POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CXXIII

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A Bing–Borsuk retract which contains a 2-dimensional absolute retract

WARSZAWA 1975
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

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1. Introduction*

In [1], Bing and Borsuk described a certain 3-dimensional compact metric absolute retract Q^* such that Q^* contains no 2-dimensional disc, and they ask whether Q^* contains any 2-dimensional absolute retract.

The retract Q^* is the decomposition space associated with a certain upper semicontinuous decomposition M of a 3-dimensional ball Q. The elements of M consist of countably many arcs $a_1, a_2, a_3, \ldots$ in Q together with the singleton subsets of $Q - \bigcup_{i=1}^{\infty} a_i$. See [1], Section 8, for a description of M.

By a *Bing–Borsuk retract* we shall mean any retract obtained from a 3-dimensional ball Q by using an upper semicontinuous decomposition M satisfying the conditions imposed on M in [1], Sections 2–8 (and a technical condition described below). Each such space is a compact metric absolute retract of dimension 3 that contains no disc [1].

The main result of this paper is that *there exists a Bing–Borsuk retract which contains a 2-dimensional absolute retract.*

Recently, Singh [4] has given an example of a 3-dimensional AR which contains no 2-dimensional AR. Singh's construction is quite similar to that of [1]. We shall discuss this point further in Section 13.

We shall now describe some notation and terminology to be used in this paper. In this paper, all spaces are metric and by a retract we shall always understand a metric retract. Following Borsuk [2], we shall use the notation AR to denote a compact absolute retract.

For a definition of *upper semicontinuous decomposition* and for basic facts concerning such decompositions, see [5]. If M is an upper semicontinuous decomposition of a space X, then X/M denotes the associated decomposition space, and $\varphi\colon X \to X/M$ denotes the projection map.

Q denotes the unit solid ball, centered at the origin, in E^3; $Q = \{x: x \in E^3 \textit{ and } d(x, 0) \leqslant 1\}$. S denotes the boundary of Q.

A sequence $M_1, M_2, \ldots$ of sets in a metric space is a *null sequence* if and only if the sequence of diameters $(\operatorname{diam} M_1), (\operatorname{diam} M_2), \ldots$ converges to 0.

In this paper, we use *complex* to include infinite complexes, and *polyhedron* to include non-compact polyhedra. If K is a complex, then $|K|$

* Research supported in part by National Science Foundation grant no. GP-30796.

denotes the *carrier* (or *polyhedron*) of K. For any complex K and any non-negative integer n, K^n denotes the *n-skeleton* of K, i.e.,

$$K^n = \{\sigma\colon \ \sigma \in K^n \text{ and } \dim \sigma \leqslant n\}.$$

2. Antoine's necklaces

In Section 5, we shall give a brief description of the construction of Bing–Borsuk retracts. In this section, and the next, we shall describe certain sets to be used in the construction. We follow [1], Section 3–5, closely; our aim is primarily that of indicating our notation.

In this section, let i denote some fixed positive integer.

First we shall describe the construction of an Antoine's necklace M^i. Suppose that A^i is a polyhedral solid torus in $\operatorname{Int} Q$. Let m_i be an even positive integer with $m_i \geqslant 8$. Let $\mathscr{L}$ be a chain of solid tori $\{L_1^i, L_2^i, \ldots, L_{m_i}^i\}$ in $\operatorname{Int} A^i$. By a *chain* of solid tori, we shall, in this paper, mean a chain constructed as in [1], Section 3. This chain circles A^i exactly once, and if s and t are integers such that $1 \leqslant s \leqslant m_i$ and $1 \leqslant t \leqslant m_i$, then L_s^i and L_t^i are linked if and only if $|s-t| = 1$. The sets $L_1^i, L_2^i, \ldots, L_{m_i}^i$ are polyhedral solid tori and are the *links* of the chain $\mathscr{L}$; we say they are the links of the *first stage* of the construction of M^i, and $\mathscr{L}$ is the chain of solid tori of the *first stage*.

Suppose j is a positive integer such that $j \leqslant m_i$. Let $\mathscr{L}_j$ be a chain of solid tori $\{L_{j1}^i, L_{j2}^i, \ldots, L_{jm_i}^i\}$ in $\operatorname{Int} L_j^i$, constructed relative to L_j^i as $\mathscr{L}$ is constructed relative to A^i. Then $L_{j1}^i, L_{j2}^i, \ldots, L_{jm_i}^i$ are the *links* of the chain $\mathscr{L}_j$; we say they are links of the *second stage* of the construction of M^i.

Let this process be continued. Suppose that k is a positive integer and suppose that if each of $j_1, j_2, \ldots, j_k$ is a positive integer no greater than m_i, then there has been defined a link $L_{j_1j_2\ldots j_k}^i$. In each such link $L_{j_1j_2\ldots j_k}^i$, we define a chain $\mathscr{L}_{j_1j_2\ldots j_k}$ of solid tori $\{L_{j_1j_2\ldots j_k1}^i, L_{j_1j_2\ldots j_k2}^i, \ldots, L_{j_1j_2\ldots j_km_i}^i\}$ in fashion analogous to that described above. The resulting links are the *links* of the $(k+1)$st stage of the construction of M^i.

We require further that for any positive number ε, there is a positive integer N such that for any positive integer n greater than N, the maximum diameter of the links at the nth stage should be less than ε.

For each positive integer k, let N_k^i denote the union of the links of the kth stage in the construction of M^i. Let M^i denote $\bigcap\limits_{k=1}^{\infty} N_k^i$; M^i is an Antoine's necklace.

Now we shall describe some notation and terminology for indexes. Suppose k is a positive integer. Then a is a *stage k index for M^i* if and only if there exist positive integers $j_1, j_2, \ldots, j_k$, each no greater than m_i, such that $a = j_1j_2\ldots j_k$.

3. Wreaths

We shall need the notion of winding number of an arc in a solid torus. Suppose that a is an arc in a solid torus T. Let $\tilde{T}$ denote the universal covering space of T. If D is a meridional disc of T, then in $\tilde{T}$ there exist infinitely many (mutually disjoint) copies both of D and of a. Let $\tilde{a}$ be one of these copies of a. For each meridional disc D of T, there is an integer k_D such that $\tilde{a}$ intersects exactly k_D distinct copies of D in $\tilde{T}$. It is easily seen that k_D is independent of the choice of $\tilde{a}$. Then let

$$\omega(a, T) = \min\{k_D\colon D \textit{ is a meridional disc of } T\}.$$

We shall call $\omega(a, T)$ the *winding number* of a in T.

We return now to the construction of the preceding section; we use the notation of Section 2.

Suppose j is a positive integer such that $j \leqslant m_i$. Let M_j^i denote $M^i \cap L_j^i$; M_j^i is also an Antoine's necklace. We have the following lemma whose proof is well known.

LEMMA 3.1. *For each positive integer j such that $j \leqslant m_i$, there is an arc J_j^i such that $J_j^i \subset \operatorname{Int} L_j^i$, $M_j^i \subset J_j^i$, and J_j^i has winding number* 1 *in L_j^i.*

For each positive integer j such that $j \leqslant m_i$, let J_j^i be an arc in $\operatorname{Int} L$ containing M_j^i and having winding number 1 in L_j^i. If $W_i = \bigcup_{j=1}^{m_i} J_j^i$, then W_i is a *wreath substituting for the solid torus A^i*, and the arcs $J_1^i, J_2^i, \ldots, J_{m_i}^i$ are the links of the wreath W_i.

The term "wreath" is accordingly used in this paper in a more restricted sense than in [1], the difference being that we require that each arc J_j^i which is a link of the wreath should have winding number 1 in the solid torus L_j^i.

4. Construction of discs

We describe a construction which yields, for a specified Antoine's necklace A, a disc having A on its boundary. Constructions of this type are well known. Later in this paper, we shall make use of the process described here. We use the notation and terminology of Section 3.

Suppose that L is a polyhedral solid torus and $\mathscr{L}$ is a chain of solid tori $\{L_1, L_2, \ldots, L_m\}$ in $\operatorname{Int} L$. Then F is a *flange for L and $\mathscr{L}$* if and only if F is a polyhedral disc in L such that (1) $F \cap \operatorname{Bd} L$ is an arc on $\operatorname{Bd} F$ and (2) F is disjoint from each link of $\mathscr{L}$. The arc $F \cap \operatorname{Bd} L$ is the *edge of F* on L.

Suppose that F is a flange for L and $\mathscr{L}$. Then $R_1, R_2, \ldots, R_m$ are *strips for L, $\mathscr{L}$, and F* if and only if $R_1, R_2, \ldots, R_m$ are mutually disjoint polyhedral discs in $\operatorname{Int} L$ such that if $t = 1, 2, \ldots, m$, then (1) $R_t \cap F = \operatorname{Bd} R_t \cap \operatorname{Bd} F$ and is an arc, (2) $R_t \cap L_t = \operatorname{Bd} R_t \cap \operatorname{Bd} L_t$ and is an arc, and (3) if $s = 1, 2, \ldots, m$ and $t \neq s$, then R_t and L_s are disjoint.

Suppose i is some positive integer. Let A^i be a polyhedral solid torus. M^i will denote an Antoine's necklace in A^i constructed as in Section 3, and for each positive integer k and each stage k index a, (1) L_a^i will denote a link of the kth stage in the construction of M^i, and (2) $\mathscr{L}_a^i$ will denote the chain of solid tori in $\operatorname{Int} L_a^i$. If j is a positive integer such that $j \leqslant m_i$, let M_j^i denote $M^i \cap L_j^i$.

Suppose that $j = 1, 2, \ldots, m_i$, and suppose that λ_j^i is a polygonal arc on $\operatorname{Bd} L_j^i$. We shall describe a disc D_j^i such that $\lambda_j^i \subset \operatorname{Bd} D_j^i$, $D_j^i - \lambda_j^i \subset \operatorname{Int} L_j^i$, and $M_j^i \subset \operatorname{Bd} D_j^i$. Further, D_j^i is locally polyhedral modulo M_j^i.

Let F_j^i be a flange for L_j^i and $\mathscr{L}_j^i$ having λ_j^i as its edge on $\operatorname{Bd} L_j^i$, and let $R_{j1}^i, R_{j2}^i, \ldots, R_{jm_i}^i$ be strips for L_j^i, $\mathscr{L}_j^i$, and F_j^i. If $t = 1, 2, \ldots, m_i$, let λ_{jt}^i be $L_{jt}^i \cap R_{jt}^i$; λ_{jt}^i is an arc on $\operatorname{Bd} L_{jt}^i$. Let Δ_j^i denote

$$F_j^i \cup \Big(\bigcup_{t=1}^{m_i} R_{jt}^i\Big).$$

Suppose that p is a positive integer, a is a stage p index for M^i, and the polygonal arc λ_a^i on $\operatorname{Bd} L_a^i$ has been constructed. Then there exist (1) a flange F_a^i for L_a^i and $\mathscr{L}_a^i$ having λ_a^i as its edge on $\operatorname{Bd} L_a^i$ and (2) strips $R_{a1}^i, R_{a2}^i, \ldots, R_{am_i}^i$ for L_a^i, $\mathscr{L}_a^i$, and F_a^i. If $t = 1, 2, \ldots, m_i$, let λ_{at}^i denote $R_{at}^i \cap L_{at}^i$. Let Δ_a^i denote

$$F_a^i \cup \Big(\bigcup_{t=1}^{m_i} R_{at}^i\Big).$$

Let $D_j^i = \Delta_j^i \cup \big(\bigcup\{\Delta_a^i\colon\ a \text{ is an index for } M^i\}\big) \cup M_j^i$. It is easy to show that D_j^i is a disc, and that $M_j^i \subset \operatorname{Bd} D_j^i$. It is clear that $\lambda_j^i \subset \operatorname{Bd} D_j^i$ and $D_j^i - \lambda_j^i \subset \operatorname{Int} L_j^i$.

Suppose that if $j = 1, 2, \ldots, m_i$, μ_j^i is an arc on $\operatorname{Bd} D_j^i$ containing M_j^i and lying in $\operatorname{Int} L_j^i$.

LEMMA 4.1. *If $j = 1, 2, \ldots, m_i$, μ_j^i has winding number 1 in L_j^i.*

Proof. Let Ψ_j^i be a polyhedral meridional disc in L_j^i which does not intersect λ_j^i. We may modify Ψ_j^i to obtain a meridional disc Φ_j^i in L_j^i such that (1) $\Phi_j^i \cap J_j^i \subset M_j^i$ and (2) D_j^i lies, except for $J_j^i \cap \Phi_j^i$, entirely "to one side" of Φ_j^i; that is, there is a neighborhood N of Φ_j^i in L_j^i such that $N - \Phi_j^i$ has exactly two components and D_j^i intersects only one of them. Φ_j^i may be constructed from Ψ_j^i by repeatedly pushing tubes over the strips and flanges used to construct D_j^i. We may suppose Φ_j^i to be locally polyhedral modulo M_j^i.

It follows that if in the universal covering space $\tilde{L}_j^i$ of L_j^i, $\tilde{D}_j^i$ is a copy of D_j^i, then $\tilde{D}_j^i$ intersects at most one of the copies of Φ_j^i in $\tilde{L}_j^i$. Accordingly, $\omega(\mu_j^i, L_j^i) \leqslant 1$. Since μ_j^i contains M_j^i, it can be shown that $\omega(\mu_j^i, L_j^i) \neq 0$. Hence $\omega(\mu_j^i, L_j^i) = 1$.

5. Construction of Bing–Borsuk retracts

In this section we shall describe the construction of Bing–Borsuk retracts. We follow [1] closely. For terms not defined in this paper, see [1]. Recall that Q is a 3-dimensional ball and $S = \mathrm{Bd}Q$.

A *chord* of Q is a closed segment both of whose endpoints belong to S. In [1], it is shown that there is a sequence $K_1, K_2, \ldots$ of mutually disjoint chords of Q which form a null sequence and are *dense* on S, i.e., for each open subset U of S, there is a positive integer ν such that both endpoints of K_ν are in U.

A sequence $A^1, A^2, \ldots$ of solid tori in $\mathrm{Int}Q$ is *dense* in Q if and only if for each simple closed curve C in $\mathrm{Int}Q$, there is a positive integer n such that A^n misses C and some core of A^n is (homologically) linked with C (relative to the integers).

The results of [1], together with Lemma 3.1, yield the following: Suppose that $K_1, K_2, \ldots$ is a null family of mutually disjoint chords of B, dense on S. Then there exists a sequence $A^1, A^2, \ldots$ of polyhedral solid tori in $(\mathrm{Int}Q) - \bigcup_{j=1}^{\infty} K_j$ such that (1) $A^1, A^2, \ldots$ is dense in Q, (2) if $i = 1, 2, \ldots$, the inner radius of A^i is less than $1/i$, (3) if $i = 1, 2, \ldots$, there is a wreath W_i in A^i substituting for A^i such that (a) each link of W_i has diameter less than $1/i$, and (b) W_i misses $\bigcup_{j=1}^{\infty} K_j$, and (4) the sets $W_1, W_2, \ldots$ are mutually disjoint.

We may now describe the Bing–Borsuk retracts. First we introduce a special type of decomposition of Q. By a *Bing–Borsuk decomposition of* Q we shall mean a decomposition M of Q consisting of (a) the arcs of a null sequence $K_1, K_2, \ldots$ of mutually disjoint chords of Q which are dense on S, (b) the arcs which are the links of wreaths $W_1, W_2, \ldots$ obtained as in the paragraph immediately preceding this one, and (c) the singleton subsets of $Q - \big[\big(\bigcup_{j=1}^{\infty} K_j\big) \cup \big(\bigcup_{i=1}^{\infty} W_i\big)\big]$. By construction, the arcs of M form a null sequence, and hence M is upper semicontinuous.

The statement that a space Q^* is a *Bing–Borsuk retract* means that there exists a Bing–Borsuk decomposition M of Q such that $Q^* = Q/M$. By the results of [1], each Bing–Borsuk retract is a 3-dimensional compact AR that contains no disc.

We may now indicate how the remainder of this paper is organized. In Section 10, we shall describe the construction of a particular Bing–Borsuk decomposition of Q. The associated decomposition space Q^* is thus a Bing–Borsuk retract. At the same time, we construct a 2-dimensional compact absolute retract X in Q such that the image X^* of X under projection is a 2-dimensional AR in Q^*. In Section 11, we shall show that X is a 2-dimensional AR, and in Section 12 that X^* is a 2-dimensional AR.

Sections 6–9 are devoted to material preliminary to the construction described in Section 10.

6. Sets in solid tori

Suppose that L is a polyhedral solid torus in $\operatorname{Int} Q$, m is a positive integer such that $m \geqslant 8$, $\mathscr{L}$ is a circular chain $\{L_1, L_2, \ldots, L_m\}$ of linked polyhedral unknotted solid tori in $\operatorname{Int} L$, as described in Section 2, and $j = 1, 2, \ldots, m$.

The statement that Δ is a *special meridional disc of L relative to $\mathscr{L}$ and j* means that Δ is a polyhedral meridional disc in L such that (1) $L \cap L_j$ is the union of two disjoint polyhedral meridional discs in L_j, and (2) if $t = 1, 2, \ldots, j-1, j+1, \ldots, m_i$, Δ and L_t are disjoint. See Figure 1.

Suppose Δ is a special meridional disc in L relative to $\mathscr{L}$ and j. Then V is a *special neighborhood of Δ in L relative to $\mathscr{L}$* if and only if V is a polyhedral 3-cell in L such that (1) $\operatorname{Bd} \Delta \subset \operatorname{Bd} V$ and $\operatorname{Int} \Delta \subset \operatorname{Int} V$, (2) $\operatorname{Bd} V \cap \cap \operatorname{Bd} L$ is an annulus having $\operatorname{Bd} \Delta$ as a centerline, (3) $L_j \cup L_{j+1} \subset \operatorname{Int} V$ and each of $L_{j-1} \cap V$ and $L_{j+2} \cap V$ is a 3-cell, and (4) if $t = 1, 2, \ldots, j-2, j+3, \ldots, m$, V and L_t are disjoint. See Figure 1.

