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SOME DIFFERENTIAL OPERATORS CONNECTED WITH QUASICONFORMAL DEFORMATIONS ON MANIFOLDS

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Introduction

The theory of quasiconformal mappings and deformations has lately been fruitfully extended to Riemannian manifolds. Let us e.g. mention papers by Suominen [19], Lelong-Ferrand [11], Kiernan [10], Goldberg, Ishihara and Petridis [8], Shibata and Mohri [18] and others. Riemannian manifolds are one of the most natural spaces in which geometrical properties of mappings come to light. Especially, both dependence on the metric structure and some relations of the notions of conformality or quasiconformality to other ones (e.g. harmonicity) could be exposed. It is well known that conformality and harmonicity have the common source: holomorphic mappings in the complex plane C^1. Therefore, some relationships between quasiconformality and harmonicity might be expected also in spaces of higher dimensions or even on manifolds. For example, Goldberg, Ishihara and Petridis [8] obtained a generalization of the Schwarz–Ahlfors lemma on a distance decreasing property for quasiconformal harmonic mappings of Riemannian manifolds.

In this paper we shall introduce differential operators related to quasiconformal deformations of a Riemannian manifold. We derive some their properties: strong ellipticity (Theorem 1), dependence on the metric structure (on the Ricci tensor) and a relation to the Laplace–Beltrami operator (Theorem 2), and obtain some transformation formulas (Theorem 3).

The case of Euclidean spaces

Quasiconformal deformations, an infinitesimal version of quasiconformal mappings, turned out to be a useful tool for studying several problems of quasiconformality in R^n $(n \geqslant 2)$.

Before we pass to the case of manifolds, let us shortly recall some notions and facts from the n-dimensional theory.

A quasiconformal mapping F in $\mathbf{R}^n$ satisfies the following n-dimensional version of the Beltrami equation:

$$(1)\qquad (J_F^{-1/n} DF)^* (J_F^{-1/n} DF) = G,$$

where DF denotes the differential matrix of F, J_F the Jacobian of F, i.e. $J_F = |\det DF|$, and G is a symmetric matrix field of bounded norm. Without loss of generality we may assume that $\det G = 1$. The system (1) is nonlinear and, for $n > 2$, overdetermined: the number of independent equations is $\frac{1}{2}n(n+1)-1$.

For details we refer the reader to the paper of Bojarski and Iwaniec [6].

A linearization of the system (1) leads to the following system:

$$(2)\qquad SZ_{ij} = \frac{1}{2}\left(\frac{\partial Z^i}{\partial x_j} + \frac{\partial Z^j}{\partial x_i}\right) - \frac{1}{n}\sum_{k=1}^{n} \frac{\partial Z^k}{\partial x_k}\delta_{ij}, \qquad i, j = 1, \ldots, n.$$

The notion of quasiconformal deformation defined to be a field Z such that the norm of the matrix field (2) is bounded was introduced and systematically studied by Ahlfors in a series of papers [1]–[5] and in the papers of Reimann [15], Semenov [17], Sarvas [16] and others.

The differential operator S defined by (2) (called *Ahlfors' operator*) from the space of vector fields (deformations) into the space of symmetric matrix fields of zero trace is, therefore, connected in a natural way with quasiconformal deformations. S is interesting in its own right. It was investigated by several authors (cf. [2], [4], [16]).

S has a conjugate operator S^* of the form

$$S^* \varphi_i = \sum_j \frac{\partial \varphi_{ij}}{\partial x_j}.$$

It is worth noticing that in the case $n = 3$ the equation

$$S^* SZ = V$$

is the classical equation of the theory of elasticity (cf. [2], [20]).

The case of Riemannian manifolds

Let M be a Riemannian manifold of dimension n with a Riemannian metric g (g is a symmetric positive definite tensor field of type (0, 2) on M). We assume that all manifolds and mappings in question are smooth, i.e. of class C^∞. Denote by ∇ the Riemannian connection of the metric g and by $T_p(M)$ and $T_p^*(M)$, $p \in M$, the tangent and cotangent space at p, respectively.

Define the operator S from the space $\mathfrak{X}(M)$ of all vector fields on M

into the space $\mathfrak{M}(M)$ of all symmetric tensor fields of type (0, 2) and of zero trace (with respect to g) as follows:

$$SZ = \mathscr{L}_Z g - \frac{2}{n} \operatorname{div} Z g, \qquad Z \in \mathfrak{X}(M), \tag{3}$$

where $\mathscr{L}_Z$ denotes the Lie derivative in the direction Z and $\operatorname{div} Z$ is the divergence of Z defined by $\operatorname{div} Z = \operatorname{tr}(X \mapsto \nabla_X Z)$.

