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Separation axioms, covering properties and inverse limits generated by developable topological spaces

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Für Petra und Björn

Introduction

Following R. H. Bing [1951] a topological space X is called *developable* if it has a *development*, i.e. a sequence $(\mathcal{U}_n)_{n<\omega}$ of open covers of X such that for each point x in X the collection $\{St(x, \mathcal{U}_n) \mid n < \omega\}$ forms a neighborhood base of x, where $St(x, \mathcal{U}_n) = \bigcup\{U \in \mathcal{U}_n \mid x \in U\}$. The notion of a development, however, was introduced already in 1916 as a part of R. L. Moore's axioms for an abstract theory of convergence and continuity (Moore [1916]). Based on these axioms, *Moore spaces*, i.e. developable regular T_1-spaces, were intensively studied by Moore and his school. Many of their results are collected in R. L. Moore's book "Foundations of Point Set Theory" ([1962]; first edition 1931). Independently, developments were also considered by P. S. Aleksandrov and P. Urysohn [1923].

One of the most interesting problems in recent research in set-theoretic topology originated from F. B. Jones' work on the metrization of Moore spaces. Assuming $2^\omega < 2^{\omega_1}$ he could show in [1937] that every separable normal Moore space is metrizable. Since then the so-called *Normal Moore Space Conjecture*, i.e. the question whether every normal Moore space is metrizable, has been the source of hundreds of research papers (according to M. E. Rudin [1975]). Around 1967/68 J. H. Silver and F. D. Tall proved that Martin's axiom together with the assumption $\omega_1 < 2^\omega$ yields a non-metrizable separable normal Moore space (see Tall [1977]), thereby showing that the existence of a non-metrizable separable normal Moore space is consistent with and independent of the usual axioms of set theory. For non-separable normal Moore spaces the situation turned out to be more complicated. In [1978] P. J. Nyikos proved that the *Product Measure Extension Axiom* (PMEA) implies that every normal Moore space is metrizable. Since it was known that in order to establish the consistency of PMEA, the consistency of the existence of a measurable cardinal is needed (Solovay [1971]), his result connected the Normal Moore Space Conjecture with an old and deep problem of set theory. Finally, the important paper [1983] of W. G. Fleissner clarified how close this connection really is. "Either the existence of a measurable cardinal is inconsistent with the axioms of set theory or any proof of the Normal Moore Space Conjecture must start with a large cardinal assumption" (Fleissner [1984], page 757). Interestingly, D. K. Burke has recently shown that PMEA also implies that every countably paracompact Moore space is metrizable (Burke [1984a]).

As a result of the intensive work on the Normal Moore Space Conjecture and the work of the Moore school of topologists, Moore spaces and, more generally, developable spaces belong to the best investigated classes of topological spaces. Among the high-lights of the theory are M. E. Rudin's example of a non-completable Moore space (Rudin [1950]), R. H. Bing's paper [1951] on the metrization of topological spaces, the influential paper of H. H. Wicke and J. M. Worrell, Jr. [1965] on developable spaces, and a series of more than twenty papers on Moore spaces by G. M. Reed, culminating with the proof that every locally compact, locally connected, normal Moore space is metrizable (Reed and Zenor [1976]).

Accepting developable spaces as an important and useful generalization of metrizable topological spaces we have initiated the systematic investigation of separation axioms and covering properties generated by developable spaces in (Brandenburg [1978]). The basic idea was to apply the formation of certain hulls (e.g. the *epireflective hull*, the *normality hull*, the *paracompactness hull*, and the *left-fitting hull*), which have been studied mainly in connection with metrizable or compact spaces, to the class of developable T_1-spaces, and to search for internal characterizations of the resulting spaces. This procedure led to a number of new concepts, new insights, and new problems. Since then many important contributions have been made by J. Chaber [1983], [1983a], [1984], [1984a], R. W. Heath [1984], N. C. Heldermann [1980], [1981], M. Hušek (Brandenburg and Hušek [1987]), A. Mysior [1980], (Brandenburg and Mysior [1984]), S.-H. Sun and Y.-M. Wang [1988], and the author [1980a], [1981], [1983], [1985], [1986], [1988].

One aim of this paper is to provide an up to date survey on this area. Of course, some stress will be layed on the work of the author, i.e. with a few exceptions only those results will be proved which are due to the author (some of them in collaboration with M. Hušek respectively A. Mysior). The results on inverse limits of developable spaces (§ 7) are new and appear here for the first time; but there are also some new results in other sections. The second aim of the paper is to stimulate further research. For this reason, nineteen research problems are mentioned (A–S) which might serve as starting-points for new interesting investigations. Some of them may be easy, others will be harder.

Instead of describing the contents of the subsequent sections, we conclude these introductory remarks by presenting two diagrams which illustrate the relationships between various (more or less) well-known classes of topological spaces and the classes of spaces which will be studied here (the latter are underlined). These diagrams are meant for the expert, i.e. they contain some entries which will not be defined here(1). We hope they will explain why we will

(1) Most of the relevant definitions can be found in (Engelking [1977]), (Burke [1984]), or (Gruenhage [1984]).

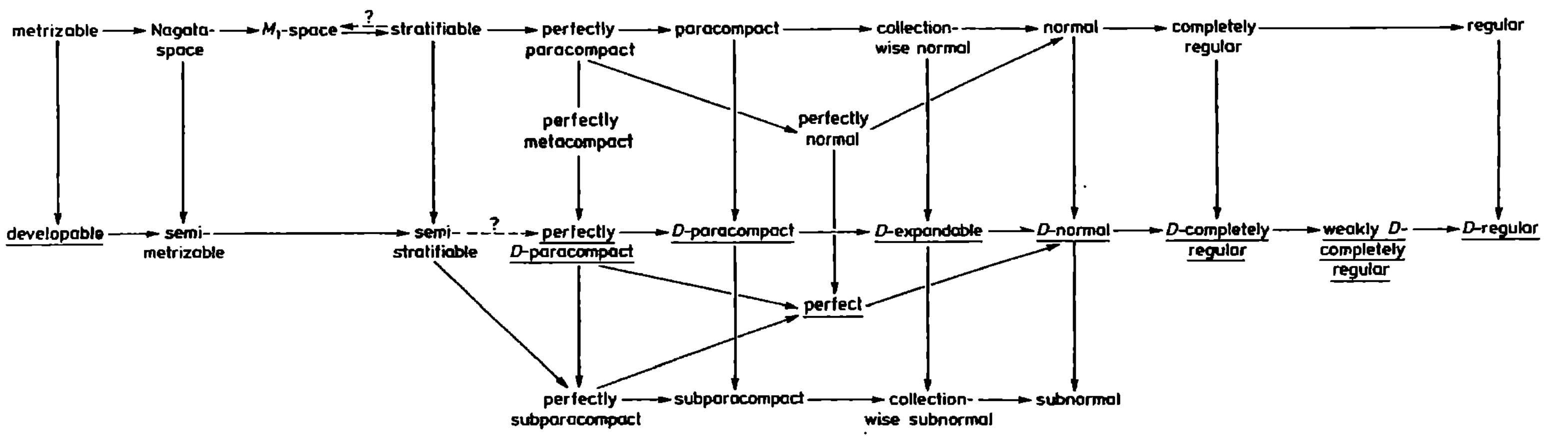

metrizable
Nagata-space
M_1-space
?
stratifiable
perfectly paracompact
paracompact
collection-wise normal
normal
completely regular
regular
perfectly metacompact
perfectly normal
developable
semi-metrizable
semi-stratifiable
?
perfectly D-paracompact
D-paracompact
D-expandable
D-normal
D-completely regular
weakly D-completely regular
D-regular
perfect
perfectly subparacompact
subparacompact
collection-wise subnormal
subnormal

Diagram I

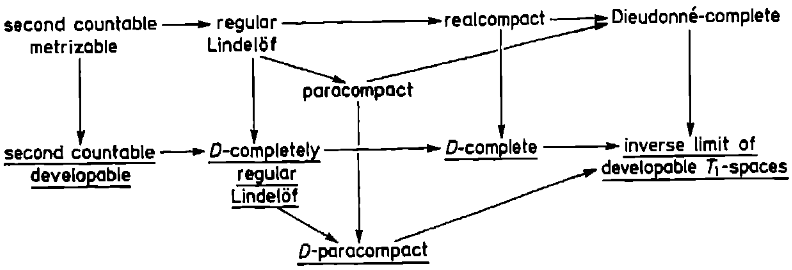


Diagram II

rarely present (counter-)examples witnessing that certain implications are not valid. In most cases the latter is quite obvious in view of the validity of other implications. For simplicity all spaces occurring in the diagrams are assumed to be at least T_1-spaces.

Notation

Throughout this paper no separation axioms are assumed unless explicitly stated. According to this convention compact spaces, paracompact spaces, normal spaces, completely regular spaces, and regular spaces are not necessarily T_1-spaces. D-completely regular spaces are the exception of the rule. Their definition includes the T_1-axiom (see § 2). For notions from topology which are used here without definition we refer to (Engelking [1977]). However, the reader should always be aware of the above convention.

Our set-theoretic notation is fairly standard and can be found, e.g., in (Kunen [1980]). An *ordinal* (number) is the set of all ordinals which precede it. Thus $\alpha \in \beta$ and $\alpha < \beta$ are the same. A *cardinal* (number) is an initial ordinal, i.e. an ordinal which cannot be mapped bijectively onto any smaller ordinal; ω is the smallest infinite cardinal, while ω_1 denotes the smallest uncountable cardinal. If $\varkappa$ is a cardinal, $\varkappa^+$ is its successor cardinal, whereas $\varkappa + 1 = \{\varkappa\} \cup \varkappa$ denotes its successor ordinal. Whenever an ordinal is considered as a topological space without specific mention of a topology, it is assumed that it carries the topology induced by the natural well-order. Thus, 2 is the discrete space $\{0, 1\}$, ω is the discrete space $\{0, 1, 2, \ldots\}$, and $\omega+1$ is the one-point compactification of ω, etc.

The following symbols will be used frequently:

$|X|$ cardinality of X;
$\mathscr{P}(X)$ power set of X;
$[X]^{\varkappa}$ the set of all subsets of X of cardinality $\varkappa$;
$[X]^{\leqslant\varkappa}$ the set of all subsets of X of cardinality $\leqslant \varkappa$;
$[X]^{<\varkappa}$ the set of all subsets of X of cardinality $< \varkappa$;
X_Y the set of all mappings from X into Y.

If the set of real numbers is considered as a topological space, it will always carry the Euclidean topology. A family or collection of sets will be denoted, for example, by $\mathcal{U}$, or by $(U_i)_{i \in I}$, but also by $(U(i))_{i \in I}$. If $\mathcal{U} = (U_i)_{i \in I}$ is a family of subsets of X, we will rarely distinguish between $(U_i)_{i \in I}$ and $\{U_i \mid i \in I\}$. Thus $U \in \mathcal{U}$ usually means that there exists an $i \in I$ such that $U = U_i$. $\mathcal{U}$ is a *cover* of X, if $\bigcup\{U \mid U \in \mathcal{U}\} = X$.

If X is a set, $x \in X$, $A \subset X$, and $\mathcal{U}$ and $\mathcal{V}$ are families of subsets of X, then

$\mathcal{V}$ *refines* $\mathcal{U}$ (or $\mathcal{V}$ is a *refinement* of $\mathcal{U}$) if $\bigcup\{V \mid V \in \mathcal{V}\} = \bigcup\{U \mid U \in \mathcal{U}\}$, and for each $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ with $V \subset U$;

$$\mathcal{U} \wedge \mathcal{V} = \{U \cap V \mid U \in \mathcal{U},\ V \in \mathcal{V}\};$$

$$\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid A \cap U \neq \emptyset\};$$

$$\mathrm{St}(x, \mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid x \in U\};$$

$$\mathrm{ord}(x, \mathcal{U}) = |\{U \in \mathcal{U} \mid x \in U\}|.$$

Finally, all classes of topological spaces are assumed to be *homeomorphism-closed*, i.e. they contain every homeomorphic copy of its members.

§ 1. The spaces D and D_1

For every nonempty class **E** of topological spaces there exists a smallest class EH(**E**) of topological spaces which contains **E** and is closed with respect to the formation of arbitrary products and subspaces. EH(**E**) is called the *epireflective hull* of **E**. It consists of all *E-completely regular* spaces, i.e. of those spaces which can be embedded into products of spaces belonging to **E**. Thus, if **M** is the class of metrizable spaces, a topological space is M-completely regular if and only if it is a completely regular T_1-space.([2])

The investigation of *D-completely regular* spaces, where **D** is the class of developable T_1-spaces, was initiated in (Brandenburg [1978])([3]). At that time we did not expect that there might exist a nice developable T_1-space which could serve as an analogue, for the theory of D-completely regular spaces, of the real line. However, in [1980] A. Mysior pointed out that the class of D-completely regular spaces is *simple*, i.e. that there exists a single developable T_1-space X such that $\mathrm{EH}(\{X\}) = \mathrm{EH}(\mathbf{D})$. Shortly later, N. C. Heldermann [1980] found an ingenious construction of a second countable developable T_1-space generating all D-completely regular spaces. This contruction was simplified in (Chaber [1983a]) and, in a different way, in (Brandenburg

(2) For more on epireflective hulls see (Herrlich [1968], [1971], [1983]).

(3) In (Brandenburg [1978]) we have called these spaces D-regular. This term will now be used for a more general class of spaces (see 2.14).

[1983]). The space D_1 which we are going to define now is a modification of the latter construction. In this section we will present its basic properties following (Brandenburg [1983]).

Let (0) be the unique mapping from 1 onto $0 \in \omega$ and consider

$$S = \{(0)\} \cup \bigcup \{{}^k(\omega \setminus \{0\}) \mid 0 < k < \omega\}.$$

The elements of S will be denoted by $(n_0, \ldots, n_{k-1})$. For each $(n_0, \ldots, n_{k-1}) \in S$ let $p_{(n_0,\ldots,n_{k-1})}\colon {}^S\omega \to \omega$ be the projection, i.e. $p_{(n_0,\ldots,n_{k-1})}(x) = x(n_0, \ldots, n_{k-1})$ for each $x \in {}^S\omega$. Moreover, for each $n < \omega$ let $n \in {}^S\omega$ be the constant mapping which maps S onto $\{n\} \subset \omega$. Inductively define subsets $A(n_0, \ldots, n_{k-1})$ of ${}^S\omega$ as follows:

If $k = 1$, set

$$A(n_0) = \begin{cases} \{0\} & \text{if } n_0 = 0, \\ p_{(0)}^{-1}[\{0, \ldots, n_0\}] \setminus A(0) & \text{if } n_0 > 0. \end{cases}$$

If $k > 1$ and $A(n_0, \ldots, n_{k-2})$ is already defined, set

$$A(n_0, \ldots, n_{k-1}) = p_{(n_0,\ldots,n_{k-2})}^{-1}[\{0, \ldots, n_{k-1}\}] \setminus (A(0) \cup A(n_0, \ldots, n_{k-2})).$$

Take $\mathscr{A} = \{A(n_0, \ldots, n_{k-1}) \mid (n_0, \ldots, n_{k-1}) \in S\}$ as a subbase for the closed sets of a topology on ${}^S\omega$ and call the resulting space $\boldsymbol{D}$.

1.1. THEOREM. *$\boldsymbol{D}$ is a developable space of weight ω.*

Proof. For each $(n_0, \ldots, n_{k-1}) \in \bigcup \{{}^k(\omega \setminus \{0\}) \mid 0 < k < \omega\}$ consider the open cover $\mathscr{V}_{(n_0,\ldots,n_{k-1})}$ of $\boldsymbol{D}$ defined by

$$\mathscr{V}_{(n_0,\ldots,n_{k-1})} = \begin{cases} \{D \setminus A(0),\ D \setminus A(n_0)\} & \text{if } k = 1, \\ \{D \setminus A(0),\ D \setminus A(n_0, \ldots, n_{k-2})\} \wedge \{D \setminus A(n_0, \ldots, n_{k-2}),\ D \setminus A(n_0, \ldots, n_{k-1})\} & \text{if } k > 1. \end{cases}$$

If $\varphi\colon \omega \to \bigcup \{{}^k(\omega \setminus \{0\}) \mid 0 < k < \omega\}$ is a bijection, then it is easily seen that $(\mathscr{U}_n)_{n<\omega}$ is a development of $\boldsymbol{D}$, where $\mathscr{U}_0 = \mathscr{V}_{\varphi(0)}$, and $\mathscr{U}_n = \mathscr{U}_{n-1} \wedge \mathscr{V}_{\varphi(n)}$ whenever $0 < n < \omega$. Obviously, $w(\boldsymbol{D}) = \omega$. ∎

Now let $\pi\colon \boldsymbol{D} \to \boldsymbol{D}_1$ be the T_0-reflection of $\boldsymbol{D}$, i.e. $\boldsymbol{D}_1$ is the quotient space obtained from $\boldsymbol{D}$ by identifying all points which have the same neighborhoods in $\boldsymbol{D}$ and π is the corresponding quotient mapping. Then $\boldsymbol{D}_1$ is a T_1-space of weight ω which is also developable (see Worrell, Jr. [1965] for a more general result). The following somewhat technical lemma is our main tool for constructing continuous mappings into $\boldsymbol{D}$ respectively $\boldsymbol{D}_1$.

1.2. LEMMA. *Let X be a topological space and consider the set $\mathscr{E}(X)$ of all mappings $E\colon S \to \{A \subset X \mid A \text{ closed}\}$ satisfying*

(E.1) $X \setminus E(0) = \bigcup \{E(n) \mid 0 < n < \omega\}$, *and* $X \setminus (E(0) \cup E(n_0, \ldots, n_{k-1})) = \bigcup \{E(n_0, \ldots, n_{k-1}, n) \mid 0 < n < \omega\}$ *whenever* $(n_0, \ldots, n_{k-1}) \in S \setminus \{(0)\}$;

(E.2) *$E(n) \subset E(n+1)$ and $E(n_0, \ldots, n_{k-1}, n) \subset E(n_0, \ldots, n_{k-1}, n+1)$ whenever $0 < n < \omega$ and $(n_0, \ldots, n_{k-1}) \in S \setminus \{(0)\}$. For each $E \in \mathscr{E}(X)$ let f_E: $X \to \boldsymbol{D}$ be defined by $x \mapsto f_E(x)$, where $f_E(x) = 0 \in \boldsymbol{D}$ whenever $x \in E(0)$, and*

$$f_E(x)(n_0, \ldots, n_{k-1})$$
$$= \begin{cases} \min\{0 < n < \omega \mid x \in E(n)\} & \text{if } (n_0, \ldots, n_{k-1}) = (0), \\ 0 & \text{if } (n_0, \ldots, n_{k-1}) \neq (0) \\ & \text{and } x \in E(n_0, \ldots, n_{k-1}), \\ \min\{0 < n < \omega \mid x \in E(n_0, \ldots, n_{k-1}, n)\} & \text{if } (n_0, \ldots, n_{k-1}) \neq (0) \\ & \text{and } x \notin E(n_0, \ldots, n_{k-1}), \end{cases}$$

whenever $x \in X \setminus E(0)$ and $(n_0, \ldots, n_{k-1}) \in S$. Then the following holds:

(i) *$f_E^{-1}[A(n_0, \ldots, n_{k-1})] = E(n_0, \ldots, n_{k-1})$ for each $E \in \mathscr{E}(X)$ and for each $(n_0, \ldots, n_{k-1}) \in S$.*

(ii) *The correspondence $E \mapsto f_E$ defines an injective mapping from $\mathscr{E}(X)$ into $C(X, \boldsymbol{D}) = \{f \in {}^X\boldsymbol{D} \mid f$ continuous$\}$.*

(iii) *The correspondence $E \mapsto \pi \circ f_E$ defines a surjective mapping from $\mathscr{E}(X)$ onto $C(X, \boldsymbol{D}_1) = \{f \in {}^X\boldsymbol{D}_1 \mid f$ continuous$\}$.*

Proof. (i) Consider an arbitrary $E \in \mathscr{E}(X)$. We proceed by induction.