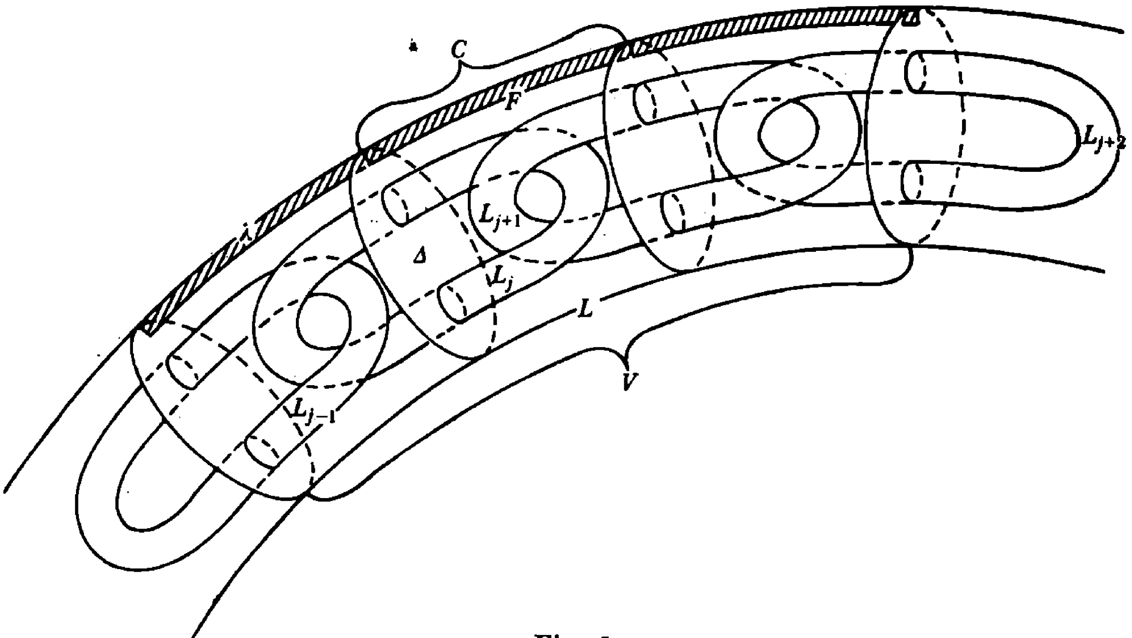

Fig. 1

Suppose, in addition, that V is a special neighborhood of Δ in L relative to $\mathscr{L}$. Then λ is a *special arc for* L, Δ, *and* V if and only if (1) λ is a polygonal arc spanning the annulus $\mathrm{Bd}\,V \cap \mathrm{Bd}\,L$ and (2) λ intersects $\mathrm{Bd}\,\Delta$ in only one point. See Figure 1.

Recall that F is a *flange for* L if and only if F is a polyhedral disc in L such that $F \cap \mathrm{Bd}L$ is an arc in $\mathrm{Bd}F$; the arc $F \cap \mathrm{Bd}L$ is the *edge* of F on $\mathrm{Bd}L$.

If Δ is a special meridional disc of L relative to $\mathscr{L}$ and j, and V is a special neighborhood of Δ in L relative to $\mathscr{L}$, then a flange F for L is a *special flange for* L, $\mathscr{L}$, Δ, *and* V if and only if (1) $F \subset V$, (2) $F \cap \Delta$ is an arc, (3) $F \cap \mathrm{Bd}\,V$ is an arc on $\mathrm{Bd}F$ which contains, in its interior, the arc $F \cap \mathrm{Bd}L$, and (4) F is disjoint from each link of $\mathscr{L}$. See Figure 1.

C is an *associated* 3-*cell for* L, $\mathscr{L}$, Δ, *and* V if and only if C is a polyhedral 3-cell in V such that (1) $\Delta \subset \mathrm{Bd}C$, (2) $(\mathrm{Bd}C) \cap (\mathrm{Bd}\,V)$ is an annulus in $\mathrm{Int}[(\mathrm{Bd}\,V) \cap (\mathrm{Bd}T)]$, (3) $(\mathrm{Bd}C) - (\Delta \cup \mathrm{Bd}\,V)$ is a disc spanning V, (4) each of $C \cap L_j$ and $C \cap L_{j+1}$ is a 3-cell, and (5) C is disjoint from $L_{j-1} \cup L_{j+2}$. See Figure 1.

Now suppose that i, m_i, k, and r are positive integers with $m_i \geqslant 8$, and if $t = 1, 2, \ldots, k$, j_t is a positive integer such that $j_t \leqslant m_i$. Let a denote $j_1 j_2 \ldots j_k$.

Suppose that L^i_a is a polyhedral solid torus in $\mathrm{Int}Q$, $\mathscr{L}^i_a$ is a chain of solid tori $\{L^i_{a1}, L^i_{a2}, \ldots, L^i_{am_i}\}$ in i, and $j = 1, 2, \ldots, m_i$. Suppose that Δ^{ir}_a is a special meridional disc of L^i_a relative to $\mathscr{L}^i_a$ and j, V^{ir}_a is a special neighborhood of Δ^{ir}_a in L^i_a relative to $\mathscr{L}^i_a$, C^{ir}_a is an associated 3-cell for L^i_a, $\mathscr{L}^i_a$, Δ^{ir}_a, and V^{ir}_a, and λ^{ir}_a is a special arc for L^i_a, Δ^{ir}_a, and V^{ir}_a. See Figure 2.

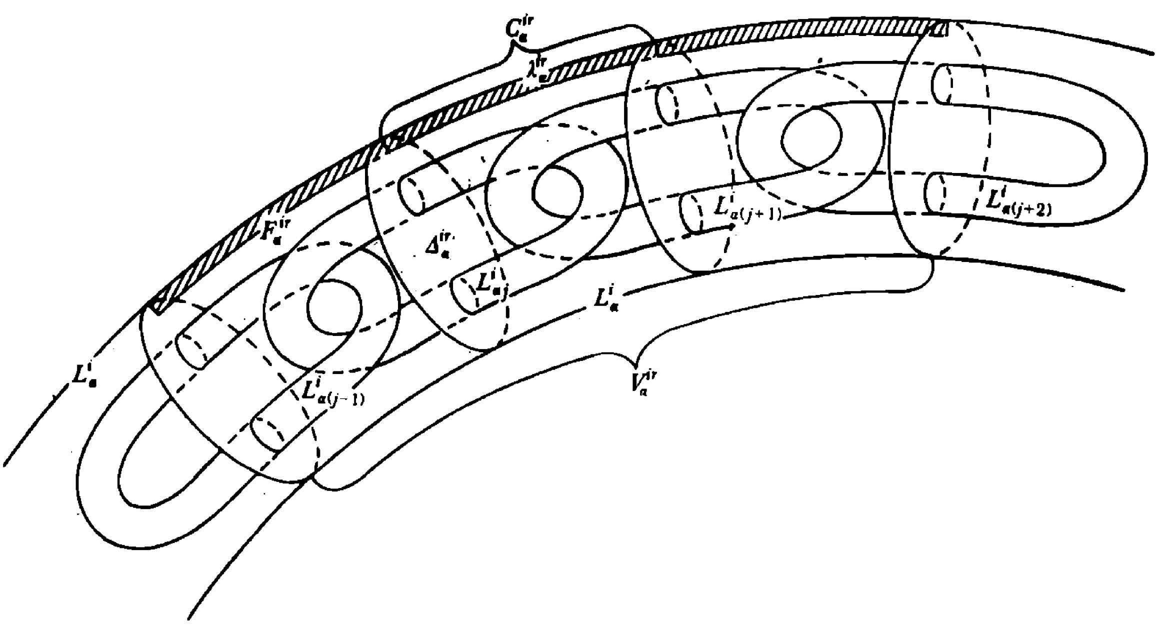

Fig. 2

Now we shall construct certain sets in L^i_a. Let F^{ir}_a be a special flange for L^i_a, $\mathscr{L}^i_a$, Δ^{ir}_a with edge λ^{ir}_a on $\mathrm{Bd}\,L^i_a$. Let

$$R^{ir}_{a1}, R^{ir}_{a2}, \ldots, R^{ir}_{a(j-1)}, R^{ir}_{a(j+2)}, \ldots, R^{ir}_{am_i}$$

be mutually disjoint polyhedral discs in L^i_a such that (1) if $t = 1, 2, \ldots, m_i$ (a) R^{ir}_{at} intersects F^{ir}_a in an arc on $\mathrm{Bd}\,F^{ir}_a$, and otherwise F^{ir}_a and F^{ir}_{at} are disjoint, (b) $R^{ir}_{at}\cap L^i_{at}$ is an arc λ^{ir}_{at} in $\mathrm{Bd}\,R^{ir}_{at}\cap\mathrm{Bd}\,L^i_{at}$, and (c) R^{ir}_{at} is disjoint from Δ^{ir}_a, and (2) $R^{ir}_{a(j-1)}$ and $R^{ir}_{a(j+2)}$ lie in V^{ir}_a, and if $t = 1, 2, \ldots, j-2, j+3, \ldots, m_i$, $R^{ir}_{at}\cap V^{ir}_a$ is a disc. See Figure 3.

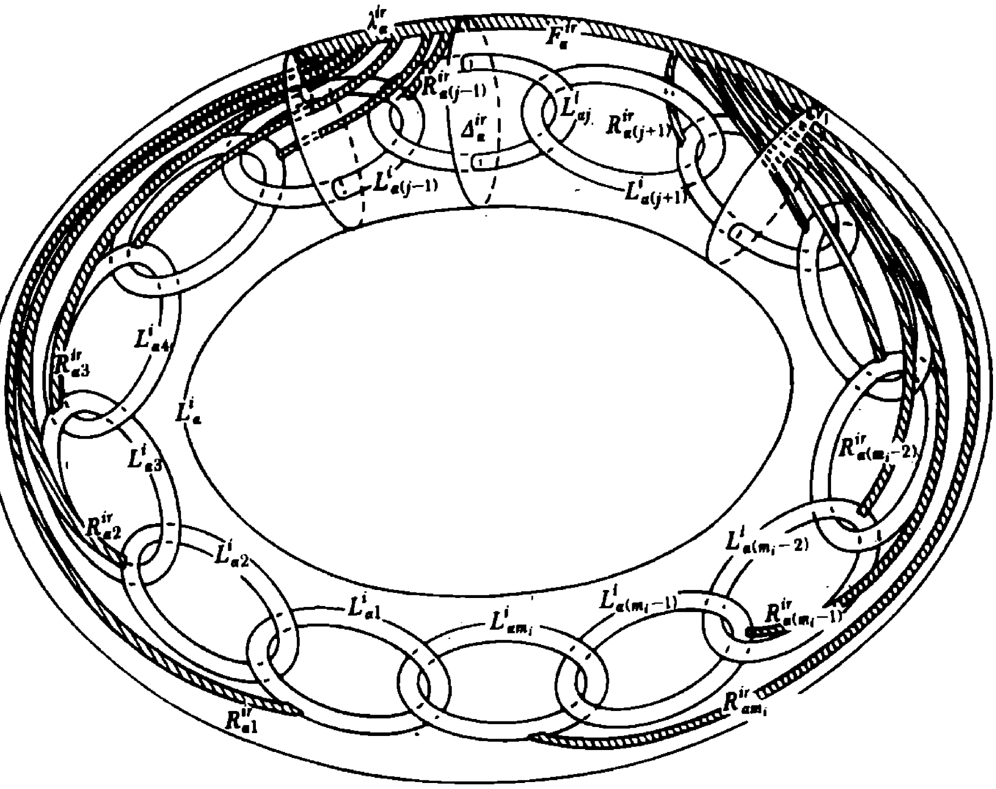

Fig. 3

Let Σ^i_{aj} denote $(\mathrm{Bd}\,L^a_{ij})\cap C^{ir}_a$, and let Λ^{ir}_{aj} be a polyhedral disc attached to $\Delta^{ir}_a\cup\Sigma^i_{aj}$ as shown in Figure 4, and such that $\Lambda^{ir}_{aj}\cap L^i_{a(j+1)}$ is a polyhedral meridional disc $\Delta^{ir}_{a(j+1)}$.

Now $(\mathrm{Bd}\,\Lambda^{ir}_{aj})\cap\Sigma^i_{aj}$ is an arc λ^{ir}_{aj}. Let S^{ir}_{aj} be a polyhedral disc in C^{ir}_a such that $S^{ir}_{aj}\cap F^{ir}_a$ is an arc in $(\mathrm{Bd}\,S^{ir}_{aj})\cap(\mathrm{Bd}\,F^{ir}_a)$, $S^{ir}_{aj}\cap\Delta^{ir}_a$ is an arc spanning $\Delta^{ir}_a - \mathrm{Int}(L^i_{aj}\cap\Delta^{ir}_a)$ as shown in Figure 4, and $S^{ir}_{aj}\cap\Sigma^{ir}_{aj}$ is a subarc of λ^{ir}_{aj}.

Let V^{ir}_{aj} be $L^i_{aj}\cap C^{ir}_a$. Let $\hat{\Delta}^{ir}_a$ be $\Delta^{ir}_a - \mathrm{Int}(\Delta^{ir}_a\cap L^i_{aj})$. $\hat{\Delta}^{ir}_a$ is a disc with two holes. Let Δ^{ir}_{aj} be a polyhedral meridional disc in L^i_{aj} such that $\mathrm{Bd}\,\Delta^{ir}_{aj} \subset \mathrm{Bd}\,L^i_{aj}$ and $\mathrm{Int}\,\Delta^{ir}_{aj} \subset \mathrm{Int}\,V^{ir}_{aj}$. See Figure 5. Let $V^{ir}_{a(j+1)}$ be $L^i_{a(j+1)}\cap C^{ir}_a$.

Let $S^{ir}_{a(j+1)}$ be a polyhedral disc in C^{ir}_a and near Λ^{ir}_{aj} such that (1) $\mathrm{Int}\,S^{ir}_{a(j+1)}$ is disjoint from L^i_{aj}, $L^i_{a(j+1)}$, and $L^i_{a(j+2)}$, (2) $S^{ir}_{a(j+1)}\cap\mathrm{Bd}\,L^i_{aj}$

is a subarc of $\lambda^{ir}_{\alpha j}$ with one endpoint the point $x^{ir}_{\alpha(j+1)}$ of $\Delta^{ir}_{\alpha j}\cap\lambda^{ir}_{\alpha j}$, (3) $S^{ir}_{\alpha(j+1)}\cap\mathrm{Bd}\,L^{i}_{\alpha(j+1)}$ is the point $y^{ir}_{\alpha(j+1)}$ common to $\Lambda^{ir}_{\alpha j}$ and $\lambda^{ir}_{\alpha(j+1)}$, and (4) $S^{ir}_{\alpha(j+1)}\cap\Lambda^{ir}_{\alpha j}$ is an arc from $x^{ir}_{\alpha(j+1)}$ to $y^{ir}_{\alpha(j+1)}$. See Figure 5.

Let X^{ir}_{α} denote

$$F^{ir}_{\alpha}\cup\hat{\Delta}^{ir}_{\alpha}\cup\Sigma^{ir}_{\alpha j}\cup\Lambda^{ir}_{\alpha j}\cup S^{ir}_{\alpha j}\cup\Delta^{ir}_{\alpha j}\cup$$
$$\cup\Sigma^{ir}_{\alpha(j+1)}\cup S^{ir}_{\alpha(j+1)}\cup\left(\bigcup\{R^{ir}_{\alpha t}\colon\ t=1,2,\ldots,j-1,j+2,\ldots,m_i\}\right).$$

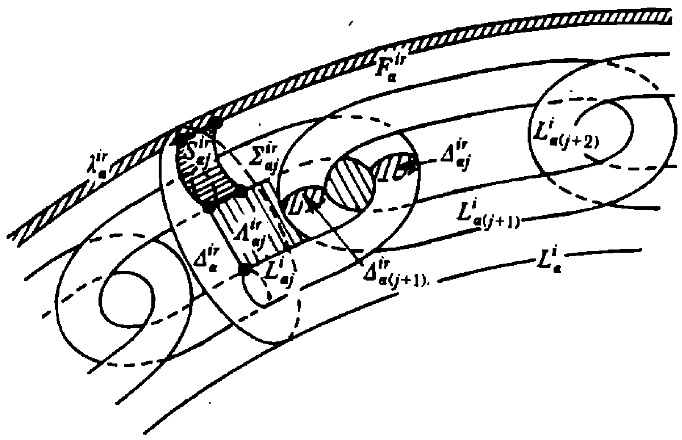

Fig. 4

Suppose then that there has been specified a solid torus L^{ir}_{α} and chain $\mathscr{L}^{ir}_{\alpha}$ of solid tori. If the sets Δ^{ir}_{α}, V^{ir}_{α}, and λ^{ir}_{α} are also given, we say that X^{ir}_{α} is *based on* Δ^{ir}_{α}, V^{ir}_{α}, *and* λ^{ir}_{α}.

PROPOSITION 6.1. *X^{ir}_{α} is a contractible 2-complex, and hence is an* AR.

