^2\lambda})^{-1}\cdot G\left(\frac{1}{\lambda}\right). \tag{14}$$

From this, together with (10), the same result (8) follows.

References

[1] M. Kac, *Random walk in the presence of absorbing barriers*, Ann. Math. Statist. **16** (1945), 62–67.

[2] H. Langer, G. Maibaum, and P. H. Müller, *Zu einem Satz über Verteilungen quadratischer Formen in Hilberträumen*, Math. Nachr. **61** (1974), 175–179.

[3] P. S. Liptser and A. N. Shiryayev, *Statistics of Stochastic Processes*, Springer-Verlag, Berlin–Heidelberg–New York 1978.

[4] R. Magiera, *Sequential estimation for the spectral density parameter of a stationary Gaussian process*, this volume.

[5] A. J. Markushevitch, *Theory of Analytic Functions*, vol. 2, Moscow 1968, in Russian.

[6] D. E. Varberg, *Equivalent Gaussian measures with a particularly simple Radom–Nikodym derivative*, Ann. Math. Statist. **38** (1967), 1027–1030.

Presented to the semester
Sequential Methods in Statistics
September 7–December 11, 1981