If $k = 1$, then $x \in f_E^{-1}[A(0)]$ if and only if $f_E(x) = 0 \in \boldsymbol{D}$. From the definition of f_E we see that the latter condition is equivalent to $x \in E(0)$. Similarly, if $n_0 > 0$, then $x \in f_E^{-1}[A(n_0)]$ if and only if $x \in X \setminus E(0)$ and $f_E(x)(0) \in \{1, \ldots, n_0\}$, which is equivalent to $x \in E(n_0)$ (because of (E.1) and (E.2)).

If $k > 1$ and the claim is true for all $(n_0, \ldots, n_{k-2}) \in S$, then $x \in f_E^{-1}[A(n_0, \ldots, n_{k-1})]$ implies that $x \in X \setminus E(0)$ and $f_E(x)(n_0, \ldots, n_{k-2}) \in \{0, \ldots, n_{k-1}\}$. More precisely, it cannot happen that $f_E(x)(n_0, \ldots, n_{k-2}) = 0$, for otherwise we would have $x \in E(n_0, \ldots, n_{k-2}) = f_E^{-1}[A(n_0, \ldots, n_{k-2})]$, i.e. $f_E(x) \in A(n_0, \ldots, n_{k-2}) \cap A(n_0, \ldots, n_{k-1})$, which is impossible. Hence $f_E(x)(n_0, \ldots, n_{k-2}) \in \{1, \ldots, n_{k-1}\}$, which shows that $x \in E(n_0, \ldots, n_{k-1})$ (because of (E.2)). Consequently $f_E^{-1}[A(n_0, \ldots, n_{k-1})] \subset E(n_0, \ldots, n_{k-1})$. In order to prove the reverse inclusion, note that if $x \in E(n_0, \ldots, n_{k-1})$, then $x \in X \setminus E(0)$ and $f_E(x)(n_0, \ldots, n_{k-2}) \in \{0, \ldots, n_{k-1}\}$. Moreover, $f_E(x)(n_0, \ldots, n_{k-2}) \notin A(n_0, \ldots, n_{k-2})$ for otherwise $x \in f_E^{-1}[A(n_0, \ldots, n_{k-2})] = E(n_0, \ldots, n_{k-2})$ which is impossible (because of (E.1)). Hence $x \in f_E^{-1}[A(n_0, \ldots, n_{k-1})]$ and therefore $E(n_0, \ldots$ $\ldots, n_{k-1}) \subset f_E^{-1}[A(n_0, \ldots, n_{k-1})]$.

(ii) That $f_E \in C(X, \boldsymbol{D})$ for each $E \in \mathscr{E}(X)$ is an immediate consequence of (i). If D, $E \in \mathscr{E}(X)$ are distinct, then $D(n_0, \ldots, n_{k-1}) \neq E(n_0, \ldots, n_{k-1})$ for some $(n_0, \ldots, n_{k-1}) \in S$. Without loss of generality we may assume that there is an $x \in D(n_0, \ldots, n_{k-1}) \setminus E(n_0, \ldots, n_{k-1})$. But then $f_D(x)(n_0, \ldots, n_{k-1}) = 0$ while $f_E(x)(n_0, \ldots, n_{k-1}) > 0$. Hence f_D and f_E are also distinct.

(iii) Consider an arbitrary $f \in C(X, \boldsymbol{D}_1)$. Since the T_0-reflection π: $\boldsymbol{D} \to \boldsymbol{D}_1$ is a closed mapping, we obtain an $E \in \mathscr{E}(X)$ by defining $E(n_0, \ldots, n_{k-1})$

$=f^{-1}[\pi[A(n_0,\ldots,n_{k-1})]]$ for each $(n_0,\ldots,n_{k-1})\in S$. We claim that $\pi o f_E = f$. Indeed, if $\pi o f_E \neq f$, there would exist an $x\in X$ such that $\pi o f_E(x) \neq f(x)$. Since $\mathscr{A}$ is a subbase for the closed sets of $\boldsymbol{D}$, this would imply that $f(x)\in\pi[A(n_0,\ldots,n_{k-1})]$ but $\pi o f_E(x)\notin\pi[A(n_0,\ldots,n_{k-1})]$ for some $A(n_0,\ldots,n_{k-1})\in\mathscr{A}$, i.e. that simultaneously

$$x\in f^{-1}[\pi[A(n_0,\ldots,n_{k-1})]] = E(n_0,\ldots,n_{k-1})$$

and

$$x\notin f_E^{-1}[A(n_0,\ldots,n_{k-1})] = E(n_0,\ldots,n_{k-1}). \blacksquare$$

As a first application of this technique we can now relate two of the basic notions introduced in (Brandenburg [1978]) to the spaces $\boldsymbol{D}$ and $\boldsymbol{D}_1$.

1.3. Definition. A collection $\mathscr{B}$ of closed subsets of a topological space X is called a *G_δ-collection*, if for every B in $\mathscr{B}$ there exists a sequence $(B_n)_{n<\omega}$ in $\mathscr{B}$ such that $X\backslash B = \bigcup\{B_n \mid n<\omega\}$. A subset of X is said to be *D-closed*, if it belongs to some G_δ-collection in X. Complements of D-closed sets are called *D-open*.

The equivalence of (i) and (iv) in the following proposition was proved (implicitly) in (Brandenburg [1978]), while the equivalence of (i) and (iii) is essentially due to N. C. Heldermann [1980].

1.4. Proposition. *For a subset B of a topological space X the following conditions are equivalent:*

(i) *B is D-closed.*

(ii) *$B = g^{-1}[A(0)]$ for some continuous mapping g from X into $\boldsymbol{D}$.*

(iii) *There exists a continuous mapping f from X into $\boldsymbol{D}_1$ such that $B = f^{-1}[\{\pi(0)\}]$.*

(iv) *There exist a continuous mapping f from X into some developable space Y and a closed subset A of Y such that $B = f^{-1}[A]$.*

Proof. (i) implies (ii): If B is D-closed, then we may assume without loss of generality that B belongs to a (countable) G_δ-collection $\mathscr{B}$ which is closed with respect to the formation of finite unions (Brandenburg [1978], 3.1.17). Therefore, for each $(n_0,\ldots,n_{k-1})\in S$ we can define a set $E(n_0,\ldots,n_{k-1})\in\mathscr{B}$ as follows.

If $k=1$, set

$$E(n_0) = \begin{cases} B & \text{if } n_0 = 0, \\ \bigcup\{B_n \mid n \leqslant n_0\} & \text{if } n_0 > 0, \end{cases}$$

where $(B_n)_{n<\omega}$ is a sequence in $\mathscr{B}$ satisfying $X\backslash B = \bigcup\{B_n \mid n<\omega\}$.

If $k>1$ and for each $(n_0,\ldots,n_{k-2})\in S$, $E(n_0,\ldots,n_{k-2})\in\mathscr{B}$ is already defined, choose a sequence $(E(n_0,\ldots,n_{k-2})_n)_{n<\omega}$ in $\mathscr{B}$ such that $X\backslash(E(0)\cup \cup E(n_0,\ldots,n_{k-2})) = \bigcup\{E(n_0,\ldots,n_{k-2})_n \mid n<\omega\}$ and set

$$E(n_0,\ldots,n_{n-1}) = \bigcup\{E(n_0,\ldots,n_{k-2})_n \mid n \leqslant n_{k-1}\}.$$

Since the so-defined mapping E from S into the closed sets of X belongs to $\mathscr{E}(X)$, the induced mapping f_E from X into D (1.2) is continuous and $B = E(0) = f_E^{-1}[A(0)]$.

Clearly, (ii) implies (iii), for if $B = g^{-1}[A(0)]$ for some continuous mapping g from X into D, then $f = \pi \circ g$ is a continuous mapping from X into D_1 such that $B = f^{-1}[\{b\}]$, where $b = \pi(0)$.

Obviously, (iii) implies (iv). If f is a continuous mapping from X into a developable space Y such that $B = f^{-1}[A]$ for some closed subset A of Y, then $\mathscr{B} = \{f^{-1}[C] \mid C \subset Y \text{ closed}\}$ is a G_δ-collection in X containing B, which proves that (iv) implies (i). ∎

We can now prove the main results of this section.

1.5. Theorem (Brandenburg [1983]). *If A is a D-closed subset of a topological space X and $f\colon A \to D$ is continuous, then there exists a continuous mapping $g\colon X \to D$ such that $g(x) \in \operatorname{cl}\{f(x)\}$ whenever $x \in A$.*

Proof. Let $\mathscr{B}$ be a G_δ-collection in X containing A. By virtue of (Brandenburg [1978], 3.1.17) we may assume that $\mathscr{B}$ is closed with respect to finite unions. Therefore we can define a closed set $B(n_0, \dots, n_{k-1}) \subset X \backslash A$ for each $(n_0, \dots, n_{k-1}) \in S \backslash \{(0)\}$ as follows.

If $k = 1$, choose a sequence $(B(n))_{0<n<\omega}$ in $\mathscr{B}$ such that $X \backslash A = \bigcup \{B(n) \mid 0 < n < \omega\}$ and $B(n) \subset B(n+1)$ whenever $0 < n < \omega$. If $k > 1$ and $B(n_0, \dots, n_{k-2}) \in \mathscr{B}$ is already defined, let $(B(n_0, \dots, n_{k-2}, n))_{0<n<\omega}$ be a sequence in $\mathscr{B}$ such that

$$X \backslash (B(n_0, \dots, n_{k-2}) \cup A) = \bigcup \{B(n_0, \dots, n_{k-2}, n) \mid 0 < n < \omega\} \quad \text{and}$$

$$B(n_0, \dots, n_{k-2}, n) \subset B(n_0, \dots, n_{k-2}, n+1) \quad \text{whenever } 0 < n < \omega.$$

With the help of these sets we define, for each $(n_0, \dots, n_{k-1}) \in S$, another closed subset $E(n_0, \dots, n_{k-1})$ of X by

$$E(n_0, \dots, n_{k-1}) = \begin{cases} f^{-1}[A(0)] & \text{if } (n_0, \dots, n_{k-1}) = (0), \\ f^{-1}[A(n_0, \dots, n_{k-1})] \cup B(n_0, \dots, n_{k-1}) & \text{if } (n_0, \dots, n_{k-1}) \neq (0). \end{cases}$$

Since it is easily seen that the so-defined mapping E from S into the closed subsets of X belongs to $\mathscr{E}(X)$, the induced mapping f_E from X into D (see 1.2) is continuous. Suppose that there exists an $x \in A$ such that $f_E(x) \notin \operatorname{cl}\{f(x)\}$. Since $\mathscr{A}$ is a subbase for the closed sets of D, there must be an $A(n_0, \dots, n_{k-1}) \in \mathscr{A}$ such that $f(x) \in A(n_0, \dots, n_{k-1})$ and $f_E(x) \notin A(n_0, \dots, n_{k-1})$. On the other hand it follows from 1.2 (i) that

$$x \in f^{-1}[A(n_0, \dots, n_{k-1})] \subset E(n_0, \dots, n_{k-1}) = f_E^{-1}[A(n_0, \dots, n_{k-1})],$$

i.e. $f_E(x) \in A(n_0, \dots, n_{k-1})$ — a contradiction! Hence $f_E(x) \in \operatorname{cl}\{f(x)\}$ for each $x \in A$, which completes the proof. ∎

Let us say that a subset A of a topological space X is *D-embedded* in X, if every continuous mapping from A into D_1 is the restriction of a continuous mapping from X into D_1. The preceding theorem implies:

1.6. COROLLARY. *Every D-closed subset of a topological space is D-embedded.*

Proof. If A is a D-closed subset of a topological space X and f is a continuous mapping from A into D_1, then there is an $E \in \mathscr{E}(A)$ such that $\pi \circ f_E = f$ (1.2 (iii)). By virtue of Theorem 1.5 there exists a continuous mapping g from X into D such that $g(x) \in \mathrm{cl}\,\{f_E(x)\}$ for each $x \in A$. Since $\pi \circ g$ is a continuous mapping such that $\pi \circ g \restriction A = f$, the proof is complete. ∎

1.7. Remarks. (a) It is well known that a subset B of a topological space X is a *zero-set* if and only if there exist a continuous mapping f from X into some metrizable topological space Y and a closed subset A of Y such that $B = f^{-1}[A]$. Therefore, Proposition 1.4 shows that D-closed sets are the proper analogue of zero-sets for the theory of developable spaces. From this point of view it is somewhat surprising that Corollary 1.6. holds, for the corresponding statement concerning zero-sets is known to be false (i.e. not every zero-set is C-embedded; see (Gillman and Jerison [1960], Example 3 K)).

(b) In view of the simple internal characterization of D-closed sets given by 1.3 and 1.4, it is quite natural to ask for a similar characterization of zero-sets. It turns out that such a characterization was already proved in [1958] by J. Kerstan. Let us call a collection $\mathscr{B}$ of closed subsets of a topological space X a *strong G_δ-collection*, if for every B in $\mathscr{B}$ there exist two sequences $(A_n)_{n<\omega}$ and $(B_n)_{n<\omega}$ such that $X \backslash B = \bigcup \{X \backslash A_n \mid n < \omega\}$ and $X \backslash A_n \subset B_n \subset X \backslash B$ for each $n < \omega$. Obviously, every strong G_δ-collection is a G_δ-collection (1.3). Kerstan proved that *a subset of a topological space is a zero-set if and only if it belongs to some strong G_δ-collection.* Unfortunately this nice result is little known. Quite recently an internal characterization of completely regular T_1-spaces was rediscovered by G. Reynolds [1979], which is an immediate consequence of Kerstan's theorem. One reason, why this theorem is not widely known, might be its lengthy proof in (Kerstan [1958]). Therefore the following short proof (due to Brandenburg and Mysior [1984a]) may be of some interest.

Note first that the collection of all zero-sets of a topological space is a strong G_δ-collection. Conversely, if a subset B of a topological space X belongs to a strong G_δ-collection $\mathscr{B}$ in X, one may assume without loss of generality that $\mathscr{B}$ is countable. Let $X_{\mathscr{B}}$ be the topological space obtained by supplying X with the topology generated by $\mathscr{B}$ as a subbase for the closed sets. Since $\mathscr{B}$ is a strong G_δ-collection, $X_{\mathscr{B}}$ is regular but not necessarily T_1. However, identifying points in $X_{\mathscr{B}}$ which have equal closures yields a second countable regular T_1-space Y. If $f\colon X_{\mathscr{B}} \to Y$ denotes the corresponding quotient map and $A = f[B]$, then $f\colon X \to Y$ is continuous, A is closed in Y, and $B = f^{-1}[A]$. By virtue of the Urysohn metrization theorem, Y is metrizable. Therefore A, being a closed subset of a metrizable space, is a zero-set. Consequently B is a zero-set in X.

(c) The spaces $\boldsymbol{D}$ and $\boldsymbol{D}_1$ are not compact. In fact, $(\pi[A(n, 1)])_{0<n<\omega}$ is a centered family of closed subsets of $\boldsymbol{D}_1$ which has an empty intersection.

(d) Both spaces, $\boldsymbol{D}$ and $\boldsymbol{D}_1$, have cardinality 2^{ω}. ∎

§ 2. D-Completely regular spaces

Recall from § 1 that a topological space is D-completely regular if and only if it can be embedded into a product of developable T_1-spaces. In (Brandenburg [1978]) we have used the theory of nearness spaces (Herrlich [1974]) as the main tool for proving some interesting characterizations of these spaces. Here we choose a different approach, based on the results of § 1, which yields in particular that the class of D-completely regular spaces is simply generated by the space $\boldsymbol{D}_1$. In order to formulate the first main theorem we need some notions from the theory of generalized metric spaces.

2.1. Definition. A topological space X is called

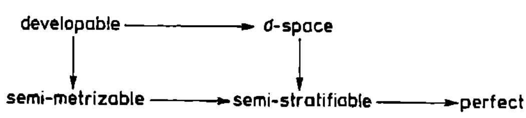
(i) a *σ-space* (Okuyama [1967]) if it has a σ-closure preserving closed network.

(ii) *semi-metrizable* if there exists a distance function $d\colon X \times X \to \boldsymbol{R}$ such that $d(x, y) \geqslant 0$, $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ for all $x, y \in X$, and $\operatorname{cl} A = \{x \in X \mid d(x, A) = 0\}$ for each $A \subset X$, where $d(x, A) = \inf\{d(x, a) \mid a \in A\}$.

(iii) *semi-stratifiable* (Creede [1970]) if for each open set U in X there exists a sequence $(U_n)_{n<\omega}$ of closed sets such that $U = \bigcup\{U_n \mid n < \omega\}$ and $U_n \subset V_n$ whenever $U \subset V$ and $n < \omega$.

(iv) *perfect* (e.g. Heath and Michael [1971]) if every open subset is an $F_{\sigma}^{\bullet}$-set.

The relationships between these classes of spaces are summarized in the following diagram.[4] For simplicity all spaces are assumed to be T_1-spaces.



Let us call a (sub)base for the closed sets of a topological space a G_δ-*(sub)base* if it is a G_δ-collection in the sense of 1.3. In (Brandenburg [1978]) we have shown that conditions (i)–(vii) in the following theorem are equivalent.

[4] For more on generalized metric spaces see (Gruenhage [1984]).

The equivalences (viii) and (ix) are essentially due to N. C. Heldermann [1980]. Note, however, that he used a slightly different space $\boldsymbol{D}_1$ and that (viii) is somewhat stronger than in his theorem.

2.2. THEOREM. *For a T_1-space X the following conditions are equivalent:*

(i) *X is D-completely regular.*

(ii) *X can be embedded into a product of T_1 σ-spaces.*

(iii) *X can be embedded into a product of semi-metrizable spaces.*

(iv) *X can be embedded into a product of semi-stratifiable T_1-spaces.*

(v) *X can be embedded into a product of perfect T_1-spaces.*

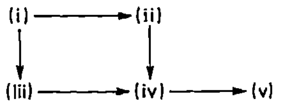
(vi) *The D-open subsets of X form a base for the topology of X.*

(vii) *X has a G_δ-(sub)base for the closed sets.*

(viii) *For every closed set A in X, for every point $x \in X \backslash A$ and for every pair a, b of distinct points in $\boldsymbol{D}_1$ there exists a continuous mapping f from X into $\boldsymbol{D}_1$ such that $f[A] \subset \{a\}$ and $f(x) = b$.*

(ix) *X can be embedded into ${}^{\varkappa}\boldsymbol{D}_1$, where $\varkappa = w(X)$ and ${}^{\varkappa}\boldsymbol{D}_1$ carries the natural product topology.*

Proof. The implications



follow immediately from the preceding diagram. In order to prove that (v) implies (vi) note first that every closed subset of a perfect space is D-closed, for the collection of all closed subsets of a perfect space is a G_δ-collection (1.3). In view of this observation it is not difficult to show that the collection of all D-open subsets of a subspace of a product of perfect spaces forms a base for the open sets.

Obviously, (vi) implies (vii). If a space has a G_δ-subbase for the closed sets, then it also has a G_δ-base (Brandenburg [1978], 3.1.17). Therefore, in order to verify that (vii) implies (viii) we may assume that there is a G_δ-base $\mathscr{B}$ for the closed sets of X. Thus, if A is a closed set in X and $x \in X \backslash A$, there exists a $B \in \mathscr{B}$ such that $A \subset B$ and $x \in X \backslash B$. Moreover, there is a $C \in \mathscr{B}$ such that $x \in C \subset X \backslash B$. If a, b are distinct points in $\boldsymbol{D}_1$, we can define a continuous mapping g: $B \cup C \to \boldsymbol{D}_1$ such that $g[B] \subset \{a\}$ and $g[C] \subset \{b\}$. By virtue of Corollary 1.6 there exists a continuous mapping f from X into $\boldsymbol{D}_1$ such that $f \restriction (B \cup C) = g$. Clearly, f has the desired properties. To show that (viii) implies (ix) is a matter of routine. Since (i) is formally weaker than (ix), the proof is complete. ∎

A topological space X is said to be a *universal space for the class* $\mathbf{E}$ of topological spaces, if $X \in \mathbf{E}$, and every space from $\mathbf{E}$ can be embedded into X. The following corollary shows that ${}^{\omega}\boldsymbol{D}_1$ is a universal space for the class of second countable developable T_1-spaces.