Proof. Clearly X^{ir}_{α} is a 2-complex. Let $X^{ir}_{\alpha}(1)$ denote

$$\Delta^{ir}_{\alpha}\cup\Sigma^{ir}_{\alpha j}\cup\Lambda^{ir}_{\alpha j}\cup\Delta^{ir}_{\alpha j}.$$

Then $X^{ir}_{\alpha}(1)$ is contractible. Let $X^{ir}_{\alpha}(2)$ denote

$$F^{ir}_{\alpha}\cup\left(\bigcup\{R^{ir}_{\alpha t}\colon\ t=1,2,\ldots,j-1,j+2,\ldots,m_i\}\right).$$

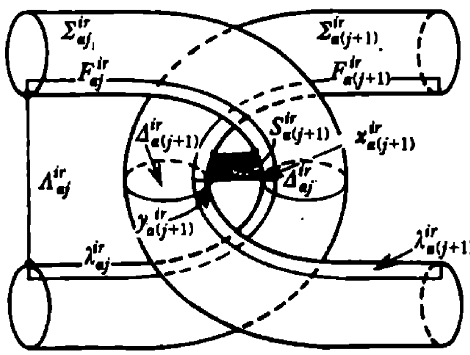

Fig. 5

Then $X_a^{ir}(2)$ is a disc and $X_a^{ir}(1)\cap X_a^{ir}(2)$ is a spanning arc of $X_a^{ir}(2)$, and thus if $X_a^{ir}(3) = X_a^{ir}(1)\cup X_a^{ir}(2)$, $X_a^{ir}(3)$ is contractible. $\Sigma_{a(j+1)}^{ir}$ is an annulus that intersects $X_a^{ir}(3)$ in a simple closed curve $\mathrm{Bd}\,\Delta_{a(j+1)}^{ir}$ which is a center-line of $\Sigma_{a(j+1)}^{ir}$. Thus if $X_a^{ir}(4) = X_a^{ir}(3)\cup\Sigma_{a(j+1)}^{ir}$, $X_a^{ir}(4)$ is contractible. Finally, S_{aj}^{ir} and $S_{a(j+1)}^{ir}$ are disjoint discs which intersect $X_a^{ir}(4)$ only in arcs on their boundaries. Since $X_a^{ir} = X_a^{ir}(4)\cup S_{aj}^{ir}\cup S_{a(j+1)}^{ir}$, it follows that X_a^{ir} is contractible.

The following proposition can be proved by a simple modification of the proof of Proposition 6.1.

PROPOSITION 6.2. *$X_a^{ir}\cap V_a^{ir}$ is an* AR.

PROPOSITION 6.3. *Let Y_a^{ir} denote*

$$X_a^{ir}\cup V_{aj}^{ir}\cup V_{a(j+1)}^{ir}.$$

Then Y_a^{ir} is an AR.

Proof. Since $V_{aj}^{ir}\cap X_a^{ir} = \Sigma_{aj}^{ir}\cup\Delta_{aj}^{ir}$, $V_{a(j+1)}^{ir}\cap X_a^{ir} = \Sigma_{a(j+1)}^{ir}\cup\Delta_{a(j+1)}^{ir}$, and X_a^{ir} is contractible, it follows that Y_a^{ir} is contractible. Since Y_a^{ir} is a 3-complex, it is thus an AR.

We point out now how to construct certain discs that are useful. At the next step of the construction, the process described above is repeated in each of the links L_{aj}^i and $L_{a(j+1)}^i$. In particular, we construct a flange F_{aj}^{ir} in L_{aj}^i with edge λ_{aj}^{ir} on $\mathrm{Bd}\,L_{aj}^i$, and a flange $F_{a(j+1)}^{ir}$ in $L_{a(j+1)}^i$ with edge $\lambda_{a(j+1)}^{ir}$ on $\mathrm{Bd}\,L_{a(j+1)}^i$.

Let E_a^{ir} denote

$$F_a^{ir}\cup\left[\bigcup\{R_{at}^{ir}\colon t = 1,2,\ldots,j-1,j+2,\ldots,m_i\}\right]\cup \cup S_{aj}^{ir}\cup F_{aj}^{ir}\cup S_{a(j+1)}^{ir}\cup F_{a(j+1)}^{ir}.$$

Then E_a^{ir} is a disc in L_a^i such that (1) $E_a^{ir}\cap\mathrm{Bd}\,L_a^i$, (2) if $t = 1,2,\ldots,j-1$, $j+2,\ldots,m_i$, $E_a^{ir}\cap L_{at}^i = \lambda_{at}^{ir}$, and (3) $E_a^{ir}\cap L_{aj}^i = F_{aj}^{ir}$ and $E_a^{ir}\cap L_{a(j+1)}^i = F_{a(j+1)}^{ir}$.

7. Replacing discs in solid tori

Suppose that i, m_i, and r are positive integers with $m_i\geqslant 8$, A^i is a polyhedral solid torus in $\mathrm{Int}\,Q$, $\mathscr{L}^i$ is a chain of solid tori $\{L_1^i, L_2^i,\ldots,L_{m_i}^i\}$ in $\mathrm{Int}\,A^i$, and $j = 1,2,\ldots,m_i$. Suppose that Δ^{ir} is a special meridional disc of A^i relative to $\mathscr{L}^i$ and j, V^{ir} is a special neighborhood of Δ^{ir} in L relative to $\mathscr{L}^i$, and C^{ir} is an associated 3-cell for A^i, $\mathscr{L}^i$, Δ^{ir}, and V^{ir}. See Figure 6.

Let Δ_j^{ir} denote a polyhedral meridional disc in L_j^i as shown in Figure 7, let Λ_j^{ir} denote a polyhedral disc in A^i, as shown in Figure 7, with $\Lambda_j^{ir}\cap L_{j+1}^i$ a polyhedral meridional disc Δ_{j+1}^{ir} in L_{j+1}^i. Let V_j^{ir} be $C^{ir}\cap L_j^i$ and let

V^{ir}_{j+1} be $C^{ir} \cap L^i_{j+1}$; V^{ir}_j and V^{ir}_{j+1} are special 3-cell neighborhoods of Δ^{ir}_j and Δ^i_{j+1}, respectively, in L^i_j.

Let Σ^{ir}_j be $(\operatorname{Bd} L^i_j) \cap V^{ir}_j$ and let Σ^{ir}_{j+1} be $(\operatorname{Bd} L^i_{j+1}) \cap V^{ir}_{j+1}$. Let $\hat{\Delta}^{ir}$ be $\Delta^{ir} - \operatorname{Int}(\Delta^{ir} \cap L^i_j)$; $\hat{\Delta}^{ir}$ is a disc with two holes.

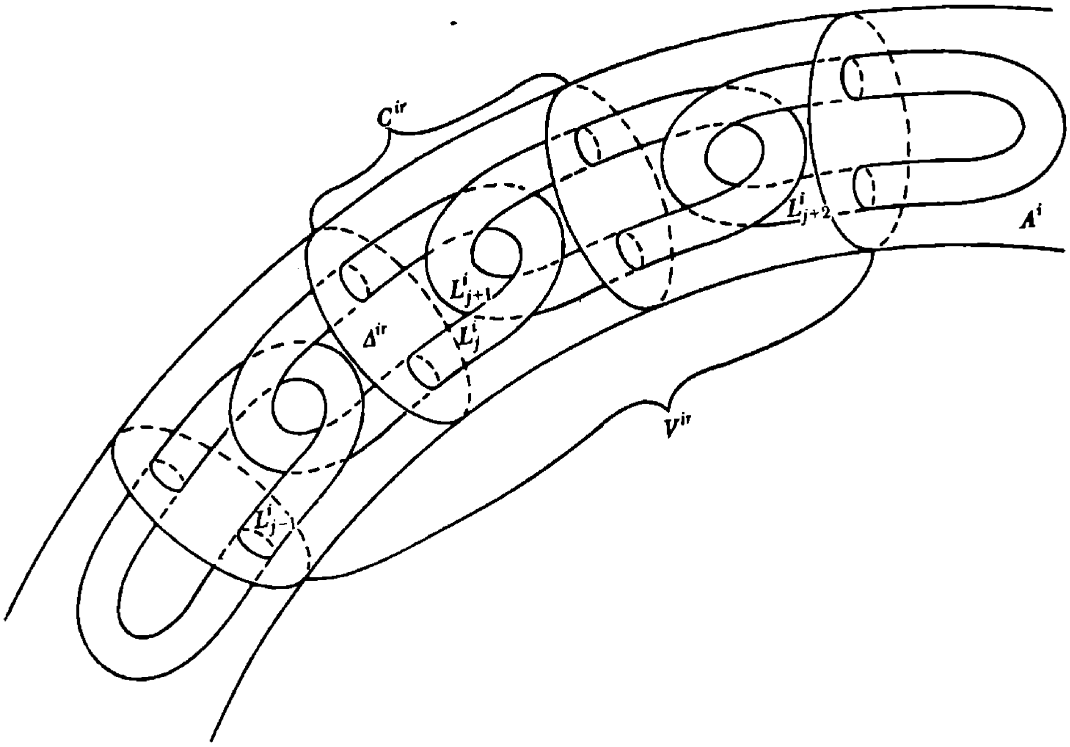

Fig. 6

Let X^{ir}_0 denote

$$\hat{\Delta}^{ir} \cup \Sigma^{ir}_j \cup \Delta^{ir}_j \cup \Lambda^{ir}_j \cup \Sigma^{ir}_{j+1}.$$

It is easy to see that X^{ir}_0 is a contractible 2-complex.

If L is a polyhedral solid torus in $\operatorname{Int} Q$, Δ is a polyhedral meridional disc in L, V is a polyhedral 3-cell neighborhood of Δ in L with $V \cap \operatorname{Bd} L$ an annulus having $\operatorname{Bd} \Delta$ as a centerline, and λ is a polygonal arc spanning

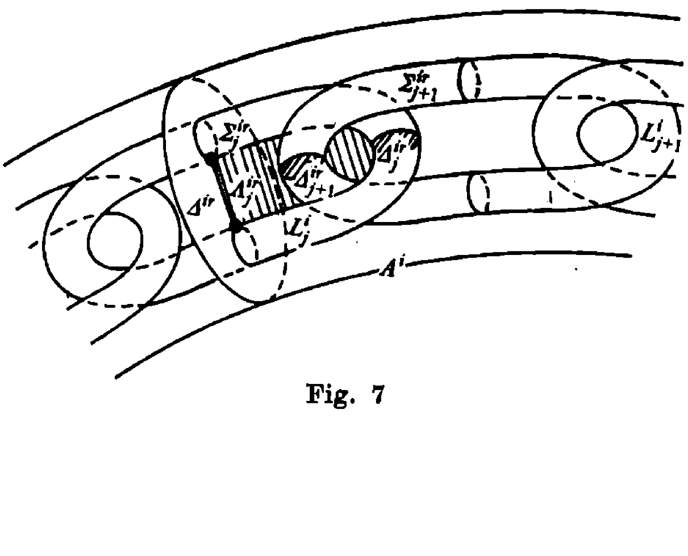

Fig. 7

$V \cap \mathrm{Bd}\, L$ and intersecting $\mathrm{Bd}\, \Delta$ in only one point, then (Δ, V, λ) is an *admissible triple for L.*

If (Δ, V, λ) is an admissible triple for a solid torus L, and m is any even positive integer with $m \geqslant 8$, it is easily seen that there exists a chain $\mathscr{L}$ of solid tori in $\mathrm{Int}\, L$ such that Δ is a special meridional disc in L relative to $\mathscr{L}$ and 1, V is a special neighborhood of Δ relative to $\mathscr{L}$, and λ is a special arc for L, Δ, V.

If l is j or $j+1$, let λ_l^{ir} be a special arc for L_l^i, Δ_l^{ir} and V_l^{ir}. Then $(\Delta_l^{ir}, V_l^{ir}, \lambda_l^{ir})$ is an admissible triple for L_l^i.

If $t = 1, 2, \ldots, m_i$, let $\mathscr{L}_t^i$ be a chain solid tori $\{L_{t1}^i, L_{t2}^i, \ldots, L_{tm_i}^i\}$ in $\mathrm{Int}\, T_t^i$ such that (1) if l is j or $j+1$, Δ_l^{ir} is a special meridional disc in L_l^i relative to $\mathscr{L}_l^i$ and 1, and V_l^{ir} is a special neighborhood of Δ_l^{ir} relative to $\mathscr{L}_l^i$. Since $m_i \geqslant 8$, clearly such chains exist.

If l is either j or $j+1$, then, by the results of Section 6, there exists a contractible 2-complex X_l^{ir} such that (1) $X_l^{ir} \subset L_l^i$, (2) $X_l^{ir} \cap \mathrm{Bd}\, L_l^i = \lambda_l^{ir} \cup \cup \mathrm{Bd}\, \Delta_l^{ir}$, (3) X_l^{ir} is based on Δ_l^{ir}, V_l^{ir}, and λ_l^{ir} (4) if $t = 1, 2, \ldots, j-1, j+2, \ldots, m_i$, then $X_l^{ir} \cap L_{lt}^i$ is a polygonal arc λ_{lt}^i on $\mathrm{Bd}\, L_{lt}^i$.

Let X_1^{ir} denote

$$\left(X_0^{ir} - (\Delta_j^{ir} \cup \Delta_{j+1}^{ir})\right) \cup (X_j^{ir} \cup X_{j+1}^{ir}).$$

It is easy to see that X_1^{ir} is a contractible 2-complex in A^i, and that $X_1^{ir} \cap \cap \mathrm{Bd}\, A^i = \mathrm{Bd}\, \Delta^{ir}$. Note that X_0^{ir} and X_1^{ir} differ only in $L_j^i \cup L_{j+1}^i$, and that $X_1^{ir} \subset \Delta^{ir} \cup L_j^i \cup L_{j+1}^i$. Note also that if l is either j or $j+1$, and t is either 1 or 2, then there exist a meridional disc Δ_{lt}^{ir} in L_{lt}^i, a neighborhood V_{lt}^{ir} of Δ_{lt}^{ir} in L_{lt}^i, and an arc λ_{lt}^{ir} on $\mathrm{Bd}\, L_{lt}^i$ such that (a) $(\Delta_{lt}^{ir}, V_{lt}^{ir}, \lambda_{lt}^{ir})$ is an admissible triple for L_{lt}^i and (b) $X_1^{ir} \cap L_{lt}^i = \Delta_{lt}^{ir} \cup \lambda_{lt}^{ir}$.

At this point we need a restricted type of index. If k is a positive integer, then a special index of stage k is a finite sequence $j_1 j_2 \ldots j_k$ of positive integers such that if $n = 1$, j_1 is j or $(j+1)$, and if $n = 2, 3, \ldots, k$, j_n is 1 or 2.

Suppose k is a positive integer. Let $\mathscr{I}_k$ denote the set of all indexes of stage k in the construction of M^i having first term either j or $j+1$. Let $\mathscr{A}_k$ denote the set of all indexes of $\mathscr{I}_k$ that are not special indexes (of stage k). Finally, let $\mathscr{B}_k$ denote the set of all special indexes of stage k.

Suppose now that p is a positive integer and that the construction in A^i relative to Δ^{ir} has been completed through step p. Then the following exist:

(1) For each positive integer q less than $p+1$ and each stage q index a for M^i, a polyhedral torus L_a^i and a chain $\mathscr{L}_a^i$ of solid tori in $\mathrm{Int}\, L_a^i$. These chains form the first p stages in the construction of an Antoine's necklace.

(2) A contractible 2-complex X_p^{ir} in A^i such that (a) $X_p^{ir} \cap \mathrm{Bd}\, A^i = \mathrm{Bd}\, \Delta^{ir}$, (b) if β is a special index of stage $(p+1)$, then there exist a

meridional disc Δ_β^{ir} of L_β^i, a neighborhood V_β^{ir} of Δ_β^{ir} in L_β^i, and an arc λ_β^{ir} on $\mathrm{Bd}\,L_\beta^i$ such that $(\Delta_\beta^{ir}, V_\beta^{ir}, \lambda_\beta^{ir})$ is an admissible triple for L_β^i and $X_p^{ir}\cap L_\beta^i = \Delta_\beta^{ir}\cup\lambda_\beta^{ir}$, and (c) if $\alpha\in\mathscr{A}_{p+1}$ and $t = 1, 2, \ldots, m_i$, $X_p^{ir}\cap L_{\alpha t}^i$ is a polygonal arc $\lambda_{\alpha t}^{ir}$ on $\mathrm{Bd}\,L_{\alpha t}^i$.

We shall now describe the $(p+1)$st step of the construction in A^i relative to Δ^{ir}. For each stage $(p+1)$ index α, there is a chain $\mathscr{L}_\alpha^i$ of m_i solid tori in L_α^i such that if α is, in fact, a special index of stage $(p+1)$, then Δ_α^{ir} is a special meridional disc in L_α^i relative to $\mathscr{L}_\alpha^i$ and 1, V_α^{ir} is a special neighborhood of Δ_α^{ir} relative to $\mathscr{L}_\alpha^i$, and λ_α^{ir} is a special arc for L_α^i, Δ_α^{ir}, and V_α^{ir}.

First suppose that β is a special index of stage $(p+1)$. By the results of Section 6, there exists a contractible 2-complex X_β^{ir} such that (1) $X_\beta^{ir}\subset L_\beta^i$ and $X_\beta^{ir}\cap\mathrm{Bd}\,L_\beta^i = (\mathrm{Bd}\,\Delta_\beta^{ir})\cup\lambda_\beta^{ir}$, (2) X_β^{ir} is based on Δ_β^{ir}, V_β^{ir}, and λ_β^{ir}, and (3) if $t = 3, 4, \ldots, m_i$, $X_\beta^{ir}\cap L_{\beta t}^i$ is an arc $\lambda_{\beta t}^{ir}$ on $\mathrm{Bd}\,L_{\beta t}^i$.

