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Banach-space-valued stationary processes and their linear prediction

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Contents

0. Introduction

In this paper[1] we consider problems of infinite-dimensional linear analysis arising in connection with the correlation theory of Banach-space-valued stationary (in the wide sense) processes. The attempt to extend the theory for finite-vector-valued processes developed by Ju. A. Rozanov [14] and N. Wiener and P. Masani [23] to infinite-dimensional processes has resulted in several papers in recent years. For example, for the case of Hilbert spaces see [3], [7], [11] and for the case of Banach spaces [12], [16], [19].

We observe that a random vector of the second order $x(\omega)$ taking its values in a linear topological space Y may be considered as a linear transformation of the topological dual space $Y^* = X$ into the Hilbert space of random variables having the second absolute moments finite $H = L_2(\Omega, B, P)$;

$$(\xi x^*)(\omega) = x^*\big(x(\omega)\big) \in H.$$

By a stochastic process of the second-order $(\xi_t)_{t \in T}$ we mean a curve in the space of linear operators $L(X, H)$. Such a process is stationary if the operator

$$R_{t,s} = \xi_t^* \xi_s \in \bar{L}(X, X^*)$$

depends only on the difference $t-s$, where $\bar{L}(X, X^*)$ denotes the space of anti-linear operators with domain X and range in X^*. In the finite-dimensional case, the above operator model of stochastic processes corresponds to the "real process", but does not yet in the infinite-dimensional case. However, we use this model for its convenience for the prediction problem. For example, the problem of a characterization of regular (purely non-deterministic) Banach-space-valued stationary sequences is equivalent to the problem of a factorization of positive operator-valued functions (cf. Section 10).

In [8], A. N. Kolmogorov studied second-order scalar-valued stationary sequences as curves in a Hilbert space. By generalizing the concept of the Hilbert space to allow the inner product to take values which are no longer scalars but elements of more general topological vector space, it is possible to extend Kolmogorov's model to sequences of random

[1] The present paper was written during the second author's stay at the Tbilisi State University in the academic year 1972/73.

elements, with values in a suitable abstract space which are second-order stationary in a generalized but natural sense. In our recent paper [22], we studied this idea using as a such "generalized Hilbert space" — an LVH-space (pseudo-Hilbertian space) which first appeared in [10], by R. M. Loynes. We have obtained spectral representations and an ergodic theorem for such processes. However, in view of a difference from the Hilbert space situation (there are no teorems guaranteeing the existence of the projection onto a subspace) the description of the linear prediction problem causes certain difficulties.

It is known (cf. [22], Theorem 2) that the space $L(X, H)$ is an LVH-space. Hence the operator model of stationary processes taking values in a Banach space is a special case of the above natural generalization of Kolmogorov's model. In this case, the difficulties connected with the projection onto a subspace do not arise, because for the linear prediction problem it suffices to define the orthogonal projection on a closed right ideal in $L(X, H)$. But, according to Yood's result, this projection may be well defined (cf. Section 7).

Except in Section 10, we consider processes indexed by the elements of any locally compact Abelian (LCA) group, instead of integers. Although the ideas and concepts used here are similar to those used earlier by other authors in the classical prediction theory, some of the techniques are different, since the integers are ordered and individually generated, whereas an arbitrary group need not be. The idea of having a process indexed by the elements of a group, instead of by integers, has attracted the attention of several mathematicians in recent years, see for example [1], [2], [6], [16], [20].

This paper can, in general, be divided into three main parts. The first part (Sections 1–3) develops the correlation theory of Banach-space-valued random elements which first appeared in N.N. Vakhania (cf. [18], Section 4). The second part (Sections 4–6) gives traditional spectral representation theorems for Banach-space-valued stationary processes and their correlation functions, as well as ergodic theorems. The third part (Sections 7–10) is devoted to the study of the linear prediction problem in an infinite-dimensional case.

The authors wish to express their sincere gratitude to Professor N. N. Vakhania for many helpful discussions.

1. Linear operators generated by random elements

Fix a probability system (Ω, B, P). All random elements are considered to be defined over it. Let Y be a complex Banach space with the dual space Y^*. A function x: $\Omega \to Y$ is said to be *weakly measurable* if

$x^*(x(\cdot))$ is measurable for every linear functional $x^* \in Y^*$. A weakly measurable function x from a probability space (Ω, B, P) into Y is said to be a *random element* (a random Y-variable). We say that $x: \Omega \to Y$ is *Pettis integrable* if it is weakly integrable, i.e., $x^*(x(\cdot))$ is integrable for every $x^* \in Y^*$, and for every measurable set $A \subset \Omega$ there exists an element $y_A \in Y$ satisfying

$$x^*(y_A) = \int_A x^*(x(\omega))P(d\omega)$$

for every $x^* \in Y^*$. Then y_A is called the *Pettis integral* of $x(\cdot)$ over A and is written as

$$y_A = (\text{Pettis}) \int_A x(\omega)P(d\omega).$$

The *expectation* m of a random element $x(\cdot)$ is defined by the Pettis integral

$$m = (\text{Pettis}) \int_\Omega x(\omega)P(d\omega).$$

A random element $x(\omega)$ is said to have a *weak p-order* $(1 \leqslant p \leqslant \infty)$ if $x^*(x(\omega)) \in L_p(\Omega, B, P)$ for every $x^* \in Y^*$ and to have a *strong p-order* if $\|x(\omega)\| \in L_p(\Omega, B, P)$. Obviously, if a random element has a strong p-order, then it has also a weak p-order. The converse implication does not hold (cf. [18], p. 124).

As a direct consequence of the well-known theorem of Pettis, we have the following sufficient condition for the existence of the expectation of random elements.

PROPOSITION 1.1. *Let Y be a separable Banach space and let the Y-valued random element $x(\omega)$ have a weak p-order $(p > 1)$. Then the expectation of $x(\omega)$ exists.*

Another sufficient condition was given by N. N. Vakhania (cf. [18], p. 128).

With every random element $x(\omega)$ in Y we may correspond, in a natural way, a linear mapping ξ on the dual space Y^* to the space of all measurable functions:

$$(\xi x^*)(\omega) = x^*(x(\omega)). \tag{1.1}$$

We say that the operator ξ in (1.1) is *generated by a random element* $x(\omega)$ or that it is *decomposable.*

PROPOSITION 1.2. *The operator ξ generated by a random element $x(\omega)$ of a weak p-order $(1 \leqslant p \leqslant \infty)$ is a linear continuous operator on the dual space Y^* to $L_p(\Omega, B, P)$. If $x(\omega)$ has a strong p-order $(1 \leqslant p < \infty)$, then ξ is a compact operator.*

The first part of this proposition follows from the fact that ξ has a closed graph (cf. [18], p. 140). The second part is proved in [17].

In Section 0 we emphasized that we will in this paper consider a stochastic process as a curve in the space of linear operators $L(X, H)$, and that in this infinite-dimensional case a real process corresponding to it does not exist in general. This is a consequence of the fact that there exist linear mappings on a Banach space into the space of measurable functions which are not generated by any random element (i.e., for which relation (1.1) does not hold). A simple example yields the identical operator in the space $L_2([0, 1])$.

The following two propositions, which give sufficient conditions for operators to be generated by a random element, directly follow from the general form of operators from a Banach space into L_∞ and $C(S)$, respectively.

PROPOSITION 1.3. *Let Y be a reflexive separable Banach space. If ξ is a linear continuous operator from the dual space Y^* into $L_p(\Omega, B, P)$ for an arbitrary $p \geqslant 1$, such that*

$$\operatorname{vraisup} |(\xi x^*)(\omega)| < \infty$$

for every $x^* \in Y^*$, then ξ is generated by a random element on Y.

PROPOSITION 1.4. *Let Y be a reflexive separable Banach space, S be a compact Hausdorff topological space with the Borelian σ-algebra of subsets, and $C(S)$ be the space of all continuous functions on S.*

Then every continuous linear operator ξ: $Y^* \to C(S)$ is generated by a random element on Y.

We next present another condition in terms of a factorization.

PROPOSITION 1.5. *Let Y be a reflexive separable Banach space, and the linear operator ξ: $Y^* \to L_2(\Omega, B, P)$ admit a factorization $\xi = \varphi\psi$, where ψ is a linear continuous operator from the dual space Y^* into a Hilbert space H and φ: $H \to L_2(\Omega, B, P)$ is a Hilbert–Schmidt operator. Then the operator ξ is generated by a random element of the strong second-order on Y.*

Proof. According to the assumption, φ: $H \to L_2(\Omega, B, P)$ is a Hilbert–Schmidt operator. Hence

$$(\varphi g)(\omega) = \big(g, h(\omega)\big), \qquad g \in H,$$

where $h(\omega)$ is an H-valued random element of the strong second-order. Now, consider the random element $x(\omega) = \psi^* h(\omega)$. Clearly, it takes values in the Banach space Y and

$$x^*\big(x(\omega)\big) = x^*\big(\psi^* h(\omega)\big) = \big(\psi x^*, h(\omega)\big) = (\varphi\psi x^*)(\omega) = (\xi x^*)(\omega).$$

Since $h(\omega)$ has a strong second-order, the random element $x(\omega)$ also has a strong second-order. □

The decomposability of operators ξ: $Y^* \to L_p(\Omega, B, P)$ $(1 < p < \infty)$ may be characterized also in terms of p-absolutely summing operators.

The following two propositions are direct consequences of the result given by S. Kwapień in [9] (cf. Theorem 1, 2).

PROPOSITION 1.6. *Let ξ: $Y^* \to L_p(\Omega, B, P)$ be a linear operator generated by a random element of a strong p-order. Then ξ is p-absolutely summing.*

PROPOSITION 1.7. *Let Y be a reflexive separable Banach space and ξ: $Y^* \to L_p(\Omega, B, P)$ be such that its adjoint ξ^* is p-absolutely summing. Then the operator ξ is generated by a random element of the strong p-order.*

The condition on a factorization by a Hilbert–Schmidt operator in Proposition 1.5 is equivalent to the fact that the adjoint operator ξ^* is 1-absolutely summing (cf. [9], Corollary 2), and hence Proposition 1.5 is a corollary of Proposition 1.7. In Propositions 1.3, 1.4, 1.5, and 1.7, the assumption of reflexivity of the Banach space Y may be ommited, if the decomposability of ξ is meant in the more general sense: ξ is generated by a random element on Y^{**}.

2. Covariance operator of generalized random elements

In view of Proposition 1.2, an operator generated by a random element of a weak p-order is linear and continuous. Thus the following definition appears in a natural way.

DEFINITION 2.1. A linear continuous operator ξ: $Y^* \to L_p(\Omega, B, P)$ is said to be a *generalized random element* of p-order on Y.

The expectation of a generalized random element ξ on Y is defined as an element $m \in Y$, if one exists, such that

$$M(\xi x^*)(\omega) = \int_{\Omega} (\xi x^*)(\omega) P(d\omega) = x^*(m)$$

for every $x^* \in Y^*$.

DEFINITION 2.2. Let ξ be a generalized random element of the second-order. The operator $R = \xi^* \xi$ is called the *covariance operator* of the generalized random element ξ, where ξ^*: $L_2(\Omega, B, P) \to Y^{**}$ is the adjoint(1) of ξ.

It is easy to see that the covariance operator may be equivalently defined by the relation

$$(Rx^*)(y^*) = M\overline{(\xi x^*)(\omega)}(\xi y^*)(\omega), \quad x^*, y^* \in Y^*,$$

(1) By the adjoint operator ξ^* of an operator ξ from a Banach space X into a Hilbert space H we mean the operator defined as follows: $(\xi^* h)(x) = (\xi x, h)$, $x \in X$, $h \in H$.

or if ξ is generated by a random element $x(\omega)$, by the relation

$$(Rx^*)(y^*) = M\overline{x^*(x(\omega))}\,y^*(x(\omega)), \qquad x^*, y^* \in Y^*.$$

From Definition 2.2, we immediately obtain the following properties of the covariance operator:

1° the operator R is a *continuous anti-linear*(1) mapping from Y^* into Y^{**},

2° the operator R is *self-adjoint*, i.e., for arbitrary $x^*, y^* \in Y^*$,

$$(Rx^*)(y^*) = \overline{(Ry^*)(x^*)},$$

where the symbol $(Rx^*)(y^*)$ denotes a value of the functional $Rx^* \in Y^{**}$ on an element $y^* \in Y^*$,

3° the operator R is *non-negative*, i.e., for arbitrary $x^* \in Y^*$,

$$(Rx^*)(x^*) \geqslant 0.$$

We note that property 2° follows from 1° and 3°. This fact is analogous to that for bilinear forms in a complex Hilbert space, and follows from the polarization formula.

The following theorem states that properties 1°–3° are sufficient for an operator R to be the covariance operator of a generalized random element. Before we proceed to this result, we shall make a remark concerning notation. The space of all continuous anti-linear operators from a Banach space X into its dual space X^* will be denoted by $\bar{L}(X, X^*)$.

THEOREM 2.3. *An operator $R \in \bar{L}(X, X^*)$ is the covariance operator of a generalized random element of the second-order if and only if it is non-negative.*

Proof. The necessity is a consequence of the above consideration. The sufficiency follows directly from the following lemma on factorization. □

This lemma will be used repeatedly, and is of interest in itself.