2.3. COROLLARY (Brandenburg [1983]). *For a T_1-space X the following conditions are equivalent:*

(i) *X has a countable G_δ-(sub)base for the closed sets.*

(ii) *X can be embedded into ${}^{\omega}D_1$.*

(iii) *X is a second countable developable space.* ∎

Let us mention some facts which indicate that the behaviour of developable Hausdorff spaces is essentially different.

2.4. THEOREM (i) (Mysior [198?]). *There is no universal space for the class of developable Hausdorff spaces of weight ω.*

(ii) (van Douwen [1979]). *There is no universal space for the class of separable developable Hausdorff spaces.*

(iii) (van Douwen [1979]). *There is no universal space for the class of separable Moore spaces.* ∎

On the other hand it is well known that for every infinite cardinal $\varkappa$ there exists a universal space for the class of metrizable spaces of weight $\varkappa$ (Kowalsky [1957]). Hence the following question arises which is well-known among people working in generalized metric spaces.

PROBLEM A. Let $\varkappa > \omega$ be a cardinal. Does there exist a developable T_1-space $D(\varkappa)$ of weight $\varkappa$ such that every developable T_1-space of weight $\varkappa$ can be embedded into $D(\varkappa)$?

Quite recently J. Chaber has answered the corresponding question concerning metacompact developable T_1-spaces in the affirmative.

2.5. THEOREM (Chaber [1983a], [1984a]). (i) *For every cardinal $\varkappa \geqslant \omega$ there exists a universal space for the class of metacompact developable T_1-spaces of weight $\varkappa$.*

(ii) *For every cardinal $\varkappa \geqslant \omega$ there exists a single orthocompact*(5) *developable T_1-space of weight $2^{\varkappa}$ which contains a copy of every orthocompact developable T_1-space of weight $\varkappa$.* ∎

Since completely regular T_1-spaces are characterized externally as

– (up to homeomorphism) the subspaces of compact Hausdorff spaces (Tikhonov [1930]) and

– the T_1-spaces which are uniformizable (Weil [1938]),

it is quite natural to wonder whether D-complete regularity can be characterized in a similar way. Of course, in order to get satisfactory answers one first has to find suitable generalizations of compact Hausdorff spaces respectively uniform spaces. In this context the following notion turns out to be useful.

(5) A topological space is called *orthocompact*, if every open cover has an interior-preserving open refinement, where a collection $\mathcal{U}$ of open sets is *interior-preserving*, if $\bigcap\{U \mid U \in \mathcal{U}'\}$ is open for each $\mathcal{U}' \subset \mathcal{U}$.

2.6. DEFINITION. A topological space is called *D-compact* if every open cover has a finite refinement consisting of open F_σ-sets.

Evidently, every compact Hausdorff space is D-compact. The co-finite topology on ω yields an example of a D-compact T_1-space which is not Hausdorff, while the co-finite topology on ω_1 defines a compact T_1-space which is not D-compact.

Whenever $\mathscr{U}$ is an open cover of a D-completely regular space X, one can find, for each $x \in X$, an open F_σ-set V_x and a $U_x \in \mathscr{U}$ such that $x \in V_x \subset U_x$ (2.2(vi)). If additionally X is assumed to be compact, there exists an $A \in [X]^{<\omega}$ such that $X = \bigcup\{V_x \mid x \in A\}$. This simple observation proves that (ii) implies (i) in the following proposition. The converse implication will be established in 4.3.

2.7. PROPOSITION. *For a T_1-space X the following conditions are equivalent:*

(i) *X is D-compact.*

(ii) *X is compact and D-completely regular.* ■

Using this proposition, we can now prove what might be called the analogue, for D-complete regularity, of Tikhonov's characterization of completely regular T_1-spaces.

2.8. THEOREM. *For a topological space X the following conditions are equivalent:*

(i) *X is D-completely regular.*

(ii) *X is homeomorphic to a subspace of a D-compact T_1-space.*

Proof. Because of 2.7 it suffices to show that (i) implies (ii). We claim that the following holds.

A. Every developable T_1-space of weight ω can be embedded as a dense subspace into a compact developable T_1-space.

Once we have proved Claim A, we may argue as follows. If X is D-completely regular, there is some cardinal $\varkappa$ such that X can be embedded into ${}^{\varkappa}\boldsymbol{D}_1$ (2.2.(ix)). By virtue of A, $\boldsymbol{D}_1$ has a developable T_1-compactification $\alpha\boldsymbol{D}_1$. Hence X is homeomorphic to a subspace of the D-compact space ${}^{\varkappa}(\alpha\boldsymbol{D}_1)$.

For the proof of Claim A consider an arbitrary developable T_1-space Y of weight ω and let $\mathscr{B}$ be a countable G_δ-base for the closed sets of Y (2.3). Moreover, let αY be the Wallman–Frink-compactification of Y with respect to $\mathscr{B}$, i.e. αY is the set of all $\mathscr{B}$-ultrafilters (see 7.3) supplied with the topology which has the collection $\mathscr{B}^* = \{B^* \mid B \in \mathscr{B}\}$ as a base for the closed sets, where $B^* = \{\mathscr{F} \in \alpha Y \mid B \in \mathscr{F}\}$. Since αY is a compact T_1-space, $\mathscr{B}^*$ is countable, and the correspondence $y \mapsto \mathscr{F}_y = \{B \in \mathscr{B} \mid y \in B\}$ defines an embedding from Y into αY (e.g. see Steiner [1968]), it suffices to show that αY is D-compact, for then the developability of αY follows from 2.7 and 2.3. But in order to prove that αY is D-compact it is enough to note that $\alpha Y \setminus B^* = \bigcup\{A^* \mid A \in \mathscr{B},\ A \cap B = \emptyset\}$ is an F_σ-set in αY for each $B \in \mathscr{B}$. ■

As a consequence of the preceding theorem every *D*-completely regular space has a T_1 *D*-compactification. We will see later (7.15) that certain *D*-completely regular spaces (including all countably compact *D*-completely regular spaces) have a *D*-compactification with a universal property similar to the Čech–Stone compactification.

Let us now explain the important relationship between *D*-complete regularity and the theory of nearness spaces.

2.9. DEFINITION. Let μ be a non-empty collection of covers of a set X.

(i) For each subset A of X,

$$\mathrm{int}_\mu A = \{x \in A \mid \mathrm{St}(x, \mathscr{U}) \subset A \text{ for some } \mathscr{U} \in \mu\}$$

is called the *interior of A with respect to* μ.

(ii) (Herrlich [1974]). The pair (X, μ) is called a *nearness space* (and μ is a *nearness structure* on X) if the following conditions are satisfied:

(N.1) If $\mathscr{U} \in \mu$ and $\mathscr{U}$ refines $\mathscr{V}$, then $\mathscr{V} \in \mu$.
(N.2) If $\mathscr{U}, \mathscr{V} \in \mu$, then $\mathscr{U} \wedge \mathscr{V} \in \mu$.
(N.3) If $\mathscr{U} \in \mu$, then $\mathrm{int}_\mu \mathscr{U} = \{\mathrm{int}_\mu U \mid U \in \mathscr{U}\} \in \mu$.

Any subcollection β of a nearness structure μ on X with the property that for each $\mathscr{V} \in \mu$ there is a $\mathscr{U} \in \beta$ which refines $\mathscr{V}$ is called a *base* of μ. It is a *subbase* of μ if the collection of all covers of the form $\mathscr{U}_0 \wedge \quad \wedge \mathscr{U}_{n-1}$, $n < \omega$, $\mathscr{U}_i \in \beta$, forms a base of μ. Since arbitrary nearness spaces are too general for our purpose, we have to impose some suitable restrictions.

2.10. DEFINITION (Brandenburg [1978]). (i) A non-empty collection β of covers of a set X is called *kernel-normal* if for each $\mathscr{U} \in \beta$ there is a $\mathscr{V} \in \beta$ which refines $\mathrm{int}_\beta \mathscr{U} = \{\mathrm{int}_\beta U \mid U \in \mathscr{U}\}$.

(ii) A nearness space (X, μ) is said to be *para-uniform* if for each $\mathscr{U} \in \mu$ there exists a countable kernel-normal subcollection $\beta_{\mathscr{U}}$ of μ such that $\mathscr{U} \in \beta_{\mathscr{U}}$.

Clearly, every uniform (nearness) space (Herrlich [1974]) is para-uniform, for it is easily seen that every normal cover in the sense of J. W. Tukey [1940] is kernel-normal. The collection of all covers of ω which are open with respect to the co-finite topology on ω forms a base for a para-uniform nearness structure on ω which is not uniform. Since every nearness structure μ on a set X induces a topology $\tau_\mu = \{U \subset X \mid \mathrm{int}_\mu U = U\}$, it is natural to ask for a characterization of those topological spaces (X, τ) which are *para-uniformizable*, i.e. for which there is a para-uniform nearness structure μ on X such that $\tau_\mu = \tau$. It turns out that a T_1-space is para-uniformizable if and only if it is *D*-completely regular. For a proof of this fact we need the following basic observation concerning *kernel-normal open covers* of a topological space.

2.11. LEMMA (Brandenburg [1978], [1981]). *For an open cover $\mathcal{U} = (U_i)_{i \in I}$ of a topological space (X, τ) the following conditions are equivalent:*

(i) *$\mathcal{U}$ is kernel-normal, which by definition means that there is a countable kernel-normal collection β of open covers of X such that $\mathcal{U} \in \beta$.*

(ii) *There exists a developable topology $\tau' \subset \tau$ and a τ'-open cover $\mathcal{V} = (V_i)_{i \in I}$ of X such that $V_i \subset U_i$ for each $i \in I$.*

Proof. (i) implies (ii): Assume that $\mathcal{U}$ is kernel-normal. Then there exists a countable kernel-normal collection β of open covers of X such that $\mathcal{U} \in \beta$. Denote by μ the collection consisting of all open covers of the form $\mathcal{U}_0 \wedge \quad \wedge \mathcal{U}_{n-1}$, where $\{\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}\} \in [\beta]^{<\omega}$. It is easy to see that $\tau' = \{U \subset X \mid \mathrm{int}_\mu U = U\}$ is a topology on X such that $\tau' \subset \tau$. We claim that (X, τ') is developable. To prove this assertion it suffices to verify:

A. For each $\mathcal{H} \in \mu$ the collection $\mathrm{int}_\mu \mathcal{H}$ is a τ'-open cover of X.

Indeed, since μ is countable, A implies that $\{\mathrm{int}_\mu \mathcal{H} \mid \mathcal{H} \in \mu\}$ is a development of (X, τ').

In order to verify Claim A consider an arbitrary $\mathcal{H} \in \mu$. There exist $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1} \in \beta$ such that $\mathcal{H} = \mathcal{U}_0 \wedge \quad \wedge \mathcal{U}_{n-1}$. At first we show that $\mathrm{int}_\mu \mathcal{H}$ is a cover of X. Since β is kernel-normal, $\mathrm{int}_\beta \mathcal{U}_i$ is a cover of X for each $i < n$. Thus, given an arbitrary point $x \in X$, there exist $U_0 \in \mathcal{U}_0, \ldots, U_{n-1} \in \mathcal{U}_{n-1}$ such that $x \in \mathrm{int}_\beta U_0 \cap \ldots \cap \mathrm{int}_\beta U_{n-1}$. For each $i < n$ there is a $\mathcal{W}_i \in \beta$ such that $\mathrm{St}(x, \mathcal{W}_i) \subset U_i$. Hence $\mathrm{St}(x, \mathcal{W}_0 \wedge \quad \wedge \mathcal{W}_{n-1}) \subset U_0 \cap \ldots \cap U_{n-1}$, i.e. $x \in \mathrm{int}_\mu H$ where $H = U_0 \cap \ldots \cap U_{n-1} \in \mathcal{H}$. Next we prove that for each $H \in \mathcal{H}$, $\mathrm{int}_\mu H$ is τ'-open, i.e. that $\mathrm{int}_\mu H \subset \mathrm{int}_\mu(\mathrm{int}_\mu H)$. To this end consider an arbitrary $H \in \mathcal{H}$ and a point $x \in \mathrm{int}_\mu H$. There exist $\mathcal{E}_0, \ldots, \mathcal{E}_{k-1} \in \beta$ such that $\mathrm{St}(x, \mathcal{E}_0 \wedge \ldots \wedge \mathcal{E}_{k-1}) \subset H$. Since β is kernel-normal, there is an $\mathcal{F}_i \in \beta$ for each $i < k$ such that $\mathcal{F}_i$ refines $\mathrm{int}_\beta \mathcal{E}_i$. We claim that $\mathrm{St}(x, \mathcal{F}_0 \wedge \ldots \wedge \mathcal{F}_{k-1}) \subset \mathrm{int}_\mu H$. In fact, if $z \in \mathrm{St}(x, \mathcal{F}_0 \wedge \quad \wedge \mathcal{F}_{k-1})$, there exist $E_0 \in \mathcal{E}_0, \ldots, E_{k-1} \in \mathcal{E}_{k-1}$ such that $x, z \in \mathrm{int}_\beta E_0 \cap \ldots \cap \mathrm{int}_\beta E_{k-1}$. For each $i < k$ there is a $\mathcal{G}_i \in \beta$ such that $\mathrm{St}(z, \mathcal{G}_i) \subset E_i$. Now

$$\mathrm{St}(z, \mathcal{G}_0 \wedge \quad \wedge \mathcal{G}_{k-1}) \subset E_0 \cap \ldots \cap E_{k-1} \subset \mathrm{St}(x, \mathcal{E}_0 \wedge \quad \wedge \mathcal{E}_{k-1}) \subset H,$$

which proves that $z \in \mathrm{int}_\mu H$. Hence we have shown that $\mathrm{St}(x, \mathcal{F}_0 \wedge \quad \wedge \mathcal{F}_{k-1}) \subset \mathrm{int}_\mu H$, i.e., $x \in \mathrm{int}_\mu(\mathrm{int}_\mu H)$, which completes the proof of Claim A.

For each $i \in I$ define now $V_i = \mathrm{int}_\mu U_i$. Since $\mathcal{U}$ belongs to β, A implies that $(V_i)_{i \in I}$ is a τ'-open cover with the desired property.

(ii) implies (i): Assume that there is a developable topology $\tau' \subset \tau$ and a τ'-open cover $\mathcal{V}$ of X which refines $\mathcal{U}$. If $(\mathcal{V}_n)_{n<\omega}$ is a development of (X, τ'), then $\beta = \{\mathcal{U}\} \cup \{\mathcal{V}_n \mid n < \omega\}$ is a countable kernel-normal collection of τ-open covers of X containing $\mathcal{U}$, i.e. $\mathcal{U}$ is kernel-normal. ∎

Our second lemma on kernel-normal open covers will be applied in Sections 4 and 5.

2.12. LEMMA (Brandenburg [1978], [1981]). *If μ is a non-empty collection of open covers of a topological space X such that*

(i) *$\mathscr{U} \wedge \mathscr{V} \in \mu$ whenever $\mathscr{U} \in \mu$ and $\mathscr{V} \in \mu$,*

(ii) *for each $\mathscr{U} \in \mu$ there is a sequence $\beta = (\mathscr{U}_n)_{0<n<\omega}$ in μ containing a $\mathscr{U}_n$ such that $\mathrm{int}_\beta \mathscr{U}_n$ refines $\mathscr{U}$,*

then every $\mathscr{U} \in \mu$ is kernel-normal.

Proof. Consider an arbitrary $\mathscr{U} \in \mu$. For technical reasons define $\mathscr{U}_{(0,n)} = \{X\}$ whenever $0 < n < \omega$. Using complete induction and conditions (i) and (ii) it is easy to construct an open cover $\mathscr{U}_{(k,k)} \in \mu$, $0 < k < \omega$, and a sequence $\beta(k) = (\mathscr{U}_{(k,n)})_{n>k}$ in μ such that

(a) $\mathscr{U} = \mathscr{U}_{(1,1)}$ and $\mathscr{U}_{(k,k)} = \mathscr{U}_{(k-1,k)}$ for each $k > 1$;

(b) $\mathscr{U}_{(k,k+1)}$ refines $\mathrm{int}_{\beta(k)} \mathscr{U}_{(k,k)}$ whenever $0 < k < \omega$;

(c) $\mathscr{U}_{(k,n)}$ refines $\mathscr{U}_{(k-1,n)}$ whenever $0 < k < \omega$ and $n \geqslant k$.

We set $\xi = (\mathscr{U}_{(k,k)})_{0<k<\omega}$ and claim that ξ is kernel-normal. To prove this assertion note first that $\mathrm{int}_{\beta(k)} A \subset \mathrm{int}_\xi A$ whenever $A \subset X$ and $0 < k < \omega$. For if $x \in \mathrm{int}_{\beta(k)} A$, there exists an $n > k$ such that $\mathrm{St}(x, \mathscr{U}_{(n,k)}) \subset A$. Condition (c) implies that

$$\mathrm{St}(x, \mathscr{U}_{(n,n)}) \subset \mathrm{St}(x, \mathscr{U}_{(n-1,n)}) \subset \quad \subset \mathrm{St}(x, \mathscr{U}_{(k,n)}).$$

Therefore $x \in \mathrm{int}_\xi A$. In particular, it follows that $\mathrm{int}_{\beta(k)} \mathscr{U}_{(k,k)}$ refines $\mathrm{int}_\xi \mathscr{U}_{(k,k)}$ for all k such that $0 < k < \omega$. By virtue of (a) and (b) $\mathscr{U}_{(k+1,k+1)} = \mathscr{U}_{(k,k+1)}$ refines $\mathrm{int}_{\beta(k)} \mathscr{U}_{(k,k)}$. Hence $\mathscr{U}_{(k+1,k+1)}$ refines $\mathrm{int}_\xi \mathscr{U}_{(k,k)}$ whenever $0 < k < \omega$, which proves that ξ is kernel-normal. Since $\mathscr{U} \in \xi$, the proof is complete. ∎

We can now prove the characterization of para-uniformizable topological spaces which actually was our starting-point for the investigation of D-completely regular spaces in (Brandenburg [1978]). The equivalence (ii) was added by N. C. Heldermann in [1980]. The equivalence (iii) is new and will turn out to be important later (see 7.6).

2.13. THEOREM. *For a topological space $X = (X, \tau)$ the following conditions are equivalent:*

(i) *X has a G_δ-(sub)base for the closed sets.*

(ii) *For each closed set A in X and for each $x \in X \setminus A$ there exists a continuous mapping f from X into $\boldsymbol{D}$ such that $f(x) \notin \mathrm{cl} f[A]$.*

(iii) *The collection of all countable kernel-normal open covers of X is a base for a para-uniform nearness structure μ_ω on X satisfying $\tau_{\mu_\omega} = \tau$.*

(iv) *The collection of all kernel-normal open covers of X is a base for a para-uniform nearness structure μ_f on X satisfying $\tau_{\mu_f} = \tau$.*(6)

(v) *X is para-uniformizable.*

Proof. (i) implies (ii): Let $\mathscr{B}$ be a G_δ-base for the closed sets of X. We may assume that $\mathscr{B}$ is closed with respect to finite unions (Brandenburg [1978],

(6) Note that μ_f is the finest among all para-uniform nearness structures on X which induce τ.

3.1.17). If A is a closed set in X and $x \in X \backslash A$, there exist $B, C \in \mathscr{B}$ such that $A \subset B$ and $x \in C \subset X \backslash B$. Hence, if b, c are points in $\boldsymbol{D}$ such that $\mathrm{cl}\{b\} \cap \mathrm{cl}\{c\} = \emptyset$, we can define a continuous mapping g from $B \cup C$ into $\boldsymbol{D}$ in such a way that $g[B] \subset \{b\}$ and $g[C] \subset \{c\}$. Since $B \cup C$ is D-closed, there exists a continuous mapping f from X into $\boldsymbol{D}$ such that $f(z) \in \mathrm{cl}\{g(z)\}$ whenever $z \in B \cup C$ (1.5). In particular, $f(x) \notin \mathrm{cl} f[A]$.