Now suppose that $\alpha\in\mathscr{A}_{p+1}$ and $t = 1, 2, \ldots, m_i$. Then there exist (1) a flange F_α^{ir} in L_α^i with edge λ_α^{ir} on $\mathrm{Bd}\,L_\alpha^i$ and (2) strips $R_{\alpha 1}^{ir}, R_{\alpha 2}^{ir}, \ldots, R_{\alpha m_i}^{ir}$ for L_α^i, $\mathscr{L}_\alpha^i$, and F_α^{ir}. Let E_α^{ir} denote

$$F_\alpha^{ir}\cup\Big(\bigcup_{s=1}^{m_i} R_{\alpha s}^{ir}\Big).$$

Let X_{p+1}^{ir} denote

$$\big[X_p^{ir}-\bigcup\{\Delta_\beta^{ir}\colon \beta\in\mathscr{B}_{p+1}\}\big]\cup\big[\bigcup\{X_\beta^{ir}\colon \beta\in\mathscr{B}_{p+1}\}\big]\cup\big[\bigcup\{E_\alpha^{ir}\colon \alpha\in\mathscr{A}_{p+1}\}\big].$$

It is easily verified that X_{p+1}^{ir} is a contractible 2-complex in A^i such that $X_{p+1}^{ir}\cap\mathrm{Bd}\,A^i = \mathrm{Bd}\,\Delta^{ir}$. Now for each special index β of stage $(p+2)$, let Δ_β^{ir} denote $X_{p+1}^{ir}\cap L_\beta^i$.

We may easily define, for each such index β, a neighborhood V_β^{ir} of Δ_β^{ir} in L_β^i such that $(\Delta_\beta^{ir}, V_\beta^{ir}, \lambda_\beta^{ir})$ is an admissible triple for L_β^i. Further, if $\alpha\in\mathscr{A}_{p+2}$ and $t = 1, 2, \ldots, m_i$, then $X_{p+1}^{ir}\cap L_{\alpha t}^i = \lambda_{\alpha t}^{ir}$.

Note that X_{p+1}^{ir} and X_p^{ir} differ only in

$$\bigcup\{L_\gamma^i\colon \gamma\in\mathscr{I}_{p+1}\},$$

and that

$$X_{p+1}^{ir}\subset X_p^{ir}\cup\big[\bigcup\{L_\gamma^i\colon \gamma\in\mathscr{I}_{p+1}\}\big].$$

Clearly the construction at the $(p+1)$st step yields the sets needed to repeat the process. Hence we obtain, by repetition of the construction, (1) a sequence $X_1^{ir}, X_2^{ir}, \ldots$, (2) for each positive integer n, a finite collection of chains of solid tori $\{\mathscr{L}_\alpha^i\colon \alpha\in\mathscr{I}_n\}$, and (3) for each positive integer n, a finite collection of mutually disjoint 3-cells, $\{V_\beta^{ir}\colon \beta\in\mathscr{B}_n\}$. Clearly we may carry out the construction such that as n increases without bound, $(\max\{\mathrm{diam}\,L_\alpha^{ir} = \alpha\in\mathscr{I}_n\})$ approaches 0. Recall that for each special index β considered, $V_\beta^{ir}\subset L_\beta^{ir}$.

For each non-negative integer n, let Ω_n^{ir} denote

$$X_n^{ir} \cup \left[\bigcup \{L_a^i\colon\ a \in \mathscr{I}_{n+1}\}\right].$$

LEMMA 7.1. *For each positive integer n, Ω_n^{ir} is a compact connected set and $\Omega_{n+1}^{ir} \subset \Omega_n^{ir}$.*

Proof. For each positive integer n, X_n^{ir} is contractible, and hence it is clear that Ω_n^{ir} is connected; it is easy to see that Ω_n^{ir} is compact.

It follows from the construction described above that if n is a positive integer, X_{n+1}^{ir} and X_n^{ir} differ only in $\bigcup \{L_a^i\colon\ a \in \mathscr{I}_n\}$. Since

$$\bigcup \{L_\mu^i\colon\ \mu \in \mathscr{I}_{n+2}\} \subset \bigcup \{L_\nu^i\colon\ \nu \in \mathscr{I}_{n+1}\},$$

it follows from the definitions of Ω_{n+1}^{ir} and Ω_n^{ir} that $\Omega_{n+1}^{ir} \subset \Omega_n^{ir}$.

Let X^{ir} denote $\bigcap_{n=0}^{\infty} \Omega_n^{ir}$. Since $X^{ir} \subset \Omega_0^{ir}$, $\Omega_0^{ir} = X_0^{ir} \cup L_j^i \cup L_{j+1}^i$, and $X_0^{ir} \cup L_j^i \cup L_{j+1}^i \subset V^{ir}$, it follows that $X^{ir} \subset V^{ir}$.

8. Discs in X^{ir}

By using some ideas mentioned in Section 6, we shall construct two useful discs in X^{ir}. One of these discs, D_j^i, lies in $X^{ir} \cap L_j^i$, and the other, D_{j+1}^i, lies in $X^{ir} \cap L_{j+1}^i$.

Suppose that l is j or $j+1$. Recall that, in the notation of Section 6,

$$E_l^{ir} = F_l^{ir} \cup \left[\bigcup \{R_{lt}^{ir}\colon\ t = 3, 4, \ldots, m_i\}\right] \cup S_{l1}^{ir} \cup F_{l1}^{ir} \cup S_{l2}^{ir} \cup F_{l2}^{ir}.$$

Then E_l^{ir} is a disc in L_l^i such that (1) $E_l^{ir} \cap \mathrm{Bd}\, L_l^i = \lambda_l^{ir}$, (2) if $t = 3, 4, \ldots, m_i$, $E_l^{ir} \cap L_{lt}^i = \lambda_{lt}^{ir}$, and (3) $E_l^{ir} \cap L_{l1}^i = F_{l1}^{ir}$ and $E_l^{ir} \cap L_{l2}^i = F_{l2}^{ir}$.

Suppose that β is a special index of stage p where $p > 1$. Then as in Section 6, if E_β^{ir} denotes

$$F_\beta^{ir} \cup \left[\bigcup \{R_{\beta t}^{ir}\colon\ t = 3, 4, \ldots, m_i\}\right] \cup S_{\beta 1}^{ir} \cup F_{\beta 1}^{ir} \cup S_{\beta 2}^{ir} \cup F_{\beta 2}^{ir},$$

E_β^{ir} is a disc in L_β^i such that (1) $E_\beta^{ir} \cap \mathrm{Bd}\, L_\beta^i = \lambda_\beta^{ir}$, (2) if $t = 3, 4, \ldots, m_i$, $F_\beta^{ir} \cap L_{\beta t}^i = \lambda_{\beta t}^{ir}$, and (3) $E_\beta^{ir} \cap L_{\beta 1}^i = F_{\beta 1}^{ir}$ and $E_{\beta 2}^{ir} \cap L_{\beta 2}^i = F_{\beta 2}^{ir}$.

Suppose that a is an index of stage p not a special index but with first term l. Then in Section 7, we let E_a^{ir} denote

$$F_a^{ir} \cup \left[\bigcup_{s=1}^{m_i} R_{as}^{ir}\right].$$

Then E_a^{ir} is a disc in L_a^{ir} such that (1) $E_a^{ir} \cap \mathrm{Bd}\, L_a^{ir} = \lambda_a^{ir}$ and (2) if $t = 1, 2, \ldots, m_i$, $E_a^{ir} \cap L_{at}^{ir} = \lambda_{at}^{ir}$.

Recall that in the construction described in Section 7, the chains of solid tori in L_l^i satisfy certain conditions described in Section 3, and accordingly they define an Antoine's necklace M_l^i in $\mathrm{Int}\, L_l^i$.

Notice that $X^{ir}-(M_j^i \cup M_{j+1}^i)$ is locally a 2-complex.

Let D_l^i denote

$$M_l^i \cup \left[\bigcup \{E_a^{ir}\colon\ a \text{ is an index for } M^i \text{ with first term } l\}\right].$$

It is easily verified that (1) D_l^i is a disc in L_l^i, (2) $D_l^i \cap \mathrm{Bd}\, L_l^i = \lambda_l^{ir}$, and (3) $M_l^i \subset \mathrm{Bd}\, D_l^i$.

If l is j or $j+1$, then we define $\hat{J}_l^i$ to be the arc

$$(\mathrm{Bd}\, D_l^{ir}) - (\mathrm{Int}\, \lambda_l^{ir}).$$

Define J_l^i to be an arc in $\mathrm{Int}\, \hat{J}_l^i$ containing M_l^i; note that $J_l^i \subset \mathrm{Int}\, L_l^i$.

It follows from Lemma 4.1 that if l is j or $j+1$, then J_l^i has winding number 1 in L_l^i.

9. X^{ir} is an AR

In this section we shall prove that the set X^{ir} constructed in Section 7 is an AR. We retain the notation and terminology of Sections 6–8.

PROPOSITION 9.1. *X^{ir} has dimension 2.*

Proof. This follows from the facts that (1) $X^{ir}-(M_j^i \cup M_{j+1}^i)$ is locally a 2-complex and (2) $\dim(M_j^i \cup M_{j+1}^i) = 0$.

For each positive integer n, let U_n^{ir} denote

$$X^{ir} \cup \left[\bigcup \{V_\beta^{ir}\colon\ \beta \in \mathscr{B}_{n+1}\}\right].$$

Clearly for each positive integer n, U_n^{ir} is compact. Further if β is a special index of stage $(n+1)$ and l is either 1 or 2, $V_{\beta l}^{ir} \subset V_\beta^{ir}$. It then follows that for each positive integer n, $U_{n+1}^{ir} \subset U_n^{ir}$. Since, as n increases without bound, $(\max\{\mathrm{diam}\, V_\beta^{ir}\colon\ \beta \in \mathscr{B}_{n+1}\})$ approaches 0, it follows that

$$X^{ir} = \bigcap_{n=1}^{\infty} U_n^{ir}.$$

LEMMA 9.2. *For each positive integer n, U_n^{ir} is an* AR.

Proof. First notice that U_n^{ir} may be obtained from X_n^{ir} in the following way: first define $U_n^{ir}(1)$ to be

$$X^{ir} \cup \left[\bigcup \{V_\beta^{ir}\colon\ \beta \in \mathscr{B}_{n+1}\}\right].$$

By the construction of X_n^{ir}, if β is a special index of stage $(n+1)$, $X_n^{ir} \cap V_\beta^{ir}$ is a disc spanning V_β^{ir}. It follows, since X_n^{ir} is contractible, that $U_n^{ir}(1)$ is contractible. Clearly $U_n^{ir}(1)$ is a finite 3-complex, and hence (see [2]) is an AR.

It follows from the construction of X^{ir} that U_n^{ir} is obtained from $U_n^{ir}(1)$ by attaching to $U_n^{ir}(1)$ finitely many mutually disjoint discs, each of which intersects $U_n^{ir}(1)$ exactly in an arc on its boundary. It follows from [2] that U_n^{ir} is an AR.

LEMMA 9.3. *If n is any positive integer and β is a special index of stage n, then $V_\beta^{ir} \cap X_n^{ir}$ is a contractible 2-complex.*

Proof. This follows from the construction of X_n^{ir}.

LEMMA 9.4. *If n is a positive integer and β is a special index of stage n, then $V_\beta^{ir} \cap U_n^{ir}$ is an* AR.

Proof. By Lemma 9.3, $V_\beta^{ir} \cap X_n^{ir}$ is an AR. Now $V_\beta^{ir} \cap U_n^{ir}$ is obtained from $V_\beta^{ir} \cap X_n^{ir}$ as follows: First, attach to $V_\beta^{ir} \cap X_n^{ir}$ the two 3-cells $V_{\beta 1}^{ir}$ and $V_{\beta 2}^{ir}$. There results a contractible 3-complex which is then an AR. Second, attach to this contractible 3-complex certain discs that lie in $X^{ir} \cap V_\beta^{ir}$. These discs intersect the 3-complex exactly in arcs on their boundaries. There results $V_\beta^{ir} \cap U_n^{ir}$, and hence $V_\beta^{ir} \cap U_n^{ir}$ is an AR.

Recall that in Section 8 we constructed a certain arc J_j^i in L_j^i and a certain arc J_{j+1}^i in L_{j+1}^i.

LEMMA 9.5. *If l is j or $j+1$, p is a point of J_l^i, and U is any neighborhood of p in Q, then for some positive integer n, there is a closed neighborhood V of p in U_n^{ir} such that $V \subset U$, V is an* AR, *and for any special index β of stage n such that V_β^{ir} intersects V, $V_\beta^{ir} \subset V$.*

Proof. We shall consider three cases.

Case 1: p is a point of $J_l^i - M_l^i$. In this case, X^{ir} is, by construction, locally a 2-complex at p. Let n be a positive integer such that for any special index β of stage $(n+1)$, $p \notin V_\beta^{ir}$. Then it is easy to construct a closed neighborhood V of p in U_n^{ir} as required.

Now let Z^{ir} denote $\bigcap_{s=1}^{\infty} [\bigcup \{V_\beta^{ir}\colon \beta \in \mathscr{B}_s\}]$. Z^{ir} is a Cantor set, and $Z^{ir} \subset M_j^i \cup M_{j+1}^i$.

Case 2: p is a point of $M_l^i - Z^{ir}$. In this case, there exists a sequence $j_1, j_2, j_3, \ldots$ of positive integers such that $j_1 = l$, each of $j_2, j_3, j_4, \ldots$ is less than or equal to m_i, the sequence $j_1, j_2, j_3, \ldots$ does not terminate in 1's and 2's, and $p \in \bigcap_{s=1}^{\infty} L_{j_1 j_2 \ldots j_s}$.

Then there is a positive integer n such that (1) $L_{j_1 j_2 \ldots j_n j_{n+1}} \subset U$ and (2) if β is any special index of stage $(n+1)$, p is not in V_β^{ir}. The fact that $p \notin Z^{ir}$ yields (2) of the preceding statement. Let a_{n+1} denote $j_1 j_2 \ldots j_n j_{n+1}$.

By the construction of X_n^{ir}, $L_{a_{n+1}} \cap X_n^{ir}$ is an arc on $\operatorname{Bd} L_{a_{n+1}}$. In the construction of X^{ir} from X_n^{ir}, we add a disc in $L_{a_{n+1}}$ to $L_{a_{n+1}} \cap X_n^{ir}$. It follows that $L_{a_{n+1}} \cap X^{ir}$ is a disc, hence an AR, and is a neighborhood of p in U_n^{ir} as required.

Case 3: p is a point of Z^{ir}. In this case, there exist a positive integer n and a special index β of stage n such that $p \in \operatorname{Int} V_\beta^{ir}$ and $V_\beta^{ir} \subset U$. By Lemma 9.4, $V_\beta^{ir} \cap U_n^{ir}$ is an AR. Since $p \in U_n^{ir}$, then $U_n^{ir} \cap V_\beta^{ir}$ is a closed neighborhood of p as required.

LEMMA 9.6. *If $k = 0$, 1, or 2, X^{ir} is $k-C$.*

Proof. Suppose that $f\colon S^k \to X^{ir}$ is a map. By Lemma 9.2, U_1^{ir} is an AR, and since $X^{ir} \subset U_1^{ir}$, there is an extension $F\colon B^{k+1} \to U_1^{ir}$ of f to B^{k+1}.

Now we shall construct a retraction $R_1\colon U_1^{ir} \to U_2^{ir}$. If $j_1 = j$ or $j+1$ and $j_2 = 1$ or 2, then by Lemma 9.4, $V_{j_1j_2}^{ir} \cap U_2^{ir}$ is an AR, and hence there is a retraction $r_{j_1j_2}\colon V_{j_1j_2}^{ir} \to V_{j_1j_2}^{ir} \cap U_2^{ir}$. Let $R_1 = U_1^{ir} \to U_2^{ir}$ be the map such that if $j_1 = j$ or $j+1$, and $j_2 = 1$ or 2, then $R_1|V_{j_1j_2}^{ir} = r_{j_1j_2}$, and elsewhere on U_1^{ir}, R_1 is the identity. Then R_1 is a retraction of U_1^{ir} onto U_2^{ir}.

Let $G_1 = R_1F$; then $G_1\colon B^{j+1} \to U_2^{ir}$, and since $X^{ir} \subset U_2^{ir}$, it follows by the construction of R_1 that G_1 extends f.

By applying Lemma 9.4 to U_3^{ir}, we may construct a retraction $R_2\colon U_2^{ir} \to U_3^{ir}$ such that if $x \in \big(U_2^{ir} - \bigcup\{V_\beta^{ir}\colon \beta \in \mathscr{B}_3\}\big)$, then $R_2(x) = x$, and for each special index β of stage 3, $R_2[V_\beta^{ir}] \subset V_\beta^{ir}$. Let $G_2 = R_2G_1$. Then $G_2\colon B^{k+1} \to U_3^{ir}$ and since $X^{ir} \subset U_3^{ir}$, G_2 extends f. Further, it follows that

$$||G_1, G_2|| < (\max\{\operatorname{diam} V_\nu^{ir}\colon \nu \in \mathscr{B}_2\}).$$

Suppose now that q is a positive integer, maps $G_1, G_2, \ldots, G_q$ have been defined, and if $s = 1, 2, \ldots, q$, then $G_s\colon B^{k+1} \to U_{s+1}^{ir}$, G_s extends f, and if $t = s+1, s+2, \ldots, q$,

$$||G_s, G_t|| < (\max\{\operatorname{diam} V_\mu^{ir}\colon \mu \in \mathscr{B}_{s+1}\}).$$

By applying Lemma 9.4 to U_{q+1}^{ir}, we may construct a retraction $R_{q+1}\colon U_{q+1}^{ir} \to U_{q+2}^{ir}$ such that if σ is a special index of stage $q+2$, then $R_{q+1}[V_\sigma^{ir}] \subset V_\sigma^{ir}$, and if $x \in \big(U_{q+1}^{ir} - \bigcup\{V_\sigma^{ir}\colon \sigma \in \mathscr{B}_{q+2}\}\big)$, then $R_{q+1}(x) = x$. Let $G_{q+1} = R_{q+1}G_q$. Then G_{q+1} is a map from B^{k+1} to U_{q+2}^{ir}, and G_{q+1} extends f. Further it follows from the construction that if $s = 1, 2, \ldots, q$, then

$$||G_{q+1}, G_s|| < (\max\{\operatorname{diam} V_\sigma^{ir}\colon \sigma \in \mathscr{B}_{s+1}\}).$$

In this way, we must construct a sequence $G_1, G_2, \ldots$ of extensions of f, each having domain B^{q+1} and each into Q. Now $(\max\{\operatorname{diam} V_\tau^{ir}\colon \tau \in \mathscr{B}_t\})$ approaches 0 as t increases without bound. This implies that the sequence $G_1, G_2, \ldots$ converges uniformly to a map G. Since $X^{ir} = \bigcap_{n+1}^{\infty} U_n^{ir}$, it follows that $G\colon B^{k+1} \to X^{ir}$. Since each of $G_1, G_2, \ldots$ extends f, G extends f.