Observe that in the special case $M = \boldsymbol{R}^n$ and g the Euclidean metric in $\boldsymbol{R}^n$, the tensor field SZ is represented (up to a factor $\frac{1}{2}$) by the matrix field (2).

The operator S is strictly related to quasiconformal deformations: If Z is an arbitrary deformation (i.e. a vector field on M), then the norm of SZ measures the quasiconformality of Z. Namely, Z is called a *k-quasiconformal deformation* ($k \geqslant 0$) if $\|SZ\| \leqslant k$ on M.

One can prove (cf. [13] and [14]) that the norm of S is conformally invariant and that a k-quasiconformal (complete) deformation generates a one parameter family F_t, $t \in \boldsymbol{R}$, of transformations of M whose rank of quasiconformality can be estimated: F_t is an $\exp(\frac{1}{2}k^2|t|)$-quasiconformal transformation. Furthermore, S is an elliptic operator of rank 1 (in the sense of the injectivity of its symbol).

S has an adjoint operator S^* from $\mathfrak{M}(M)$ into $\mathfrak{X}(M)$ which is of a very concise form: S^* is the divergence operator (cf. [14]). More precisely,

$$g(S^* \varphi, Z) = 2 \operatorname{div} \varphi(Z), \qquad \varphi \in \mathfrak{M}(M),\ Z \in \mathfrak{X}(M). \tag{4}$$

Recall that the divergence of a (0, 2)-type tensor field φ is the 1-form locally defined by

$$\operatorname{div} \varphi_k = \nabla^i \varphi_{ik}. \tag{5}$$

The operators S and S^* are adjoint to each other in the following sense. If $Z \in \mathfrak{X}(M)$, $\varphi \in \mathfrak{M}(M)$ and Z or φ has a compact support then

$$\langle S^* \varphi, Z \rangle = -\langle \varphi, SZ \rangle, \tag{6}$$

where the scalar products $\langle\ ,\ \rangle$ in $\mathfrak{X}(M)$ and in $\mathfrak{M}(M)$ are both generated by the Riemannian metric g.

The third operator related to quasiconformal deformations, S^*S, is simply the composition of the above two operators S and S^*. S^*S acts from the space $\mathfrak{X}(M)$ into itself and has many interesting properties.

First we derive some inequalities for its symbol $\sigma = \sigma_{S^*S}$. Let $p \in M$ and let $\omega \in T_p^*(M)$. The symbol at the point p is, by definition, the mapping $\sigma_p(\omega)\colon T_p(M) \to T_p(M)$ of the form

$$\sigma_p(\omega) v = S^* S (f^2 Z)_p, \qquad v \in T_p(M), \tag{7}$$

where f is a function and Z a vector field in a neighbourhood of p such that

$$f(p) = 0, \qquad df(p) = \omega, \qquad Z_p = v. \tag{8}$$

Of course, the definition (7) does not depend on the choice of f and Z.

THEOREM 1. *At each point $p \in M$ the symbol σ of the differential operator $S^* S$ satisfies the inequality*

$$4\|\omega\|^2\|v\|^2 \leqslant g(\sigma_p(\omega)v, v) \leqslant \frac{8(n-1)}{n}\|\omega\|^2\|v\|^2. \tag{9}$$

Consequently, $S^* S$ is a strongly elliptic operator of the second order.

Proof. Let $p \in M$, $v \in T_p(M)$, $\omega \in T_p^*(M)$. Let f be a function and Z a vector field such that (8) holds. Then one can check (cf. [14]) that

$$S(f^2 Z) = f^2 SZ + 2fdf \otimes g(\cdot, Z) + 2fg(Z, \cdot) \otimes df - \frac{4}{n} df(Z)g. \tag{10}$$

On the other hand, one can check that if φ is a tensor field of type (0, 2) and a a function on M, then

$$\operatorname{div} a\varphi = a \operatorname{div} \varphi + \varphi(\operatorname{grad} a, \cdot), \tag{11}$$

where grad a is the vector field defined by $g(\operatorname{grad} a, X) = da(X)$, $X \in \mathfrak{X}(M)$.