(ii) implies (iii): It is easily seen that

$$\mu_\omega = \{\mathscr{V} \mid \mathscr{V} \text{ is a cover of } X \text{ which is refined by some countable kernel-normal open cover of } X\}$$

satisfies conditions (N.1) and (N.2) for a nearness structure on X (see 2.9(ii)). In order to prove that it also satisfies (N.3) consider an arbitrary $\mathscr{V} \in \mu_\omega$. Let $\mathscr{U}$ be a countable kernel-normal open cover of X such that $\mathscr{U}$ refines $\mathscr{V}$. By virtue of 2.11 there exist a developable topology $\tau' \subset \tau$ and a τ'-open cover $\mathscr{W} = (W_U)_{U \in \mathscr{U}}$ of X such that $W_U \subset U$ for each $U \in \mathscr{U}$. For each $U \in \mathscr{U}$ let f_U be a continuous mapping from (X, τ') into $\boldsymbol{D}$ such that $X \backslash W_U = f^{-1}[A(0)]$ (1.4(ii)). Since $\mathscr{U}$ is countable, the initial topology τ'' on X with respect to $(f_U\colon (X, t') \to \boldsymbol{D})_{U \in \mathscr{U}}$ is developable and second countable. We will show that $\mathscr{W} \in \mu_\omega$ refines $\{\mathrm{int}_{\mu_\omega} U \mid U \in \mathscr{U}\}$, which immediately implies that $\{\mathrm{int}_{\mu_\omega} V \mid V \in \mathscr{V}\}$ belongs to μ_ω. To this end consider a $U \in \mathscr{U}$ and a point $x \in W_U$. Since (X, τ'') is developable and second countable, there is a τ''-open cover $\mathscr{G}$ such that $\mathrm{St}(x, \mathscr{G}) \subset W_U \subset U$. Since $\mathscr{G} \in \mu_\omega$, it follows that $x \in \mathrm{int}_{\mu_\omega} U$, i.e. that $W_U \subset \mathrm{int}_{\mu_\omega} U$ for each $U \in \mathscr{U}$, which completes the verification of (N.3).

Evidently, $\tau_{\mu_\omega} \subset \tau$. For the proof of the reverse inclusion consider an arbitrary $B \in \tau$ and a point $x \in B$. By virtue of (ii) there exists a continuous mapping f from X into $\boldsymbol{D}$ such that $f(x) \in \boldsymbol{D} \backslash \mathrm{cl} f[X \backslash B]$. If $\tau_f \subset \tau$ is the initial topology on X with respect to f, we can find a countable τ_f-open cover $\mathscr{H}$ of X such that $\mathrm{St}(x, \mathscr{H}) \subset B$. Since $\mathscr{H} \in \mu_\omega$, it follows that $x \in \mathrm{int}_{\mu_\omega} B$, i.e. that $B = \mathrm{int}_{\mu_\omega} B$. Consequently, $B \in \tau_{\mu_\omega}$, i.e. $\tau_{\mu_\omega} = \tau$.

Since an obvious modification of the argument used to verify that μ_ω satisfies (N.3) shows that (X, μ_ω) is para-uniform, the proof of this implication is complete.

It is easily seen that (iii) implies (iv). Therefore it only remains to verify that (v) implies (i). To this end consider a para-uniform nearness structure μ on X such that $\tau_\mu = \tau$. For every $\mathscr{V} \in \mu$ let $\xi(\mathscr{V})$ be a countable kernel-normal subcollection of μ containing $\mathscr{V}$ Note that every $\xi(\mathscr{V})$ is a subbase for a nearness structure $\mu(\mathscr{V}) \subset \mu$ on X. We claim that

$$\mathscr{B} = \{X \backslash U \mid \text{there is a } \mathscr{V} \in \mu \text{ such that } \mathrm{int}_{\tau_{\mu(\mathscr{V})}} U = U\}$$

is a G_δ-base for the closed sets of X.

Consider a closed subset A of X and a point $x \in X \backslash A$. There is a $\mathscr{V} \in \mu$ such that $\mathrm{St}(x, \mathscr{V}) \subset X \backslash A$. It follows that $x \in \mathrm{int}_{\mu(\mathscr{V})} V \subset A$ for some $V \in \mathscr{V}$, i.e. we have shown that $\mathscr{B}$ is a base for the closed sets of X. If $X \backslash U \in \mathscr{B}$, where

$\operatorname{int}_{\tau_{\mu(\mathscr{V})}} U = U$ for some $\mathscr{V} \in \mu$, consider a countable base $\beta(\mathscr{V})$ of $\mu(\mathscr{V})$. For $\mathscr{W} \in \beta(\mathscr{V})$ define

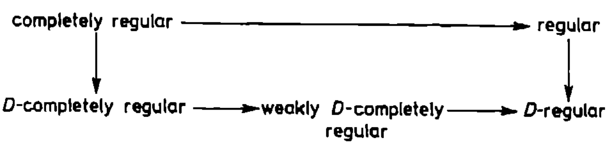
$$U(\mathscr{W}) = \bigcup \{\operatorname{int}_{\mu(\mathscr{V})} W \mid W \in \mathscr{W},\ \operatorname{int}_{\mu(\mathscr{V})} W \cap (X \backslash U) \neq \emptyset\}.$$

Then $\{X \backslash U(\mathscr{W}) \mid \mathscr{W} \in \beta(\mathscr{V})\}$ is a countable subcollection of $\mathscr{B}$ satisfying $U = \bigcup \{X \backslash U(\mathscr{W}) \mid \mathscr{W} \in \beta(\mathscr{V})\}$, which shows that $\mathscr{B}$ is a G_δ-collection and completes the proof. ∎

In [1981] N. C. Heldermann has studied the following generalizations of D-completely regular spaces.

2.14. DEFINITION. A topological space is called *weakly D-completely regular* if it has a base consisting of open F_σ-sets. It is called *D-regular* if every point has a neighborhood base of (not necessarily open) F_σ-sets.(7)

For T_1-spaces the following implications hold:



The main contribution of (Heldermann [1981]) is the construction of (somewhat involved) examples witnessing that there are no other implications between these notions. In particular, it is shown that there are a regular T_1-space which is not weakly D-completely regular, and a regular weakly D-completely regular T_1-space which is not D-completely regular.

2.15. Remarks. (a) (Weak) D-complete regularity and D-regularity are productive, hereditary, additive, and preserved by inverse limits. However, none of these properties is preserved by quotients. That D-complete regularity is not invariant with respect to perfect mappings was recently shown by S.-H. Sun and Y.-M. Wang [1988].

(b) Para-uniformizable topological spaces form precisely the *bireflective hull* BH(**D**) (e.g. see Marny [1979]) of the class **D** of developable T_1-spaces. For a topological space X the corresponding bireflection is given by id: $X \to X_D$, where X_D is the space obtained by supplying the underlying set of X with the topology which has the collection of all D-open subsets of X as a base (Brandenburg [1978], Heldermann [1981]).

(c) For a topological space X the epireflection corresponding to the class of D-completely regular spaces is given by e: $X \to eX$, where eX is the image of X under the mapping e from X into ${}^{C(X,\boldsymbol{D}_1)}\boldsymbol{D}_1$ which maps each $x \in X$ onto $e(x) \in {}^{C(X,\boldsymbol{D}_1)}\boldsymbol{D}_1$, defined by $e(x)(f) = f(x)$ for each $f \in C(X, \boldsymbol{D}_1)$. Of course, both spaces $C(X, \boldsymbol{D}_1)$ and ${}^{C(X,\boldsymbol{D}_1)}\boldsymbol{D}_1$ are supposed to carry the topology of pointwise convergence.

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(7) Our terminology here differs from that of (Heldermann [1981]).

(d) While the class of D-completely regular spaces is simply generated by the space D_1, it is a consequence of a theorem of H. Herrlich [1965] that the class of D-regular spaces is not simple (see also 3.2). That the class of weakly D-completely regular spaces is not simple was shown by A. Mysior [1980].

(e) In (Brandenburg [1988]) it is shown that a nearness space is para-uniform if and only if it can be embedded into a product of nearness spaces having a countable base. It is an open problem whether the Herrlich completion (Herrlich [1974]) of a para-uniform nearness space is necessarily para-uniform.

(f) Alternatively, Claim A in the proof of Theorem 2.8 can be verified using (Chaber [1984a], Proposition 3.1).

(g) A T_1-space is completely regular if and only if it has a strong G_δ-(sub)base (1.7(b)) for the closed sets. This little known internal characterization of complete regularity due to J. Kerstan [1958] corresponds to our 2.2 (vii).

(h) Since every metrizable space can be embedded into ${}^{\varkappa}\boldsymbol{R}$ for some cardinal $\varkappa$, it is worth mentioning that 2.2 implies that every developable T_1-space can be embedded in ${}^{\varkappa}R$ for some cardinal $\varkappa$, where R denotes the reals supplied with the co-finite topology. To verify this fact it suffices to recall that every T_1-space of cardinality $\leqslant 2^{\omega}$ can be embedded into ${}^{\lambda}R$ for some cardinal λ (see Engelking [1977], Problem 2.7.8(b)), and to apply 2.2. ■

Recall that a T_1-space is called *quasi-metrizable*, if there exists a distance function $d\colon X \times X \to \boldsymbol{R}$ such that $d(x, y) \geqslant 0$, $d(x, y) = 0$ if and only if $x = y$, $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$, and $(U(x, \varepsilon))_{\varepsilon > 0}$ forms a neighborhood base for each $x \in X$, where $U(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$. In view of 2.2(iii) the following problem arises naturally.

PROBLEM B. Characterize internally the spaces belonging to the epireflective hull of the class $\mathbf{Q}$ of quasi-metrizable T_1-spaces.

§ 3. On the epireflective hull of Moore spaces

For those topologists who are used to work with spaces which are at least regular and T_1 the situation would be much nicer if results similar to those obtained in § 1 and § 2 could be proved for the spaces belonging to the epireflective hull of the class of Moore spaces. However, in response to a problem raised by H. Herrlich, we will show in this section that the epireflective hull of the class of Moore spaces is not simple. Consequently, there is little hope that for this class analogues of Theorems 2.2(vii), 2.8, and 2.13 can be obtained. In fact, the corollary to the following theorem shows that even for developable Hausdorff spaces the situation is essentially different from the T_1-case (compare with 2.3 and 2.4). All results in this section are from the paper (Brandenburg and Mysior [1984]).

3.1. THEOREM. *For every Hausdorff space Y there exists a non-trivial metacompact Moore space on which all continuous mappings into Y are constant.*

3.2. COROLLARY. *Whenever* $\mathbf{E}$ *is a class of Hausdorff spaces containing all metacompact Moore spaces,* $\mathbf{E}$ *is not simple, i.e. there is no Hausdorff space X such that* $\mathbf{E} = \mathrm{EH}(\{X\})$. *In particular, neither the epireflective hull of the class of developable Hausdorff spaces nor the epireflective hull of the class of Moore spaces is simple.* ∎

Another consequence of Theorem 3.1 is that in the special case $Y = \boldsymbol{R}$ one obtains a metacompact Moore space ([8]) on which every real-valued continuous mapping is constant. The first example of a Moore space which admits only constant real-valued mappings was found by S. Armentrout [1961] based on a non-completely regular Moore space constructed by F. B. Jones [1958]. Other examples were given by J. N. Younglove [1969] and P. Roy (cited in Jones [1973]). Although their spaces are not metacompact they do have other nice properties. On the other hand, we believe that the construction we are going to present now is simpler.

The main step in the proof of Theorem 3.1 is Lemma 3.4 below. For its verification we use the following result of Bernstein type which seems to be a part of the set-theoretical folklore. However, for the sake of completeness, we include a brief argument.

3.3. LEMMA. *Let λ be an uncountable cardinal. If $\mathscr{G}$ is a family of subsets of a set X such that $|\mathscr{G}| = \lambda$ and $|G| = \lambda$ for each $G \in \mathscr{G}$, then there exists a partition $\mathscr{Z}$ of X such that $|\mathscr{Z}| = \omega$ and $|G \cap Z| = \lambda$ for each $G \in \mathscr{G}$ and $Z \in \mathscr{Z}$.*

Proof. For every $G \in \mathscr{G}$ there exists a partition $\mathscr{F}_G$ of G into λ sets of cardinality λ. The set $\mathscr{F} = \bigcup\{\mathscr{F}_G \mid G \in \mathscr{G}\}$ can be written as $\mathscr{F} = \{F_\gamma \mid \gamma < \lambda\}$. Using transfinite induction one can choose, for every $\gamma < \lambda$, a countable set

$$Z_\gamma = \{z_{\gamma,n} \mid n < \omega\} \subset F_\gamma \setminus \bigcup\{Z_\alpha \mid \alpha < \gamma\}.$$

If $Z_n = \{z_{\gamma,n} \mid \gamma < \lambda\}$ for each $n \geqslant 1$ and $Z_0 = X \setminus \bigcup\{Z_n \mid 1 \leqslant n < \omega\}$, then $\mathscr{Z} = \{Z_n \mid n < \omega\}$ is the desired partition of X. ∎

3.4. LEMMA. *For every Hausdorff space Y there exists a metacompact Moore space R containing two distinct points a, b such that $f(a) = f(b)$ for every continuous mapping f from R into Y.*

Proof. Let Y be an arbitrary Hausdorff space and let $\varkappa$ be a cardinal such that $|Y| \leqslant \varkappa$ and $\varkappa^\omega = 2^\varkappa$. Note, for example, that the cardinal $|Y| + 2^{|Y|} + 2^{2^{|Y|}} + \quad$ satisfies the hypotheses on $\varkappa$ (see Sierpiński [1965], page 154). Moreover, let $X = (X, d)$ be a metric space of weight $\varkappa$ such that every

([8]) Note that metacompact Moore spaces are identical with T_1-spaces having a *uniform base* in the sense of P. S. Aleksandrov [1960].

nonempty open subset of X has cardinality $2^\varkappa$, e.g. the countable power of a discrete space of cardinality $\varkappa$ with the usual product metric. Denote by $\mathscr{G}$ the collection of all subsets of X which have cardinality $2^\varkappa$ and are intersections of $\leqslant \varkappa$ F_σ-sets in X. Since $\mathscr{G}$ is of cardinality $2^\varkappa$, it follows from Lemma 3.3 that there exists a partition $\{Z_i \mid i \in Z\}$ of X such that $|G \cap Z_i| = 2^\varkappa$ for each integer i and for each G in $\mathscr{G}$.

The space R is obtained by defining the following topology on $X \times X \cup \{a, b\}$, where a, b are two distinct points not contained in $X \times X$. All points in $X \times X$ with distinct coordinates are isolated. For a point x in Z_i a basic neighborhood system of (x, x) in R consists of all sets

$$U_n(x, x) = \{(x, z) \mid z \in Z_i \cup Z_{i+1},\ d(x, z) < 1/n\} \cup \{(z, x) \mid z \in Z_i \cup Z_{i+1},\ d(x, z) < 1/n\},$$

where $0 < n < \omega$. Finally, basic neighborhoods of a and b are of the form

$$U_n(a) = \{a\} \cup \bigcup \{U_n(x, x) \mid x \in \bigcup \{Z_i \mid i \leqslant -n\}\}$$

and

$$U_n(b) = \{b\} \cup \bigcup \{U_n(x, x) \mid x \in \bigcup \{Z_i \mid i \geqslant n\}\},$$

respectively. Since basic neighborhoods of points from $X \times X$ are clopen in R and $\mathrm{cl}_R\, U_{n+1}(a) \subset U_n(a)$, $\mathrm{cl}_R\, U_{n+1}(b) \subset U_n(b)$ whenever $0 < n < \omega$, the space R is regular and T_1. Moreover, R is a metacompact Moore space, for the open covers

$$\mathscr{U}_n = \{U_n(a), U_n(b)\} \cup \{U_n(x, x) \mid x \in X\} \cup \{\{(y, z)\} \mid (y, z) \in X \times X \setminus \bigcup \{U_n(x, x) \mid x \in X\}\}$$

are point-finite and form a development of R (see Engelking [1977], 5.4.7).

Now let f be an arbitrary continuous mapping from R into Y. To prove that $f(a) = f(b)$ it suffices to show that $f(b) \in V$ for every open neighborhood V of $f(a)$ in Y. To this end consider an arbitrary open set V in Y containing $f(a)$. For each $y \in Y$ denote by $A(y)$ the set of all points $x \in X$ such that $f(x, x) = y$. By the continuity of f there exists an integer i_0 such that $\{A(y) \mid y \in V\}$ covers Z_{i_0}. Consequently, since $|V| \leqslant \varkappa$ and $|Z_{i_0}| = 2^\varkappa$, there must exist a $y_0 \in V$ such that $|A(y_0) \cap Z_{i_0}| = 2^\varkappa$. We claim that $f(b) = y_0$. Note that this is proved once we have shown that every neighborhood of b meets $f^{-1}[\{y_0\}]$ which, however, is an immediate consequence of the following claim.

A. $|A(y_0) \cap Z_i| = 2^\varkappa$ for every $i \geqslant i_0$.

To prove Claim A we proceed by induction. Assume $i \geqslant i_0$ and $|A(y_0) \cap Z_i| = 2^\varkappa$. To see that $|A(y_0) \cap Z_{i+1}| = 2^\varkappa$ it suffices to construct a set G in $\mathscr{G}$ such that $G \cap Z_{i+1} \subset A(y_0) \cap Z_{i+1}$. For this purpose we define for every point y in Y distinct from y_0 an F_σ-set $F(y)$ in X containing $A(y_0)\cap Z_i$ and disjoint from $A(y)\cap Z_{i+1}$. Observe that $G = \bigcap\{F(y) \mid y\in Y,\ y\neq y_0\}$ then has the desired properties.

Let y be an arbitrary point in Y distinct from y_0. There exists an open neighborhood W of y_0 which does not contain y in its closure. For each $0<n<\omega$ denote by A_n the set of all points $x\in A(y_0)\cap Z_i$ such that $f[U_n(x, x)] \subset W$. Since $F(y) = \bigcup\{\mathrm{cl}_X A_n \mid 0<n<\omega\}$ is an F_σ-set in X containing $A(y_0)\cap Z_i$, it only remains to prove that $\mathrm{cl}_X A_n$ and $A(y)\cap Z_{i+1}$ are disjoint whenever $0<n<\omega$. To this end consider an arbitrary n such that $0<n<\omega$ and a point $z\in A(y)\cap Z_{i+1}$. Since $f(z, z)$ does not belong to the closure of W in Y, there exists an $m\geqslant n$ such that $f[U_m(z, z)]\cap W=\emptyset$. It follows that $d(x, z)\geqslant 1/m$ for each $x\in A_n$, for otherwise there would exist an $x\in A_n$ such that $U_m(x, x)\cap U_m(z, z)\neq\emptyset$ which is impossible. Hence $z\notin \mathrm{cl}_X A_n$ which completes the proof. ∎

Proof of Theorem 3.1. Let Y be an arbitrary Hausdorff space. Our first aim is to construct for every Moore space M a Moore space $R(M)$ containing M as a closed subspace in such a way that every continuous mapping from $R(M)$ into Y is constant on M. Additionally $R(M)$ will be metacompact if and only if M is metacompact.