LEMMA 2.4 (on factorization) (cf. [25] or [18], p. 135). *Let X be a Banach space. If $R \in \bar{L}(X, X^*)$ is non-negative, then there exist a Hilbert space H and an operator $A \in L(X, H)$, such that R can be factorized as follows:*

(2.1)
$$\begin{array}{ccc} X & \xrightarrow{\;R\;} & X^* \\ & {\scriptstyle A}\searrow \quad \nearrow{\scriptstyle A^*} & \\ & H & \end{array}$$

where the operator A is uniquely (up to an isometry equivalence) defined, i.e.,

(1) An operator $T\colon X \to Y$ is said to be *anti-linear* if $T(a_1x_1 + a_2x_2) = \bar{a}_1T(x_1) + \bar{a}_2T(x_2)$ for all $x_1, x_2 \in X$ and complex numbers a_1, a_2.

if the operator R admits two factorizations (2.1): $R = A_1^* A_1$ *and* $R = A_2^* A_2$, *where* $A_1 \in L(X, H_1)$ *and* $A_2 \in L(X, H_2)$, *respectively, then there exists an isometry* $U: H_1 \to H_2$ *such that* $A_2 = UA_1$.

It would be interesting to know when the Hilbert space in the factorization (2.1) may be chosen as a separable space. The answer is given in the following lemma.

LEMMA 2.5. *The necessary and sufficient condition for the space H in the factorization* (2.1) *to be separable is that the image $R(X)$ is separable.*

Proof. The necessity is clear. To prove sufficiency, we consider a ball S in the space X. It suffices to prove that the set $A(S)$ is separable. From the assumption of the theorem, there exists a sequence $\{y_k\}$ in S such that the sequence $\{Ry_k\}$ is dense in $R(S)$. We shall show that the sequence $\{Ay_k\}$ is dense in $A(S)$. Let h be an arbitrary element of $A(S)$. Then there exists an element $y \in S$ such that $Ay = h$. Now we choose a subsequence $\{y_{n_k}\}$ such that $\{Ry_{n_k}\}$ converges to Ry. Hence we obtain

$$\|h - Ay_{n_k}\|^2 = \|A(y - y_{n_k})\|^2 = \big(R(y - y_{n_k})\big)(y - y_{n_k})$$
$$\leqslant \|R(y - y_{n_k})\| \, \|y - y_{n_k}\| \to 0,$$

if k tends to ∞. Thus $A(S)$ is separable. □

The operator A in the factorization (2.1) is analogous to the square root of non-negative operators in a Hilbert space. Therefore the next definition naturally follows.

DEFINITION 2.6. The operator A in (2.1) is said to be a *square root* of the non-negative operator R.

From Lemma 2.4, it follows that a square root is uniquely defined up to an isometric operator. We remark that if R is the covariance operator of a generalized random element ξ, then ξ is a square root of R.

In the correlation theory of infinite-dimensional processes it would be interesting to know when the covariance operator coresponds to a "real" random element of the second-order, i.e., to an operator which is generated by a random element of the second-order. Since a decomposable operator and a non-decomposable operator may have this same covariance operator, the above fact cannot be characterized in terms of the covariance operator. However, sufficient conditions may be obtained in such terms.

PROPOSITION 2.7. *Let Y be a Banach space and an operator $R \in \bar{L}(Y^*, Y^{**})$ admit the representation $R = A^* K A$, where A is a linear continuous operator from Y^* into a Hilbert space H, such that $A^*(H) \subset Y$ and K is a non-negative nuclear operator in H. Then every generalized random element with the covariance operator R is generated by a random element of the strong second-order.*

Proof. Let ξ be a generalized random element with the covariance operator R, i.e., ξ is a linear continuous operator from Y^* into $L_2(\Omega, B, P)$ and $R = \xi^* \xi$. Consider the operator $\eta = TA$, where $T: H \to H$ is a square root of the non-negative nuclear operator K. Therefore T is a Hilbert–Schmidt operator. Then ξ and η are square roots of the operator R. From Lemma 2.4, there exists an isometric operator $U: H \to L_2(\Omega, B, P)$ such that $\xi = U\eta = UTA$. But an operator $UT: H \to L_2(\Omega, B, P)$ is a Hilbert–Schmidt operator. Therefore we may choose an H-valued random element $h(\omega)$ of the strong second-order, such that

$$(UTf)(\omega) = (f, h(\omega))_H$$

for each $f \in H$. Now consider the random element $x(\omega) = A^* h(\omega)$. Obviously, $x(\omega)$ is a Y-valued random element of the strong second-order, and the generalized random element ξ is generated by $x(\omega)$. □

Let now $x(\omega)$ be a Y-valued random element of the strong second-order, and $\xi: Y^* \to L_2(\Omega, B, P)$ be the generalized random element generated by $x(\omega)$ with the covariance operator $R = \xi^* \xi$. By Proposition 1.6, the operator ξ is 2-absolutely summing. Let η be a square root of R. Then from Lemma 2.4, $\eta = U\xi$, where U is an isometry. But ξ is 2-absolutely summing, and so η is also 2-absolutely summing. Thus we have the following proposition.

PROPOSITION 2.8. *Let R be the covariance operator of a random element of the stong second-order. Then every square root of R is 2-absolutely summing.*

3. The space of generalized random elements of the second-order as an LVH-space

In the set of all generalized random elements of the second-order, we may introduce an algebraic-topological structure. In fact, it is easy to see that this set is a linear space. For generalized random elements ξ, η we define an *inner product* by

$$[\xi, \eta] = R_{\xi\eta} = \eta^* \xi, \tag{3.1}$$

where the operator $R_{\xi\eta}$ is the correlation operator of these elements. It is clear that $R_{\xi\eta} \in \bar{L}(Y^*, Y^{**})$. In this section we study the properties of a pseudo-Hilbertian space obtained in this way. We note that while a generalized random element of the second-order was defined as a continuous linear operator from Y^* into $L_2(\Omega, B, P)$, for further consideration it is not important that this operator is mapping the dual space of a Banach space into the Hilbert space of random variables having the second absolute moments finite. Therefore, by a *generalized random* *element of the "second-order"* we shall mean an element of the space $L(X, H)$, where X is a Banach space and H is a Hilbert space.

By $\mathscr{X}$ we denote the space $\bar{L}(X, X^*)$ in which the inner product of generalized random elements, defined as in (3.1), takes its values. A basic set of neighbourhoods of the origin in $\mathscr{X}$ is defined by

$$N_\theta(\varepsilon, A, B) = \{R;\ |(Rx)(y)| < \varepsilon, x \in A, y \in B\},$$

where $\varepsilon > 0$, A, B are finite subsets in X. With the above topology, the space $\mathscr{X}$, the *admissible space* for the inner product (3.1), has the following properties:

I. *$\mathscr{X}$ is a complete locally convex Hausdorff linear topological space.*

From the definition of the topology in $\mathscr{X}$, it follows that $\mathscr{X}$ is a locally convex Hausdorff linear topological space. Suppose that R_a is a generalized Cauchy sequence of elements in $\mathscr{X}$. This means that for each $\varepsilon > 0$ and $x, y \in X$ there exists a_0, such that for all $a, \beta \succ a_0$ we have

$$\big|\big((R_a - R_\beta)y\big)(x)\big| < \varepsilon.$$

By the uniform boundedness principle for a family of operators, we conclude that this sequence is convergent to a bounded linear operator and so $\mathscr{X}$ is complete.

II. *There is a closed convex cone $\mathscr{C}$ in $\mathscr{X}$ such that $\mathscr{C} \cap -\mathscr{C} = 0$; then we define a partial ordering in $\mathscr{X}$ by writing $R_1 \succ R_2$ if $R_1 - R_2 \in \mathscr{C}$.*

It suffices to take as $\mathscr{C}$ a set of all non-negative operators, i.e., $R \in \mathscr{C}$ if $(Rx)(x) \geqslant 0$ for all $x \in X$.

III. *The topology in $\mathscr{X}$ is compatible with the partial ordering, in the sense that there is a basic set $\{N'_\theta\}$ of convex neighbourhoods of the origin, such that if $R_1 \in N'_\theta$ and $R_1 \succ R_2 \succ 0$, then $R_2 \in N'_\theta$.*

It is easy to verify that the following basic set:

$$N'_\theta(\varepsilon, A) = \{R;\ |(Rx)(x)| < \varepsilon, x \in A\},$$

where $\varepsilon > 0$ and A is an arbitrary finite subset of X, satisfies property III.

IV. *$\mathscr{X}$ has an involution.*

For each operator $R \in \mathscr{X}$ we define an operator $R^* \in \mathscr{X}$, such that for all $x, y \in X$,

$$(Ry)(x) = \overline{(R^*x)(y)}.$$

The operator R^* will be called the *adjoint operator*(1) of R.

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(1) The adjoint operator of an operator $R \in L(X, X^*)$ is an operator $R^+ : X^{**} \to X^*$ defined by the relation $(R^+ y^{**}) = \overline{y^{**}(Rx)}$. It is easy to see that R^* is obtained by restricting R^+ to the space X. (X is regarded as a natural imbedding into the space X^{**}.)

The following properties of an involution are simple:

$$(R^*)^* = R, \quad (a_1 R_1 + a_2 R_2)^* = \bar{a}_1 R_1^* + \bar{a}_2 R_2^*.$$

V. *If* $R \succcurlyeq 0$, *then* $R^* = R$.

This means that the operator R is self-adjoint, which follows from a more general property: if $(Rx)(x)$ assumes real values, then $(Ry)(x) = \overline{(Rx)(y)}$. This last fact may be proved by using essentially such arguments as in the case of Hilbert spaces.

VI. *If* $R_1 \succcurlyeq R_2 \succcurlyeq \ldots \succcurlyeq 0$, *then the sequence* $\{R_n\}$ *is convergent.*

According to the property *I*, the space $\mathscr{X}$ is complete. Hence, by the relation

$$\left|\big((R_m - R_n)x\big)(y)\right|^2 \leqslant \big((R_m - R_n)x\big)(x)\big((R_m - R_n)y\big)(y) \to 0$$

for $m, n \to \infty$ $(m < n)$, the sequence $\{R_n\}$ is convergent. The above relation is a consequence of the generalized Schwarz inequality which may be proved as in the case of a Hilbert space.

Now we denote by $\mathscr{H}$ the space $L(X, H)$ with topology defined by a basic set of neighbourhoods of the origin

$$\mathcal{O}(N_\theta) = \{\xi\colon [\xi, \xi] \in N_\theta\},$$

where $N_\theta = N_\theta(\varepsilon, A, B)$ is a basic neighbourhood of the origin in $\mathscr{X}$. Then $\mathscr{H}$ has the following two properties:

VII. $\mathscr{H}$ *is a complete locally convex Hausdorff linear topological space.*

It suffices to prove that $\mathscr{H}$ is complete. But this fact is a direct consequence of the fact that the above topology in $\mathscr{H}$ coincides with the strong operator topology in the space of continuous linear operators from X into H, which is complete.

VIII. *The inner product on* $\mathscr{H}$, *defined by* (3.1), *is a map* $\xi, \eta \to [\xi, \eta]$ *from* $\mathscr{H} \times \mathscr{H}$ *into the admissible space* $\mathscr{X}$, *and has the following properties*:

(i) $[\xi, \xi] \succcurlyeq 0$ *and* $[\xi, \xi] = 0$ *implies* $\xi = 0$,

(ii) $[\xi, \eta] = [\eta, \xi]^*$,

(iii) $[a_1\xi_1 + a_2\xi_2, \eta] = a_1[\xi_1, \eta] + a_2[\xi_2, \eta]$.

We note that properties I–VIII for an arbitrary pair of linear topological spaces $\mathscr{H}, \mathscr{X}$, characterize an LVH-*space* $\mathscr{H}$ (pseudo-Hilbertian space) *with an admissible space* $\mathscr{X}$, which first appeared in R. M. Loynes [10]. It is clear that in any of these space concepts analogous to those found useful in the case of a Hilbert space may be defined: the most important are subspaces, operators and their adjoints, and various special types of operators, such as projection, self-adjoint and unitary operators. The adjoint of an operator T in an LVH-space $\mathscr{H}$ is an operator T^*, if one exists, such that

$$[T\xi, \eta] = [\xi, T^*\eta],$$

for all $\xi, \eta \in \mathscr{H}$; P is a projection if P^* exists and

$$P = P^* = P^2,$$

and U is unitary if U^* exists and

$$UU^* = U^*U = I.$$

Note, however, that there are no theorems guaranteeing the existence of the adjoint of an operator, or of the projection onto a subspace. This is the main difference from the Hilbert space situation and gives rise to certain difficulties. For our present purpose, a special LVH-space is of main interest; $\mathscr{H} = L(X, H)$ is the space of all generalized random elements of the second-order.