By the definition (4) of the adjoint operator S^* we have

$$g(\sigma_p(\omega)v, v) = 2 \operatorname{div}(Sf^2 Z)(Z)_p.$$

Therefore, we have to combine formulas (10) and (11). After calculations in which we use (8) and the following equalities: $\operatorname{grad} f^2 = 2f \operatorname{grad} f$, $df(\operatorname{grad} f)_p = \|\omega\|^2$, $g(Z, Z)_p = \|v\|^2$ and $g(\operatorname{grad} f, Z)_p = \omega(v)$, we obtain

$$g(\sigma_p(\omega)v, v) = 4\left(\|\omega\|^2\|v\|^2 + \frac{n-2}{n}\omega(v)^2\right).$$

Consequently, since $n \geqslant 2$ and $\omega(v)^2 \leqslant \|\omega\|^2\|v\|^2$, we obtain the desired assertion (9).

Next we are going to decompose $S^* S$ into three components to emphasize both its dependence on the geometry of M (more precisely, on the Ricci tensor R) and its relationship with the Laplace–Beltrami operator Δ. Namely, we prove the following:

THEOREM 2. *For an arbitrary deformation $Z \in \mathfrak{X}(M)$*

$$g(S^* SZ, \cdot) = 4R(Z, \cdot) - 2\Delta\alpha - \frac{2n-4}{n} d\delta\alpha \tag{12}$$

or, equivalently,

$$g(S^* SZ, \cdot) = -2\delta d\alpha - [(4n-4)/n] d\delta\alpha + 4R(Z, \cdot), \tag{13}$$

where R is the Ricci tensor, $\Delta = \delta d + d\delta$ is the Laplace–Beltrami operator and α is the 1-form dual to Z in the sense $\alpha(X) = g(Z, X)$, $X \in \mathfrak{X}(M)$.

Proof. The calculation will be done in a local neighbourhood with coordinates $(x^1, \ldots, x^n)$ on M. In such a neighbourhood we have (by the definition (3))

$$SZ_{ij} = (\mathcal{L}_Z g)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) - \frac{2}{n}\operatorname{div} Zg\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Since $\mathcal{L}_Z g(X, Y) = Zg(X, Y) - g(\mathcal{L}_Z X, Y) - g(X, \mathcal{L}_Z Y)$, $X, Y \in \mathfrak{X}(M)$, using the equality $\mathcal{L}_Z X = \nabla_Z X - \nabla_X Z$ we obtain

$$\mathcal{L}_Z g_{ij} = Z^k g_{lj} \Gamma^l_{ki} + \frac{\partial Z^k}{\partial x_i} g_{kj} + Z^k g_{il} \Gamma^l_{kj} + \frac{\partial Z^k}{\partial x_i} g_{ik},$$

where $Z = Z^k(\partial/\partial x_k)$, $\nabla_i(\partial/\partial x_k) = \Gamma^l_{ik}(\partial/\partial x_l)$ and $g_{ik} = g(\partial/\partial x_i, \partial/\partial x_k)$. Since $\alpha_i = g_{ik} Z^k$, one can check similarly that

$$\nabla_i \alpha_j = Z^k g_{lj} \Gamma^l_{ki} + \frac{\partial Z^k}{\partial x_i} g_{kj}.$$

Therefore,

$$\mathcal{L}_Z g_{ij} = \nabla_i \alpha_j + \nabla_j \alpha_i.$$

Analogously, one can calculate that

$$\operatorname{div} Z = -\delta\alpha,$$

where δ is the codifferential operator: $\delta\alpha = -\nabla^i \alpha_i$. Consequently,

$$SZ_{ij} = \nabla_i \alpha_j + \nabla_j \alpha_i + \frac{2}{n}\delta\alpha g_{ij}.$$

Using the definition (5) of divergence we obtain

$$\operatorname{div} SZ_j = \nabla^i \nabla_i \alpha_j + \nabla^i \nabla_j \alpha_i + \frac{2}{n} d\delta\alpha_j.$$

Now, applying the reasoning of [12], pp. 126–127, we derive

$$\operatorname{div} SZ_j = 2R_{ij} Z^i - \Delta\alpha_j - \frac{n-2}{n} d\delta\alpha_j,$$

which implies our assertions (12) and (13).

The form of the operator $S^* S$ obtained in (13) suggests that this operator should have some invariance (or rather quasi-invariance) properties under conformal transformations. This follows from the invariance properties obtained for the operators $\tilde{\Box} = k\delta d + l d\delta$ (k and l are some real constants) in paper [7] to which my attention was called by B. Ørsted. That invariance holds (cf. [7], Theorem 1.1) if we add to $\tilde{\Box}$ a zeroth order differential operator depending on the Ricci tensor R, which is just the case in our situation but, on the other hand, we have other coefficients k and l.