Let M be an arbitrary Moore space and let $(\mathscr{V}_n)_{n<\omega}$ be a development of M such that $\mathscr{V}_{n+1}$ refines $\mathscr{V}_n$ for each $n<\omega$. The underlying set of $R(M)$ is $\{p\}\cup(R'\times M)\cup M$, where R is the space from Lemma 3.4, $R' = R\setminus\{a, b\}$, and p is a point neither contained in $R'\times M$ nor in M. For each $n<\omega$ define a cover $R(\mathscr{V}_n)$ of $R(M)$ as follows. Let $(\mathscr{U}_n)_{n<\omega}$ be a development of R such that $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$ and $\mathscr{U}_n$ is point-finite for each $n<\omega$. Moreover, let $(U^a_n)_{n<\omega}$ and $(U^b_n)_{n<\omega}$ be sequences of open subsets of R' such that $(U^a_n\cup\{a\})_{n<\omega}$ and $(U^b_n\cup\{b\})_{n<\omega}$ are neighborhood bases in R of a and b, respectively, and $\mathrm{cl}_{R'} U^a_{n+1}\subset U^a_n$, $\mathrm{cl}_{R'} U^b_{n+1}\subset U^b_n$ for each $n<\omega$. Then $R(\mathscr{V}_n)$ consists of $\{p\}\cup(U^a_n\times M)$, all sets $U\times\{z\}$, where $U\in\mathscr{U}_n$ and $z\in M$, and all sets $R(V) = (U^b_n\times V)\cup V$, where $V\in\mathscr{V}_n$. One can easily check that taking $\bigcup\{R(\mathscr{V}_n)\mid n<\omega\}$ as a base yields a regular T_1-space $R(M)$ for which $(R(\mathscr{V}_n))_{n<\omega}$ is a development. Obviously, M is contained in $R(M)$ as a closed subspace. For further reference we note some additional properties of $R(M)$ which can be easily verified.

(i) $R(V)\cap M = V$ for each $V\in\mathscr{V}_n$;
(ii) $R(V)\cap R(W)=\emptyset$ whenever $V, W\in\bigcup\{\mathscr{V}_n\mid n<\omega\}$ and $V\cap W=\emptyset$;
(iii) $R(W)\subset R(V)$ whenever $V, W\in\bigcup\{\mathscr{V}_n\mid n<\omega\}$ and $W\subset V$;
(iv) $\mathrm{cl}_{R(M)}R(W)\subset R(V)$ whenever $\mathrm{cl}_M W\subset V$, $W\in\mathscr{V}_m$, $V\in\mathscr{V}_n$ and $n<m$;
(v) $R(\mathscr{V}_{n+1})$ refines $R(\mathscr{V}_n)$ for each $n<\omega$;
(vi) $R(\mathscr{V}_n)$ is point-finite if and only if $\mathscr{V}_n$ is point-finite.

Note that the topology on $R(M)$ is defined in such a way that for every $z\in M$ there is an obvious homeomorphism from R onto the subspace

$\{p\}\cup(R'\times\{z\})\cup\{z\}$ of $R(M)$ mapping a onto p and b onto z. Therefore $f(z)=f(p)$ for every continuous mapping f from $R(M)$ into Y, i.e. every such mapping is constant of M.

Starting with an arbitrary Moore space M_0 having a development $(\mathscr{V}_n^0)_{n<\omega}$ with every $\mathscr{V}_n^0$ point-finite and $\mathscr{V}_{n+1}^0$ refining $\mathscr{V}_n^0$, we can now define a sequence $M_0\subset M_1\subset M_2\subset\ldots$ of Moore spaces according to the rule $M_{k+1}=R(M_k)$. In the following we consider for every M_k the development $(\mathscr{V}_n^k)_{n<\omega}$ which is inductively obtained from $(\mathscr{V}_n^0)_{n<\omega}$ by $\mathscr{V}_n^{k+1}=R(\mathscr{V}_n^k)$. Our aim is to define a topology on $X=\bigcup\{M_k\,|\,k<\omega\}$ which turns X into a metacompact Moore space on which every continuous mapping into Y is constant. For this purpose we introduce the following notation.

If $V\in\mathscr{V}_n^k$ for some $n,k<\omega$, define $T(V)=\bigcup\{R^i(V)\,|\,i<\omega\}$, where $R^0(V)=V$, $R^1(V)=R(V)$, $R^2(V)=R(R(V)),\ldots$ It can be easily seen that for each $n<\omega$ the collection $\mathscr{V}_n=\{T(V)\,|\,V\in\bigcup\{\mathscr{V}_n^k\,|\,k<\omega\}\}$ is a cover of X. From (v) it follows that $\mathscr{V}_{n+1}$ refines $\mathscr{V}_n$ for each $n<\omega$. For each $x\in X$ denote by $k(x)$ the smallest $k<\omega$ such that $x\in M_k$. The crucial property of the covers $\mathscr{V}_n$ is the following:

(vii) If $T(V)\in\mathscr{V}_n$ and $x\in T(V)$, then there exists a $W\in\mathscr{V}_n^{k(x)}$ containing x such that $T(V)=T(W)$.

This can be easily verified using (i). As an immediate consequence of (vii) one obtains

(viii) $\mathrm{St}(x,\mathscr{V}_n)=\bigcup\{T(V)\,|\,V\in\mathscr{V}_n^{k(x)},\ x\in V\}$ for each $x\in X$ and $n<\omega$.

Using (ii), (iii), and (viii) it is not difficult to check that $\bigcup\{\mathscr{V}_n\,|\,n<\omega\}$ is a base for a topology which turns X into a developable T_1-space. Finally it follows from (iv) that X is a Moore space which is metacompact because of (vi) and (vii).

Now let f be an arbitrary continuous mapping from X into Y. To see that f is constant consider two distinct points x and x' in X. There exists a $k<\omega$ such that x and x' are both contained in M_k. Since it follows from (i) that M_{k+1} is a subspace of X, the restriction of f to M_{k+1} is a continuous mapping from M_{k+1} into Y which is constant on M_k because $M_{k+1}=R(M_k)$. Hence $f(x)=f(x')$, which completes the proof. ∎

3.5. Remarks. (a) A weaker notion than simplicity is that of semi-simplicity of a class of spaces: A class **A** of topological spaces is called *semi-simple* (e.g. see Herrlich [1983], 3.2.8) if there exists a *minimal* class **E** of topological spaces such that $\mathbf{A}=\mathrm{EH}(\mathbf{E})$, where minimality of **E** means that $\mathrm{EH}(\mathbf{E}\backslash\{Y\})\neq\mathbf{A}$ for every space Y in **E**. In view of Corollary 3.2 one might wonder whether the epireflective hull of the class of Moore spaces is semi-simple. We give a brief argument which shows that it is not.

Let **A** be the epireflective hull of the class of Moore spaces and consider a class **E** of spaces such that $\mathbf{A}=\mathrm{EH}(\mathbf{E})$. We claim that $\mathbf{A}=\mathrm{EH}(\mathbf{E}\backslash\{Y\})$ for each $Y\in\mathbf{E}$. To this end consider an arbitrary $Y\in\mathbf{E}$. There exists a family $(M_i)_{i\in I}$ of Moore spaces such that Y can be embedded into the product of this family.

The disjoint union M of all spaces M_i is a Moore space such that $Y \in \mathrm{EH}(\{M\})$. Therefore it suffices to prove that $M \in \mathrm{EH}(\mathbf{E}\backslash\{Y\})$. From the proof of Theorem 3.1 it follows that there exists a Moore space $R(M)$ containing M as a subspace such that all continuous mappings from $R(M)$ into Y are constant. It follows that $R(M) \in \mathrm{EH}(\mathbf{E}\backslash\{Y\})$ and, consequently, that $M \in \mathrm{EH}(\mathbf{E}\backslash\{Y\})$.

(b) The proof of Theorem 3.1 is a modification of the technique which was used by H. Herrlich [1965] to show that *for every T_1-space Y there exists a non-trivial regular T_1-space on which all continuous mappings into Y are constant.* However, the proof of the crucial Lemma 3.4 seems to be quite different from previous arguments. In fact, a modification of this construction also yields a relatively simple proof of the theorem of Herrlich [1965] and A. Ramer [1965] that *for every T_1-space Y there exists a regular T_1-space R containing two distinct points a, b such that $f(a) = f(b)$ for every continuous mapping f from R into Y*, which is the first step in the proof of the above mentioned result of Herrlich. For a description of this modification we refer to (Brandenburg and Mysior [1984]). ■

§ 4. *D*-normal spaces

Given a nonempty class $\mathbf{E}$ of topological spaces the *normality hull* $N(\mathbf{E})$ of $\mathbf{E}$ consists of all *E-normal spaces*, i.e. of those topological spaces X with the property that for every pair A, B of disjoint closed subsets of X there exists a continuous mapping f from X into some space $Y \in \mathbf{E}$ such that $\mathrm{cl} f[A] \cap \mathrm{cl} f[B] = \emptyset$. For example, if $\mathbf{M}$ is the class of metrizable topological spaces, then $\mathbf{M}$-normality is nothing but the usual normality. However, if $\mathbf{E} = \{2\}$, a topological space X is $\mathbf{E}$-normal if and only if $\mathrm{Ind}(X) = 0$.

Quite often it turns out to be a non-trivial task to characterize $\mathbf{E}$-normal spaces internally, for usually this requires to prove an analogue, depending on the class $\mathbf{E}$, of Urysohn's Lemma. Fortunately, for D-normal spaces, where as always $\mathbf{D}$ is the class of developable T_1-spaces, we can even prove an extension theorem corresponding to the classical Tietze extension theorem. Independently, J. Chaber [1983a] has got a similar result for his modification of Heldermann's space.

4.1. THEOREM (Brandenburg [1983]). *Every closed G_δ-subset of a D-normal space is D-embedded.*

Proof. Let X be a D-normal space and consider a closed G_δ-set A in X. By virtue of Corollary 1.6 it suffices to show that A is D-closed. For this purpose we define inductively, for each $(n_0, \ldots, n_{k-1}) \in S$ (9), a closed G_δ-set $E(n_0, \ldots, n_{k-1})$ of X as follows.

(9) Recall the definition of S from § 1.

If $k = 1$, set $E(0) = A$ and choose a sequence $(A_n)_{0<n<\omega}$ of closed sets such that $X\backslash E(0) = \bigcup\{A_n \mid 0 < n < \omega\}$. Since X is D-normal, we can find for each $0 < n < \omega$ a closed G_δ-set $E(n)$ such that $A_n \subset E(n) \subset X\backslash E(0)$.

If $k > 1$ and $E(n_0, \ldots, n_{k-2})$ is already defined for each $(n_0, \ldots, n_{k-2}) \in S$ such that $E(n_0, \ldots, n_{k-2})$ is a closed G_δ-set, there exists a sequence $(E(n_0, \ldots, n_{k-2})_n)_{0<n<\omega}$ of closed sets such that $X\backslash E(n_0, \ldots, n_{k-2}) = \bigcup\{E(n_0, \ldots, n_{k-2})_n \mid 0 < n < \omega\}$. By the D-normality of X again we can find a closed G_δ-set $E(n_0, \ldots, n_{k-2}, n)$ for each $0 < n < \omega$ such that $E(n_0, \ldots, n_{k-2})_n \subset E(n_0, \ldots, n_{k-2}, n) \subset X\backslash E(n_0, \ldots, n_{k-2})$, which completes the induction. Obviously, $\{E(n_0, \ldots, n_{k-1}) \mid (n_0, \ldots, n_{k-1}) \in S\}$ is a G_δ-collection in X containing A. ∎

We can now prove various characterizations of D-normal spaces which show in particular that D-normality is a rather natural generalization of normality.

4.2. Theorem. *For a topological space X the following conditions are equivalent:*

(i) *X is D-normal.*

(ii) (Brandenburg [1986]). *For every pair A, B of disjoint closed subsets of X there exists an upper-semicontinuous mapping f from X into $[0, 1]$ such that $f[A] \subset \{0\}$, $f[B] \subset \{1\}$, and $f^{-1}[\{0\}]$ is closed in X.*

(iii) (Heldermann [1980]). *Whenever $A \subset X$ is closed and $U \subset X$ is open such that $A \subset U$ there exists an open F_σ-set F such that $A \subset F \subset U$.*

(iv) (Brandenburg [1978]). *For every pair A, B of disjoint closed subsets of X there exist disjoint closed G_δ-sets F, G in X such that $A \subset F$ and $B \subset G$.*

(v) (Heldermann [1980], Brandenburg [1983]). *For every pair A, B of disjoint closed subsets of X and for every pair a, b of distinct points in $\mathbf{D}_1$ there exists a continuous mapping f from X into $\mathbf{D}_1$ such that $f[A] \subset \{a\}$ and $f[B] \subset \{b\}$.*

Proof. (i) implies (ii): Let A, B be disjoint closed subsets of X. By virtue of (i) there exists a continuous mapping g from X into some developable T_1-space Y such that $\mathrm{cl}\, g[A] \cap \mathrm{cl}\, g[B] = \emptyset$. Let $(\mathscr{U}_n)_{n<\omega}$ be a development of Y such that $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$ for each $n < \omega$ and define a mapping h from Y into $[0, 1]$ by

$$h(y) = \begin{cases} 0 & \text{if } y \in \mathrm{cl}\, g[A], \\ 1/2^{n(y)} & \text{if } y \in Y\backslash \mathrm{cl}\, g[A], \text{ where} \\ & n(y) = \min\{n < \omega \mid y \in \mathrm{cl}\, g[B] \cup (Y\backslash \mathrm{St}(\mathrm{cl}\, g[A], \mathscr{U}_n))\}. \end{cases}$$

If $f = h \circ g$, then $f[A] \subset \{0\}$, $f[B] \subset \{1\}$, and $f^{-1}[\{0\}] = g^{-1}[\mathrm{cl}\, g[A]]$ is closed in X. In order to prove that f is upper-semicontinuous consider an arbitrary c such that $0 < c \leqslant 1$. Since g is continuous, it suffices to show that $h^{-1}[[0, c)]$ is open in Y. But the latter follows from the observation that

$$h^{-1}[[0, c)] = \mathrm{St}(\mathrm{cl}\, g[A], \mathscr{U}_{n(c)}) \cap (Y\backslash \mathrm{cl}\, g[B]),$$

where $n(c) = \max\{n < \omega \mid c \leqslant 1/2^n\}$.

Clearly, (ii) implies (iii), and it is easily seen that (iii) implies (iv).

In order to show that (iv) implies (v) consider two disjoint closed sets A, B in X. By virtue of (iv) there exist disjoint closed G_δ-sets F, G in X such that $A \subset F$ and $B \subset G$. Therefore, if a, b are distinct points in $\boldsymbol{D}_1$, we can define a continuous mapping $g\colon F \cup G \to \boldsymbol{D}_1$ such that $g[F] \subset \{a\}$ and $g[G] \subset \{b\}$. By virtue of Corollary 1.6, g has a continuous extension $f\colon X \to \boldsymbol{D}_1$ with the desired properties. Since (i) is formally weaker than (v), the proof is complete. ∎

Obviously, every developable space and every normal space is D-normal. Moreover, from (v), respectively (iv), we obtain the following corollary which completes the proof of Proposition 2.7.

4.3. COROLLARY. (i) *Every D-compact R_0-space* ([10]) *is D-normal.*

(ii) *Every D-normal T_1-space is D-completely regular.*

(iii) *Every perfect space is D-normal.* ∎

Of course, none of these implications is reversible. An example of a completely regular T_1-space which is not D-normal will be given in 5.8. Concerning perfect spaces we know a little bit more.

4.4. THEOREM. *For a topological space X the following conditions are equivalent*:

(i) *X is perfect.*

(ii) (Brandenburg [1978]). *Every closed subset of X is D-closed.*

(iii) (Heldermann [1980]). *For every closed subset A of X there exists a continuous mapping f from X into $\boldsymbol{D}_1$, such that $A = f^{-1}[\{\pi(0)\}]$.* ∎

In view of 4.3(ii) one might ask what must be added to D-complete regularity in order to get D-normality. The following theorem from (Brandenburg [1979]) gives an answer. It is the analogue of a result due to P. Zenor [1969].

4.5. THEOREM. *For a T_1-space X the following conditions are equivalent*:

(i) *X is D-normal.*

(ii) *X is a D-completely regular space with the property that every continuous mapping from X into an arbitrary T_1-space which maps D-closed sets onto closed sets is closed.*

Proof. See (Brandenburg [1979]) for a more general result. ∎

One of the most useful properties of normal topological spaces is the fact that every point-finite open cover has a shrinking. Furthermore, every locally finite open cover of a normal space has a locally finite cozero-set refinement. Therefore it is quite natural to ask for similar characterizations of D-normal spaces. Before giving an answer we recall two definitions.

([10]) A topological space is an *R_0-space*, if every open subset is a union of closed sets.

4.6. DEFINITION. If $\mathcal{U}$ is an open cover of a topological space X, a continuous mapping f from X into a topological space Y is called a *$\mathcal{U}$-mapping* if there exists an open cover $\mathcal{V}$ of Y such that $\{f^{-1}[V] \mid V \in \mathcal{V}\}$ refines $\mathcal{U}$.

4.7. DEFINITION. (Smith [1975]) An open cover $\mathcal{U}$ of a topological space X is called a *weak $\bar{\theta}$-cover* if $\mathcal{U} = \bigcup_{n<\omega} \mathcal{U}_n$ such that

(i) for each $x \in X$ there exists an $n(x) < \omega$ such that

$$0 < \operatorname{ord}(x, \mathcal{U}_{n(x)}) < \omega;$$

(ii) the cover $(\bigcup \{U \mid U \in \mathcal{U}_n\})_{n<\omega}$ is point-finite.

The space X is said to be *weakly $\bar{\theta}$-refinable* if every open cover of X has an open refinement which is a weak $\bar{\theta}$-cover of X.

Except for the equivalences (ii) and (iii) which appear here for the first time the following theorem was already proved in (Brandenburg [1978], [1981]).

4.8. THEOREM. *For a topological space $X = (X, \tau)$ the following conditions are equivalent:*

(i) *X is D-normal.*

(ii) *For every weak $\bar{\theta}$-cover $(U_\alpha)_{\alpha<\varkappa}$ of X there exists a D-open cover*(11) *$(F_\alpha)_{\alpha<\varkappa}$ of X such that $F_\alpha \subset U_\alpha$ for each $\alpha < \varkappa$.*

(iii) *For every weak $\bar{\theta}$-cover $(U_\alpha)_{\alpha<\varkappa}$ of X there exists an open F_σ-cover*(12) *$(F_\alpha)_{\alpha<\varkappa}$ of X such that $F_\alpha \subset U_\alpha$ for each $\alpha < \varkappa$.*

(iv) *For every point-finite open cover $(U_\alpha)_{\alpha<\varkappa}$ of X there exists an open F_σ-cover $(F_\alpha)_{\alpha<\varkappa}$ of X such that $F_\alpha \subset U_\alpha$ for each $\alpha < \varkappa$.*

(v) *Every locally finite open cover of X has a locally finite open F_σ-refinement.*

(vi) *Every locally finite open cover of X is kernel-normal.*

(vii) *Every countable point-finite open cover of X is kernel-normal.*

(viii) *For every locally finite open cover $\mathcal{U}$ of X there exists a $\mathcal{U}$-mapping from X onto a developable T_1-space.*

(ix) *For every countable point-finite open cover $\mathcal{U}$ of X there exists a $\mathcal{U}$-mapping from X onto a developable T_1-space.*

Proof. (i) implies (ii): Let $\mathcal{U} = (U_\alpha)_{\alpha<\varkappa}$ be a weak $\bar{\theta}$-cover of X. There is a sequence $(A_n)_{n<\omega}$ of subsets of $\varkappa$ such that $\varkappa = \bigcup \{A_n \mid n < \omega\}$ and

(a) for each $x \in X$ there exists an $n(x) < \omega$ such that $0 < \operatorname{ord}(x, \mathcal{U}_{n(x)}) < \omega$, where $\mathcal{U}_n = \{U_\alpha \mid \alpha \in A_n\}$ for each $n < \omega$;

(b) $\mathcal{V} = (V_n)_{n<\omega}$ is point-finite, where $V_n = \bigcup \{U_\alpha \mid \alpha \in A_n\}$ for each $n \gtrsim \omega$.

(11) A *D-open* cover is a cover consisting of D-open sets (1.3).

(12) An *F_σ-cover* is a cover consisting of F_σ-sets.