4. Banach-space-valued stationary processes

By generalizing the concept of the Hilbert space to allow the inner product to take values being no longer scalars but elements of a more general admissible space, it is possible to define a Banach-space-valued stochastic process, which is stationary in a natural sense. By a Banach-space-valued stochastic process of the "second-order" we mean a family $(\xi_t)_{t \in T}$ of elements belonging to a special LVH-space $\mathscr{H} = L(X, H)$, which was considered in the previous section, i.e., such a process is a curve in the space $L(X, H)$. We will consider processes indexed by the elements of any LCA group G. Therefore we recall for further reference some notation and result on LCA groups.

Let G be an LCA group. The set of all characters of G, i.e., a continuous homomorphism of G into the circle group, forms a group Γ, the dual group of G. The dual group is also an LCA group under compact-open topology. In view of the duality between G and Γ (Pontryagin's duality theorem) we will denote the characters of G by $\langle g, \gamma \rangle$, $g \in G$, $\gamma \in \Gamma$ (cf. [15]). The Borel σ-algebra of the LCA group is the minimal σ-algebra generated by the closed subsets. On every LCA group there exists a non-negative measure, finite on compact sets and positive on non-empty open sets, the so-called Haar measure of the group, which is translation — invariant. We will denote Haar measures on G and Γ by dg and $d\gamma$, respectively.

DEFINITION 4.1. A Banach-space-valued process of the second-order over an LCA group G is called *stationary* if the function of two variables (1)

(1) The symbol $[\cdot, \cdot]$ denotes the inner product in the LVH-space $\mathscr{H} = L(X, H)$, cf. Section 3.

$R(g, h) = [\xi_g, \xi_h]$ depends only on gh^{-1}, and the correlation function

$$R(g) = [\xi_g, \xi_e]$$

is weakly continuous (i.e., for each $x \in X$ the scalar-valued function $\big(R(g)x\big)(x)$ is continuous).

Now we give several simple examples of such processes.

EXAMPLE 4.2. Let $X = H = \mathscr{K}$ be a Hilbert space and $(U_g)_{g \in G}$ a weakly continuous unitary representation of an LCA group G in $\mathscr{K}$. If we put

$$\xi_g = U_g, \quad g \in G,$$

then, of course, $(\xi_g)_{g \in G}$ is a $\mathscr{K}$-valued stationary process.

EXAMPLE 4.3. Let X be a Banach space and η be a linear continuous operator from X to a Hilbert space H. Put

$$\xi_g = U_g \eta, \quad g \in G,$$

where $(U_g)_{g \in G}$ is a weakly continuous unitary representation of an LCA group G. We have

$$(U_h \eta)^* (U_g \eta) = \eta^* U_h^* U_g \eta = \eta^* U_{gh^{-1}} \eta,$$

thus $(\xi_g)_{g \in G}$ is an X-valued stationary process. This example is more general. Each Banach-space-valued stationary process is of the form described in this example (see (4.2)).

EXAMPLE 4.4. Let ψ_g be an $L(X, H)$-valued function on a discrete Abelian group G, such that

$$(\psi_g x, \psi_h y)_H = \begin{cases} 0 & \text{if } g \neq h, \\ (Ky)(x) & \text{if } g = h, \end{cases}$$

where K is a positive anti-linear operator belonging to $\bar{L}(X, X^*)$. Obviously, the process $(\psi_g)_{g \in G}$ is stationary. It is called an *orthogonal process*.

EXAMPLE 4.5. Let $(\psi_n)_{n \in Z}$ be a Banach-space-valued orthogonal process indexed by integers. Let A_k, $k \in Z$, be a sequence of operators belonging to $L(X, X)$, such that for each $n \in Z$ the series $\sum_{k=0}^{\infty} \psi_{n-k} A_k$ is strongly convergent. Put

$$\xi_n = \sum_{k=0}^{\infty} \psi_{n-k} A_k, \quad n \in Z.$$

We have

$$\xi_n^* \xi_m = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} A_s^* \psi_{n-s}^* \psi_{m-k} A_k = \sum_{k=0}^{\infty} A_{m-n-k}^* K A_k.$$

Hence the process $(\xi_n)_{n \in Z}$ is stationary. It is obtained from an orthogonal process $(\psi_n)_{n \in Z}$ by one-side moving averages.

PROPOSITION 4.6. *A Banach-space-valued process $(\xi_g)_{g\in G}$ over an LCA grup G, of the second-order, is stationary if and only if for each $x\in X$, the one-dimensional process $(\xi_g x)_{g\in G}$ is stationary.*

Proof. The necessity immediately follows from Definition 4.1. To prove sufficiency we consider the following form;

$$\big(R(g,h)x\big)(y) = (\xi_g y, \xi_h x)_H, \qquad x, y\in X.$$

By the polarization formula we obtain

$$\big(R(g,h)x\big)(y) = \tfrac{1}{4}\Big\{\big(\xi_g(x+y), \xi_h(x+y)\big)_H - \big(\xi_g(x-y), \xi_h(x-y)\big)_H + \\ + i\big(\xi_g(x+iy), \xi_h(x+iy)\big)_H - i\big(\xi_g(x-iy), \xi_h(x-iy)\big)_H\Big\}.$$

From the stationarity of $(\xi_g x)_{g\in G}$ for each $x\in X$, it follows that the function $R(g,h)$ of two arguments depends only on gh^{-1}. Thus the process $(\xi_g)_{g\in G}$ is stationary. □

Let H_ξ denote the *vector-time-domain* of a Banach-space-valued stationary process $(\xi_g)_{g\in G}$, i.e., the closed subspace of the Hilbert space H spanned over the elements $\xi_g x$, $g\in G$ and $x\in X$.

The stationary process $(\xi_g)_{g\in G}$ defines in the space H_ξ a unitary representation of the group G. Namely, the suitable unitary operators U_h are defined by the formula

$$U_h \xi_g x = \xi_{gh} x, \qquad g, h\in G, \ x\in X,$$

and for the remaining points of the space H_ξ, the operators U_h are defined by a natural extension. By the generalized theorem of Stone (cf. [4]), for the operators U_h we have the spectral representation

$$U_h = \int_\Gamma \langle g, \gamma\rangle E(d\gamma), \tag{4.1}$$

where $E(\cdot)$ is a resolution of the identity in H_ξ, i.e., a regular, normed and orthogonal spectral family of projectors in H_ξ, defined on the Borel σ-algebra of the dual group Γ. Since

$$\xi_h = U_h \xi_e, \tag{4.2}$$

by (4.1) we obtain the following result:

THEOREM 4.7. *Let $(\xi_g)_{g\in G}$ be a Banach-space-valued stationary process over an LCA group G. Then*

$$\xi_g = \int_\Gamma \langle g, \gamma\rangle \Phi(d\gamma), \qquad g\in G, \tag{4.3}$$

where $\Phi(d\gamma)$ is the $L(X,H)$-valued strongly σ-additive measure on the Borel σ-algebra of the dual group Γ, given by the relation $\Phi(d\gamma) = E(d\gamma)\xi_e$; $E(\cdot)$ is a resolution of the identity in H_ξ.

The spectral representation (4.3) was obtained in [16] and also in [12], where Banach-space-valued stationary processes over a semi-group are considered.

In concluding this section we give the following definition:

Definition 4.8. The operator-valued measure $\Phi(d\gamma)$ in the spectral representation (4.3) is called the *random measure* of the process $(\xi_g)_{g\in G}$. It follows from the above construction that the measure is orthogonal in the following sense:

$$\Phi^*(\Delta)\Phi(\Delta') = 0 \quad \text{if } \Delta\cap\Delta' = \emptyset.$$

5. Correlation function and operator-valued measures

The correlation function of Banach-space-valued stationary processes takes its values in the admissible space $\mathfrak{X} = \bar{L}(X, X^*)$ of the LVH-space $\mathscr{H} = L(X, H)$ (see Definition 4.1). The traditional notion, for Bochner's theorem, of the positive-definite function in the case of an $\bar{L}(X, X^*)$-valued function may be defined in several ways.

Definition 5.1. Let X be a Banach space. An $\bar{L}(X, X^*)$-valued function $R(g)$ on a group G is called:

(a) *positive-definite* if

$$\sum_{i,j=1}^{N}\big(R(g_i g_j^{-1})x_i\big)(x_j) \geqslant 0$$

for each N, $g_1, g_2, \ldots, g_N \in G$ and $x_1, x_2, \ldots, x_N \in X$;

(b) *positive-definite, in the strong sense,* if

$$\sum_{i,j=1}^{N} A_i^* R(g_i g_j^{-1}) A_j \succcurlyeq 0$$

for each N, $g_1, g_2, \ldots, g_N \in G$ and $A_1, A_2, \ldots, A_N \in L(X, X)$;

(c) *positive-definite, in the weak sense,* if

$$\sum_{i,j=1}^{N} a_i \bar{a}_j R(g_i g_j^{-1}) \succcurlyeq 0$$

for each N, $g_1, g_2, \ldots, g_N \in G$ and complex numbers $a_1, a_2, \ldots, a_N$.

In the following theorem, a characterization of the class of all correlation functions of Banach-space-valued stationary processes is given. This result for the case of processes indexed by the additive group of integers with discrete topology and by the additive group of reals with natural topology, was proved, in another way, in the recent paper [25].

THEOREM 5.2. *If $R(g)$ is a weakly continuous $\bar{L}(X, X^*)$-valued function on an LCA group G, then the following are equivalent:*

(i) *$R(g)$ is the correlation function of a Banach-space-valued stationary process over the group G.*

(ii) *$R(g)$ is positive-definite in the strong sense,*

(iii) *$R(g)$ is positive-definite,*

(iv) *$R(g)$ is positive-definite in the weak sense,*

(v) *there is a unique positive $\bar{L}(X, X^*)$-valued regular Borel measure F on the dual group Γ such that for all $g \in G$*

$$R(g) = \int_{\Gamma} \langle g, \gamma \rangle F(d\gamma),$$

where the integral is meant in the weak sense.

Proof. Using the idea proposed in [21], we prove this theorem in the direction: (i)$\Leftrightarrow$(iii) and (ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(ii).

(i)$\Leftrightarrow$(iii)(1). From the definition of the correlation function of a Banach-space-valued stationary process (see Definition 4.1), it immediately follows that $R(g)$ is positive-definite. Conversely, consider the scalar functions $K(\tau, \sigma)$ of two variables $\tau, \sigma \in T = X \times G$ given by the following relation

$$K(\tau, \sigma) = \big(R(gh^{-1})x\big)(y),$$

where $\tau = (x, g)$ and $\sigma = (y, h)$. Since by (iii), $R(g)$ is a positive-definite operator-valued function, $K(\tau, \sigma)$ is a positive-definite scalar-valued function. Hence, by [5] (p. 109), there exist a Hilbert space H and a family of vectors $(\eta_\tau)_{\tau \in T}$, such that

$$K(\tau, \sigma) = (\eta_\tau, \eta_\sigma)_H.$$

Thus

$$\big(R(gh^{-1})x\big)(y) = \big(\eta(x, g), \eta(y, h)\big)_H, \tag{5.1}$$

where $\eta(x, g)$ is an H-valued function of two variables $x \in X$ and $g \in G$. By (5.1), the function $\eta(x, g)$ is linear on the first variable if the second is fixed. Therefore there exists a linear operator ξ_g: $X \to H$, $g \in G$, such that

$$\xi_g x = \eta(x, g), \quad x \in X.$$

From the inequality

$$\|\xi_g x\|^2 = \big(R(e)x\big)(x) \leqslant \|R(e)\| \, \|x\|^2$$

and by the above construction, it follows that the operator ξ_g is con-

(1) The idea of the proof of this equivalence is taken from [16].

tinuous. Hence $(\xi_g)_{g\in G}$ is a Banach-space-valued stationary process with correlation function $R(g)$.

(ii) ⇒ (iii). According to condition (ii), we have the inequality

$$\sum_{i,j=1}^{N} \big(R(g_i g_j^{-1}) A_i x\big)(A_j x) \geqslant 0$$

for each $x \in X$. Now, let $x \in X$ be fixed. We define operators A_i: $X \to X$, $i = 1, 2, \ldots, N$, as follows: $A_i x = x_i$, where $x_1, x_2, \ldots, x_N$ is an arbitrary finite sequence of elements from the Banach space X. By the Hahn–Banach theorem, there exists a closed subspace $X_0 \subset X$, the complement of the subspace being spanned by the element x.

Put

$$A_i(X_0) = 0 \quad \text{and} \quad A_i(x) = x_i.$$

Then

$$\sum_{i,j=1}^{N} \big(R(g_i g_j^{-1}) A_i x\big)(A_j x) = \sum_{i,j=1}^{N} \big(R(g_i g_j^{-1}) x_i\big)(x_j) \geqslant 0.$$

Thus condition (iii) holds.

(iii) ⇒ (iv). For every finite sequence $x_1, x_2, \ldots, x_N$ we have the inequality

$$\sum_{i,j=1}^{N} \big(R(g_i g_j^{-1}) x_i\big)(x_j) \geqslant 0. \tag{5.2}$$

Let x be an arbitrary element in X and $a_1, a_2, \ldots, a_N$ be an arbitrary finite sequence of complex numbers. If we put $x_i = a_i x$ into (5.2), we obtain (iv).