Hence X^{ir} is $k-C$.

LEMMA 9.7. *If $k = 0, 1$, or 2, X^{ir} is $k-LC$.*

Proof. Suppose $p \in X^{ir}$. If $p \notin J_j^i \cup J_{j+1}^i$, then by construction, X^{ir} is locally a 2-complex at p, and hence is LC at p.

Suppose $l = j$ or $j+1$, and $p \in J_l^i$. If U is any neighborhood of p in X^{ir}, then there is a neighborhood $\hat{U}$ of p in Q such that $U = \hat{U} \cap X^{ir}$. Now by Lemma 9.5, there is a positive integer n and a closed neighborhood V of p in U_n^{ir} such that $V \subset U$, V is an AR, and for any special index β of stage n such that V_β^{ir} intersects V, $V_\beta^{ir} \subset V$.

Suppose that $k = 0, 1$, or 2 and $f\colon s^k \to V$ is a map. By an argument similar to that given for Lemma 9.6 and making use of the special properties of V relative to the cells V_β^{ir} for special indexes β of stage n, we may prove that f has a continuous extension $G = B^{k+1} \to V$. Thus for each map $f\colon S^k \to V$, there is a continuous extension $G = B^{k+1} \to U$ of f. Thus if $k = 0, 1$, or 2, X^{ir} is $k-LC$ at p.

LEMMA 9.8. *X^{ir} is a 2-dimensional* AR, *$X^{ir} \subset V^{ir}$ (and hence $X^{ir} \subset A^i$), and $X^{ir} \cap \mathrm{Bd}\, A^i = \mathrm{Bd}\, \Delta^{ir}$.*

Proof. The fact that X^{ir} is a 2-dimensional AR follows from Lemmas 9.1, 9.6, and 9.7, and [2]. The remaining facts follow from the construction of X^{ir}.

Recall that at the beginning of Section 7, we supposed that i, m_i, j, and r are positive integers with $m_i \geqslant 8$ and $j \leqslant m_i$, A^i as a polyhedral solid torus in $\mathrm{Int}\, Q$, $\mathscr{L}^i$ is a chain of solid tori $\{L_1^i, L_2^i, \ldots, L_{m_i}^i\}$ in $\mathrm{Int}\, A^i$, Δ^{ir} is a special meridional disc of A^i relative to $\mathscr{L}^i$ and j, and V^{ir} is a special neighborhood of Δ^{ir} in A^i relative to $\mathscr{L}^i$.

We then constructed the 2-dimensional AR X^{ir} in V^{ir}. We shall say that X^{ir} is a 2-dimensional AR *replacing Δ^{ir} in A^i and V^{ir}.*

We also constructed two Antoine's necklaces, M_j^i in L_j^i and M_{j+1}^i in L_{j+1}^i, and arcs J_j^i in L_j^i containing M_j^i and J_{j+1}^i in L_{j+1}^i containing M_{j+1}^i. The arcs J_j^i and J_{j+1}^i are contained in X^{ir}. We shall say that J_j^i and J_{j+1}^i are the *arcs associated with* X^{ir}. Recall that by construction, $X^{ir} - (J_j^i \cup J_{j+1}^i)$ is locally a 2-complex.

10. Construction of the decomposition M

In this section, we shall describe an upper semicontinuous decomposition M of the 3-cell Q into arcs and singletons such that the associated decomposition space Q/M is a Bing–Borsuk retract and contains a 2-dimensional absolute retract Y. We shall simultaneously construct M and the inverse image, under projection, of Y.

Recall that Q is a unit solid 3-dimensional ball in E^3 and that $S = \mathrm{Bd}\, Q$. By [1], there is a null sequence $K_1, K_2, \ldots$ of chords of Q, dense on S.

Let Q_0 be a polyhedral 3-cell in $\mathrm{Int}\, Q$ such that Q_0 is disjoint from each of $K_1, K_2, \ldots$ Such a 3-cell exists since $K_1, K_2, \ldots$ is a null sequence.

Let X_0 be a polyhedral disc in Q_0. We shall "modify" X_0 as we construct the decomposition M, obtaining a sequence $X_1, X_2, \ldots$ of sets which converge to a 2-dimensional absolute retract X in Q_0. It will follow from the construction of X that each element of M which intersects X lies in X. From these facts and [2] it follows that the image Y of X under projection φ is an AR. It is easily shown that Y has dimension 2. Thus the Bing–Borsuk retract Q/M contains the 2-dimensional absolute retract Y.

By [1], there is a sequence $C'_1, C'_2, \ldots$ of polygonal simple closed curves in Int$Q$ such that (1) if C is any simple closed curve in Int$Q$, then for some positive integer ν, C and C'_ν are (homologically) linked (relative to the integers), and (2) each simple closed curve in the sequence appears nfinitely often in the sequence.

Construction of X_1.

Adjust C'_1 to give a polygon C_1 in $[(\mathrm{Int}Q) - \bigcup_{\nu=1}^{\infty} K_\nu]$ such that C_1 and iX_0 are in general position, and in the adjustment, no point moves more than 1. Then $X_0 \cap C_1$ is finite, and C_1 and BdX_0 are disjoint.

Let A^1 be a polyhedral solid torus in Int$Q$ having C_1 as a core and such that (1) the inner radius of A^1 is less than 1, (2) if C_1 misses X_0, so does A^1, and (3) if C_1 intersects X_0, then (a) each component of $C_1 \cap X_0$ is a polyhedral meridional disc of A^1 lying in IntX_0 and (b) the number of components of $A^1 \cap X_0$ is the same as the number of points of $C_1 \cap X_0$.

Suppose first that C_1 misses X_0. Then A^1 misses X_0. By [1], and Lemma 3.1, there exist a positive integer m_1 such that $m_1 \geqslant 8$ and a wreath $\{J^1_1, J^1_2, \ldots, J^1_{m_1}\}$ substituting for A^1 such that each of $J^1_1, J^1_2, \ldots, J^1_{m_1}$ has diameter less than 1 and is disjoint from $\bigcup_{\nu=1}^{\infty} K_\nu$.

In this case, let $X_1 = X_0$. Let T_1 be a triangulation of X_1 of mesh less than 1. For each 2-simplex σ of T_1, let W^1_σ be a polyhedral 3-cell in Q_0 such that σ spans W^1_σ, $(\mathrm{diam}\, W^1_\sigma) < 2\,(\mathrm{diam}\,\sigma)$, and W^1_σ is disjoint from $\bigcup_{i=1}^{m_1} J^1_i$; further, it is to be true that if ϱ and τ are distinct 2-simplexes of T_1, $W^1_\varrho \cap W^1_\tau = \varrho \cap \tau$.

Suppose now that C_1 intersects X_0. Let p_1 be the number of points of $C_1 \cap X_0$, and let $\Delta^{11}, \Delta^{12}, \ldots, \Delta^{1p_1}$ denote the components of $X_0 \cap A^1$. Let $V^{11}, V^{12}, \ldots, V^{1p_1}$ be mutually disjoint polyhedral 3-cells in A^1 such that if $i = 1, 2, \ldots, p_1$, V^{1i} is a neighborhood of Δ^{1i} in A^1, $V^{1i} \cap \mathrm{Bd}\, A^1$ is an annulus, Bd Δ^{1i} is a centerline of $V^{1i} \cap \mathrm{Bd}\, A^1$, and $V^{1i} \subset Q_0$.

There exist a positive integer m_1 such that $m_1 \geqslant 8$ and a chain $\mathscr{L}^1$ of solid tori $\{L^1_1, L^1_2, \ldots, L^1_{m_1}\}$ in IntA^1 such that (a) each link of $\mathscr{L}^1$ has diameter less than 1, and (b) if $r = 1, 2, \ldots, p_1$, then there exists a positive integer j_{1r} such that Δ^{ir} is a special meridional disc in A^1 relative to $\mathscr{L}^1$ and j_{1r}, and V^{1r} is a special neighborhood of Δ^{1r} relative to $\mathscr{L}^1$. Let $\mathscr{R}_1 = \{j_{1r}, j_{1r}+1 \colon r = 1, 2, \ldots, p_1\}$.

If $r = 1, 2, \ldots, p_1$, let X^{1r} be a 2-dimensional AR replacing Δ^{1r} in A^1 and V^{1r}, with $X^{1r} \subset V^{1r}$ and $X^{1r} \cap \mathrm{Bd}\, A^1 = \mathrm{Bd}\, \Delta^{1r}$.

Then let

$$X_1 = \Big(X_0 - \bigcup_{r=1}^{p_1} \Delta^{1r}\Big) \cup \Big(\bigcup_{r=1}^{p_1} X^{1r}\Big).$$

It is easy to see that X_1 is a 2-dimensional AR in Q_0.

If $r = 1, 2, \ldots, p_1$, let $J^1_{j_{1r}}$ and $J^1_{(j_{1r}+1)}$ denote the two arcs in $X^{1r} \cap L^1_{j_{1r}}$ and $X^{1r} \cap L^1_{(j_{1r}+1)}$, respectively, associated with X^{1r}.

Suppose that i is a positive integer such that $i \leqslant m_1$ and $i \notin \mathscr{R}_1$. Then X_1 is also disjoint from L^1_i, and we may construct (using the integer m_1) an Antoine's necklace M^1_i in $\operatorname{Int} L^1_i$ as in Section 3; we may assume that M^1_i is disjoint from $\bigcup_{\nu=1}^{\infty} K_\nu$. Let J^1_i be an arc in $\operatorname{Int} L^1_i$ containing M^1_i, and disjoint from $\bigcup_{\nu=1}^{\infty} K_\nu$, and such that $\omega(J^1_i, L^1_i) = 1$. Note that J^1_i is disjoint from X.

Clearly, in this case, $\{J^1_1, J^1_2, \ldots, J^1_{m_1}\}$ is a wreath substituting for A^1 such that each link has diameter less than 1.

In either case, let $\mathscr{J}_1$ denote $\{J_1, J_2, \ldots, J^1_{m_1}\}$. Notice that if an arc of $\mathscr{J}_1$ intersects X_1, it is contained in X_1.

Recall that, by construction, $(X_1 - \bigcup_{r=1}^{m_1} J^1_r)$ is locally a 2-complex. Let T_1 be a triangulation of $(X_1 - \bigcup_{r=1}^{m_1} J^1_r)$ of mesh less than 1 and such that for any positive number ε, there is a neighborhood N_ε of $X_1 \cap \left(\bigcup_{r=1}^{m_1} J^1_r\right)$ such that any simplex of T_1 intersecting N_ε has diameter less than ε.

For each 2-simplex σ of T_1, let W^1_σ be polyhedral 3-cell in Q_0 such that σ spans W^1_σ, $(\operatorname{diam} W^1_\sigma) < 2(\operatorname{diam}\sigma)$, and W^1_σ is disjoint from $\bigcup_{i=1}^{m_1} J^1_i$; further, it is to be true that if ϱ and τ are distinct 2-simplexes of T_1, then $W^1_\varrho \cap W^1_\tau = \varrho \cap \tau$.

It follows from the construction of the sets W^1_σ that in *either* case, if σ is any 2-simplex of T_1, $\operatorname{Int} W^1_\sigma$ is disjoint from both $|T^1_1|$ and $\bigcup_{i=1}^{m_1} J^1_i$.

Construction of X_2.

Adjust C'_2 to obtain a polygon C_2 in $\operatorname{Int} Q$ such that C_2 misses C_1 and $\bigcup_{i=1}^{m_1} J^1_i$, C_2 and the (possibly noncompact) polyhedron $X_1 - \bigcup_{i=1}^{m_1} J^1_i$ (regarded as the carrier of T_1) are in general position, and in this adjustment, no point is moved by more than $1/2$. Then $C_2 \cap X_1$ is finite, and for each point x of $C_2 \cap X_1$, there is a 2-simplex σ of T_1 such that $x \in \operatorname{Int}\sigma$.

Let A^2 be a polyhedral solid torus in $\operatorname{Int} Q$ having C_2 as a core and such that (1) the inner radius of A^2 is less than $1/2$, (2) if C_1 misses X_1, then so does A^2, and (3) if C_2 intersects X_1, then (a) each component of $A^2 \cap X_1$ is a meridional disc of A^2, (b) the number of components of $A^2 \cap X_1$ is equal to the number of points of $C_2 \cap X_1$, and (c) each component of $A^2 \cap X_1$ is contained, for some 2-simplex σ of T_1, in $\operatorname{Int}\sigma$.

Suppose that C_2 misses X_1. Then A^2 misses X_1. By [1], and Lemma 3.1, there exist a positive integer m_2 and a wreath $\{J^2_1, J^2_2, \ldots, J^2_{m_2}\}$ substituting for A^2 such that each of $J_1^2, J_2^2, \ldots, J_{m_2}^2$ has diameter less than $1/2$, and is disjoint from $\bigcup_{\nu=1}^{\infty} K_\nu$.

In this case let $X_2 = X_1$. Let T_2 be a triangulation of $(X_2 - \bigcup_{i=1}^{m_1} J_i^1)$ of mesh less than $1/2$ and such that T_2 is a subdivision of T_1. For each 2-simplex σ of T_2, let W_σ^2 be a polyhedral 3-cell such that σ spans W_σ^2, $(\operatorname{diam} W_\sigma^2) < 2(\operatorname{diam}\sigma)$, W_σ^2 is disjoint from $\left(\bigcup_{t=1}^{2}\bigcup_{i=1}^{m_t} J_i^t\right)$, and if $\tau \in T_1$ and $\sigma \subset \tau$, then $W_\sigma^2 \subset W_\tau^1$. It is also to be true that if ϱ and τ are distinct 2-simplexes of T_2, then $W_\varrho^2 \cap W_\tau^2 = \varrho \cap \tau$.

Suppose now that C_2 intersects X_1. Let p_2 be the number of points of $C_2 \cap X_1$, and let $\Delta^{21}, \Delta^{22}, \ldots, \Delta^{2p_2}$ denote the components of $A^2 \cap X_1$. Let $V^{21}, V^{22}, \ldots, V^{2p_2}$ be mutually disjoint polyhedral 3-cells in A^2 such that if $i = 1, 2, \ldots, p_2$, V^{2i} is a neighborhood of Δ^{2i} in A^2, $V^{2i} \cap \operatorname{Bd} A^2$ is an annulus, $\operatorname{Bd}\Delta^{2i}$ is a centerline of $V^{2i} \cap \operatorname{Bd} A^2$, and if τ_{2i} is the 2-simplex of T_1 containing Δ^{2i}, then $V^{2i} \subset W_{\tau_{2i}}^1$. It follows that if $i = 1, 2, \ldots, p_2$, $V^{2i} \subset Q_0$ and has diameter less than 1.

There exist a positive integer m_2 such that $m_2 \geqslant 8$ and a chain $\mathscr{L}^2$ of solid tori $\{L_1^2, L_2^2, \ldots, L_{m_2}^2\}$ in $\operatorname{Int} A^2$ such that (a) each link of $\mathscr{L}^2$ has diameter less than $1/2$, (b) if $r = 1, 2, \ldots, p_2$, then there exists a positive integer j_{2r} such that Δ^{2r} is a special meridional disc in A^2 relative to $\mathscr{L}^2$ and j_{2r}, and V^{2r} is a special neighborhood of Δ^{2r} relative to $\mathscr{L}^2$. Let $\mathscr{R}_2 = \{j_{2r}, j_{2r}+1\colon r = 1, 2, \ldots, p_2\}$.

If $r = 1, 2, \ldots, p_2$, let X^{2r} be a 2-dimensional AR replacing Δ^{2r} in A^2 and V^{2r}, with $X^{2r} \subset V^{2r}$ and $X^{2r} \cap \operatorname{Bd} A^2 = \operatorname{Bd}\Delta^{2r}$.

Let

$$X_2 = \left(X_1 - \bigcup_{r=1}^{p_2} \Delta^{2r}\right) \cup \left(\bigcup_{r=1}^{p_2} X^{2r}\right).$$

It is easy to see that X_2 is a 2-dimensional AR in Q_0.

If $r = 1, 2, \ldots, p_2$, let $J_{j_{2r}}^2$ and $J_{(j_{2r}+1)}^2$ denote the two arcs in $L_{j_{2r}}^2 \cap X^{2r}$ and $L_{(j_{2r}+1)}^2 \cap X^{2r}$, respectively, associated with X^{2r}.