We claim that the following holds:

A. For each m, k such that $0 < m < \omega$ and $0 < k < \omega$ there exists a collection $\mathscr{F}(m, k) = (F(m, k, \alpha))_{\alpha < \varkappa}$ of D-open sets in X such that the following conditions are satisfied:
(c) $F(m, k, \alpha) \subset U_\alpha$ for each $\alpha < \varkappa$;
(d) if $\operatorname{ord}(x, \mathscr{V}) = m$ and $\operatorname{ord}(x, \mathscr{U}_n) = k$ for some $n < \omega$, then $x \in \bigcup \{F \in \mathscr{F}(m', k') \mid (m', k') \leqslant (m, k)\}$, where $(m', k') \leqslant (m, k)$ if either $m' < m$ or $m' = m$ and $k' \leqslant k$.

Note that once we have proved Claim A it follows from (a)–(d) that $(F_\alpha)_{\alpha < \varkappa}$ is a D-open cover of X such that $F_\alpha \subset U_\alpha$ for each $\alpha < \varkappa$, where $F_\alpha = \bigcup \{F(m, k, \alpha) \mid 0 < m < \omega,\ 0 < k < \omega\}$.

In order to verify Claim A we proceed by induction. Assume that $m = 1$ and $k = 1$. Then for each $n < \omega$ and for each $\alpha \in A_n$

$$E(1, 1, n, \alpha) = \bigcap \{X \setminus U_\beta \mid \beta \in A_n \setminus \{\alpha\}\} \cap \bigcap \{X \setminus V_l \mid l \in \omega \setminus \{n\}\}$$

is a closed set contained in U_α. Hence, by virtue of (i), there exists a D-open set $F(1, 1, n, \alpha)$ such that $E(1, 1, n, \alpha) \subset F(1, 1, n, \alpha) \subset U_\alpha$. If $F(1, 1, \alpha) = \bigcup \{F(1, 1, n, \alpha) \mid n < \omega,\ \alpha \in A_n\}$ for each $\alpha < \varkappa$, then $\mathscr{F}(1, 1) = (F(1, 1, \alpha))_{\alpha < \varkappa}$ satisfies (c) and (d).

Now let $(1, 1) < (m, k)$ and assume that for each $(m', k') < (m, k)$ a collection $\mathscr{F}(m', k')$ of D-open sets satisfying (c) and (d) is already defined. For each $N \in [\omega]^m$, $n \in N$, and $B \in [A_n]^k$ define

$$E(m, k, N, n, B) = \bigcap \{X \setminus U_\alpha \mid \alpha \in A_n \setminus B\} \cap \bigcap \{X \setminus V_l \mid l \in \omega \setminus N\}$$
$$\cap \bigcap \{X \setminus F \mid F \in \mathscr{F}(m', k'),\ (m', k') < (m, k)\}.$$

Then the following holds:

B. Every $E(m, k, N, n, B)$ is contained in $\bigcap \{U_\beta \mid \beta \in B\}$.

C. Every $\mathscr{E}(m, k, N, n) = \{E(m, k, N, n, B) \mid B \in [A_n]^k\}$ is a discrete collection of closed sets.

In fact, if $z \in E(m, k, N, n, B)$ and $N(z) = \{l < \omega \mid z \in V_l\}$, it follows from the definition of $E(m, k, N, n, B)$ and from the induction hypothesis that $N = N(z)$. In particular, $0 < \operatorname{ord}(z, \mathscr{U}_n) \leqslant |B| = k$. But $\operatorname{ord}(z, \mathscr{U}_n) < k$ is impossible by the induction hypothesis. Hence $z \in \bigcap \{U_\beta \mid \beta \in B\}$, which proves Claim B.

In order to verify Claim C consider an arbitrary $\mathscr{E}(m, k, N, n)$ and an $x \in X$. If $\operatorname{ord}(x, \mathscr{V}) > m$, then $\bigcap \{V_l \mid l < \omega, x \in V_l\}$ is a neighborhood of x which meets no member of $\mathscr{E}(m, k, N, n)$. If $\operatorname{ord}(x, \mathscr{V}) < m$ or $\operatorname{ord}(x, \mathscr{V}) = m$ and $\operatorname{ord}(x, \mathscr{U}_n) < k$, then, by the induction hypothesis, $\bigcup \{F \in \mathscr{F}(m', k') \mid (m', k') < (m, k)\}$ is a neighborhood of x which meets no member of $\mathscr{E}(m, k, N, n)$. If $\operatorname{ord}(x, \mathscr{U}_n) > k$, there exist $k+1$ sets in $\mathscr{U}_n$ containing x. Their intersection is a neighborhood of x which has an empty intersection with every member of $\mathscr{E}(m, k, N, n)$. Finally, if $\operatorname{ord}(x, \mathscr{V}) = m$ and $\operatorname{ord}(x, \mathscr{U}_n) = k$, define $N(x) = \{l < \omega \mid x \in V_l\}$ and $B(x) = \{\alpha \in A_n \mid x \in U_\alpha\}$. Then $\bigcap \{U_\alpha \mid \alpha \in B(x)\}$ is a neighborhood of x which meets at most one member of $\mathscr{E}(m, k, N, n)$, namely $E(m, k, N(x), n, B(x))$ in case that $N = N(x)$.

For each $N \in [\omega]^m$, $n \in N$, and $B \in [A_n]^k$ define now

$$\alpha(m, k, N, n, B) = \min\{\alpha < \varkappa \mid E(m, k, N, n, B) \subset U_\alpha\},$$

which exists by virtue of Claim B. It follows from Claim C that for every $\alpha < \varkappa$

$$E(m,k, N, n, \alpha) = \bigcup \{E(m, k, N, n, B) \mid B \in [A_n]^k,\ \alpha(m, k, N, n, B) = \alpha\}$$

is a closed set contained in U_α. Since X is D-normal, there exist D-open sets $F(m, k, N, n, \alpha)$, $\alpha < \varkappa$, such that

$$E(m, k, N, n, \alpha) \subset F(m, k, N, n, \alpha) \subset U_\alpha.$$

If

$$F(m, k, \alpha) = \bigcup \{F(m,k, N, n, \alpha) \mid N \in [\omega]^m,\ n \in N\} \text{ for each } \alpha < \varkappa,$$

then $\mathscr{F}(m, k) = (F(m, k, \alpha))_{\alpha < \varkappa}$ is a collection of D-open sets satisfying (c) and (d), which completes the induction.

Obviously, (ii) implies (iii), (iii) implies (iv), and (iv) implies (v). In order to prove that (v) implies (vi) let μ be the collection of all locally finite open covers of X. By virtue of Lemma 2.12 it suffices to show that for each $\mathscr{U} \in \mu$ there exists a sequence $\beta = (\mathscr{U}_n)_{n<\omega}$ in μ such that $\mathscr{U}_n$ refines $\operatorname{int}_\beta \mathscr{U}$ for some $n < \omega$. To this end consider an arbitrary $\mathscr{U} \in \mu$. Assuming (v) there exists a locally finite open F_σ-refinement $\mathscr{V} = (V_\alpha)_{\alpha<\varkappa}$ of $\mathscr{U}$. For each $\alpha < \varkappa$ let $(V(\alpha, n))_{n<\omega}$ be a sequence of closed sets such that $V_\alpha = \bigcup\{V(\alpha, n) \mid n < \omega\}$. If $n < \omega$ and $x \in X(n) = \bigcup \{V(\alpha, n) \mid \alpha < \varkappa\}$, define

$$U(x, n) = \bigcap \{V_\alpha \mid \alpha < \varkappa,\ x \in V(\alpha, n)\} \cap (X \setminus \bigcup \{V(\alpha, n) \mid \alpha < \varkappa,\ x \notin V(\alpha, n)\}).$$

Since $\mathscr{V}$ is locally finite, every $U(x, n)$ is an open set containing x. Therefore, for each $n < \omega$ the collection

$$\mathscr{U}_n = \{V_\alpha \setminus X(n) \mid \alpha < \varkappa\} \cup \{U(x, n) \mid x \in X(n)\}$$

is a locally finite open cover of X, i.e. $\beta = (\mathscr{U}_n)_{n<\omega}$ is a sequence in μ. Since $\mathscr{U}_0$ refines $\operatorname{int}_\beta \mathscr{U}$, we have reached our aim.

By virtue of Lemma 2.11, (vi) implies that for a given locally finite open cover $\mathscr{U}$ of X there exists a developable topology $\tau' \subset \tau$ and a τ'-open cover $\mathscr{V}$ of X which refines $\mathscr{U}$. If Y is the space obtained from (X, τ') by identifying points which have identical closures, then Y is a developable T_1-space and the corresponding quotient mapping defines a $\mathscr{U}$-mapping from (X, τ) onto Y. Therefore (vi) implies (viii). Similarly, (vii) implies (ix). Clearly, both (viii) and (ix) imply (i). Hence the proof is complete once we have shown that (ii) implies (vii).

To this end consider a point-finite open cover $\mathscr{U} = (U_n)_{n<\omega}$ of X. Assuming (ii) there exists a D-open cover $\mathscr{V} = (V_n)_{n<\omega}$ of X such that $V_n \subset U_n$ for each $n < \omega$. By virtue of Proposition 1.4 there exists, for each $n < \omega$, a continuous mapping f_n from X into D such that $X \setminus V_n = f_n^{-1}[A(0)]$. If τ' is the initial topology on X with respect to $(f_n\colon X \to D)_{n<\omega}$, then $\tau' \subset \tau$ is developable and $\mathscr{V}$ is τ'-open. Hence $\mathscr{U}$ is kernel-normal (2.11), which completes the proof. ∎

Let us call a topological space X *F_σ-shrinkable* if for every open cover $(U_\alpha)_{\alpha<\varkappa}$ of X there exists an open F_σ-cover $(F_\alpha)_{\alpha<\varkappa}$ of X such that $F_\alpha \subset U_\alpha$ for each $\alpha < \varkappa$. The preceding theorem yields:

4.9. COROLLARY. *Every weakly $\bar{\theta}$-refinable D-normal space is F_σ-shrinkable.* ∎

The Dowker space constructed by M. E. Rudin [1971] is an example of a normal, hence D-normal, T_1-space containing a countable open cover which has no open F_σ-shrinking. We do not know whether there exists a simpler T_1-space witnessing that D-normal spaces need not be F_σ-shrinkable.

Recall that for a family $(X_i)_{i\in I}$ of topological spaces and a point $a \in \prod_{i\in I} X_i$ the subspace

$$\Sigma(a) = \{x \in \prod_{i\in I} X_i \mid |\{j \in I \mid x_j \neq a_j\}| \leqslant \omega\} \quad \text{of } \prod_{i\in I} X_i$$

is called a *Σ-product* of the spaces $(X_i)_{i\in I}$ with respect to a (Corson [1959]). In [1983] M. E. Rudin has shown that every Σ-product of metrizable spaces has the *shrinking property*, which means that every open cover $(U_\alpha)_{\alpha<\varkappa}$ has an open refinement $(V_\alpha)_{\alpha<\varkappa}$ such that $\operatorname{cl} V_\alpha \subset U_\alpha$ for each $\alpha < \varkappa$. This observation motivates our next research problem.

PROBLEM C. Does every Σ-product of developable T_1-spaces have the F_σ-shrinking property, i.e. is every Σ-product of developable T_1-spaces F_σ-shrinkable?

The following generalization of D-normality was studied by T. R. Kramer [1973] and J. Chaber [1979].

4.10. DEFINITION. A topological space X is called *subnormal* if for every pair A, B of disjoint closed subsets of X there exists a pair F, G of disjoint G_δ-sets such that $A \subset F$ and $B \subset G$.

4.11. EXAMPLE (Mysior [1980]). Let $A(\omega_1) = X(\omega_1) \cup \{a\}$ be the Aleksandrov one-point compactification of a discrete space $X(\omega_1)$ of cardinality ω_1 and consider the quotient space X obtained from $(A(\omega_1) \times (\omega+1)) \setminus \{(a, \omega)\}$ by identifying the points in $\{a\} \times \omega$. Then X is a subnormal Hausdorff space which is not even D-completely regular. ∎

Using the fact that ${}^{\omega_1}\omega$ is not normal (Stone [1948]), N. Noble [1971] has shown that, more generally, ${}^{\varkappa}X$ is not normal for all non-compact Hausdorff spaces X provided that $\varkappa \geqslant \max\{\omega_1, w(X)\}$, where $w(X)$ denotes the weight of X. For short proofs of Noble's theorem see (Franklin and Walker [1972]), (Keesling [1972]), and (Polkowski [1979]), for an application, e.g., (Herrlich and Strecker [1971]). Since R. Pol and E. Puzio-Pol [1976] improved Stone's result by showing that ${}^{\omega_1}\omega$ is not even subnormal, the following problem arises.

PROBLEM D. Suppose that X is an arbitrary non-compact Hausdorff space. Does there exist a cardinal $\varkappa$ such that ${}^{\varkappa}X$ is not subnormal?

In the remainder of this section we will show that the corresponding question for D-normality can be answered affirmatively, thereby improving on Noble's theorem. More precisely, we will verify the following theorem from (Brandenburg and Hušek [1987]).

4.12. THEOREM. *If X is a non-compact Hausdorff space and $\varkappa \geqslant \max\{\omega_1, w(X)\}$, then ${}^{\varkappa}X$ is not D-normal.*

We can prove this theorem due to the fact that for D-normal spaces we may use the analogue 4.2(v) of Urysohn's lemma which enables us to mimic Keesling's proof of Noble's theorem. In order to describe this method we need some notation.

A continuous mapping f from a cartesian product $\prod_{i\in I} X_i$ of topological spaces into a topological space Y is said to *depend on a subset J of I* if $f(x) = f(y)$ whenever $x, y \in \prod_{i\in I} X_i$ and $p_J(x) = p_J(y)$, where p_J denotes the natural projection from $\prod_{i\in I} X_i$ onto $\prod_{j\in J} X_j$. If f depends on a countable subset J of I, it is said to *depend on countably many coordinates*. Similarly, a subset A of $\prod_{i\in I} X_i$ is said to depend on countably many coordinates, if there exists a countable subset J of I such that $p_J^{-1}[p_J[A]] = A$. The following proposition generalizes Keesling's argument from [1972]. For the sake of completeness we give a proof.

4.13. PROPOSITION. *Suppose that* **E** *is a nonempty class of T_1-spaces which has the following property:*

(*) *Whenever X is a non-compact Hausdorff space such that ${}^{\varkappa}X$ is countably compact, then every continuous mapping from ${}^{\varkappa}X$ into a space from* **E** *depends on countably many coordinates.*

If ${}^{\omega_1}\omega$ is not **E**-*normal, then, more generally, ${}^{\varkappa}X$ is not* **E**-*normal for every non-compact Hausdorff space X and for each $\varkappa \geqslant \max\{\omega_1, w(X)\}$.*

Proof. Consider an arbitrary non-compact Hausdorff space X and suppose that ${}^{\varkappa}X$ is **E**-normal for some $\varkappa \geqslant \max\{\omega_1, w(X)\}$. Then ${}^{\varkappa}X$ is countably compact, for othervise ${}^{\varkappa}X$ would contain a closed subspace homeomorphic to ${}^{\omega_1}\omega$, which is impossible. It follows that X and ${}^{w(X)}X$ are also countably compact. Since X is not compact, there exists a centered family $(F_\alpha)_{\alpha < w(X)}$ of closed subsets of X such that $\bigcap\{F_\alpha \mid \alpha < w(X)\} = \emptyset$. Moreover, by the **E**-normality of ${}^{w(X)}X$ there exists a continuous mapping f from ${}^{w(X)}X$ into a space from **E** such that $f[\prod_{\alpha < w(X)} F_\alpha] \cap f[\Delta] = \emptyset$, where $\Delta \subset {}^{w(X)}X$ is the diagonal. By virtue of (∗), f depends on countably many coordinates. Hence $\bigcap\{F_\alpha \mid \alpha\in A\} = \emptyset$ for some countable set A. It follows that $\bigcap\{F_\alpha \mid \alpha\in B\} = \emptyset$ for a finite subset B of A, for X is countably compact — a contradiction! ∎

Thus, in order to prove Theorem 4.12 we only have to convince ourself that the class **D** of developable T_1-spaces satisfies condition (∗) in the previous proposition. We could do this directly, but we prefer to consider the more general problem to characterize internally those product spaces on which every continuous mapping into a developable T_1-space depends on countably many coordinates.This leads to the following theorem from (Brandenburg and Hušek [1987]) which is interesting in itself (see 4.18(j)).

4.14. THEOREM. *For a product space $\prod_{i\in I} X_i$ the following conditions are equivalent:*

(i) *Every continuous mapping from $\prod_{i\in I} X_i$ into a subdevelopable space* ([13]) *depends on countably many coordinates.*

(ii) *Every continuous mapping from $\prod_{i\in I} X_i$ into a developable T_1-space depends on countably many coordinates.*

(iii) *Every continuous mapping from $\prod_{i\in I} X_i$ into $\boldsymbol{D}_1$ depends on countably many coordinates.*

(iv) *Every D-closed subset of $\prod_{i\in I} X_i$ depends on countably many coordinates.*

Proof. Obviously, (i) implies (ii), and (ii) implies (iii). If A is a D-closed subset of $\prod_{i\in I} X_i$, there exist a point $a\in \boldsymbol{D}_1$ and a continuous mapping f from $\prod_{i\in I} X_i$ into $\boldsymbol{D}_1$ such that $A = f^{-1}[\{a\}]$ (1.4). Assuming (iii) f depends on countably many coordinates $J\subset I$. Since $p_J^{-1}[p_J[A]] = A$, we have shown that (iii) implies (iv).

In order to prove that (iv) implies (i) suppose that there exists a continuous mapping f from $\prod_{i\in I} X_i$ into a subdevelopable space (Y,τ) which does not depend on countably many coordinates. Without loss of generality we may assume that f is surjective. Let $\tau'\subset\tau$ be a T_1-topology which is developable. We will first show that it suffices to verify the following claim.

A. (Y,τ') is second countable.

In fact, once we have proved A we may choose, for every set B from a countable base $\mathscr{B}$ for the closed sets of (Y,τ'), a countable subset I_B of I on which $f^{-1}[B]$ depends. Then f depends on $\bigcup\{I_B \mid B\in\mathscr{B}\}$ which is countable — the desired contradiction!

([13]) We call a topological space (X,τ) *subdevelopable* if there exists a developable T_1-topology $\tau'\subset\tau$.

Suppose now that Claim A is false. Then there exist a development $(\mathcal{U}_n)_{n<\omega}$ of τ' such that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for each $n < \omega$ and a subset $D = \{y_\alpha \mid \alpha < \omega_1\}$ of Y which is closed and discrete with respect to τ' [14]. For every subset A of $\prod_{i \in I} X_i$ define

$$J(A) = \{j \in I \mid p_{I\setminus\{j\}}^{-1}[p_{I\setminus\{j\}}[A]]\setminus A \neq \emptyset\}.$$

Then the following holds [15]:

B. Every closed subset A of $\prod_{i\in I} X_i$ depends on $J(A)$.

C. If $A \subset \prod_{i \in I} X_i$ depends on $J \subset I$, then $J(A) \subset J$.

In particular, if $I(P) = J(f^{-1}[\{y_\alpha \mid \alpha \in P\}])$ for each subset P of ω_1, then, assuming (iv), every $I(P)$ is countable and $f^{-1}[\{y_\alpha \mid \alpha \in P\}]$ depends on $I(P)$.

D. There exists an uncountable family $\mathcal{P}$ of pairwise disjoint nonempty subsets of ω_1 such that $\bigcup\{I(P) \mid P \in \mathcal{P}\}$ is countable.

In order to verify this claim we may assume that $\bigcup\{I(\{\alpha\}) \mid \alpha < \omega_1\}$ is uncountable, i.e. that $\bigcup\{I(\{\alpha\}) \mid \alpha < \omega_1\} = \{i_\beta \mid \beta < \omega_1\}$. If

$$\gamma_0 = \min\{\gamma < \omega_1 \mid I(\omega_1) \subset \{i_\beta \mid \beta \leqslant \gamma\}\},$$

then, for each γ such that $\gamma_0 \leqslant \gamma < \omega_1$, the set $\mathcal{P}_\gamma = \{P_\gamma(\delta) \mid \delta < \omega_1\}$ forms a partition of ω_1, where

$$P_\gamma(\delta) = \bigcap\{P \subset \omega_1 \mid \delta \in P,\ I(P) \subset \{i_\beta \mid \beta \leqslant \gamma\}\}.$$

Since $\bigcup\{I(P_\gamma(\delta)) \mid \delta < \omega_1\}$ is countable, it suffices to show that at least one $\mathcal{P}_\gamma$ is uncountable.