(iv) ⇒ (v). Let $R(g)$ be a positive-definite, in the weak sense, function on the LCA group G. Thus the scalar-valued function $\big(R(g)x\big)(x)$ for every $x \in X$ is a continuous positive-definite function on the group G. The well-known Bochner's theorem (cf. [15], p. 19) implies that for every $x \in X$ there is a unique positive regular Borel measure F_{xx} on the dual group Γ, such that for all $g \in G$,

$$\big(R(g)x\big)(x) = \int_{\Gamma} \langle g, \gamma\rangle F_{xx}(d\gamma).$$

Using the polarization formula

$$\big(R(g)x\big)(y) = \tfrac{1}{4}\Big\{\big(R(g)(x+y)\big)(x+y) - \big(R(g)(x-y)\big)(x-y) + \\ + i\big(R(g)(x+iy)\big)(x+iy) - i\big(R(g)(x-iy)\big)(x-iy)\Big\},$$

we obtain

$$\big(R(g)x\big)(y) = \int_{\Gamma} \langle g, \gamma\rangle F_{xy}(d\gamma),$$

where $F_{xy}(\Delta)$ is the bilinear form for fixed Δ. From the Schwarz inequality for a positive form

$$|F_{xy}(\Delta)|^2 \leqslant F_{xx}(\Delta) F_{yy}(\Delta)$$

and the inequality

$$F_{xx}(\Delta) \leqslant F_{xx}(\Gamma) = \int_{\Gamma} F_{xx}(d\gamma) = \big(R(e)x\big)(x),$$

it follows that $F_{xy}(\Delta)$ is bounded. Therefore there exists a continuous anti-linear operator $F(\Delta)$, such that $F_{xy}(\Delta) = \big(F(\Delta)x\big)(y)$. Hence

$$\big(R(g)x\big)(y) = \int_{\Gamma} \langle g, \gamma\rangle \big(F(d\gamma)x\big)(y), \tag{5.3}$$

thus (v) holds. The uniqueness of the operator-valued measure F follows from the uniqueness of the scalar-valued measure F_{xx} and the above construction.

(v) $\Rightarrow$(ii). Let the function $R(g)$ admit the representation (5.3). Then from elementary properties of characters we obtain

$$\begin{aligned} \sum_{i,j=1}^{N} \big(A_i^* R(g_i g_j^{-1}) A_j x\big)(x) &= \sum_{i,j=1}^{N} \Big(A_i^* \int_{\Gamma} \langle g_i g_j^{-1}, \gamma\rangle F(d\gamma) A_j x\Big)(x) \\ &= \int_{\Gamma} \sum_{i,j=1}^{N} \big(\langle g_i, \gamma\rangle A_i^* F(d\gamma) A_j \overline{\langle g_j, \gamma\rangle} x\big)(x). \end{aligned} \tag{5.4}$$

From the definition of the Lebesgue integral we may write the last integral in the following form:

$$\sup_{\{\Delta_1, \Delta_2, \ldots, \Delta_M\}} \Big\{ \sum_{k=1}^{M} \inf_{\gamma \in \Delta_k} \sum_{i,j=1}^{N} \langle g_i, \gamma\rangle \big(A_i^* F(\Delta_k) A_j x\big)(x) \overline{\langle g_j, \gamma\rangle} \Big\}, \tag{5.5}$$

where $\{\Delta_1, \Delta_2, \ldots, \Delta_M\}$ is a measurable dissection of Γ. The measure F takes values in the set of all positive operators from $\bar{L}(X, X^*)$. Then by Lemma 2.4, for an arbitrary set Δ_k there exist a Hilbert space H_k and a linear bounded operator T_k: $X \to H_k$, such that $F(\Delta_k) = T_k^* T_k$ and $T_k^* T_j = 0$ for $k \neq j$. Hence (5.5) may be represented as follows

$$\sup_{\{\Delta_1, \Delta_2, \ldots, \Delta_M\}} \Big\{ \sum_{k=1}^{M} \inf_{\gamma \in \Delta_k} \Big\| \sum_{i=1}^{N} \langle g_i \gamma\rangle T_k A_i x \Big\|^2 \Big\}.$$

It is implied by (5.4) and (5.5) that the function $R(g)$ is positive-definite in the strong sense. □

This theorem has numerous interesting consequences, a few of which will now be presented.

DEFINITION 5.3. The $\bar{L}(X, X^*)$-valued measure $F(d\gamma)$ in condition (v) of the previous theorem is called the *spectral measure* of a Banach-space-valued stationary process.

From the uniqueness in condition (v) of Theorem 5.2, there immediately follows the close relation between the spectral measure and the random measure of a process.

COROLLARY 5.4. *If $F(d\gamma)$ is the spectral measure and $\Phi(\Delta)$ is the random measure of a Banach-space-valued stationary process, then for each Borel subset of the dual group Γ we have*

$$F(\Delta) = \Phi^*(\Delta)\Phi(\Delta) = [\Phi(\Delta), \Phi(\Delta)].$$

The equivalence of conditions (i) and (iv) in Theorem 5.2 yields the following result on a factorization of an operator-valued function.

COROLLARY 5.5. *Let $f(g)$ be a weakly continuous positive-definite $\bar{L}(X, X^*)$-valued function on an LCA group G. Then there exist a Hilbert space H, a linear operator $A \in L(X, H)$ and a weakly continuous unitary representation U_g of the group G in H, such that*

$$f(g) = A^* U_g A.$$

From the equivalence of conditions (i) and (v) of Theorem 5.2, we obtain the following corollary on a factorization of an operator-valued measure.

COROLLARY 5.6. *Let $P(\Delta)$ be a positive $\bar{L}(X, X^*)$-valued measure on the Borel σ-algebra of an LCA group. Then there exist a Hilbert space H and an orthogonal $L(X, H)$-valued measure $M(\Delta)$ on this σ-algebra, such that*

$$P(\Delta) = M^*(\Delta)M(\Delta) = [M(\Delta), M(\Delta)].$$

We note that the last result holds in the more general case; if these measures are defined on an arbitrary measurable space. This follows, for example, from Theorem 4 of the recent paper [12].

We shall close this section with a remark on the absolute continuity for operator-valued measures. It is well known that in the finite-dimensional case an absolute continuous operator-valued measure $F(d\gamma)$ has an *operator-valued density* $f(\gamma)$:

$$F(\Delta) = \int_{\Delta} f(\gamma)\, d\gamma.$$

In the infinite-dimensional case this fact does not hold. For all $x \in X$ we may write

$$\big(F(\Delta)x\big)(x) = \int_{\Delta} f_x(\gamma)\, d\gamma,$$

where $f_x(\gamma)$ is a scalar-valued function of γ which depends on x. But it is not true that the function $f_x(\gamma)$ may be represented in the form

$$f_x(\gamma) = \big(f(\gamma)x\big)(x),$$

where for each $\gamma \in \Gamma$ the operator-valued function $f(g)$ takes values in the space $\bar{L}(X, X^*)$. In fact, let us consider the Hilbert space $H = L_2 \times \times ([-\pi, +\pi])$ and put

$$\big(F(\Delta)x(\cdot)\big)(\lambda) = \chi_\Delta(\lambda)x(\lambda), \quad x(\cdot) \in L_2([-\pi, +\pi]),$$

where $\chi_\Delta(\lambda)$ is the characteristic function of a measurable subset $\Delta \subset [-\pi, +\pi]$. Then $F(\Delta)$ is the spectral measure of the stationary process considered in Example 4.2 in the case of a group of integers. The suitable unitary operator U is given by the relation $Ux(\lambda) = e^{i\lambda}x(\lambda)$, $x(\lambda) \in L_2([-\pi, +\pi])$.

Now we shall find the matrix representation of the operator $F(\Delta)$. Let

$$e_n = \frac{1}{\sqrt{2\pi}} e^{in\lambda}, \quad n = 0, \pm 1, \pm 2, \ldots,$$

be the orthonormal base for the Hilbert space $L_2([-\pi, +\pi])$. We have

$$F_{nm}(\Delta) = \big(F(\Delta)e_n, e_m\big) = \frac{1}{2\pi} \int_\Delta e^{i(n-m)\lambda} d\lambda. \tag{5.6}$$

If the measure $F(\Delta)$ has an operator-valued density $f(\lambda)$, then by (5.6)

$$\{f_{n,m}(\lambda)\} = \{e^{i(n-m)\lambda}\} \quad \text{a.e.} \tag{5.7}$$

But for each λ, the matrix (5.7) does not define a linear operator in the space $L_2([-\pi, +\pi])$. Therefore the operator-valued density of the measure F, with respect to the Lebesgue measure, does not exist.

Returning to the general case, we give the following definition.

DEFINITION 5.7. We say that the spectral measure $F(d\gamma)$ of a Banach-space-valued stationary process, which is absolutely continuous with respect to the Haar measure $d\gamma$, has an operator-valued density (or that the stationary process has a spectral density) if there exists a weakly continuous $\bar{L}(X, X^*)$-valued function $f(\gamma)$ defined on the dual group Γ, such that

$$\big(F(\Delta)x\big)(y) = \int_\Delta \big(f(\gamma)x\big)(y)\, d\gamma, \quad x, y \in X.$$

6. Ergodic properties

Now we study ergodic properties of Banach-space-valued stationary processes over an LCA group. It is convenient to begin with the special case of processes indexed by integers, i.e., $G = Z$. Before we prove the ergodic theorem, we need the following lemma which immediately follows from the well-known ergodic theorem of F. Riesz (cf. [13]).

LEMMA 6.1. *Let $(\xi_n)_{n\in Z}$ be a Banach-space-valued process of the second-order (not necessarily stationary) admitting a representation $\xi_n = A^n \xi_0$, where A is a linear continuous operator from a Hilbert space H into itself and there exists a positive constant C such that $\|A^n\| < C$ for all $n \in Z$. Then for every $x \in X$,*

$$\lim \frac{1}{n-m} \sum_{k=m}^{n} \xi_k x = P_0 \xi_0 x,$$

where $(n-m) \to \infty$, the limit is meant in norm topology of the Hilbert space H and P_0 is an orthogonal projection on a set of all fixed points of the operator A.

The ergodic theorem for Banach-space-valued stationary sequences is as follows:

THEOREM 6.2. ([19]). *Let $(\xi_n)_{n\in Z}$ be a Banach-space-valued stationary sequence. Then a "time mean" $\frac{1}{n-m} \sum_{k=m}^{n} \xi_k$ is convergent to $\Phi(\{0\})$ if $(n-m) \to \infty$ in strong operator topology of the space $L(X, H)$, where $\Phi(d\gamma)$ is the random measure of the sequence $(\xi_n)_{n\in Z}$.*

Proof. Let $(\xi_n)_{n\in Z}$ be a Banach-space-valued stationary sequence with the shift operator U. Thus, according to (4.2), $\xi_n = U^n \xi_0$ and $\|U^n\| = 1$ for every $n \in Z$. Hence from Lemma 6.1,

$$\frac{1}{n-m} \sum_{k=m}^{n} \xi_k x \to P_0 \xi_0 x \quad \text{for every } x \in X,$$

where P_0 is the orthogonal projection on the set N of all fixed-points of the unitary operator U. If $E(d\lambda)$ is a resolution of the identity in H which corresponds to the unitary operator U by Stone's theorem, then the set N coincides with the image of the operator $E(\{0\})$. Hence, $P_0 = E(\{0\})$. But by Theorem 4.7, $E(\{0\})\xi_0 = \Phi(\{0\})$, which completes the proof. □

The example given in [22] shows that the strong operator convergence in the above theorem cannot be replaced by uniform operator convergence.

Now we shall extend Theorem 6.2 to the general case of Banach-space-valued stationary processes over an LCA group. We need the following notion of a "mean value over the group" for operator-valued functions, which represents a "time mean". This notion for vector-valued functions arises in the theory of almost periodic functions.

DEFINITION 3.6. Let T_g be an operator-valued function on a group G, taking values in the space $L(X, Y)$, where X is a linear topological space and Y is a linear normed space. Then function T_g is said to possess the *mean value over the group G* equal to $\hat{M}$ ($\hat{M}$ belongs to $L(X, Y)$) if for every $x \in G$ and $\varepsilon > 0$ there exists a finite system of elements of the group G, $g_1, g_2, \ldots, g_N$, such that

$$\sup_{g \in G} \left\| \frac{1}{N} \sum_{i=1}^{N} T_{gg_i} x - \hat{M}_x \right\| < \varepsilon,$$

where the symbol $\|\cdot\|$ denotes the norm in Y. The operator $\hat{M} \in L(X, Y)$ is then uniquely defined and will be denoted by $M_G T$.

We are now ready to give a group analogue of Theorem 6.2.