Suppose that $i = 1, 2, \ldots, m_2$, and $i \notin \mathscr{R}_2$. Then X_2 is disjoint from L_i^2, and we may construct (using the integer m_2) an Antoine's necklace M_i^2 in $\operatorname{Int} L_i^2$ as in Section 3. Let J_i^2 be an arc in $\operatorname{Int} L_i^2$ containing M_i^2. Note that J_i^2 is disjoint from X_2.

In this case, $\{J_1^2, J_2^2, \ldots, J_{m_2}^2\}$ is a wreath substituting for A^2 such that each link has diameter less than $1/2$.

In either case, let $\mathscr{J}_2$ denote $\mathscr{J}_1 \cup \{J_1^2, J_2^2, \ldots, J_{m_2}^2\}$. Notice that if an arc of $\mathscr{J}_2$ intersects X_2, it lies in X_2.

By construction, $\left(X_2 - \bigcup_{t=1}^{2}\bigcup_{r=1}^{m_t} J_r^t\right)$ is locally a 2-complex. Let T_2 be a triangulation of $\left(X_2-\bigcup_{t=1}^{2}\bigcup_{r=1}^{m_t}J_r^t\right)$ such that T_2 has mesh less than $1/2$, T_2 is a subdivision of T_1, $\left(\bigcup_{r=1}^{p_2}\operatorname{Bd}\Delta^{2r}\right)$ lies in $|T_2^1|$, and for any positive number ε, there is a neighborhood N_ε of $X_2\cap\left(\bigcup_{t=1}^{2}\bigcup_{r=1}^{m_t}J_r^t\right)$ such that any simplex of T_2 that intersects N_ε has diameter less than ε.

For each 2-simplex σ of T_2, let W_σ^2 be a polyhedral 3-cell such that σ spans W_σ^2, $(\operatorname{diam} W_\sigma^2) < 2(\operatorname{diam}\sigma)$, and W_σ^2 is disjoint from $\left(\bigcup_{t=1}^{2}\bigcup_{r=1}^{m_t}J_r^t\right)$; further, if ϱ and τ are distinct 2-simplexes of T_2, then $W_\varrho^2\cap W_\tau^2 = \varrho\cap\tau$. Also, it is to be true that if σ is a 2-simplex of T_2 then (1) if there is a 2-simplex τ of T_1 such that $\sigma\subset\tau$, then $W_\sigma^2\subset W_\tau^1$, and (2) if there is some positive integer r such that $r\leqslant p_2$ and $\sigma\subset X^{2r}$, and τ is the 2-simplex of T_1 such that $\sigma\subset\tau$, then $W_\sigma^2\subset W_\tau^1$. From the construction of X_2, it follows that for each 2-simplex σ of T_2, either (1) there is a 2-simplex τ of T_1 such that $\sigma\subset\tau$ or (2) $\sigma\subset\left(\bigcup_{r=1}^{p_2}X^{2r}\right)$; recall that $\left(\bigcup_{r=1}^{p_2}\operatorname{Bd}\Delta^{2r}\right)$ lies in $|T_2^1|$.

It follows from the construction that in *either* case, if σ is any 2-simplex of T_2, $\operatorname{Int} W_\sigma^2$ is disjoint from both $|T_2^1|$ and $\left(\bigcup_{t=1}^{2}\bigcup_{r=1}^{m_t}J_r^t\right)$.

Construction of X_{k+1}.

Suppose that k is a positive integer greater than 1, and that we have constructed the following: (1) Polyhedral solid tori $A^1, A^2, \ldots, A^k$ with cores $C_1, C_2, \ldots, C_k$, respectively. (2) Wreaths $\{J_1^1, J_2^1, \ldots, J_{m_1}^1\}$, $\{J_1^2, J_2^2, \ldots, J_{m_2}^2\}, \ldots, \{J_1^k, J_2^k, \ldots, J_{m_k}^k\}$ substituting for $A^1, A^2, \ldots, A^k$, respectively, such that if $t = 1, 2, \ldots, k$, and $i = 1, 2, \ldots, m_t$, then $(\operatorname{diam} J_i^t) < 1/t$ and J_i^t is disjoint from $\bigcup_{\nu=1}^{\infty}K_\nu$. Let

$$\mathscr{J}_k = \{J_i^t\colon t = 1, 2, \ldots, k, \textit{ and } i = 1, 2, \ldots, m_t\}.$$

(3) An absolute retract X_k such that (a) if $t = 1, 2, \ldots, k$, and $i = 1, 2, \ldots, m_t$, and J_i^t intersects X_k, then J_i^t lies in X_k, and (b) $\left(X_k-\bigcup_{t=1}^{k}\bigcup_{i=1}^{m_t}J_i^t\right)$ is locally a 2-complex. (4) A triangulation T_k of $\left(X_k-\bigcup_{t=1}^{k}\bigcup_{i=1}^{m_t}J_i^t\right)$ such that $(\operatorname{mesh} T_k) < 1/k$, and for each positive number ε, there is a nieghborhood N_ε of $X\cap\cap\left(\bigcup_{t=1}^{k}\bigcup_{i=1}^{m_t}J_i^t\right)$ such that any simplex of T_k that intersects N_ε has diameter less than ε. (5) For each 2-simplex σ of T_k, a polyhedral 3-cell W_σ^k in Q_0 such that (a) σ spans W_σ^k, (b) $(\operatorname{diam} W_\sigma^k) < 2(\operatorname{diam}\sigma)$, and (c) W_σ^2 is disjoint from $\left(\bigcup_{t=1}^{k}\bigcup_{i=1}^{m_t}J_i^t\right)$. Further, if ϱ and τ are distinct 2-simplexes of T_k, then $W_\varrho^k\cap W_\tau^k = \varrho\cap\tau$.

Adjust C'_{k+1} to obtain a polygon C_{k+1} in Int$Q$ such that C_{k+1} misses each of $C_1, C_2, \ldots, C_k$, and $\left(\bigcup_{t=1}^{k}\bigcup_{i=1}^{m_t} J_i^t\right)$, C_{k+1} and the (possibly noncompact) polyhedron $\left(X_k - \bigcup_{t=1}^{k}\bigcup_{i=1}^{m_k} J_i^t\right)$ (regarded as the carrier of T_k) are in general position, and in this adjustment, no point is moved by more than $1/(k+1)$. Then $C_{k+1} \cap X_k$ is finite, and if $x \in C_{k+1} \cap X_k$, there is a simplex σ of T_k such that $x \in \operatorname{Int}\sigma$.

Let A^{k+1} be a polyhedral solid torus in Int$Q$ having C_{k+1} as a core and such that (1) the inner radius of A^{k+1} is less than $1/(k+1)$, (2) if C_{k+1} misses X_k, so does A^{k+1}, and (3) if C_{k+1} intersects X_k, then (a) each component of $A^{k+1} \cap X_k$ is a meridional disc of A^{k+1}, (b) the number of components of $A^{k+1} \cap X_k$ is equal to the number of points of $C_{k+1} \cap X_k$, and (c) each component of $A^{k+1} \cap X_k$ is contained, for some 2-simplex σ of T_k, in $\operatorname{Int}\sigma$.

Suppose that C_{k+1} misses X_k. Then A^{k+1} misses X_k. There exist a positive integer m_{k+1} with $m_{k+1} \geqslant 8$ and a wreath $\{J_1^{k+1}, J_2^{k+1}, \ldots, J_{m_{k+1}}^{k+1}\}$ substituting for A^{k+1} such that each of $J_1^{k+1}, J_2^{k+1}, \ldots, J_{m_{k+1}}^{k+1}$ has diameter less that $1/(k+1)$, is disjoint from $\bigcup_{\nu=1}^{\infty} K_\nu$, and such that if $i = 1, 2, \ldots, m_{k+1}$, $\omega(J_i^{k+1}, L_i^{k+1}) = 1$.

In this case let $X_{k+1} = X_k$. Let T_{k+1} be a subdivision of T_k of mesh less than $1/(k+1)$. For each 2-simplex σ of T_{k+1}, let W_σ^{k+1} be a polyhedral 3-cell such that σ spans W_σ^{k+1}, $(\operatorname{diam} W_\sigma^{k+1}) < 2(\operatorname{diam}\sigma)$, W_σ^{k+1} is disjoint from $\left(\bigcup_{t=1}^{k+1}\bigcup_{i=1}^{m_t} J_i^t\right)$, and if $\tau \in T_k$ and $\sigma \subset \tau$, then $W_\sigma^{k+1} \subset W_\tau^k$. It is also to be true that if ϱ and τ are distinct 2-simplexes of T_{k+1}, then $W_\varrho^{k+1} \cap W_\tau^{k+1} = \varrho \cap \tau$.

Suppose now that C_{k+1} intersects X_k. Let p_{k+1} be the number of points of $C_{k+1} \cap X_k$, and let $\Delta^{k+1,1}, \Delta^{k+1,2}, \ldots, \Delta^{k+1,p_{k+1}}$ denote the components of $A^{k+1} \cap X_1$. Let $V^{k+1,1}, V^{k+1,2}, \ldots, V^{k+1,p_{k+1}}$ be mutually disjoint polyhedral 3-cells in A^{k+1} such that if $i = 1, 2, \ldots, p_{k+1}$, $V^{k+1,i}$ is a neighborhood of $\Delta^{k+1,i}$ in A^{k+1}, $V^{k+1,i} \cap \operatorname{Bd} A^{k+1}$ is an annulus, $\operatorname{Bd}\Delta^{k+1,i}$ is a centerline of $V^{k+1,i} \cap \operatorname{Bd} A^{k+1}$, and if $\tau_{k+1,i}$ is the 2-simplex of T_k containing $\Delta^{k+1,i}$, then $V^{k+1} \subset W_{\tau_{k+1,i}}^k$. It follows that if $i = 1, 2, \ldots, m_{k+1}$, $V^{k+1,i} \subset Q_0$ and has diameter less than $2/(k+1)$.

There exists a positive integer m_{k+1} such that $m_{k+1} \geqslant 8$ and a chain $\mathscr{L}^{k+1}$ of solid tori $\{L_1^{k+1}, L_2^{k+1}, \ldots, L_{m_{k+1}}^{k+1}\}$ in $\operatorname{Int} A^{k+1}$ such that (a) each link of $\mathscr{L}^{k+1}$ has diameter less than $1/(k+1)$, (b) if $r = 1, 2, \ldots, p_{k+1}$, then there exists a positive integer $j_{k+1,r}$ such that $\Delta^{k+1,r}$ is a special meridional disc in A^{k+1} relative to $\mathscr{L}^{k+1}$ and $j_{k+1,r}$, and $V^{k+1,r}$ is a special neighborhood of $\Delta^{k+1,r}$ relative to $\mathscr{L}^{k+1}$.

If $r = 1, 2, \ldots, p_{k+1}$, let $X^{k+1,r}$ be a 2-dimensional AR replacing

$\Delta^{k+1,r}$ in A^{k+1} and $V^{k+1,r}$, with $X^{k+1,r} \subset V^{k+1,r}$ and $X^{k+1,r} \cap \mathrm{Bd}\, A^{k+1} = \mathrm{Bd}\, \Delta^{k+1,r}$.

Let

$$X_{k+1} = \Big(X_k - \bigcup_{r=1}^{p_{k+1}} \Delta^{k+1,r}\Big) \cup \Big(\bigcup_{r=1}^{p_{k+1}} X^{k+1,r}\Big).$$

It is easy to see that X_{k+1} is a 2-dimensional AR in Q_0.

If $r = 1, 2, \ldots, p_{k+1}$, let $J^{k+1}_{j_{k+1,r}}$ and $J^{k+1}_{(j_{k+1,r}+1)}$ denote the two arcs in $L^{k+1}_{j_{k+1,r}} \cap X^{k+1,r}$ and $L^{k+1}_{(j_{k+1,r}+1)} \cap X^{k+1,r}$ respectively, associated with $X^{k+1,r}$. Let

$$\mathscr{R}_{k+1} = \{j_{k+1,r}, j_{k+1,r}+1 \colon r = 1, 2, \ldots, p_{k+1}\}.$$

Suppose that $i = 1, 2, \ldots, m_{k+1}$, and $i \notin \mathscr{R}_{k+1}$. Then X_{k+1} is disjoint from L^{k+1}_i, and we may construct (using the integer m_{k+1}) an Antoine's necklace M^{k+1}_i in $\mathrm{Int}\, L^{k+1}_i$ as in Section 3; we may assume M^{k+1}_i is disjoint from $\bigcup_{\nu=1}^{\infty} K_\nu$. Let J^{k+1}_i be an arc in $\mathrm{Int}\, L^{k+1}_i$ containing M^{k+1}_i and disjoint from $\bigcup_{\nu=1}^{\infty} K_\nu$, and such that $\omega(J^{k+1}_i, L^{k+1}_i) = 1$. Note that J^{k+1}_i is disjoint from X_{k+1}.

In either case, $\{J^{k+1}_1, J^{k+1}_2, \ldots, J^{k+1}_{m_{k+1}}\}$ is a wreath substituting for A^{k+1} such that each link has diameter less than $1/(k+1)$ and is disjoint from $\bigcup_{\nu=1}^{\infty} K_\nu$. Let $\mathscr{J}_{k+1}$ denote $\mathscr{J}_k \cup \{J^{k+1}_1, J^{k+1}_2, \ldots, J^{k+1}_{m_{k+1}}\}$. Notice that if an arc of $\mathscr{J}_{k+1}$ intersects X_{k+1}, it lies in X_{k+1}.

By construction, $\big(X_{k+1} - \bigcup_{t=1}^{k+1} \bigcup_{r=1}^{m_t} J^t_r\big)$ is locally a 2-complex. Let T_{k+1} be a triangulation of $\big(X_{k+1} - \bigcup_{t=1}^{k+1} \bigcup_{r=1}^{m_t} J^t_r\big)$ such that $(\mathrm{mesh}\, T_{k+1}) < 1/(k+1)$, T_{k+1} is a subdivision of T_k, $\big(\bigcup_{r=1}^{p_{k+1}} \mathrm{Bd}\, \Delta^{k+1,r}\big)$ lies in $|T^1_{k+1}|$, and for any positive number ε, there is a neighborhood N_ε of $X_{k+1} \cap \big(\bigcup_{t=1}^{k+1} \bigcup_{r=1}^{m_t} J^t_r\big)$ such that any simplex of T_{k+1} that intersects N_ε has diameter less than ε.

For each 2-simplex σ of T_{k+1}, let W^{k+1}_σ be a polyhedral 3-cell such that σ spans W^{k+1}_σ, $(\mathrm{diam}\, W^{k+1}_\sigma) < 2(\mathrm{diam}\, \sigma)$, and W^{k+1}_σ is disjoint from $\big(\bigcup_{t=1}^{k+1} \bigcup_{r=1}^{m_t} J^t_r\big)$; further, if ϱ and τ are distinct 2-simplexes of T_{k+1}, then $W^{k+1}_\varrho \cap W^{k+1}_\tau = \varrho \cap \tau$. Also, it is to be true that if σ is a 2-simplex of T_{k+1}, then (1) if there is a 2-simplex τ of T_k such that $\sigma \subset \tau$, then $W^{k+1}_\sigma \subset W^k_\tau$, and (2) if for some positive integer r with $r \leqslant p_{k+1}$, $\sigma \subset X^{k+1,r}$, and τ is the 2-simplex of T_k such that $\sigma \subset \tau$, then $W^{k+1}_\sigma \subset W^k_\tau$.

It follows from the construction that in either case, if σ is any 2-simplex of T_{k+1}, $\mathrm{Int}W_\sigma^{k+1}$ is disjoint from both $|T_{k+1}^1|$ and $\left(\bigcup_{t=1}^{k+1}\bigcup_{r=1}^{m_t} J_r^t\right)$. Notice that in either case, if $t = 1, 2, \ldots, k+1$, $r = 1, 2, \ldots, p_t$, and J_r^t intersects X_{k+1}, then $J_r^t \subset X_{k+1}$.

Conclusion. By induction, it follows that there exist the following: (1) A sequence $A^1, A^2, \ldots$ of polyhedral solid tori in $\mathrm{Int}Q$, such that for each positive integer i, A^i has C_i as a core. It follows as in [1] that the sequence $A^1, A^2, \ldots$ is dense in Q. (2) Wreaths $\{J_1^1, J_2^1, \ldots, J_{m_1}^1\}$, $\{J_1^2, J_2^2, \ldots, J_{m_2}^2\}, \ldots$ substituting for $A^1, A^2, \ldots$, respectively, such that if $t = 1, 2, \ldots$ and $i = 1, 2, \ldots, m_t$, then $(\mathrm{diam} J_i^t) < 1/t$. For each positive integer n,

$$\mathscr{J}_n = \{J_i^t\colon\ t = 1, 2, \ldots, n,\ and\ i = 1, 2, \ldots, m_n\}.$$

(3) A sequence $X_1, X_2, \ldots$ of absolute retracts in Q_0. It is true that (a) if $t = 1, 2, \ldots$ and $i = 1, 2, \ldots, m_t$, $s = 1, 2, \ldots$, $t \leqslant s$, and J_i^t intersects X_s, then $J_i^t \subset X_s$, and (b) if $s = 1, 2, \ldots$, then $\left(X_s - \bigcup_{t=1}^{s}\bigcup_{i=1}^{m_t} J_i^t\right)$ is locally a 2-complex. (4) For each positive integer s, a triangulation T_s of $\left(X_s - \bigcup_{t=1}^{s}\bigcup_{i=1}^{m_t} J_i^t\right)$ as described above. (5) For each positive integer s and each 2-simplex σ of T_s, a 3-cell W_σ^s in Q_0 as described above.