Suppose that every $\mathcal{P}_\gamma$ is countable. Then, for every γ such that $\gamma_0 \leqslant \gamma < \omega_1$, we can find two disjoint subsets R_γ, S_γ of ω_1 such that

(1) $|R_\gamma \cap P| = 1$ and $|S_\gamma \cap P| = 1$ for each uncountable $P \in \mathcal{P}_\gamma$;

(2) $R_\gamma \cap P = S_\gamma \cap P = \emptyset$ for each countable $P \in \mathcal{P}_\gamma$;

(3) $R = \bigcup\{R_\gamma \mid \gamma_0 \leqslant \gamma < \omega_1\}$ has cardinality ω_1 and $R \cap \bigcup\{S_\gamma \mid \gamma_0 \leqslant \gamma < \omega_1\} = \emptyset$.

Since there exists a γ_1 such that $\gamma_0 \leqslant \gamma_1 < \omega_1$ and $I(R) \subset \{i_\beta \mid \beta \leqslant \gamma_1\}$, it follows that $P \subset R$ for some uncountable $P \in \mathcal{P}_{\gamma_1}$. But then $\bigcup\{S_\gamma \mid \gamma_0 \leqslant \gamma < \omega_1\}$ must have a nonempty intersection with P, which contradicts (3).

Having established D we continue the proof of Claim A by considering an arbitrary family $\mathcal{P} = (P_\xi)_{\xi<\omega_1}$ of pairwise disjoint nonempty subsets of ω_1 such that $I(\mathcal{P}) = \bigcup\{I(P_\xi) \mid \xi < \omega_1\}$ is countable. We may assume that there is a subset $\{i_\xi \mid \xi < \omega_1\}$ of $I\setminus I(\mathcal{P})$ such that every X_{i_ξ} contains a non-trivial D-closed subset F_{i_ξ}, i.e. $\emptyset \neq F_{i_\xi} \neq X_{i_\xi}$. (For otherwise there would exist

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[14] This follows from the fact that a developable T_1-space is second countable if and only if each of its closed discrete subspaces is at most countable.

[15] This is well-known and easy to prove.

a countable subset K of $I\backslash I(\mathscr{P})$ such that for every $i \in I\backslash(K \cup I(\mathscr{P}))$ every continuous mapping from X_i into (Y, τ') is constant. In this case f would depend on the countable set $K \cup I(\mathscr{P})$, contrary to our assumption.) By virtue of Proposition 1.4 there exist a point b in $\boldsymbol{D}_1$ and, for every $\xi < \omega_1$, a continuous mapping f_ξ from X_{i_ξ} into $\boldsymbol{D}_1$ such that $F_{i_\xi} = f_\xi^{-1}[\{b\}]$. For every $\xi < \omega_1$ let g_ξ be the continuous mapping from $A_\xi = f^{-1}[\{y_\alpha \mid \alpha \in P_\xi\}]$ into $\boldsymbol{D}_1$ defined by $g_\xi = f_\xi \circ p_{i_\xi} \upharpoonright A_\xi$. Our plan is to use these mappings to construct a continuous mapping g from $\prod_{i \in I} X_i$ into a space Z containing a D-closed subset whose preimage under g does not depend on countably many coordinates.

As the underlying set of Z we take the disjoint union $(Y\backslash D) \cup \bigcup \{D_\xi \mid \xi < \omega_1\}$, where each D_ξ is a copy of $\boldsymbol{D}_1$. For each $\xi < \omega_1$ let $(\mathscr{U}_n^\xi)_{n<\omega}$ be a development of D_ξ such that $\mathscr{U}_{n+1}^\xi$ refines $\mathscr{U}_n^\xi$ for each $n < \omega$. We supply Z with the topology generated by $\mathscr{V} = \bigcup_{n<\omega} \mathscr{V}_n$ as a base, where

$$\mathscr{V}_n = \{U\backslash D \mid U \in U_n\} \cup \{U^\xi \cup (\mathrm{St}(\{y_\alpha \mid \alpha \in P_\xi\}, \mathscr{U}_n)\backslash \{y_\alpha \mid \alpha \in P_\xi\}) \mid U^\xi \in \mathscr{U}_n^\xi,\ \xi < \omega_1\}.$$

If $g\colon \prod_{i \in I} X_i \to Z$ is defined by

$$g(x) = \begin{cases} f(x) & \text{if } f(x) \in Y\backslash D, \\ g_\xi(x) & \text{if } x \in A_\xi, \end{cases}$$

it is easily seen that g is continuous. Since Z is developable, $A = g^{-1}[\{b_\xi \mid \xi < \omega_1\}]$ is a D-closed subset of $\prod_{i \in I} X_i$, where every b_ξ is the copy of $b \in \boldsymbol{D}_1$ in D_ξ. We claim that A does not depend on countably many coordinates.

In fact, we show that if J is a subset of I such that $p_J^{-1}[p_J[A]] = A$, then $\{i_\xi \mid \xi < \omega_1\}$ is contained in J. To this end assume the contrary, i.e. that there exists a $\xi_0 < \omega_1$ such that $i_{\xi_0} \in I\backslash J$. Then we can find points x, y in $\prod_{i \in I} X_i$ satisfying

(4) $$p_{I\backslash\{i_{\xi_0}\}}(x) = p_{I\backslash\{i_{\xi_0}\}}(y);$$

(5) $$p_{i_{\xi_0}}(x) \in F_{i_{\xi_0}};$$

(6) $$p_{i_{\xi_0}}(y) \in X_{i_{\xi_0}} \backslash F_{i_{\xi_0}};$$

(7) $$p_{I(\mathscr{P})}(x) \in p_{I(\mathscr{P})}[A_{\xi_0}].$$

Observe that (7) implies that $x \in A_{\xi_0}$, and (5) implies that $x \in A$. By virtue of (4) we conclude that $y \in A$ (since $i_{\xi_0} \in I\backslash J$). On the other hand, (4), (7), and (6) imply that $g(y) \in Z\backslash\{b_\xi \mid \xi < \omega_1\}$ – a contradiction! Therefore $\{i_\xi \mid \xi < \omega_1\} \subset J$, i.e. A does not depend on countably many coordinates which, however, contradicts our assumption (iv). Consequently Claim A is true, which completes the proof. ∎

Now, if A is a closed G_δ-subset of a product space $\prod_{i \in I} X_i$ which does not depend on countably many coordinates, we can find an uncountable subset J of I, a closed set $B \subset \prod_{i \in I} X_i \backslash A$, and points $x^j \in A$, $y^j \in B$ such that $p_{I\backslash\{j\}}(x^j) = p_{I\backslash\{j\}}(y^j)$ for each $j \in J$ (recall Claim B in the preceding proof). It follows that $\{x^j \mid j \in J\} \cup \{y^j \mid j \in J\}$ is uncountable. If in addition $\prod_{i \in I} X_i$ is assumed to be countably compact, then this set must have an accumulation point z in $\prod_{i \in I} X_i$. But for every basic neighborhood U of z there exists a $j \in J$ such that $x^j \in U$ and $y^j \in U$. Hence it follows that $z \in A \cap B$ which, however, is impossible since A and B are disjoint. This contradiction shows that every closed G_δ-subset, in particular every D-closed subset, of a countably compact product space depends on countably many coordinates. In view of 4.13, 4.14, and the result of Pol and Puzio-Pol [1976] cited above, this observation completes the proof of Theorem 4.12.

In connection with 4.12–4.14 there are many open problems. Note, for example, that Problem D would be solved if the following question could be answered negatively.

PROBLEM E. Does there exist a subnormal Hausdorff space X such that ${}^{\varkappa}X$ is countably compact for each cardinal $\varkappa$, but X is not D-normal?

It follows from Lemma 2.11 that a topological space X is subdevelopable if and only if it has a sequence $(\mathscr{U}_n)_{n<\omega}$ of kernel-normal open covers such that $\bigcap \{\mathrm{St}(x, \mathscr{U}_n) \mid n < \omega\} = \{x\}$ for each $x \in X$. In (Chaber [1983a], Remark 6) it is pointed out that every T_1 σ-space is subdevelopable. But we do not know the answer to the following question.

PROBLEM F. Is every semi-metrizable topological space (2.1(ii)) necessarily subdevelopable?

All we can show is that "small" semi-metrizable spaces are always subdevelopable. For this purpose we recall the notion of a perfectly subparacompact space.

4.15. DEFINITION (Burke [1969]). A topological space is called *subparacompact* if every open cover has a σ-discrete closed refinement. It is called *perfectly subparacompact* if it is perfect (2.1(iv)) and subparacompact.

4.16. THEOREM. *Let X be a perfectly subparacompact space with G_δ-diagonal* ([16]). *If $|X| \leqslant 2^\omega$, then X is subdevelopable.*

Proof. ([17]) Since X has a G_δ-diagonal, there exists a sequence $(\mathscr{U}_n)_{n<\omega}$ of open covers of X such that

$$(*) \qquad \bigcap \{\mathrm{St}(x, \mathscr{U}_n) \mid n < \omega\} = \{x\}$$

for each $x \in X$ (Ceder [1961]). By the subparacompactness of X every $\mathscr{U}_n$ has a σ-discrete closed refinement $\mathscr{A}_n = \bigcup_{k<\omega} \mathscr{A}_{n,k}$. Since $|X| \leqslant 2^\omega$, we can find, for all $n, k < \omega$, an injective mapping $f_{n,k}$ from $\mathscr{A}_{n,k}$ into $\boldsymbol{D}_1$ (1.7(d)). Let $\mathscr{B}$ be a countable base for the open sets of $\boldsymbol{D}_1$ and define $E(B, n, k) = \bigcup \{A \in \mathscr{A}_{n,k} \mid f_{n,k}(A) \in B\}$ for each $B \in \mathscr{B}$ and $n, k < \omega$. By virtue of 4.4 there exist

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([16]) A topological space X is said to have a G_δ-diagonal if $\{(x, x) \mid x \in X\}$ is a G_δ-set in $X \times X$.

([17]) This proof is another application of a technique due to G. M. Reed and P. Zenor [1976].

a point $a \in D_1$ and continuous mappings $g_{(B,n,k)}$ from X into D_1 such that $E(B, n, k) = g^{-1}_{(B,n,k)}[\{a\}]$. Now let $g\colon X \to {}^{\mathscr{B}\times\omega\times\omega}D_1$ be the mapping defined by $g(x)(B, n, k) = g_{(B,n,k)}(x)$ for all $x \in X$ and $(B, n, k) \in \mathscr{B}\times\omega\times\omega$. Since g is continuous and ${}^{\mathscr{B}\times\omega\times\omega}D_1$ is a developable T_1-space, it suffices to show that g is injective.

To this end consider two distinct points x, y in X. By virtue of (*) there is an $n_0 < \omega$ such that $y \notin \mathrm{St}(x, \mathscr{U}_{n_0})$. Moreover, there is a $k_0 < \omega$ and an $A \in \mathscr{A}_{n_0,k_0}$ such that $x \in A$. Suppose that $g(x) = g(y)$. Then $g_{(B,n,k)}(y) = g_{(B,n,k)}(x)$ *for each* $(B, n, k) \in \mathscr{B}\times\omega\times\omega$, i.e.

$$\begin{aligned} y &\in \bigcap \{g^{-1}_{(B,n,k)}[\{g_{(B,n,k)}(x)\}] \mid (B, n, k) \in \mathscr{B}\times\omega\times\omega\} \\ &\subset \bigcap \{g^{-1}_{(B,n_0,k_0)}[\{g_{(B,n_0,k_0)}(x)\}] \mid B \in \mathscr{B},\ f_{n_0,k_0}(A) \in B\} \\ &\subset \bigcap \{g^{-1}_{(B,n_0,k_0)}[\{a\}] \mid B \in \mathscr{B},\ f_{n_0,k_0}(A) \in B\} \\ &\subset \bigcap \{E(B, n_0, k_0) \mid B \in \mathscr{B},\ f_{n_0,k_0}(A) \in B\} \\ &\subset A \subset \mathrm{St}(x, \mathscr{U}_{n_0}), \end{aligned}$$

which is impossible. Hence $g(x) \neq g(y)$, which completes the proof. ■

4.17. COROLLARY. *Every semi-metrizable space of cardinality* $\leqslant 2^{\omega}$ *is subdevelopable.* ■

We conclude this section with some remarks and two more problems.

4.18. Remarks. (a) For classes $\mathbf{E}$ consisting of a single space the normality hull $N(\mathbf{E})$ was already studied by S. Mrówka [1968], for arbitrary classes $\mathbf{E}$ by G. Preuß [1970]. However, the concept of E-normality introduced by H. Herrlich [1967] (for classes $\mathbf{E}$ of Hausdorff spaces) is more general than our notion of E-normality.

(b) It is not true that a closed subset of a D-normal space is always D-embedded (compare with 4.1). In fact, if M is the Michael line, i.e. the space obtained from the reals by isolating all irrationals, one can easily define a continuous mapping from the closed subspace consisting of all rationals into D_1 which has no continuous extension to all of M (Chaber [1984a]).

(c) Let us say that a subset A of a topological space is *d-embedded* in X if for every D-closed subset F of A there is a D-closed subset G of X such that $F = G \cap A$. Then a topological space X *is D-normal if and only if every closed subset of* X *is d-embedded in* X (Brandenburg [1978], 3.2.3).

(d) In [1984] J. Chaber has pointed out that a topological space *is D-normal if and only if any two disjoint closed sets can be separated by disjoint subsets of which the first is open and the second is a* G_δ*-set.*

(e) A subset A of a topological space X is said to be *normally situated* in X if for every open subset U of X containing A there exists an open subset V of X such that $A \subset V \subset U$ and $V = \bigcup \{V_\alpha \mid \alpha < \varkappa\}$, where $(V_\alpha)_{\alpha<\varkappa}$ is a family, locally finite in V, of open F_σ-sets of X. In (Brandenburg [1981]) it is shown that every normally situated subspace of a D-normal space is D-normal, in particular every F_σ-subset. However, the subspace $(\omega_1+1)\times(\omega+1)\setminus\{(\omega_1, \omega)\}$ of $(\omega_1+1)\times(\omega+1)$ witnesses that D-normality is not hereditary. For an internal characterization of hereditarily D-normal spaces see (Brandenburg [1981]).

(f) Even the product of two paracompact Hausdorff spaces need not be D-normal. In fact, in [1972] K. Alster and R. Engelking have constructed an example of a paracompact Hausdorff P-space ([18]) X such that $X\times X$ is a non-normal P-space. Since every subnormal P-space is normal, $X\times X$ cannot even be subnormal.

(g) If $(X_n)_{n<\omega}$ is a family of non-trivial Hausdorff spaces (i.e. $|X_n|\geqslant 2$ for each $n<\omega$), then $\prod_{n<\omega}X_n$ is hereditarily D-normal if and only if $\prod_{n<\omega}X_n$ is perfect (Brandenburg [1981]).

(h) Clearly, quotients of D-normal spaces need not be D-normal, for every topological space is a quotient of a paracompact Hausdorff space (Isbell [1969]). But D-normality is additive.

(i) Closed images of D-normal spaces are subnormal, closed-and-open images of D-normal spaces are D-normal (Brandenburg [1981]). However, perfect preimages of D-normal spaces are not necessarily D-normal (Chaber [1984]).

(j) E. V. Ščepin [1976] has raised the problem to characterize internally those product spaces $\prod_{i\in I}X_i$ on which every continuous mapping into a space with G_δ-diagonal depends on countably many coordinates. This might be a difficult task, for the usual techniques in this area require that the range space is Hausdorff. ([19]) Therefore Theorem 4.14 seems to be a step in the right direction. ∎

For a topological space X let $\exp X$ be the space obtained by supplying the set of all nonempty closed subsets of X with the Vietoris topology. By a theorem of J. Keesling [1970] (assuming CH) and N. V. Veličko [1975] X is compact provided that $\exp X$ is a normal T_1-space. We do not know whether the compactness of X is already a consequence of the D-normality of $\exp X$:

PROBLEM G. Let X be a Hausdorff space such that $\exp X$ is D-normal. Is X necessarily compact? In particular, prove that $\exp\omega$ is not D-normal.

Let us finally note that it follows from Theorem 4.2 that the normality hull of the class of semi-metrizable spaces coincides with the class of D-normal spaces, a fact which motivates our last problem in this section.

PROBLEM H. Find internal characterizations of Q-normal spaces, where **Q** is the class of quasi-metrizable T_1-spaces.

([18]) A topological space is a *P-space* if every G_δ-set is open.

([19]) For surveys on this area see (Engelking [1966]) and (Hušek [1976]).

§ 5. D-paracompact spaces

The *paracompactness hull* $P(\mathbf{E})$ of a class $\mathbf{E}$ of topological spaces was introduced by C. N. Maxwell [1961] and studied by V. Šedivá-Trnková [1964]. It consists of all topological spaces X with the property that for every open cover $\mathscr{U}$ of X there is a $\mathscr{U}$-mapping (4.6) from X into some space belonging to $\mathbf{E}$. For example, for the class $\mathbf{M}$ of metrizable spaces $P(\mathbf{M})$ is precisely the class of fully normal spaces (Tukey [1940]). Thus, by Stone's theorem [1948] $P(\mathbf{M})\cap\{\text{Hausdorff spaces}\}$ is the class of paracompact Hausdorff spaces, a fact which was shown directly by C. H. Dowker [1948]. *D-paracompact spaces*, i.e. the spaces belonging to the paracompactness hull $P(\mathbf{D})$ of the class $\mathbf{D}$ of developable T_1-spaces, were first studied by C. M. Pareek who proved a (somewhat technical) internal characterization in [1972] (see 5.12(a)). These investigations were continued in (Brandenburg [1978] and [1985]) and (Chaber [1984]).

The approach chosen in (Brandenburg [1978]) can roughly be described as follows. By definition, D-paracompactness generalizes full normality as well as paracompactness. Thus, in order to characterize D-paracompact spaces internally it is first necessary to find suitable generalizations of normal open covers in the sense of Tukey [1940] respectively of locally finite open (F_σ-) covers. If this is done properly, the covering properties defined by means of these generalizations should coincide (remember Stone's theorem!) and characterize D-paracompactness. Since Lemma 2.11 indicates that we have already found the "right" generalization of normal open covers, it remains to look for a suitable weakening of local finiteness.

5.1. Definition (Brandenburg [1978]). A collection $\mathscr{U} = (U(i))_{i\in I}$ of subsets of a topological space X is called *dissectable* if there exists a *dissection* of $\mathscr{U}$, i.e. a mapping D from $I\times\omega$ into $\mathscr{P}(X)$ satisfying

(D.1) $U(i) = \bigcup\{D(i, n)\mid n<\omega\}$ for each $i\in I$;

(D.2) $\{D(i, n)\mid i\in I\}$ is a closure-preserving collection of closed sets for each $n<\omega$;

(D.3) for each $n<\omega$ and for each $x\in\bigcup\{D(i, n)\mid i\in I\}$ the set $\bigcap\{U(i)\mid i\in I,\ x\in D(i, n)\}$ is a neighborhood of x.

$\mathscr{U}$ is called *σ-dissectable*, if $\mathscr{U} = \bigcup_{n<\omega}\mathscr{U}_n$, where every $\mathscr{U}_n$ is dissectable.

Obviously, every locally finite collection of F_σ-sets is dissectable. Quite recently we have noticed that dissectability can be reformulated in terms of real-valued upper semicontinuous mappings.