THEOREM 6.4. *If $(\xi_g)_{g \in G}$ is a Banach-space-valued stationary process over an LCA group G, then $(\xi_g)_{g \in G}$ has the mean value over the group G equal to $\Phi(\{1\})$, where $\Phi(d\gamma)$ is the random measure of the process and* 1 *is the unit character.*

Proof. For every $x \in X$, $(\xi_g x)_{g \in G}$ is a one-dimensional stationary process over G. Let ψ_x denote its random measure. According to the result of R. Jajte (cf. [6]), for every $\varepsilon > 0$ there exists a system of elements of the group G, $g_1, g_2, \ldots, g_N$, such that

$$\sup_{g \in G} \left\| \frac{1}{N} \sum_{i=1}^{N} \xi_{gg_i} x - \psi_x(\{1\}) \right\| < \varepsilon. \tag{6.1}$$

Clearly, $\psi_x(\Delta) = \Phi(\Delta)x$ for every $x \in X$ and a Borelian set Δ in Γ. Hence, by (6.1) and by Definition 6.3, we have $M_G \xi = \Phi(\{1\})$. □

Now we assume that the process $(\xi_g)_{g \in G}$ is a curve in the space $L\big(Y^*, L_2(\Omega, B, P)\big)$ and that the expectation of the generalized random element ξ_g exists and does not depend of g (see Section 2).

Let $m \in Y$ be the expectation of the Banach space-valued stationary process $(\xi_g)_{g \in G}$. The expectation operator $V\colon Y^* \to L_2(\Omega, B, P)$ is defined by the relation

$$Vx^* = x^*(m), \qquad x^* \in Y^*,$$

i.e., the value of V on an element $x^* \in Y^*$ is the function in $L_2(\Omega, B, P)$ identically equal to the value of the functional x^* on the element m.

DEFINITION 6.5. A Banach-space-valued stationary process $(\xi_g)_{g \in G}$ over an LCA group G is called *ergodic* if $M_G \xi = V$.

From Theorem 6.4 it follows that a prosess $(\xi_g)_{g \in G}$ is ergodic if and only if for all $x^* \in Y^*$,

$$\Phi(\{1\})x^* = x^*(m). \tag{6.2}$$

Let $(\xi_g)_{g\in G}$ be an ergodic process. Then formula (6.2) and Corollary (5.4), imply

$$(6.3)\qquad \big(F(\{1\})x^*\big)(y^*) = M\overline{\Phi(\{1\})x^*}\,\Phi(\{1\})\,y^* = \overline{x^*(m)}y^*(m)$$

for all $x^*, y^*\in Y^*$. Conversly, let formula (6.3) hold for arbitrary $x^*, y^*\in Y^*$. By Corollary 5.4, we obtain $M|\Phi(\{1\})x^*|^2 = |x^*(m)|^2$. Observe further that due to the stationarity of the process $(\xi_g)_{g\in G}$, we have

$$M\xi_g x^* = \int_\Gamma \langle g,\gamma\rangle M\Phi(d\gamma)x^* = x^*(m),\qquad g\in G,$$

and hence $M\big(\Phi(\{1\})x^*\big) = x^*(m)$. Finally, from the relation

$$M|\Phi(\{1\})x^* - x^*(m)|^2 = M|\Phi(\{1\})x^*|^2 - |x^*(m)|^2 = 0$$

it follows that $\Phi(\{1\})x^* = x^*(m)$. Thus we obtain the following result.

THEOREM 6.6. *Let $(\xi_g)_{g\in G}$ be a Banach-space-valued stationary process over an LCA group G, with the expectation $m\in Y^*$. Then the following conditions are equivalent:*

(i) *$(\xi_g)_{g\in G}$ is ergodic,*

(ii) *$\Phi(\{1\})x^* = x^*(m)$ for every $x^*\in Y^*$,*

(iii) *$\big(F(\{1\})x^*\big)(y^*) = \overline{x^*(m)}y^*(m)$ for every $x^*, y^*\in Y^*$, where $\Phi(d\gamma)$ is the random measure and $F(d\gamma)$ is the spectral measure of the process $(\xi_g)_{g\in G}$.*

7. Linear prediction problem

We now proceed to a description of the linear prediction problem. The reason we define the Banach-space-valued stationary process as a curve in the space of generalized random elements of the second-order, is as follows — this space is complete and the best linear prediction belongs to this space.

Let $(\xi_g)_{g\in G}$ be a Banach-space-valued stationary process over an LCA group G. Denote by $\mathfrak{M}_\xi$ the *operator-time-domain* of the process $(\xi_g)_{g\in G}$, i.e., $\mathfrak{M}_\xi$ is a subspace of the space $L(X, H)$ spanned over the elements

$$\xi_g A,\quad g\in G,\quad A\in L(X, X).$$

Let S be a non-empty subset of the group G. By $\mathfrak{M}_\xi(S)$ denote a subspace of the space $\mathfrak{M}_\xi$ spanned over the elements $\xi_g A$, $g\in S$, $A\in L(X, X)$. Of course, the spaces $\mathfrak{M}_\xi$ and $\mathfrak{M}_\xi(S)$ are right ideals in $L(X, H)$, i.e., if $\eta\in\mathfrak{M}_\xi$, then also $\eta A\in\mathfrak{M}_\xi$ for $A\in L(X, X)$.

The following theorem will be used repeatedly. It directly follows from Yood's theorem on the closure of ideals of transformations between Banach spaces (cf. [24], Theorem 5.2).

THEOREM 7.1 *Let $\mathfrak{N}$ be a right ideal in the space $L(X, H)$ which is closed in strong operator topology. Then there exists in H a closed subspace N such that the ideal $\mathfrak{N}$ contains only operators $\eta \in L(X, H)$, the range of which is included in N, i.e.*

$$\mathfrak{N} = \{\eta \in L(X, H);\ \eta x \in N,\ x \in X\}.$$

By Theorem 7.1, there exists a close relation (one-to-one correspondence) between right ideals in $L(X, H)$ and closed subsets in H. We will say that the closed subspace N defines the right ideal $\mathfrak{N}$ or that N coresponds to $\mathfrak{N}$. Now, let H_ξ and $H_\xi(S)$ denote closed subspaces in H which correspond to the closed ideals $\mathfrak{M}_\xi$ and $\mathfrak{M}_\xi(S)$, respectively. The space H_ξ is called the *vector-time-domain* of the Banach-space-valued stationary process $(\xi_g)_{g \in G}$ (see Section 4).

We noted in Section 3 that not every subspace in the space $L(X, H)$ has an orthogonal complement. However, if a subspace $\mathfrak{N}$ of the space $L(X, H)$ is a closed right ideal in $L(X, H)$, then the orthogonal complement $\mathfrak{N}^\perp$ exists. Indeed, let $N \subset H$ be the closed subset in the Hilbert space H, which defines the closed right ideal $\mathfrak{N}$. The orthogonal complement $\mathfrak{N}^\perp$ is now defined as the closed right ideal in $L(X, H)$ which corresponds to $N^\perp$, where $N^\perp$ is the orthogonal complement of the closed subspace N in the Hilbert space H. Now it is possible to define an orthogonal projection on a closed right ideal in $L(X, H)$ in a natural way. Let $\mathfrak{N} \subset L(X, H)$ be a closed right ideal and let $N \subset H$ be the closed subspace which corresponds to $\mathfrak{N}$. Let $P_N\colon H \to H$ be the orthogonal projection on N. The operator $\boldsymbol{P}\colon L(X, H) \to L(X, H)$, given by the relation

$$\boldsymbol{P}\eta = P_N \cdot \eta, \qquad \eta \in L(X, H),$$

will be called the *orthogonal projection* on $\mathfrak{N}$.

The following properties of this operator directly follow from the above definition:

(i) the operator $\boldsymbol{P}$ is continuous and linear,

(ii) if $\eta \in \mathfrak{N}$, then $\boldsymbol{P}\eta = \eta$,

(iii) if $\eta \in \mathfrak{N}^\perp$, then $\boldsymbol{P}\eta = 0$,

(iv) $\boldsymbol{P}^2 = \boldsymbol{P}$,

(v) $[\boldsymbol{P}\eta, \varphi] = [\eta, \boldsymbol{P}\varphi]$, where $[\cdot, \cdot]$ denotes the inner product in $L(X, H)$ (cf. Section 3).

We are now ready to study the prediction problem.

DEFINITION 7.2. The *linear prediction* of a Banach-space-valued stationary process $(\xi_g)_{g \in G}$ at a point $g \in G$, based on a subset $S \subset G$, is an element $\hat{\xi}_{g,S} \in \mathfrak{M}_\xi(S)$ such that

$$\hat{\xi}_{g,S} = \boldsymbol{P}\,\xi_g = P_{H_\xi(S)} \cdot \xi_g,$$

where P denotes the orthogonal projection on the closed right ideal $\mathfrak{M}_\xi(S)$ in $L(X, H)$.

In the following proposition we give simple properties of the linear prediction $\hat{\xi}_{g,S}$.

PROPOSITION 7.3. *If* $\hat{\xi}_{g,S}$ *is as in Definition* 7.2, *then*

(i) $\hat{\xi}_{g,S} = \lim_{N\to\infty} \sum_{k=1}^{N} \xi_{g_k} A_k$, *where* $g_k \in S$, $A_k \in L(X, X)$, *and the limit in the formula is meant in strong operator topology,*

(ii) *for every* $x \in X$ *and* $\eta \in \mathfrak{M}_\xi(S)$,

$$\|\xi_g x - \hat{\xi}_{g,S} x\| \leqslant \|\xi_g x - \eta x\|,$$

(iii) *for every* $\eta \in \mathfrak{M}_\xi$ *and all* $x, y \in X$,

$$(\xi_g x - \hat{\xi}_{g,S} x, \eta y)_H = 0.$$

Proof. Property (i) directly follows from the fact that $\hat{\xi}_{g,S} \in \mathfrak{M}_\xi(S)$ and that the subspace $\mathfrak{M}_\xi(S)$ is the smallest subspace which includes elements of the form $\xi_g A$, $g \in S$ and $A \in L(X, X)$. To prove property (ii) we note that, by Definition 7.2, the element $\hat{\xi}_{g,S} x$ is the orthogonal projection of $\xi_g x$ on $H_\xi(S)$. Since $\eta x \in H_\xi(S)$, property (ii) follows. Clearly, $\xi_g x - \hat{\xi}_{g,S} x \perp H_\xi$. By assumption of the theorem, $\eta \in \mathfrak{M}_\xi(S)$ and consequently $\eta y \in H_\xi$ for each $y \in X$. Thus property (iii) holds. □

The linear prediction problem for a Banach-space-valued stationary process $(\xi_g)_{g\in G}$ at point $g \in G$, based on a subset $S \subset G$, may be described as follows: if the correlation function (or the spectral measure) of the process $(\xi_g)_{g\in G}$ is given, find a sequence of operators $A_1^N, A_2^N, \ldots, A_{n_N}^N$ from $L(X, X)$ and a sequence of points $g_1^N, \ldots, g_{n_N}^N \in S$, such that

1° there exists the strong limit $\hat{\xi}_{g,S} = \lim_{N\to\infty} \sum_{k=1}^{n_N} \xi_{g_k^N} A_k^N$,

2° $\|\xi_g x - \hat{\xi}_{g,S} X\| \leqslant \|\xi_g x - \eta x\|$ for every $\eta \in L(X, H_\xi(S))$ and every $x \in X$.

The operator $E_{g,S}(\xi) \in \bar{L}(X, X^*)$, defined by the relation

$$E_{g,S}(\xi) = [\xi_g - \hat{\xi}_{g,S}, \xi_g - \xi_{g,S}],$$

will be called the *prediction error operator* of the process $(\xi_g)_{g\in G}$. For a Banach-space-valued stationary process over the discrete group of integers, whose values form a basis in the space $L(X, H_\xi)$, the formula for linear prediction may be simplified. Let S be the interval $(-\infty, t]$. Then $\mathfrak{M}_\xi(S)$ is the "linear past" of the process $(\xi_n)_{n\in Z}$ up to time t. The linear prediction at time $s > t$ is given by the following formula:

$$\hat{\xi}_{s,(-\infty,t]} = \sum_{n=-\infty}^{t} \xi_n A_n, \quad A_n \in L(X, X).$$

If X is a finite-dimensional space, the above prediction problem is equivalent to the problem of the linear extrapolation of multivariate stationary processes (cf. [14] and [23]). The prediction problem for $S = (-\infty, t]$ is called an *extrapolation problem*, and for $S = Z \setminus K$, where K is a finite subset, an *interpolation problem.*

The extension of these two classical prediction problems to the case of processes indexed by the elements of any LCA group instead of integers, gives rise to certain difficulties, particulary for the extrapolation problem. In the next section, the general prediction problem for Banach-space-valued stationary processes over an LCA group is studied. In Section 9 shall consider both interpolation and extrapolation problems.

8. J-regularity and J-singularity

In the classical case (linear extrapolation), the regularity and singularity of the stationary process is determined by its behaviour on the class of intervals $(-\infty, t]$. Here, since the group is not necessarily ordered, this class, following L. A. Bruckner [1], is replaced by an arbitrary family J of non-empty subsets of the group. Regularity and singularity are then defined in terms of the behaviour of the process on the sets of J. Other concepts of regularity and singularity for processes over an LCA group have been studied in [2] and in [16].

Let J be any family of non-empty subsets of an LCA group G. We say that the family J is closed under translations if $gA \in J$ for $A \in J$ and $g \in G$.

DEFINITION 8.1. A Banach-space-valued stationary process $(\xi_g)_{g \in G}$ is called *J-singular* if

$$\mathfrak{N}_\xi = \bigcap_{S \in J} \mathfrak{M}_\xi(S) = \mathfrak{M}_\xi,$$

and *J-regular* if $\mathfrak{N}_\xi = (0)$.