Let M be the collection consisting of the chords $K_1, K_2, \ldots$, the arcs of the collection $\bigcup_{n=1}^{\infty}\mathscr{J}_n$, and singleton subsets of Q not in $\left(\bigcup_{i=1}^{\infty}K_i\right)\cup\left(\bigcup_{t=1}^{\infty}\bigcup_{i=1}^{m_t} J_i^t\right)$. By construction, if $i = 1, 2, \ldots$ and $r = 1, 2, \ldots, p_i$, then $J_{j_{ir}}^i$ and $J_{(j_{ir}+1)}^i$ are contained in Q_0. Since Q_0 is disjoint from $\bigcup_{\nu=1}^{\infty}K_\nu$, each arc of the type described above is disjoint from $\bigcup_{\nu=1}^{\infty}K_\nu$. By construction, if $i = 1, 2, \ldots$, $r = 1, 2, \ldots, m_i$, and $r \notin \mathscr{R}_i$, then J_r^i is disjoint from $\bigcup_{\nu=1}^{\infty}K_\nu$.

Now by construction, $K_1, K_2, \ldots$ is a null sequence. Further, if $t = 1, 2, \ldots$ and $i = 1, 2, \ldots, m_t$, then $(\mathrm{diam} J_i^t) < 1/t$. It then follows that M is an upper semicontinuous decomposition of Q.

The decomposition M of Q described above satisfies the conditions imposed in Section 5, and hence is a Bing–Borsuk decomposition of Q. Thus the associated decomposition space Q/M is a Bing–Borsuk retract. Following [1], we shall use Q^* to denote Q/M.

The sequence $X_1, X_2, \ldots$ converges to a compact subset X of Q_0. In the next section, it will be proved that X is, in fact, a 2-dimensional AR.

11. X is a 2-dimensional AR

We have constructed the sequence $X_1, X_2, \ldots$ in such a way that it converges to a compact set X which is a 2-dimensional AR and which contains each element of M that it intersects. It is easily shown that the image Y of X under projection has dimension 2. It follows that Y is an AR. Thus Q^* contains a 2-dimensional AR.

In this section, we shall show that X is a 2-dimensional AR. It is easier for our purposes to define X differently from what was mentioned above.

For each positive integer n, let

$$W^n = \left(\bigcup\{W^n_\sigma:\ \sigma\in T_n\}\right)\cup\left[\bigcup_{l=1}^{n}\bigcup_{i=1}^{p_l}(J^l_{j_{li}}\cup J^l_{(j_{li}+1)})\right].$$

Proposition 11.1. *For each positive integer n, W^n is compact, $W^{n+1} \subset W^n$, and $X_n \subset W^n$.*

Proof. These facts follow easily from the definitions and properties of the sets involved.

Let $X = \bigcap_{n=1}^{\infty} W^n$. Then X is compact. It can be proved that X is the limit of the sequence $X_1, X_2, \ldots$, but since we shall not use this fact we do not include a proof of it.

Lemma 11.2. *Suppose that H_0 is an* AR *in E^3, U is a proper open subset of H_0, and U is the carrier of an infinite rectilinear 2-complex T such that T is locally finite at each point of U, and if ε is any positive number, there is a neighborhood N_ε of* Bd U *such that any simplex of T that intersects N_ε has diameter less than ε. Suppose that for each 2-simplex σ of T, W_σ is a polyhedral 3-cell in E^3 such that σ spans W_σ,* $(\operatorname{diam} W_\sigma) < 2(\operatorname{diam}\sigma)$, *and W_σ is disjoint from $H_0 - U$. Suppose further that if σ and τ are distinct 2-simplexes of T, $W_\sigma\cap W_\tau = \sigma\cap\tau$. Let H denote $\left(\bigcup\{W_\sigma:\ \sigma\in T^2\}\right)\cup H_0$. Then H is an* AR.

Proof. First, H is compact. Since T is locally finite at each point of U, it follows that $\{W_\sigma:\ \sigma\in T^2\}$ is locally finite at each point of $\bigcup\{W_\sigma:\ \sigma\in T^2\}$ not on $\operatorname{Bd} U$. Thus each limit point of the set $\bigcup\{W_\sigma:\ \sigma\in T^2\}$ not in the set is in $\operatorname{Bd} U$, and hence in H_0.

Now we shall prove that H is contractible. If σ is a 2-simplex of T, there is a deformation retraction r_σ of W_σ onto σ. Let r be defined so that (1) if σ is a 2-simplex of T, $r|W_\sigma = r_\sigma$, and (2) for other points of H, r is the identity. It follows easily that r is a deformation retraction from H onto H_0. Since H_0 is contractible, so is H.

Finally, we shall prove that H is locally contractible. First, suppose $p\in\bigcup\{W_\sigma:\ \sigma\in T^2\}$. Then since both T and $\{W_\sigma:\ \sigma\in T^2\}$ are locally finite at p, it follows easily that H is locally contractible at p.

Now suppose that $p \epsilon (H - \bigcup\{W_\sigma \colon \sigma \epsilon T^2\})$. If $p \notin \mathrm{Cl}[\bigcup\{W_\sigma \colon \sigma \epsilon T^2\}]$, then $p \epsilon H_0$ and $H_0 - \mathrm{Cl}[\bigcup\{W_\sigma \colon \sigma \epsilon T^2\}]$ is a neighborhood of p in H. Since H_0 is an AR, it follows that H is locally contractible at p.

Suppose finally that no one of the preceding applies to p. Then $p \epsilon \mathrm{Bd}[\bigcup\{W_\sigma \colon \sigma \epsilon T^2\}]$ and $p \epsilon \mathrm{Bd}\, U$; in particular, $p \epsilon H_0$. Let N be a neighborhood of p in H. Since H_0 is an AR, there exists a neighborhood N' of p in H such that $(N' \cap H_0) \sim 0$ in $N \cap H_0$. Since (1) simplexes of T near $\mathrm{Bd}\, U$ are small, (2) if $\sigma \epsilon T^2$, $(\mathrm{diam}\, W_\sigma) < 2(\mathrm{diam}\, \sigma)$, and (3) $p \epsilon \mathrm{Bd}\, U$, then there is a neighborhood N'' of p in H such that $N'' \subset N'$ and for each 2-simplex σ of T such that σ intersects N'', $W_\sigma \subset N'$.

For each 2-simplex σ of T that intersects N'', let r_σ be a deformation retraction of W_σ onto σ. Let V be

$$\bigcup\{W_\sigma \colon \sigma \epsilon T^2 \textit{ and } \sigma \textit{ intersects } N''\}.$$

Let r be the function from V into $N' \cap H_0$ such that (1) if $\sigma \epsilon T^2$ and σ intersects N'', then $r | W_\sigma = r_\sigma$, and (2) elsewhere on V, r is the identity. Since for each two distinct 2-simplexes σ and τ of T, $W_\sigma \cap W_\tau = \sigma \cap \tau$, it follows that r is well-defined. Using the facts that (1) simplexes of T near $\mathrm{Bd}\, U$ are small and (2) if $\sigma \epsilon T^2$, $(\mathrm{diam}\, W_\sigma) < 2(\mathrm{diam}\, \sigma)$, it can be proved that r is continuous. Then r is a deformation retraction from V into $N' \cap H_0$. Since $N' \cap H_0 \sim 0$ in $N \cap H_0$, it follows that $V \sim 0$ in N. Thus H is locally contractible at p.

Since H has finite dimension, it follows that H is an AR.

Corollary 11.3. *For each positive integer n, W^n is an* AR.

Proof. This follows from Lemma 11.2, the fact that X_n is an AR, and properties of W^n and of the collection $\{W^n_\sigma \colon \sigma \epsilon T_n\}$.

Lemma 11.4. *If n is a positive integer and σ is a 2-simplex of T_n, then $W^n_\sigma \cap X_{n+1}$ is an* AR.

Proof. Suppose that n is a positive integer, and $\sigma \epsilon T^2_n$. First suppose that A^{n+1} intersects σ. Then there exists a positive integer s and mutually disjoint discs $\Delta^{n+1,r_1}, \Delta^{n+1,r_2}, \ldots, \Delta^{n+1,r_s}$ such that

$$W^n_\sigma \cap X_{n+1} = \Big(\sigma - \bigcup_{t=1}^{s} \Delta^{n+1,r_t}\Big) \cup \Big(\bigcup_{t=1}^{s} X^{n+1,r_t}\Big).$$

By Lemma 9.8, if $t = 1, 2, \ldots, s$, X^{n+1,r_t} is an AR. It then easily follows that $W^n_\sigma \cap X_{n+1}$ is an AR.

Now suppose that A^{n+1} and σ are disjoint. Then $W^n_\sigma \cap X_{n+1} = \sigma$, and in this case $W^n_\sigma \cap X_{n+1}$ is an AR.

Lemma 11.5. *If n is a positive integer and σ is a 2-simplex of T_n, then $W^{n+1} \cap W^n_\sigma$ is an* AR.

Proof. This follows from Lemmas 11.2 and 11.4, and facts about the construction of W^{n+1} and W^n.

LEMMA 11.6. *Suppose that n and i are positive integers, $p \in J_i^n$, and U is a neighborhood of p in E^3. Then there is a closed neighborhood V of p in W^n such that $V \subset U$ and the boundary of V relative to W^n is contained in $J_i^n \cup |T_n^1|$.*

Proof. Let U' be a closed neighborhood of p in E^3 such that $U' \subset U$ and U' does not intersects any arc of the collection $\mathscr{J}_n$ distinct from J_i^n. Since simplexes of T_n near J_i^n are small, and for each 2-simplex σ of T_n, $(\operatorname{diam} W_\sigma^n) < 2(\operatorname{diam}\sigma)$, it follows that there is a closed neighborhood U'' of p in E^3 such that if $\sigma \in T_n^2$ and σ intersects U'', then $W_\sigma^n \subset U'$.

Let $V = (\bigcup\{W_\sigma^n\colon \sigma \in T_n^2 \text{ and } \sigma \text{ intersects } U''\}) \cup (J_i^n \cap U'')$. Then (1) V is a closed neighborhood of p in W^n since $U'' \cap W^n \subset V$, and (2) $V \subset U'$ by the choice of U''.

Suppose now that x is a point of the boundary of V relative to W^n, and $x \notin J_i^n$. Since $V \subset U'$, it follows that if $t = 1, 2, \ldots, n$, and $s = 1, 2, \ldots, m_t$, then $x \notin J_s^t$. Hence there is a neighborhood N of x in W^n such that N intersects only finitely many of the sets of $\{W_\sigma^n\colon \sigma \in T_n^2\}$. Then by using facts about the construction of W^n, it may be shown that x lies in $|T_n^1|$.

LEMMA 11.7. *If k is any non-negative integer, then X is $k-C$.*

Proof. Suppose that $k = 0, 1, 2, \ldots$, and that $f\colon S^k \to X$ is a map. Now we shall construct a retraction r^1 from W^1 onto W^2 such that (1) if $\sigma \in T_1^2$, then $r^1[W_\sigma^1] \subset W_\sigma^1 \cap W^2$, and (2) r^1 is the identity map on each arc of $\mathscr{J}_1$.

Suppose that $\sigma \in T_1^2$. By Lemma 11.5, $W^2 \cap W_\sigma^1$ is an AR. Hence there is a retraction r_σ^1 from W_σ^1 onto $W^2 \cap W_\sigma^1$. Notice that $W^2 \cap \operatorname{Bd} W_\sigma^1$ is a simple closed curve and on $W^2 \cap \operatorname{Bd} W_\sigma^1$, r^1 is the identity map. Recall that if σ and τ are distinct 2-simplexes of T_1^2, $W_\sigma^1 \cap W_\tau^1 = \sigma \cap \tau$. It follows that for such simplexes σ and τ,

$$r_\sigma^1 | W_\sigma^1 \cap W_\tau^1 = r_\tau^1 | W_\sigma^1 \cap W_\tau^1.$$

Let r^1 be the map defined on W^1 such that (1) if $\sigma \in T_1^2$, then $r^1|\sigma = r_\sigma^1$, and (2) at every other point of W^1, r^1 is the identity map. By the remarks of the preceding paragraph, r^1 is well defined.

Recall that (1) $\{W_\sigma^1\colon \sigma \in T_1^2\}$ is locally finite at each point of $W^1 - \bigcup_{s=1}^{m_1} J_s^1$, (2) if ε is any positive number, there is a neighborhood N_ε of $\bigcup_{s=1}^{m_1} J_s^1$ such that every simplex of T_1 that intersects N_ε has diameter less than ε, and (3) if $\sigma \in T_1^2$, $(\operatorname{diam} W_\sigma^1) < 2(\operatorname{diam}\sigma)$. By using these facts, it is easily shown that r^1 is continuous.

Thus r^1 is a retraction from W^1 onto W^2 such that (1) if $\tau \in T_1^2$, $r^1[W_\sigma^1] \subset W_\sigma^1 \cap W^2$, and (2) r^1 is the identity on $\bigcup_{i=1}^{m_1} J_i^1$. Now by Corollary 11.3,

W^1 is an AR. Since $X \subset W^1$, then f is into W^1, and hence has a continuous extension $F\colon B^{k+1} \to W^1$. Let $G_1 = r^1F$. Then $G_1\colon B^{k+1} \to W^2$, and G_1 extends f since $r^1|W^1$ is the identity and $X \subset W^2$.

Now we shall construct a retraction r^2 from W^2 onto W^3 such that (1) if $\sigma \in T_2^2$, $r^2[W_\sigma^2] \subset W_\sigma^2 \cap W^3$, and (2) r^2 is the identity on each arc of $\mathscr{J}_2$. If $\sigma \in T_2^2$, then by Lemma 11.5, $W_\sigma^2 \cap W^3$ is an AR. Hence there is a retraction r_σ^2 from W_σ^2 onto W_σ^2. Since $r_\sigma^2|\mathrm{Bd}\,\sigma$ is the identity, we may construct a map $r^2\colon W^2 \to W^3$ as follows: (1) If $\sigma \in T_2^2$, $r^2|W_\sigma^2 = r_\sigma^2$. (2) Otherwise r^2 is the identity. It follows, as in the case of r^1 above, the r^2 is well-defined and continuous. Let $G_2 = r^2G_1$. Then $G_2\colon B^{k+1} \to W^3$ and G_2 extends f. Also, since for each 2-simplex σ of T_2, $r^2[W_\sigma^2] \subset W_\sigma^2$, it follows that if $x \in B^{k+1}$ and for some $\sigma \in T_2^2$, $G_1(x) \in W_\sigma^2$, then $G_2(x) \in W_\sigma^2$. Accordingly, $||G_1, G_2|| < 2(\mathrm{mesh}\, T_2)$.

Suppose that n is a positive integer and maps $G_1, G_2, \ldots, G_n$ have been constructed, and if $i = 1, 2, \ldots, n$, $G_i\colon B^{k+1} \to W^{i+1}$ and G_i extends f.

There is a retraction r^{n+1} from W^{n+1} onto W^{n+2} such that (1) if $\sigma \in T_{n+1}$, then $r^{n+1}[W_\sigma^{n+1}] \subset W_\sigma^{n+1} \cap W^{n+2}$, and (2) r^{n+1} is the identity on each arc of $\mathscr{J}_{n+1}$. If $\sigma \in T_{n+1}^2$, then by Lemma 11.5, $W_\sigma^{n+1} \cap W^{n+2}$ is an AR, and thus there is a retraction r_σ^{n+1} from W_σ^{n+1} onto $W_\sigma^{n+1} \cap W^{n+2}$. Since $r_\sigma^{n+1}|\mathrm{Bd}\,\sigma$ is the identity, we may define a map $r^{n+1}\colon W^{n+1} \to W^{n+2}$ as follows: (1) If $\sigma \in T_{n+1}^2$, then $r^{n+1}|\sigma = r_\sigma^{n+1}$. (2) Otherwise r^{n+1} is the identity. It follows, using facts about T_{n+1} and the sets W_σ^{n+1}, that r^{n+1} is well-defined and continuous. Let $G_{n+1} = r^{n+1}G_n$. Then $G_{n+1}\colon B^{k+1} \to W^{n+2}$ and G_{n+1} extends f. Since for each σ in T_{n+1}^2, $r^{n+1}[W_\sigma^{n+1}] \subset W_\sigma^{n+1}$, it follows that if $x \in B^{k+1}$ and for some σ in T_{n+1}^2, $G_n(x) \in W_\sigma^{n+1}$, then $G_{n+1}(x) \in W_\sigma^{n+1}$. Thus

$$||G_n, G_{n+1}|| < 2(\mathrm{mesh}\, T_{n+1}).$$

It follows from properties of $T_1, T_2, \ldots$, that if i and j are positive integers with $i < j$, $\sigma \in T_i^2$, and $\tau \in T_j^2$, then $W_\tau^j \subset W_\sigma^i$. It follows from these facts and from the construction of the retractions $r^1, r^2, \ldots, r^{n+1}$, that if i and j are positive integers with $i < j \leqslant n+1$, $x \in B^{k+1}$, $\sigma \in T_{i+1}^2$, and $G_i(x) \in W_\sigma^{i+1}$, then $G_j(x) \in W_\sigma^{i+1}$. From this it follows that for such i and j,

$$||G_i, G_j|| < 2(\mathrm{mesh}\, T_{i+1}).$$

Let this process be continued. There results a sequence $G_1, G_2, \ldots$ of maps such that (1) for each positive integer n, $G_n\colon B^{k+1} \to W^{n+1}$ and G_n extends f, and (2) for any two positive integers i and j with $i < j$,

$$||G_i, G_j|| < 2(\mathrm{mesh}\, T_{i+1}).$$

It follows that the sequence $G_1, G_2, \ldots$ converges uniformly to a continuous function $G\colon B^{k+1}\to\bigcap_{n=1}^{\infty}W^{n+1}$; clearly G extends f. Further since $X=\bigcap_{n=1}^{\infty}W^n$, G is into X. Consequently, X is $k-C$.