A study of invariance properties for $S^* S$ should rather be carried out after extending $S^* S$ to 1-forms or even to the whole exterior algebra (in our case $S^* S$ acts on vector fields only). We plan to do this in a subsequent paper.

Now, we confine ourselves to a special case. Namely, we show how S, S^* and $S^* S$ transform under a conformal change of the Riemannian metric g.

THEOREM 3. *If g and $\bar{g}$ are two conformally related Riemannian metrics on M, i.e. if $\bar{g} = ag$ for a positive function a on M, then*

$$\bar{S}Z = aSZ, \tag{14}$$

$$\overline{S^*}\,\varphi = (1/a^2)S^*\varphi + ((n-2)/a^2)V, \tag{15}$$

$$\overline{S^* S}Z = (1/a)S^* SZ + (n/a)W, \tag{16}$$

where V and W are the vector fields defined by $g(V, X) = \varphi(\operatorname{grad}(\log a), X)$ and $g(W, X) = SZ(\operatorname{grad}(\log a), X)$, $X \in \mathfrak{X}(M)$, respectively.

Proof. Let us denote by ∇ and $\bar{\nabla}$ the Riemannian connection related to g and $\bar{g}$, respectively. Then for arbitrary vector fields $X, Y \in \mathfrak{X}(M)$,

$$\bar{\nabla}_X Y = \nabla_X Y + \tfrac{1}{2}(X(b)Y + Y(b)X - g(X, Y)\operatorname{grad} b), \tag{17}$$

where $b = \log a$ (cf. [9]).

Since $\operatorname{div} Z = \operatorname{tr}(X \mapsto \nabla_X Z)$, by (17) we have

$$\overline{\operatorname{div}}\, Z = \operatorname{div} Z + \frac{n}{2}Z(b). \tag{18}$$

Since $\mathcal{L}_Z g(X, Y) = Zg(X, Y) - g(\mathcal{L}_Z X, Y) - g(X, \mathcal{L}_Z Y)$ and $\bar{g} = ag$,

$$\mathcal{L}_Z \bar{g} = a\mathcal{L}_Z g + Z(a)g. \tag{19}$$

Consequently, by (18), (19) and the definition of S, we obtain

$$\bar{S}Z = \mathcal{L}_Z \bar{g} - \frac{2}{n}\overline{\operatorname{div}}\, Z\bar{g} = a\mathcal{L}_Z g - Z(a)g - \frac{2}{n}\left(\operatorname{div} Z + \frac{n}{2}Z(b)\right)ag = aSZ,$$

i.e. the desired formula (14).

Let φ be an arbitrary tensor field of type (0, 2) on M. One can calculate, by (5) and (17), that

$$\overline{\operatorname{div}}\,\varphi = \frac{1}{a}\operatorname{div}\varphi + \frac{n-2}{2a}\varphi(\operatorname{grad} b, \cdot).$$

Consequently,

$$g(\bar{S}^*\varphi, X) = \frac{1}{a}\bar{g}(\bar{S}^*\varphi, X) = \frac{2}{a}\overline{\operatorname{div}}\,\varphi(X) = \frac{2}{a^2}\operatorname{div}\varphi + \frac{n-2}{a^2}\varphi(\operatorname{grad} b, X)$$

$$= g\left(\frac{1}{a^2}S^*\varphi + \frac{n-2}{a^2}V, X\right),$$

where V is the vector field defined by $g(V, X) = \varphi(\text{grad } b, X)$, $X \in \mathfrak{X}(M)$, which implies (15).

Finally, since $\text{div}(a\varphi) = a \text{ div } \varphi + \varphi(\text{grad } a, \cdot)$ and $(1/a)\text{grad } a = \text{grad } b$, we obtain, by (17),

$$\begin{aligned}\overline{\text{div}}\, \bar{S}Z &= \frac{1}{a}\text{div}(aSZ) + \frac{n-2}{2a} aSZ(\text{grad } b, \cdot)\\ &= \text{div } SZ + \frac{1}{a} SZ(\text{grad } a, \cdot) + \frac{n-2}{2} SZ(\text{grad } b, \cdot)\\ &= \text{div } SZ + \frac{n}{2} SZ(\text{grad } b, \cdot).\end{aligned}$$

Consequently,

$$g(\overline{S^* SZ}, X) = \frac{2}{a}\text{div } SZ + \frac{n}{a} SZ(\text{grad } b, X) = g\left(\frac{1}{a} S^* SZ + \frac{n}{a} W, X\right),$$

where W is the vector field defined by $g(W, X) = SZ(\text{grad } b, X)$, $X \in \mathfrak{X}(M)$. This implies the last assertion (16) and completes the proof.

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