Let I_ξ be a subspace of the Hilbert space H, which corresponds (in the sense of Theorem 7.1) to the right ideal $\mathfrak{N}_\xi$. Clearly, $I_\xi = \bigcap_{S \in J} H_\xi(S)$. We obtain:

PROPOSITION 8.2. *A Banach-space-valued stationary process $(\xi_g)_{g \in G}$ is J-regular if and only if $I_\xi = (0)$, and $(\xi_g)_{g \in G}$ is J-singular if and only if $I_\xi = H_\xi$.*

The properties of stationary processes are characterized in terms of their correlation functions. The following result shows that J-regularity and J-singularity may also be characterized in these terms.

THEOREM 8.3. *Let $(\xi_g)_{g\in G}$ and $(\eta_g)_{g\in G}$ be Banach-space-valued processes of the second-order (not necessarily stationary) with the correlation function $R(g,h)$. Then*

1° *there exists a unitary operator $U\colon H_\xi \to H_\eta$, such that $\eta_g = U\xi_g$ for all $g\in G$,*

2° *if the process $(\xi_g)_{g\in G}$ is J-regular (J-singular), then the process $(\eta_g)_{g\in G}$ is also J-regular (J-singular),*

3° *the prediction error operators $E_{g,S}(\xi)$ and $E_{g,S}(\eta)$ are equal.*

Proof. We define the operator U on the set of all elements $\xi_g x$, $x\in X$, $g\in G$ by the relation: $U\xi_g x = \eta_g x$. We observe that the operator U is uniquely determined. In fact, let $\xi_g x = \xi_h y$. Then

$$\begin{aligned}(U\xi_g x - U\xi_h y,\ U\xi_g x - U\xi_h y) &= (\eta_g x - \eta_h y,\ \eta_g x - \eta_h y)\\ &= \big(R(g,g)x\big)(x) - \big(R(g,h)y\big)(x) - \big(R(h,g)x\big)(y) + \big(R(h,h)y\big)(y)\\ &= (\xi_g x - \xi_h y,\ \xi_g x - \xi_h y) = 0.\end{aligned}$$

This means that $U\xi_g x = U\xi_h y$, and hence U is uniquely determined. In view of the equalities

$$(U\xi_g x,\ U\xi_h y) = (\eta_g x,\ \eta_h y) = \big(R(g,h)x\big)(y) = (\xi_g x,\ \xi_h y),$$

the operator U may be extended in a unique way to the unitary operator $U\colon H_\xi \to H_\eta$, which satisfies condition 1°.

Now let $(\xi_g)_{g\in G}$ be J-regular, i.e., $\bigcap_{S\in J} H_\xi(S) = (0)$, and by condition 1°, for each set $S\in J$, we have

$$UH_\xi(S) = H_\eta(S).$$

Hence

$$\bigcap_{S\in J} H_\eta(S) = \bigcap_{S\in J} UH_\xi(S) = U\bigcap_{S\in J} H_\xi(S) = (0),$$

and consequently the process $(\eta_g)_{g\in G}$ is J-regular.

Finally, the proof of condition 3° follows directly from 1° and therefore will be omitted. □

PROPOSITION 8.4. *Let J_1, J_2 be two families of non-empty subsets of a group G. If for arbitrary $A\in J_1$ there exists $B_A\in J_2$ such that $B_A\subset A$ (in particular, if $J_1\subset J_2$), then from J_1-regularity there follows J_2-regularity, and from J_2-singularity there follows J_1-singularity.*

The proof follows directly from the relation

$$\bigcap_{S\in J_2} H_\xi(S) \subset \bigcap_{B_A\in J_2} H_\xi(B_A) \subset \bigcap_{A\in J_1} H_\xi(A). \quad \square$$

We introduce here the following definition, which was created in this study (cf. [1]).

DEFINITION 8.5. Let $(\xi_g)_{g\in G}$ and $(\eta_g)_{g\in G}$ be Banach-space-valued stationary processes over an LCA group G. Then $(\eta_g)_{g\in G}$ is said to be *J-subordinate* to $(\xi_g)_{g\in G}$, if

(i) $(\xi_g)_{g\in G}$ and $(\eta_g)_{g\in G}$ are mutually stationary correlated,

(ii) $\mathfrak{M}_\eta(S) \subset \mathfrak{M}_\xi(S)$ for all $S\in J$,

(iii) $\mathfrak{M}_\eta \subset \mathfrak{M}_\xi$.

We will now prove the Wold decomposition theorem for Banach-space-valued stationary processes over an LCA group G. The one-dimensional version of this theorem is stated in [1], and the infinite-dimensional version for the special case (where G is the discrete group of integers or the group of reals with natural topology and J is the family of intervals $(-\infty, t]$) is stated in [19].

THEOREM 8.6. *Let J be a closed-under translation family of non-empty subsets of an LCA group G. Let $(\xi_g)_{g\in G}$ be a Banach-space-valued stationary process over the group G. Then there exists a unique decomposition of the process $(\xi_g)_{g\in G}$ with respect to J, in the form*

$$\xi_g = \varphi_g + \psi_g,$$

where

(i) *$(\varphi_g)_{g\in G}$ and $(\psi_g)_{g\in G}$ are Banach-space-valued stationary processes over G,*

(ii) *$(\varphi_g)_{g\in G}$ and $(\psi_g)_{g\in G}$ are J-subordinate to $(\xi_g)_{g\in G}$,*

(iii) *$(\varphi_g)_{g\in G}$ and $(\psi_g)_{g\in G}$ are orthogonal, i.e., $[\varphi_g, \psi_h] = 0$ for any $g, h\in G$,*

(iv) *$(\varphi_g)_{g\in G}$ is J-singular; $(\psi_g)_{g\in G}$ is J-regular.*

Proof. Let $(U_g)_{g\in G}$ be the unitary representation of the group G in the space H_ξ associated with $(\xi_g)_{g\in G}$. It is easy to show that $I_\xi = \bigcap_{S\in J} H_\xi(S)$ is an invariant subspace of the operators U_g for all $g\in G$. Since J is closed under translation, we have for each $g\in G$,

$$U_g(I_\xi) = U_g \bigcap_{S\in J} H_\xi(S) = \bigcap_{S\in J} U_g H_\xi(S) = \bigcap_{S\in J} H_\xi(gS) = I_\xi,$$

and hence

$$U_g(I_\xi^\perp) = I_\xi^\perp.$$

Let, for each $g\in G$,

$$\xi_g = P\xi_g + (I-P)\xi_g = \varphi_g + \psi_g, \tag{8.1}$$

where P denotes orthogonal projection onto the subspace I_ξ. From (8.1), by the fact that I_ξ and $I_\xi^\perp$ are invariant subspaces of U_g, we obtain

$$\xi_g = U_g\varphi_e + U_g\psi_e, \quad g\in G.$$

The image of X by the operator $U_g\varphi_e$ is lying in the subspace I_ξ and $U_g\varphi_e \perp U_g\psi_e$. Thus $U_g\varphi_e = \varphi_g$ and $U_g\psi_e = \psi_g$. This means that the processes $(\varphi_g)_{g\in G}$ and $(\psi_g)_{g\in G}$, defined by (8.1), are Banach-space-valued stationary processes over G with the same shift group $(U_g)_{g\in G}$. Since $\varphi_g \in I_\xi \subset H_\xi$, it follows that the process $(\varphi_g)_{g\in G}$ is J-subordinate to $(\xi_g)_{g\in G}$. This fact also implies that $(\psi_g)_{g\in G}$ $(\psi_g = \xi_g - \varphi_g)$ is J-subordinate to $(\xi_g)_{g\in G}$.

By the construction of the processes $(\varphi_g)_{g\in G}$ and $(\psi_g)_{g\in G}$, their mutual orthogonality follows.

From (8.1) we have

$$H_\xi(S) \subset I_\xi \oplus H_\psi(S).$$

But I_ξ and $H_\psi(S)$ are subspaces of $H_\xi(S)$, and hence

$$H_\xi(S) = I_\xi \oplus H_\psi(S) \tag{8.2}$$

for $S \in J$.

In an analogous way we obtain

$$H_\xi(S) = H_\varphi(S) \oplus H_\psi(S) \tag{8.3}$$

for $S \in J$. From (8.2) and (8.3) we have

$$H_\varphi(S) = I_\xi \tag{8.4}$$

for each $S \in J$. Since $I_\xi \subset H_\varphi$, we conclude that $I_\varphi = H_\varphi(S)$, and by Proposition 8.2, the process $(\varphi_g)_{g\in G}$ is J-singular. From (8.4), $I_\xi = I_\varphi$ and consequently $I_\psi \subset I_\varphi$, but, on the other hand, $I_\psi \perp I_\varphi$. Hence $I_\psi = (0)$, and by Proposition 8.2, the process $(\psi_g)_{g\in G}$ is J-regular.

Finally, we observe that for any decomposition of $(\xi_g)_{g\in G}$ into $(\varphi_g)_{g\in G}$ and $(\psi_g)_{g\in G}$, satisfying conditions (i)–(iv), we have $I_\varphi = I_\xi$. This relation makes the decomposition unique. □

Let $(\xi_g)_{g\in G}$ be a Banach-space-valued stationary process over G, and let P be a bounded projection in the space X. Then the process $(\xi_g P)_{g\in G}$ of the second-order is also stationary. In fact, if $R^P(g, h)$ is its correlation-function, then

$$R^P(g, h) = (\xi_g P)^*(\xi_h P) = P^* \xi_g^* \xi_h P = P^* R(gh^{-1})P. \tag{8.5}$$

DEFINITION 8.7. The process $(\xi_g P)_{g\in G}$ is called a *subprocess* of $(\xi_g)_{g\in G}$. In particular, if P is the projection on a fixed element $x \in X$, the process $(\xi_g x)_{g\in G}$ is called a *one-dimensional subprocess*.

From (8.5) the connection between the correlation function $R^P(g)$ of the subprocess $(\xi_g P)_{g\in G}$ and the correlation function of the process $(\xi_g)_{g\in G}$ is as follows:

$$R^P(g) = P^* R(g) P.$$

There follows the analogous connection between the spectral measures

$$F^P(\Delta) = P^* F(\Delta) P,$$

where Δ is an arbitrary Borel subset of the dual group Γ. In particular, for a one-dimensional subprocess we obtain that its correlation function and spectral measure are given by $\big(R(g)x\big)(x)$ and $\big(F(\Delta)x\big)(x)$, respectively.

In the following lemma, the sufficient condition for J-singularity and the necessary condition for J-regularity are given.

LEMMA 8.8. *Let $(\xi_g)_{g\in G}$ be a Banach-space-valued stationary process over an LCA group G. Then*

(a) *if $(\xi_g)_{g\in G}$ is J-regular, all subprocesses $(\xi_g P)_{g\in G}$ are J-regular,*

(b) *if all one-dimensional subprocesses $(\xi_g x)_{g\in G}$ are J-singular, $(\xi_g)_{g\in G}$ is J-singular.*

Proof (a). Denote by $H_{\xi P}(S)$ a subspace of H_ξ spanned by the elements $\xi_g Px$, $g\in S$, $x\in X$. If the Banach-space-valued stationary process $(\xi_g)_{g\in G}$ is J-regular, then $\bigcap_{S\in J} H_\xi(S) = (0)$. But $H_{\xi P}(S) \subset H_\xi(S)$. Hence $\bigcap_{S\in J} H_{\xi P}(S) = (0)$. This means that the subprocess $(\xi_g P)_{g\in G}$ is J-regular.

(b). Let each one-dimensional subprocess be J-singular. Suppose a contrario that the Banach-space-valued stationary process $(\xi_g)_{g\in G}$ is not J-singular. Then there exists a set $S_0\in J$ such that $H_\xi(S_0) \neq H_\xi$, i.e., we may find $g_0\in G$ and $x_0\in X$ such that $\xi_{g_0}x_0 \notin H_\xi(S_0)$. But the latter fact contradicts the assumption that all one-dimensional subprocesses are J-singular. □

9. J_C, J_Ψ and J_∞-singularity

In this section — using Lemma 8.8 — we give sufficient analytic conditions for Banach-space-valued stationary processes in terms of the spectral measure to be singular under some families. In the one-dimensional case, these conditions are sufficient and necessary. But in the infinite-dimensional case they are not necessary, as follows, for example, from Proposition 9.3.

Family J_C. Let G be an LCA group. By J_C denote the family of complements of all compact subsets in G. This family was considered in [20] in connection with finite-dimensional processes. If G is a discrete Abelian group, then the family J_C coincides with the family of complements of all finite subsets.

We introduce here the following definitions appearing in this study (cf. [20]). Let $\mathscr{Q}$ be a class of all complex-valued functions with compact supports defined on the group G, which are finite linear combinations of continuous positive-definite functions on G. $\mathscr{P}$ will denote a set of

Fourier transforms of all functions from the class $\mathcal{Q}$. From Lemma 8.8 and [20] (Corollary 6.3) we obtain:

PROPOSITION 9.1. *Let $(\xi_g)_{g\in G}$ be a Banach-space-valued stationary process over an LCA group G. If for each $x \in X$ and for each function $P(\gamma)\in\mathcal{P}$ the Hellinger integral*(1)

$$\int_{\Gamma} |dN_P|^2/d(Fx)(x)$$

is equal to zero or does not exist, where $N_P(\Delta) = \int_{\Delta} P(\gamma)d\gamma$, then $(\xi_g)_{g\in G}$ is J_C-singular.