LEMMA 11.8. *If k is any non-negative integer, then X is $k-LC$.*

Proof. Suppose that $p\in X$ and U is a neighborhood of p in X. Let $\hat{U}$ be a neighborhood of p in E^3 such that $U=\hat{U}\cap X$. We consider two cases.

Case 1: There are no positive integers l and m such that $x\in J_l^m$.

Now $(\operatorname{mesh}T_i)$ approaches 0 as i increases without bound, and if t is any positive integer and $\sigma\in T_t^2$, $(\operatorname{diam}W_\sigma^t)<2(\operatorname{diam}\sigma)$. It follows that there exists a positive integer n such that if $\sigma\in T_n^2$ and $p\in W_\sigma^n$, then $W_\sigma^n\subset\hat{U}$.

Let N' denote $\bigcup\{\sigma\colon \sigma\in T_n^2 \text{ and } p\in\sigma\}$. It is easy to see that N' is an AR, and is the union of finitely many 2-simplexes $\sigma_1,\sigma_2,\ldots,\sigma_q$ of T_n^2.

Let N denote $(W_{\sigma_1}^n\cup W_{\sigma_2}^n\cup\ldots\cup W_{\sigma_q}^n)$ and let V denote $X\cap N$. Since N' is an AR, it follows easily that N is an AR. We shall show that if k is any non-negative integer and $f\colon S^k\to V$ is a map, then $f\sim 0$ in U.

Suppose that $f\colon S^k\to V$ is a map. Since $V\subset N$ and N is an AR, f has a continuous extension $F\colon B^{k+1}\to N$.

By using the techniques of the proof of Lemma 11.7, we may construct a sequence $r^n, r^{n+1},\ldots$ of maps such that for any non-negative integer i, r^{n+i} is a retraction from $N\cap W^{n+i}$ onto $N\cap W^{n+i+1}$ and such that the following hold: (1) If i and j are non-negative integers, $i<j$, $\sigma\in T_{n+i}^2$, $\tau\in T_{n+j}^2$, and $\tau\subset\sigma$ (so that $W_\tau^{n+j}\subset W_\sigma^{n+i}$), then $r^{n+j}[W_\tau^{n+j}]\subset r^{n+i}[W_\sigma^{n+i}]$. (2) If i is any non-negative integer, $r^{n+i}|\big(\bigcup_{t=1}^{n+i}\bigcup_{l=1}^{m_t}J_l^t\big)$ is the identity.

Let $G_n=r^nF$, and for each positive integer i, let $G_{n+i}=r^{n+i}G_{n+i-1}$. Then for each non-negative integer i, $G_{n+i}=B^{k+1}\to N\cap W^{n+i+1}$, and since $V=N\cap X$, G_{n+i} extends f. Further, it is easily seen that if i and j are non-negative integers with $i<j$, $x\in B^{k+1}$, $\sigma\in T_{n+i}^2$, and $G_{n+i}(x)\in W_\sigma^{n+i+1}$, then $G_{n+j}(x)\in W_\sigma^{n+i+1}$. Thus for such i and j,

$$\|G_{n+i},G_{n+j}\|<2(\operatorname{mesh}T_{n+i+1}).$$

Hence the sequence $G_n, G_{n+1},\ldots$ converges uniformly to a map

$$G\colon B^{k+1}\to\bigcap_{i=0}^{\infty}(N\cap W^{n+i+1}).$$

Clearly G extends f. Since $N\cap X=\bigcap_{i=0}^{\infty}(N\cap W^{n+i+1})$, $N\cap X=V$, and $V\subset U$, it follows that G is into U. Thus $f\sim 0$ in U.

Case 2: There exist positive integers l and m such that $p\in J_l^m$.

By Lemma 11.6, there is a neighborhood N of p in W^m such that $N\subset\hat{U}$ and the boundary of N relative to W^m is contained in $J_l^m\cup|T_m^1|$

By Corollary 11.3, W^m is an AR, and hence there is a neighborhood N' of p in W^m such that $N' \sim 0$ in N. Let V be $N' \cap X$.

Suppose that f: $S^k \to V$ is a map. Since $V \subset N'$, f has a continuous extension F: $B^{k+1} \to N$.

Since the boundary of N relative to W^m lies in $J_l^m \cup |T_m^1|$, N is the union of infinitely many 2-simplexes $\sigma_1, \sigma_2, \ldots$ of T_m and a subset of J_l^m. Notice that as s increases without bound, $(\operatorname{diam} \sigma_s)$ approaches 0. As in the proofs of Lemma 11.7 and of Case 1 above, there is a retraction r^m from N onto $N \cap W^{m+1}$ such that (1) for each positive integer s, $r^m[W_{\sigma_s}^m] \subset W_{\sigma_s}^m$ and (2) $r^m | N \cap J_l^m$ is the identity. Let $G_m = r^m F$. Then G_m: $B^{k+1} \to W^{m+1} \cap N$, and G_m extends F.

As in the proofs of Lemma 11.7 and of Case 1 above, we may construct a sequence of maps $G_m, G_{m+1}, \ldots$ such that (1) for each non-negative integer s, (a) G_{m+s}: $B \to N \cap W^{m+s+1}$, and (b) G_{m+s} extends f, and (2) for any two non-negative integers s and t with $s < t$,

$$\|G_{m+s}, G_{m+t}\| < 2(\operatorname{mesh} T_{m+s+1}).$$

It follows that the sequence $G_m, G_{m+1}, \ldots$ converges uniformly to a map G: $B^{k+1} \to \bigcap_{i=0}^{\infty} (N \cap W^{m+i+1})$. Clearly G extends f. Since $\bigcap_{i=0}^{\infty} W^{m+i+1} = X$, G is into $N \cap X$, and hence is into U. Thus $f \sim 0$ in U.

Hence for each non-negative integer k, x is $k-LC$.

LEMMA 11.9. *X has dimension 2.*

Proof. Suppose $p \in X$ and U is a neighborhood of p. Let $\hat{U}$ be a neighborhood of p in E^3 such that $U = X \cap \hat{U}$. We consider two cases.

Case 1: There are no positive integers m and l such that $p \in J_l^m$.

As in the proof of Case 1 of Lemma 11.8, there is a positive integer n such that if $\sigma \in T_n^2$ and $p \in W_\sigma^n$, then $W_\sigma^n \subset \hat{U}$. Let N denote $\bigcup \{W_\sigma^n$: $\sigma \in T_n^2$ *and* $p \in W_\sigma^n\}$. Then N is a neighborhood of p in W^n and $N \subset \hat{U}$. Further, the boundary of N relative to W^n is contained in $|T_n^1|$, and thus is 1-dimensional. Let $V = N \cap X$. Since $X \subset W^n$, V is a neighborhood of p in X. Further, $V \subset U$ and the boundary of V relative to X has dimension at most 1.

Case 2: There exist positive integers m and l such that $p \in J_l^m$.

By Lemma 11.6, there is a neighborhood N of p in W^m such that $N \subset \hat{U}$ and the boundary of N relative to W^m is contained in $J_l^m \cup |T_m^1|$. Let $V = N \cap X$. It follows that V is a neighborhood of p in X, $V \subset U$, and the boundary of V relative to X has dimension at most 1.

Hence $(\dim X) \leqslant 2$. Then by Lemmas 10.7 and 10.8 and [2], X is an AR. If $\dim X = 1$, then by [2], X is a dendron. Since by construction, X contains the simple closed curve $\operatorname{Bd} X_0$, X is not a dendron. Hence $\dim X = 2$.

COROLLARY 11.10. *X is a 2-dimensional* AR.

12. Q^* contains a 2-dimensional AR

In this section we shall show that the Bing–Borsuk retract Q^* described is Section 10 contains a 2-dimensional AR. We shall, in fact, show that the image X^* of X under projection φ is a 2-dimensional AR in Q^*. To establish this, we shall use the following result of [2].

THEOREM. *Suppose that Z is an* AR, *Γ is an upper semicontinuous decomposition of Z into absolute retracts, and Z/Γ has finite dimension. Then Z/Γ is an* AR.

First we shall show that X contains each element of the decomposition M which it intersects.

LEMMA 12.1. *If k and l are any positive integers with $l \leqslant m_k$, and X intersects J_l^k, then $J_l^k \subset X$.*

Proof. Suppose k and l are positive integers with $l \leqslant m_k$. If n is a positive integer such that $n \geqslant k$, then by the construction of W^n, either $J_l^k \subset W^n$, or W^n and J_l^k are disjoint. Since $X = \bigcap_{i=1}^{\infty} W^i$, the lemma follows.

COROLLARY 12.2. *If M is an element of M and M intersects X, then $M \subset X$.*

Proof. By construction, $X \subset Q_0$, and since Q_0 is disjoint from $\bigcup_{\nu=1}^{\infty} K_\nu$, X is disjoint from $\bigcup_{\nu=1}^{\infty} K_\nu$. By Lemma 12.1, if $J \in \bigcup_{n=1}^{\infty} \mathscr{J}_n$ and J intersects X, then $J \subset X$.

Recall that φ is the projection map $\varphi\colon Q \to Q/M$ and that $X^* = \varphi[X]$.

LEMMA 12.3. *X^* has dimension* 2.

Proof. By [1], $\dim Q^* = 3$. Thus $\dim X^*$ is finite. Then by [3], $\dim X^* \leqslant 2$. Now by the construction, no nondegenerate element of G intersects $\mathrm{Bd}\, X_0$. If follows that $\varphi[\mathrm{Bd}\, X_0]$ is a simple closed curve in X^*. It then follows as in the proof of Lemma 12.1 that $\dim X^* = 2$.

We summarize the main result of this paper in the following statement.

THEOREM. *There exists a Bing–Borsuk retract which contains a 2-dimensional* AR.

13. Concluding remarks

In this paper, we have shown only that there is a Bing–Borsuk retract which contains a 2-dimensional AR. There remains open the following question: *Does every Bing–Borsuk retract contain a 2-dimensional* AR?

In the construction described in the present paper, we succeeded in constructing a 2-dimensional AR primarily by being able to choose the following two things: (1) For each positive integer i, the number

m_i of links of the chain $\mathscr{L}^i$ of solid tori (of the first stage) in A^i. (2) For positive integers i and j with $j \leqslant m_i$, the arc J_j^i.

Since we constructed the decomposition M and the set X simultaneously, we were able to make the choices indicated above so as to ensure that X is both an AR and contains each element of M that it intersects. It is however, by no means clear that if the decomposition M were given in advance, such a construction for X would be possible.

We shall now indicate the motivation for the stipulation, in the definition of Bing–Borsuk decomposition, concerning the winding numbers of the arcs J_j^i in the solid tori L_j^i.

Singh's example [4] of a 3-dimensional AR that contains no 2-dimensional AR is not a Bing–Borsuk retract. However, by modifying his construction slightly, we may establish the following result:

PROPOSITION 13.1. *There is an upper semicontinuous decomposition N of the* 3-*dimensional ball Q such that* (1) *N satisfies all the conditions expressly stated for the decomposition M of* [1], *and* (2) *the decomposition space Q/N is a* 3-*dimensional* AR *that contains no* 2-*dimensional* AR.

For the proof we need the following lemma. We use the notation of [2]; here i is some fixed positive integer.

LEMMA 13.2. *If $j = 1, 2, \ldots, m_i$, there is an arc J_j^i in* $\operatorname{Int} L_j^i$ *such that $M^i \cap L_j^i \subset J_j^i$ and J_j^i has winding number* 2 *in L_j^i.*

Proof. Suppose that $j = 1, 2, \ldots, m_i$. Then there is a chain $\mathscr{L}_j$ of solid tori $\{L_{j1}^i, L_{j2}^i, \ldots, L_{jm_i}^i\}$ in $\operatorname{Int} L_j^i$. Let M_j^i denote $M^i \cap L_j^i$. Then $M_j^i \subset \bigcup_{k=1}^{m_i} L_{jk}^i$.

We may regard L_j^i as a product $D^2 \times S^1$ where S^1 is a circle and D^2 is a disc; if $t \in S^1$, let $D_t^2 = D^2 \times \{t\}$. We may suppose that the solid tori of the chain $\mathscr{L}_j$ hit the various discs D_t^2 (where $t \in S^1$) nicely.

If $k = 1, 2, \ldots, m_i$, there is an arc δ_{jk}^i in L_{jk}^i containing $M_j^i \cap L_{jk}^i$. Let x_k and y_k be the endpoints of δ_{jk}^i.

There is an arc ε_{j1}^i in $\operatorname{Int} L_j^i$ from y_1 to x_2 that winds as indicated in Figure 8. If $k = 2, 3, \ldots, m_i - 1$, there is an arc ε_{jk}^i in $\operatorname{Int} L_j^i$ from y_k to x_{k+1} such that (1) if ε_{jk}^i hits a disc D_t^2 in the product $D^2 \times S^1$, then D_t^2 intersects $L_{jk}^i \cup L_{j(k+1)}^k$, (2) the union of all discs D_t^2 that intersect ε_{jk}^i is a 3-cell, and (3) if $t \in S^1$ and ε_{jk}^i intersects D_t^2, $\varepsilon_{jk}^i \cap D_t^2$ is a singleton. See Figure 8.

Let J_j^i be $\left[\bigcup_{k=1}^{m_i-1} (\delta_{jk}^i \cup \varepsilon_{jk}^i)\right|$. We may suppose J_j^i is an arc; clearly $M_j^i \subset J_j^i \subset \operatorname{Int} L_j^i$, and $\omega(J_j^i, L_j^i) = 2$.

We turn now to a proof of Proposition 13.1 (we follow, in this proof, the notation of [1]). In the construction described in [1], there is, for each positive integer i, a solid torus A_i, and a chain M_i^1 with links $\{L_{i1}^1, L_{i2}^1, \ldots, L_{im_i}^1\}$ substituting for A_i. In A_i, there is an Antoine's necklace M^i.

In each link L^1_{ij} of M^1_i, there is an arc J^1_{ij} containing $M^i \cap L^1_{ij}$. The wreath W_i substituting for A_i is $\bigcup_{j=1}^{m_i} J^1_{ij}$; the arcs $J^1_{i1}, J^1_{i2}, \ldots, J^1_{im_i}$ are the *links* of the wreath W_i.

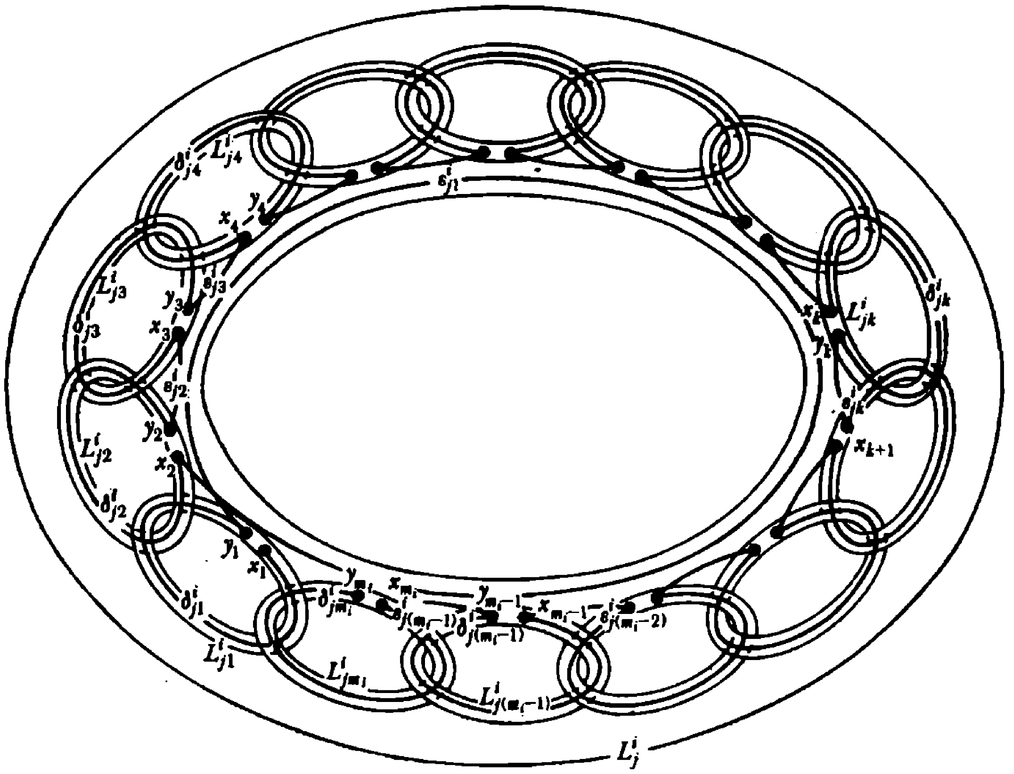

Fig. 8

Construct an upper semicontinuous decomposition of Q as in [1] but with the following modification: For each positive integer i, if $j = 1, 2, \ldots, m_i$, let J^1_{ij} be an arc in $\operatorname{Int} L^1_{ij}$ with $\omega(J^1_{ij}, L^1_{ij}) = 2$. Such an arc exists by Lemma 13.2. Let N be the resulting decomposition of Q. Then N satisfies every condition explicitly stated in [1]. Thus Q/N is a 3-dimensional AR containing no disc.

Singh's proof [4] may easily be modified to show that Q/N contains no 2-dimensional AR.

The existence of AR's as described in Proposition 13.1 lead us to include, in our definition of Bing–Borsuk retracts, the stipulation concerning winding numbers of the arcs which are the links of the wreaths. Lemma 3.1 shows that this stipulation is not excessively restrictive.

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