We observe that a family J_C arises in the linear interpolation problem. A process is J_C-singular if and only if it is interpolable (cf. [20], Lemma 6.2). Now we consider families resulting from the linear extrapolation problem.

Family J_Ψ. Let G be a discrete Abelian group, and let Ψ be an arbitrary non-trivial real-valued homomorphism having the image $\Psi(G)$ non-dense in the reals. Denote by J_Ψ a family of all translations over G, a proper subsemi-group

$$S = \{g\in G;\ \Psi(g) \leqslant 0\}.$$

S is the "non-positive half" of G with respect to the homomorphism Ψ. By [2], p. 129, it is known that then G is isomorphic to the direct product of K with the integer group Z, where K denotes the kernel $\Psi^{-1}(\{0\})$ of the homomorphism Ψ. According to the duality theory, the dual group Γ of G is the direct product of $\hat{K}$ with the circle group T, where $\hat{K}$ is the dual group of K. If τ is a measure on Borel σ-algebra of Γ such that its Fourier transform $\hat{\tau}$ is supported on K, then $\tau = \nu\times\tau\pi^{-1}$, where ν denotes the normalized Lebesgue measure on T, π denotes the canonic projection of Γ onto $\hat{K}$, and $\tau\pi^{-1}$ denotes the measure in $\hat{K}$ defined by the relation $\tau\pi^{-1}(\Delta) = \tau(\pi^{-1}(\Delta))$, $\Delta\in\hat{K}$ (see [2], p. 130). If two measures μ and τ are given, then $d\mu/d\tau$ will denote the Radon–Nikodym derivative of the absolutely continuous part of μ relative to τ.

By Lemma 8.8, and Theorem 5.1′ of [2], we obtain:

PROPOSITION 9.2. *Let $(\xi_g)_{g\in G}$ be a Banach-space-valued stationary process over a discrete Abelian group G with an arbitrary non-trivial real-valued homomorphism Ψ having a non-dense image in the reals. If for each $x\in X$ and for an arbitrary positive measure τ in Γ with the Fourier transform $\hat{\tau}$ supported on K,*

---

(1) $\int_{\Gamma} |dN_p|^2/d(Fx)(x) = \int_{\Gamma} |dN_p/dm|^2 d((Fx)(x)/dm)^{\#}dm$, where m is an arbitrary σ-finite non-negative real-valued measure with respect to which N_P and $(Fx)(x)$ are absolutely continuous and $f(\gamma)^{\#} = 1/f(\gamma)$ if $f(\gamma)\neq 0$, or $f(\gamma)^{\#} = 0$ if $f(\gamma) = 0$. For details and further references see [20], Section 2.

$$\int_\Gamma \log\big(d(Fx)(x)/d\tau\big)\,d\tau = -\infty,$$

then $(\xi_g)_{g\in G}$ *is* J_Ψ*-singular.*

Family J_∞. Let Z be the discrete group of integers, and let J_∞ be a family of all sets of the form $(-\infty, z]$, $z\in Z$. It can easily be seen that the family J_∞ is a particular case of the above family J_Ψ. It suffices to take $\Psi = I$, where I is the identical isomorphism: $Z\to Z$. Since in this case the kernel $I^{-1}(\{0\})$ is a singleton, according to the above consideration, the measure τ (such as in Proposition 9.2) is equal to the Lebesgue measure $d\lambda$ on the circle group. Hence a sufficient condition for a Banach-space-valued stationary process over Z to be J_∞-singular is for each $x\in X$, $\int_{-\pi}^{+\pi} \log\big(d(Fx)(x)/d\lambda\big)d\lambda = -\infty$. However, this condition is not necessary.

PROPOSITION 9.3. *Let* $(\xi_z)_{z\in Z}$ *be a Banach-space-valued stationary process over* Z. *The condition*

$$\int_{-\pi}^{+\pi} \log\big(d(Fx)(x)/d\lambda\big)d\lambda = -\infty$$

for all $x\in X$ *is not necessary for* J_∞*-singularity of* $(\xi_z)_{z\in Z}$.

Proof. Consider the stationary process from Example 4.2: $\xi_z = U^z$, $z\in Z$, where U is an operator defined on the space $L_2([-\pi, +\pi]) = X = H$ by $Uf = e^{i\lambda}f$. It can easily be seen that the random measure $\Phi(d\lambda)$ and the spectral measure $F(d\lambda)$ of the process $(\xi_z)_{z\in Z}$ coincide with a resolution of the identity in H:

$$\big(\Phi(\Delta)x\big)(\lambda) = \big(F(\Delta)x\big)(\lambda) = \chi_\Delta(\lambda)\cdot x(\lambda),$$

where $\chi_\Delta(\lambda)$ is the characteristic function of a measurable subset $\Delta\subset[-\pi, +\pi]$. For each function $x(\lambda)\in L_2([-\pi, +\pi])$ we have

$$\big(F(\Delta)x\big)(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \overline{\big(F(\Delta)x\big)(\lambda)}\,x(\lambda)\,d\lambda = \frac{1}{2\pi}\int_\Delta |x(\lambda)|^2 d\lambda.$$

It follows that the spectral measure $F(\Delta)$ is absolutely continuous with respect to the Lebesgue measure, and that the Radon–Nikodym derivative has the following form:

$$d(Fx)(x)/d\lambda = \frac{1}{2\pi}|x(\lambda)|^2.$$

Let us put $x(\lambda) = \sum_{k=0}^{\infty} a_k e^{-ik\lambda}$, where the last series is convergent in the space $L_2([-\pi, +\pi])$. Thus from a well-known property of the modulus of an analytic function, we obtain

$$\int_{-\pi}^{+\pi} \log|x(\lambda)|\,d\lambda > -\infty.$$

Hence

$$\int_{-\pi}^{+\pi} \log\big(d(Fx)(x)/d\lambda\big)\,d\lambda > -\infty,$$

i.e., the sufficient condition for J_∞-singularity does not hold. However, it is easy to verify that the stationary process $(\xi_z)_{z\in Z}$ is J_∞-singular. In fact, since U is a unitary operator, for each $z\in Z$, $U^z(H) = H$. Hence $H_\xi([-\infty, z]) = H$ for all $z\in Z$. It follows that $I_\xi = H$. Thus, by Proposition 8.2, $(\xi_z)_{z\in Z}$ is J_∞-singular. □

10. J_∞-regularity and factorization of operator-valued functions

In the final section we consider Banach-space-valued stationary processes over the discrete Abelian group Z of integers. We will call such processes-sequences. The results of this section were published, without proofs, in recent papers: Theorem 10.1 in [19] and Theorems 10.2 and 10.3 in [26].

Let $(\xi_z)_{z\in Z}$ be a Banach-space-valued stationary sequence. Denote by ψ_z the normal vector from ξ_z to the space $\mathfrak{M}_\xi([-\infty, z-1])$:

$$\psi_z = \xi_z - \hat{\xi}_{z,(-\infty,z-1]}.$$

It is easy to verify that $\psi_z = U^z\psi_0$, where U is the unitary operator in H, which corresponds to the sequence $(\xi_z)_{z\in Z}$ (see (4.2)). Hence the sequence $(\psi_z)_{z\in Z}$ is stationary and subordinate to $(\xi_z)_{z\in Z}$. We say that $(\psi_z)_{z\in Z}$ is the *fundamental sequence* of the sequence $(\xi_z)_{z\in Z}$. Denote by $\mathfrak{S}(\psi_z)$ a subspace of $\mathfrak{M}_\xi$ spanned by the elements

$$\psi_z A, \quad z\in Z, \quad A\in L(X, X).$$

In the following theorem, an infinite-dimensional analogue of the well-known classical representation of a purely non-deterministic stationary sequence, with respect to the one-side moving averages, is given.

THEOREM 10.1. *A Banach-space-valued stationary sequence $(\xi_z)_{z\in Z}$ is J_∞-regular if and only if it admits the representation*

$$\xi_z = \sum_{k\geqslant 0} \xi_{z,k},$$

where $\xi_{z,k}\in \mathfrak{S}(\psi_{z-k})$ *and* $U^n\xi_{z,k} = \xi_{z+n,k}$.

Proof. *Necessity.* In the space $H_\xi((-\infty, 0])$ we consider the sequence of subspaces

$$H_\xi((-\infty, -1]) \supset H_\xi((-\infty, -2]) \supset \ldots$$

Then

$$(10.1) \qquad H_\xi((-\infty, 0]) = \bigoplus_{k\geq 0} L_k \oplus \bigcap_{k\geq 0} H_\xi((-\infty, -k]),$$

where

$$L_k = H_\xi((-\infty, -k]) \cap H_\xi^\perp((-\infty, -k-1]).$$

Since $(\xi_z)_{z\in Z}$ is J_∞-singular, the second term on the direct sum in formula (10.1) is the zero-subspace. It can easily be seen that $\mathfrak{S}(\psi_{-k})$ is the right ideal in $\mathfrak{M}_\xi$, which, in view of Theorem 7.1, corresponds to the closed subspace L_k. Hence

$$\mathfrak{M}_\xi((-\infty, 0]) = \bigoplus_{k\geq 0} \mathfrak{S}(\psi_{-k}).$$

Thus the element $\xi_0 \in \mathfrak{M}_\xi((-\infty, 0])$ may be uniquely represented in the form

$$(10.2) \qquad \xi_0 = \sum_{k\geq 0} \xi_{0,k},$$

where $\xi_{0,k} \in \mathfrak{S}(\psi_{-k})$.

Hence

$$\xi_z = U^z \xi_0 = \sum_{k\geq 0} U^z \xi_{0,k} = \sum_{k\geq 0} \xi_{z,k}.$$

Obviously, $\xi_{z,k} \in \mathfrak{S}(\psi_{z-k})$ and $U^n \xi_{z,k} = \xi_{z+n,k}$.

Sufficiency. If a stationary sequence $(\xi_z)_{z\in Z}$ admits the representation $\xi_z = \sum_{k\geq 0} \xi_{z,k}$, then

$$\mathfrak{M}_\xi((-\infty, z]) \subset \bigoplus_{k=-\infty}^{z} \mathfrak{S}(\psi_k).$$

By the orthogonality of the fundamental sequence $(\psi_z)_{z\in Z}$, we deduce that

$$\bigcap_{z\in Z} \mathfrak{M}_\xi((-\infty, z]) \subset \bigcap_{z\in Z} \bigoplus_{k=-\infty}^{z} \mathfrak{S}(\psi_k) = (0).$$

Hence the sequence $(\xi_z)_{z\in Z}$ is J_∞-regular. □

The following theorem on the characterization of J_∞-regularity of Banach-space-valued stationary sequences is an analogue of the classical result on factorization of the spectral density for scalar-valued stationary sequences.

Let K be a Hilbert space, and by L_2^K denote a space of all equivalent classes of K-valued functions defined on $[-\pi, +\pi]$, such that $\int_{-\pi}^{+\pi} \|y(\gamma)\|^2 d\gamma < \infty$.

In L_2^K, introduce the norm by $\|y\| = \left(\int_{-\pi}^{+\pi} \|y(\gamma)\|^2 d\gamma\right)^{1/2}$.

THEOREM 10.2. *A Banach-space-valued stationary sequence* $(\xi_z)_{z\in Z}$ *is* J_∞*-regular if and only if*

(i) *the spectral measure* $F(d\gamma)$ *is absolutely continuous with respect to the Lebesgue measure* $d\gamma$ *on* $[-\pi, +\pi]$,

(ii) *there exist a Hilbert space* K *and a sequence of linear operators* $A_1, A_2, \ldots$ *from* $L(X, K)$, *such that the series*

$$\varphi_x(\gamma) = \sum_{k\geq 0} e^{-ik\gamma} A_k x$$

converges in L_2^K *for each* $x \in X$,

(iii) $\dfrac{d(Fx)(x)}{d\gamma} = \|\varphi_x(\gamma)\|^2$ *for each* $x \in X$.

Proof. *Necessity.* Let $(\xi_z)_{z\in Z}$ be a J_∞-regular sequence. By Theorem 10.1, it admits the representation $\xi_z = \sum_{k\geq 0} \xi_{z,k}$, where $\xi_{z,k} \in \mathfrak{S}(\psi_{z-k})$. Hence

$$(10.3) \qquad \big(R(z)x\big)(x) = (\xi_z x, \xi_0 x) = \Big(\sum_{k\geq 0} \xi_{z,k} x, \sum_{n\geq 0} \xi_{0,n} x\Big) = \sum_{k,n\geq 0} (\xi_{z,k} x, \xi_{0,n} x)\, \delta_{z-k,-n},$$

where $\delta_{k,n}$ is Kronecker's δ-symbol.

Consider the series

$$(10.4) \qquad \varphi_x(\gamma) = \frac{1}{\sqrt{2\pi}} \sum_{k\geq 0} e^{-ik\gamma} \xi_{k,k} x.$$

We shall prove that for each $x \in X$ series (10.4) is convergent and defines the function $\varphi_x(\gamma)$, which satisfies the factorization condition (iii):

$$\int_{-\pi}^{+\pi} \Big\| \sum_{n=0}^{N} e^{-in\gamma} \xi_{n,n} x - \sum_{n=0}^{M} e^{-in\gamma} \xi_{n,n} x \Big\|^2 d\gamma = \int_{-\pi}^{+\pi} \Big\| \sum_{n=M+1}^{N} e^{-in\gamma} \xi_{n,n} x \Big\|^2 d\gamma$$
$$= \sum_{k,n=M+1}^{N} \int_{-\pi}^{+\pi} e^{-i(k-n)\gamma} (\xi_{k,k} x, \xi_{n,n} x)\, d\gamma = 2\pi \sum_{n=M+1}^{N} \|\xi_{n,n} x\|^2$$
$$= 2\pi \sum_{n=M+1}^{N} (U^{-n} \xi_{n,n} x, U^{-n} \xi_{n,n} x) = 2\pi \sum_{n=M+1}^{N} \|\xi_{0,n} x\|^2.$$

Since the last series in formula (10.3) is convergent for all $z \in Z$, it is also convergent for $z = 0$. It follows that $\sum_{n=M+1}^{N} \|\xi_{0,n} x\|^2 \to 0$ for $M, N \to \infty$, and consequently series (10.4) is convergent in L_2^H for all $x \in X$. If we put

$K = H$(1) and $A_n = \xi_{n,n}$, then condition (ii) holds. It remains to prove conditions (i) and (iii). We consider the Fourier transform of the function $\|\varphi_x(\gamma)\|^2$,

$$\int_{-\pi}^{+\pi} e^{i\gamma z}\|\varphi_x(\gamma)\|^2 d\gamma = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{i\gamma z}\Big(\sum_{k\geqslant 0} e^{-ik\gamma}\xi_{k,k}x, \sum_{n\geqslant 0} e^{-in\gamma}\xi_{n,n}x\Big)d\gamma$$
$$= \frac{1}{2\pi}\sum_{k,n\geqslant 0}(\xi_{k,k}x, \xi_{n,n}x)\int_{-\pi}^{+\pi} e^{i(z-k+n)\gamma}d\gamma$$
$$= \sum_{k,n\geqslant 0}(\xi_{k,k}x, \xi_{n,n}x)\,\delta_{z-k,-n}.$$

By (10.3), it follows that

$$\int_{-\pi}^{+\pi} e^{i\gamma z}\|\varphi_x(\gamma)\|^2 d\gamma = \big(R(z)x\big)(x). \tag{10.5}$$

By the uniqueness of a Fourier transform, using the equivalence of conditions (i) and (v) of Theorem 5.2, and from (10.5), we obtain

$$\frac{d\big(F(x)\big)(x)}{d\gamma} = \|\varphi_x(\gamma)\|^2,$$

a.e. on $[-\pi, +\pi]$. Thus the necessity is proved.

Sufficiency. Let $\mathscr{D}(0)$ denote a subspace of K spanned by elements of the form; $A_k x$, $x \in X$, $k = 0, 1, \ldots$ and $A_k\colon X \to K$ are as in condition (ii). Now we decompose the orthogonal complement of the space $\mathscr{D}(0)$ in the Hilbert space K onto a countable number of orthogonal subspaces with the same dimension as the subspace $\mathscr{D}(0)$(2)

$$\mathscr{D}(1),\ \mathscr{D}(-1),\ \mathscr{D}(2),\ \mathscr{D}(-2), \ldots$$

Let us consider the shift operator $U\colon K \to K$ satisfying

$$U\mathscr{D}(n) = \mathscr{D}(n+1) \tag{10.6}$$

for each $n \in Z$. From the construction of the subspaces $\mathscr{D}(n)$ we may define such an operator as follows; for each element $e_a^{(n)}$ of the orthogonal basis in $\mathscr{D}(n)$, let

$$U' e_a^{(n)} = e_a^{(n+1)},$$

(1) We may take $K = [\psi_0(X)]$ to be the space spanned by the image of the operator ψ_0 (i.e., the value of the fundamental sequence $(\psi_z)_{z\in Z}$ in 0) and closed in the topology of the Hilbert space H.

(2) Without loss of generality, we can assume that this decomposition exists. If in the Hilbert space K it does not exists, we may include K in a greater Hilbert space K', in which this decomposition exists.

where $e_a^{(n+1)}$ is an a-element of the orthonormal basis in $\mathscr{D}(n+1)$. It can be easily seen that U' may be, in a unique way, extended to a unitary operator U in K, which satisfies (10.6).

Let us put

$$\varphi_z = \sqrt{2\pi} \sum_{k\geqslant 0} U^{z-k} A_k, \qquad z \in Z. \tag{10.7}$$

Now we observe that the series in (10.7) is convergent in strong operator topology. In fact, by condition (ii) we have

$$\int_{-\pi}^{+\pi} \Big\| \sum_{k=0}^{N} e^{-ik\gamma} A_k x - \sum_{k=0}^{M} e^{-ik\gamma} A_k x \Big\|^2 d\gamma \to 0 \qquad \text{for } M, N \to \infty.$$

It follows that

$$\sum_{n=M+1}^{N} \|A_n x\|^2 \to 0 \qquad \text{for } M, N \to \infty. \tag{10.8}$$

On the other hand, from the definition of the operator U we have

$$\Big\| \sqrt{2\pi} \sum_{k=0}^{N} U^{z-k} A_k x - \sqrt{2\pi} \sum_{k=0}^{M} U^{z-k} A_k x \Big\|^2$$

$$= \Big\| \sqrt{2\pi} \sum_{k=M+1}^{N} U^{z-k} A_k x \Big\|^2 = 2\pi \sum_{k=M+1}^{N} \|A_k x\|^2.$$

Hence, by (10.8) the series $\sum_{k\geqslant 0} U^{z-k} A_k$ is convergent in strong operator topology, and consequently formula (10.7) defines a Banach-space-valued stationary sequence. We shall prove that this sequence is J_∞-regular and that its spectral measure is F.

By (10.7) and the definition of U, we obtain

$$H_\varphi\big((-\infty, z]\big) \subset \bigoplus_{k=-\infty}^{z} \mathscr{D}(k).$$

From the orthogonality of the subspaces $\mathscr{D}(k)$, it follows that

$$\bigcap_{z\in Z} H_\varphi\big((-\infty, z]\big) \subset \bigcap_{z\in Z} \bigoplus_{k=-\infty}^{z} \mathscr{D}(k) = (0),$$

which means that the sequence $(\varphi_z)_{z\in Z}$ is J_∞-regular. The correlation function of this sequence is as follows:

$$(\varphi_z x, \varphi_0 x) = \Big(\sqrt{2\pi} \sum_{k\geqslant 0} U^{z-k} A_k x, \sqrt{2\pi} \sum_{n\geqslant 0} U^{-n} A_n x \Big)$$

$$= 2\pi \sum_{k,n\geqslant 0} (U^{z-k} A_k x, U^{-n} A_n x) = ,$$

by properties of the operator U,

$$= 2\pi \sum_{k,n\geq 0} (U^{z-k}A_k x,\ U^{-n}A_n x)\,\delta_{z-k,-n}$$

$$= \sum_{k,n\geq 0} (U^{z-k}A_k x,\ U^{-n}A_n x) \int_{-\pi}^{+\pi} e^{i(z-k-n)\gamma}\, d\gamma$$

$$= \int_{-\pi}^{+\pi} e^{iz\gamma} \Big\| \sum_{k\geq 0} e^{-ik\gamma} A_k x \Big\|^2 d\gamma =,$$

from condition (iii),

$$= \int_{-\pi}^{+\pi} e^{iz\gamma} \big(F(d\gamma)x\big)(x) = \big(R(z)x\big)(x).$$

It follows that $R(z)$ is the correlation function of the J_∞-regular Banach-space-valued sequence $(\varphi_z)_{z\in Z}$. Hence, by Theorem 8.3, the Banach-space-valued sequence $(\xi_z)_{z\in Z}$ is J_∞-regular, which completes the proof. □

In Section 5 we observed that in the infinite-dimensional case there exist absolutely continuous operator-valued measures which have no operator-valued densities. If a Banach-space-valued stationary sequence $(\xi_z)_{z\in Z}$ has an operator-valued spectral density (see Definition 5.7), the above theorem has a simple form, which we present below.

THEOREM 10.3. *Let $(\xi_z)_{z\in Z}$ be a Banach-space-valued stationary sequence which has the spectral density $p(\gamma)$. Let the Banach space X be separable*(1)*, in the definition of the sequence $(\xi_z)_{z\in Z}$. Then the sequence $(\xi_z)_{z\in Z}$ is J_∞-regular if and only if*

$$p(\gamma) = \|\varphi(\gamma)\|^2,$$

where $\varphi(\gamma)$ is an operator-valued function taking values in the space $L(X, K)$ (K being an auxiliary separable Hilbert space) given by the formula

$$\varphi(\gamma) = \sum_{k\geq 0} A_k e^{-ik\gamma}, \qquad A_k \in L(X, K).$$

Proof. *Necessity.* By Theorem 10.2, the spectral measure $F(d\gamma)$ of a J_∞-regular Banach-space-valued stationary sequence $(\xi_z)_{z\in Z}$ admits the factorization

$$\frac{d(Fx)(x)}{d\gamma} = \|\varphi_x(\gamma)\|^2,$$

where

$$\varphi_x(\gamma) = \frac{1}{\sqrt{2\pi}} \sum_{k\geq 0} \xi_{k,k} x e^{-ik\gamma}.$$

(1) The sufficiency of this condition holds without the assumption of the separability of X.

According to our assumptions, the sequence $(\xi_t)_{t\in Z}$ has the spectral density $p(\gamma)$, from which it follows that

$$\big(p(\gamma)x\big)(x) = \frac{d(Fx)(x)}{d\gamma} = \|\varphi_x(\gamma)\|^2.$$

We shall prove that $\varphi_x(\gamma) = \varphi(\gamma)x$ for each $x\in X$. The series in the definition of $\varphi_x(\gamma)$ is convergent in L_2^K. Hence

$$\int_{-\pi}^{+\pi}\left\|\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{N}\xi_{k,k}xe^{-ik\gamma}-\varphi_x(\gamma)\right\|^2 d\gamma\to 0 \quad \text{for } N\to\infty.$$

Therefore we can find a subsequence $n_s\to\infty$, of course, $\{n_s\}$ will be depend on x, such that

$$\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{n_s}\xi_{k,k}xe^{-ik\gamma}\to\varphi_x(\gamma) \quad \text{a.e.} \tag{10.9}$$

Let $S = \{x_s\}$ be a countable dense subset in X (the space X being separable). Without loss of generality, we may consider such a subsequence $\{n_s\}$ so that for $s\to\infty$ relation (10.9) holds for all $x_s\in S$. It follows that $\varphi_{x_s}(\gamma)$ as a function of x_s is linear on a subset of the full measure. Since condition (10.9) holds a.e., there exists a subset $A\subset[-\pi, +\pi]$ of the full measure, such that

$$\big(p(\gamma)x_s\big)(x_s) = \big(\varphi_{x_s}(\gamma),\ \varphi_{x_s}(\gamma)\big) \tag{10.10}$$

for all $\gamma\in A$ and $x_s\in S$. It follows that

$$\|\varphi_x(\gamma)\|^2 \leqslant \|p(\gamma)\|\,\|x\|^2, \qquad \gamma\in A,\ x\in S.$$

Hence the function $\varphi_x(\gamma)$, for all $\gamma\in A$, may be extended to a linear continuous operator $\varphi(\gamma)$ defined over the whole space X. From (10.10), we obtain

$$\big(p(\gamma)x\big)(x) = \big(\varphi(\gamma)x, \varphi(\gamma)x\big)$$

for all $x\in X$ and $\gamma\in A$, which means that

$$p(\gamma) = \varphi^*(\gamma)\varphi(\gamma). \tag{10.11}$$

It remains to prove that for all $x\in X$,

$$\varphi_x(\gamma) = \varphi(\gamma)x, \tag{10.12}$$

and to find a separable Hilbert space K for the factorization (10.11). In view of the above construction, it suffices to prove that $\varphi_x(\gamma)$, as an L_2^K-valued function of the variable x, is continuous. Let $\{x_k\}$ be a sequence of elements from X convergent to an element $x_0 \in X$. Thus by (10.5), we have

$$\int_{-\pi}^{+\pi} \|\varphi_{x_0}(\gamma) - \varphi_{x_k}(\gamma)\|^2 d\gamma = \big(R(0)(x_0 - x_k)\big)(x_0 - x_k) \to 0.$$

Hence $\varphi_x(\gamma)$ is a continuous function in L_2^K, and consequently (10.12) holds for all $x \in X$.

As we observed in the proof of the necessity of Theorem 10.2 (see footnote on p. 39), we may take $K = [\psi_0(X)]$. From the separability of X, such a space K is also separable. This completes the proof of the necessity.

The proof of the sufficiency follows the same lines as the proof of the sufficiency in Theorem 10.2. □

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