5.2. Lemma (Brandenburg [1986]). *A collection $\mathscr{U} = (U(i))_{i\in I}$ of subsets of a topological space X is dissectable if and only if there exists a family $\mathscr{F} = (f_i\colon X\to[0, 1])_{i\in I}$ of upper semicontinuous mappings satisfying*

(F.1) $f_i^{-1}[(0, 1]] = U(i)$ *for each* $i\in I$;

(F.2) *F is equi-upper semicontinuous, i.e. whenever* $x \in X$ *and* $J \subset I$ *such that* $\{f_j(x) \mid j \in J\} \subset [0, b)$, $b \leqslant 1$, *there exists a neighborhood* U *of* x *such that* $\{f_j(y) \mid j \in J,\ y \in U\} \subset [0, b)$;

(F.3) *whenever* $x \in X$ *and* $J \subset I$ *such that* $\{f_j(x) \mid j \in J\} \subset (a, 1]$, $a > 0$, *there exists a neighborhood* V *of* x *such that* $\{f_j(y) \mid j \in J,\ y \in V\} \subset (0, 1]$.

Proof. Assume first that $\mathscr{U}$ is dissectable. Then there exists a dissection D of $\mathscr{U}$ with the additional property that $D(i, n) \subset D(i, n+1)$ for each $(i, n) \in I \times \omega$ (Brandenburg [1978], 3.3.2). For each $i \in I$ define $f_i\colon X \to [0, 1]$ by

$$f_i(x) = \begin{cases} 0 & \text{if } x \in X \setminus U(i), \\ 1/2^{n(x)} & \text{if } x \in U(i), \text{ where } n(x) = \min\{n < \omega \mid x \in D(i, n)\}. \end{cases}$$

Clearly, then (F.1) is satisfied. In order to establish the equi-upper semicontinuity of $\mathscr{F}$ consider a point $x \in X$ and a subset J of I such that $\{f_j(x) \mid j \in J\} \subset [0, b)$ for some $b \leqslant 1$. If $n_b = \max\{n < \omega \mid b \leqslant 1/2^n\}$, then $x \in X \setminus D(j, n_b)$ for each $j \in J$. Since $\{D(i, n_b) \mid i \in I\}$ is closure-preserving, $U = \bigcap \{X \setminus D(j, n_b) \mid j \in J\}$ is a neighborhood of x which has the desired property that $\{f_j(y) \mid j \in J,\ y \in U\} \subset [0, b)$.

It remains to show that (F.3) is satisfied. To this end let x be a point in X and $J \subset I$ such that $\{f_j(x) \mid j \in J\} \subset (a, 1]$, where $0 < a$. Then $x \in D(j, n_a)$ for each $j \in J$, where $n_a = \min\{n < \omega \mid 1/2^n \leqslant a\}$. Hence $V = \bigcap \{U(j) \mid j \in J\}$ is a neighborhood of x which clearly has the property that $\{f_j(y) \mid j \in J,\ y \in V\} \subset (0, 1]$. Conversely, if $\mathscr{F} = (f_i\colon X \to [0, 1])_{i \in I}$ is a family of upper semicontinuous mappings satisfying (F.1)–(F.3), then it is easily seen that $(i, n) \mapsto D(i, n) = f_i^{-1}[[1/2^n, 1]]$ defines a dissection of $\mathscr{U}$. ∎

The following lemma from (Brandenburg [1978]) is our main tool for showing that an open cover of a topological space is dissectable.

5.3. Lemma. *Let* $\mathscr{U} = (U(i))_{i \in I}$ *be an open cover of a subparacompact space. If there exists a sequence* $\beta = (\mathscr{U}_n)_{n < \omega}$ *of open refinements of* $\mathscr{U}$ *such that* $U(i) = \operatorname{int}_\beta U(i)$ *for each* $i \in I$, *then, for every subset* J *of* I, *the collection* $(U(j))_{j \in J}$ *is dissectable.*

Proof. By the subparacompactness of X there exists, for each $n < \omega$, a sequence $(\mathscr{U}_n(k))_{k < \omega}$ of open refinements of $\mathscr{U}_n$ with the property that for every x in X there is a $k(x) < \omega$ such that $\operatorname{ord}(x, \mathscr{U}_n(k(x))) = 1$ (Burke [1970]). For each pair $(n, k) \in \omega \times \omega$ define $X(n, k) = \{x \in X \mid \operatorname{ord}(x, \mathscr{U}_n(k)) = 1\}$. Moreover, for each $x \in X(n, k)$ let $U(x, n, k)$ be the member of $\mathscr{U}_n(k)$ containing x and set $A(x, n, k) = X \setminus \bigcup \{U \in \mathscr{U}_n(k) \mid x \notin U\}$. It is easily seen that for every pair $(n, k) \in \omega \times \omega$ the collection $\{A(x, n, k) \mid x \in X(n, k)\}$ is closed and discrete. Additionally, the following holds:

A. $U(i) = \bigcup \{U(i, n, k) \mid (n, k) \in \omega \times \omega\}$ for each $i \in I$, where
$U(i, n, k) = \bigcup \{A(x, n, k) \mid x \in X(n, k),\ U(x, n, k) \subset U(i)\}$.

In order to verify this claim consider an arbitrary $i \in I$ and a point $z \in U(i)$. Since $U(i) = \operatorname{int}_\beta U(i)$, there is an $n(z) < \omega$ such that $\operatorname{St}(z, \mathscr{U}_{n(z)}) \subset U(i)$. Moreover, there exists a $k(z) < \omega$ such that $\operatorname{ord}(z, \mathscr{U}_{n(z)}(k(z))) = 1$. It follows that $z \in X(n(z), k(z))$ and therefore $z \in U(z, n(z), k(z)) = \operatorname{St}(z, \mathscr{U}_{n(z)} k(z)) \subset \operatorname{St}(z, \mathscr{U}_{n(z)}) \subset U(i)$, i.e. $z \in A(z, n(z), k(z)) \subset U(i, n(z), k(z))$. This shows that $U(i) \subset \bigcup \{U(i, n, k) \mid (n, k) \in \omega \times \omega\}$. That the reverse inclusion holds is clear from the definitions. Now let $m \mapsto (n(m), k(m))$ be a bijection from ω onto $\omega \times \omega$ and consider an arbitrary nonempty subset J of I. We claim that the correspondence $(j, m) \in J \times \omega \mapsto U(j, n(m), k(m))$ defines a dissection D of $(U(j))_{j \in J}$.

Indeed, Claim A shows that D satisfies condition (D.1) in 5.1. That (D.2) is also satisfied is an immediate consequence of the fact that every collection $\{A(x, n, k) \mid x \in X(n, k)\}$, $(n, k) \in \omega \times \omega$, is closed and discrete. Moreover, if $m < \omega$ and $z \in \bigcup \{U(j, n(m), k(m)) \mid j \in J\}$, there is an $x \in X(n(m), k(m))$ such that $z \in A(x, n(m), k(m))$. Then $U(x, n(m), k(m))$ is an open neighborhood of z which is contained in $\bigcap \{U(j) \mid j \in J,\ z \in U(j, n(m), k(m))\}$. Therefore D has property (D.3) from 5.1, which completes the proof. ∎

In order to formulate the main characterization theorem we need to introduce one more concept. Let us call a topological space X *D-expandable*, if for every discrete collection $(F(i))_{i \in I}$ of closed subsets of X and for every collection $(V(i))_{i \in I}$ of open subsets of X such that $F(i) \subset V(i)$ for each $i \in I$ and $F(i) \cap V(j) = \emptyset$ whenever $i, j \in I$ and $i \neq j$ there exists a dissectable collection $(U(i))_{i \in I}$ of open subsets of X such that $F(i) \subset U(i) \subset V(i)$ for each $i \in I$.

5.4. THEOREM (Brandenburg [1978], [1985], [1986]). *For a topological space $X = (X, \tau)$ the following conditions are equivalent:*

(i) *X is D-paracompact.*

(ii) *For every open cover $(U_i)_{i \in I}$ of X there exists a dissectable open cover $(V_i)_{i \in I}$ of X such that $V_i \subset U_i$ for each $i \in I$.*

(iii) *For every open cover $(U_i)_{i \in I}$ of X there exists an equi-upper semicontinuous family $\mathscr{F} = (f_i\colon X \to [0, 1])_{i \in I}$ of mappings such that $(f_i^{-1}[(0, 1]])_{i \in I}$ is a precise open refinement of $(U_i)_{i \in I}$, i.e. $f_i^{-1}[(0, 1]] \subset U_i$ for each $i \in I$, and condition* (F.3) *in* 5.2 *is satisfied.*

(iv) *X is subparacompact and D-expandable.*

(v) *X is θ-refinable* ([20]) *and D-expandable.*

(vi) *X is weakly $\bar{\theta}$-refinable* (4.7) *and D-expandable.*

(vii) *Every open cover of X has a σ-dissectable open refinement.*

(viii) *Every open cover of X is kernel-normal.*

Proof. (i) implies (ii): Let $\mathscr{U} = (U_i)_{i \in I}$ be an open cover of X. By the D-paracompactness of X there exists a $\mathscr{U}$-mapping f from X onto a developable

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([20]) A topological space X is called *θ-refinable* (Wicke and Worrell Jr. [1965]), if for every open cover $\mathscr{U}$ of X there is a sequence $(\mathscr{V}_n)_{n<\omega}$ of open refinements of $\mathscr{U}$ such that for each $x \in X$ $\operatorname{ord}(x, \mathscr{U}_{n(x)}) < \omega$ for some $n(x) < \omega$.

T_1-space Y. Consequently, there is a developable topology $\tau' \subset \tau$ and a τ'-open cover $\mathscr{V} = (V_i)_{i\in I}$ of X such that $V_i \subset U_i$ for each $i \in I$. Since developable spaces are subparacompact, it follows from Lemma 5.3 that $\mathscr{V}$ is dissectable in (X, τ'), hence also in (X, τ).

That (ii) and (iii) are equivalent follows from Lemma 5.2.

(ii) implies (iv): We will first show that X is subparacompact. To this end let $\mathscr{U} = (U_i)_{i\in I}$ be an open cover of X. By (ii) we may assume that there exists a dissection D of $\mathscr{U}$. If $\mathscr{D}_n = \{D(i, n) \mid i \in I\}$ for each $n < \omega$, then $\mathscr{D} = \bigcup_{n<\omega} \mathscr{D}_n$ is a σ-closure preserving closed refinement of $\mathscr{U}$ which proves the subparacompactness of X (Burke [1969]).

In order to show that X is D-expandable consider a discrete collection $(F(i))_{i\in I}$ of closed subsets of X and a family $\mathscr{V} = (V(i))_{i\in I}$ of open subsets of X satisfying $F(i) \subset V(i)$ for each $i \in I$ and $F(i) \cap V(j) = \emptyset$ whenever $i, j \in I$ and $i \neq j$. Then $\mathscr{V} \cup \{X \setminus \bigcup \{F(i) \mid i \in I\}\}$ is an open cover of X. By virtue of (ii) there exists an open cover $(U(i))_{i\in I} \cup \{U\}$ of X such that $(U(i))_{i\in I}$ is dissectable, $U \subset X \setminus \bigcup \{F(i) \mid i \in I\}$, and $U(i) \subset V(i)$ for each $i \in I$. Since $F(i) \subset U(i)$ for each $i \in I$, it follows that X is D-expandable.

That (iv) implies (v) is clear, since subparacompact spaces are θ-refinable (Burke [1970]), and that (v) implies (vi) is due to the fact that θ-refinable spaces are weakly $\bar{\theta}$-refinable (Smith [1975]).

(vi) implies (vii): Let $\mathscr{U} = (U_i)_{i\in I}$ be an open cover of X. By (vi) we may assume that $\mathscr{U}$ is a weak $\bar{\theta}$-cover, i.e. that $I = \bigcup \{I(n) \mid n < \omega\}$ such that

(a) for each $x \in X$ there exists an $n(x) < \omega$ such that $0 < \operatorname{ord}(x, \mathscr{U}_{n(x)}) < \omega$, where $\mathscr{U}_n = (U_i)_{i\in I(n)}$ for each $n < \omega$;

(b) $\mathscr{U}^* = (U(n))_{n<\omega}$ is point-finite, where $U(n) = \bigcup \{U_i \mid i \in I(n)\}$ for each $n < \omega$.

Then the following holds:

A. For each m, k such that $0 < m < \omega$ and $0 < k < \omega$ there exists a σ-dissectable collection $\mathscr{V}(m, k)$ of open subsets of X such that the following conditions are satisfied:

(c) every $V \in \mathscr{V}(m, k)$ is contained in some U_i;

(d) if $\operatorname{ord}(x, \mathscr{U}^*) = m$ and $\operatorname{ord}(x, \mathscr{U}_n) = k$ for some $n < \omega$, then $x \in \bigcup \{V \in \mathscr{V}(m', k') \mid (m', k') \leqslant (m, k)\}$, where $(m', k') \leqslant (m, k)$ if either $m' < m$ or $m' = m$ and $k' \leqslant k$.

Note that once this claim is proved it follows that $\mathscr{V} = \bigcup \{\mathscr{V}(m, k) \mid 0 < m < \omega,\ 0 < k < \omega\}$ is a σ-dissectable open refinement of $\mathscr{U}$.

In order to verify Claim A we proceed by induction. Assume that $m = 1$ and $k = 1$. For each $r < \omega$ and for each $j \in I(r)$ define $E(1, 1, r, j) = \bigcap \{X \setminus U_i \mid i \in I(r) \setminus \{j\}\} \cap \bigcap \{X \setminus U(n) \mid n \in \omega \setminus \{r\}\}$. Then $E(1, 1, r, j) \subset U_j$ for each $j \in I(r)$ and $U_i \cap E(1, 1, r, j) = \emptyset$ whenever $i \in I(r)$ and $i \neq j$. Since it is easily seen that $\mathscr{E}(1, 1, r) = (E(1, 1, r, j))_{j\in I(r)}$ is discrete, it follows from (vi) that for each $r < \omega$

there exists a dissectable collection $\mathscr{V}(1, 1, r) = (V(1, 1, r, j))_{j\in I(r)}$ of open subsets of X such that $E(1, 1, r, j) \subset V(1, 1, r, j) \subset U_j$ for each $j \in I(r)$. If $\mathscr{V}(1, 1) = \bigcup\{\mathscr{V}(1, 1, r) \mid r < \omega\}$, then $\mathscr{V}(1, 1)$ satisfies conditions (c) and (d).

Now let $(1, 1) < (m, k)$ and assume that for each $(m', k') < (m, k)$ a σ-dissectable collection $\mathscr{V}(m', k')$ of open subsets of X satisfying (c) and (d) is already defined. For each $N \in [\omega]^m$, $r \in N$, and $J \in [I(r)]^k$ define

$$E(m, k, N, r, J) = \bigcap\{X\backslash U_i \mid i \in I(r)\backslash J\} \cap \bigcap\{X\backslash U(n) \mid n \in \omega\backslash N\}$$
$$\cap \bigcap\{X\backslash V \mid V \in \mathscr{V}(m', k'),\ (m', k') < (m, k)\}.$$

As in the proof of Theorem 4.8, (i) implies (ii), it is easy to show that every $\mathscr{E}(m, k, N, r) = (E(m, k, N, r, J))_{J\in[I(r)]^k}$ is discrete. Since $E(m, k, N, r, J) \subset \bigcap\{U_i \mid i \in J\}$ and $\bigcap\{U_i \mid i \in J'\} \cap E(m, k, N, r, J) = \emptyset$ whenever $J' \in [I(r)]^k$ and $J \neq J'$, it follows from (vi) that for each $N \in [\omega]^m$ and for each $r \in N$ there exists a dissectable collection $\mathscr{V}(m, k, N, r) = (V(m, k, N, r, J))_{J\in[I(r)]^k}$ such that $E(m, k, N, r, J) \subset V(m, k, N, r, J) \subset \bigcap\{U_i \mid i \in J\}$. If $\mathscr{V}(m, k) = \bigcup\{\mathscr{V}(m, k, N, r) \mid N \in [\omega]^m,\ r \in N\}$, then $\mathscr{V}(m, k)$ satisfies (c) and (d), which completes the induction.

(vii) implies (viii): Let $\mathscr{U}$ be an open cover of X. Assuming (vii) there exists an open refinement $\mathscr{V} = \bigcup_{n<\omega}\mathscr{V}_n$ of $\mathscr{U}$ such that every $\mathscr{V}_n = (V(n, i))_{i\in I(n)}$ is dissectable. Let $D_n\colon I(n) \times \omega \to \mathscr{P}(X)$ be a dissection of $\mathscr{V}_n$. If $x \in \bigcup\{D_n(i, k) \mid i \in I(n)\}$, there exists an open neighborhood $U(x, n, k)$ of x such that $U(x, n, k) \subset \bigcap\{V(n, i) \mid i \in I(n),\ x \in D_n(i, k)\}$ and $U(x, n, k) \subset \bigcap\{X\backslash D_n(i, k) \mid i \in I(n),\ x \notin D_n(i, k)\}$. Set $U(V, n, k) = V \cap (X\backslash\bigcup\{D_n(i, k) \mid i \in I(n)\})$ for each $V \in \mathscr{V}$ and define

$$\mathscr{U}(n, k) = \{U(x, n, k) \mid x \in \bigcup\{D_n(i, k) \mid i \in I(n)\}\} \cup \{U(V, n, k) \mid V \in \mathscr{V}\}.$$

Clearly, every $\mathscr{U}(n, k)$ is an open cover of X which refines $\mathscr{V}$ Moreover, the following holds:

B. If $\beta = (\mathscr{U}(n, k))_{(n,k)\in\omega\times\omega}$, then $\operatorname{int}_\beta V = V$ for each $V \in \mathscr{V}$

Note that once this claim is proved it follows that every $\mathscr{U}(n, k)$ refines $\operatorname{int}_\beta \mathscr{U}$, hence $\mathscr{U}$ is kernel-normal by virtue of Lemma 2.12.

In order to verify Claim B consider an arbitrary $V \in \mathscr{V}$ There exists an $n < \omega$ such that $V = V(n, i)$, $i \in I(n)$. Since it suffices to show that $V(n, i) \subset \operatorname{int}_\beta V(n, i)$, consider an arbitrary point $x \in V(n, i)$. By property (D.1) of the dissection D_n (5.1) there exists a $k < \omega$ such that $x \in D_n(i, k)$. Claim B is proved as soon as we have shown that $\operatorname{St}(x, \mathscr{U}(n, k)) \subset V(n, i)$.

From the definiton of $U(n, k)$ it is clear that

$$\operatorname{St}(x, \mathscr{U}(n, k)) = \bigcup\{U(y, n, k) \mid y \in \bigcup\{D_n(i, k) \mid i \in I(n)\},\ x \in U(y, n, k)\}$$

so that it suffices to show that $U(y, n, k) \subset V(n, i)$ whenever $y \in \bigcup\{D_n(i, k) \mid i \in I(n)\}$ and $x \in U(y, n, k)$. Now, if $y \in \bigcup\{D_n(i, k) \mid i \in I(n)\}$ and

$x \in U(y, n, k)$, then $\{j \in I(n) \mid x \in D_n(j, k)\} \subset \{j \in I(n) \mid y \in D_n(j, k)\}$, for otherwise there would exist a $j_0 \in I(n)$ such that $x \in D_n(j_0, k)$ but $y \notin D_n(j_0, k)$, i.e.

$$x \in \bigcup \{D_n(j, k) \mid j \in I(n),\ y \notin D_n(j, k)\} \subset X \backslash U(y, n, k),$$

contradicting the fact that $x \in U(y, n, k)$. It follows that

$$\begin{aligned} U(y, n, k) &\subset \bigcap \{V(n, j) \mid j \in I(n),\ y \in D_n(j, n)\} \\ &\subset \bigcap \{V(n, j) \mid j \in I(n),\ x \in D_n(j, n)\} \\ &\subset V(n, i), \end{aligned}$$

which completes the argument.

(viii) implies (i): Assuming (viii) it follows from Lemma 2.11 that for a given open cover $\mathscr{U}$ of X there exists a developable topology $\tau' \subset \tau$ and a τ'-open cover $\mathscr{V}$ of X which refines $\mathscr{U}$. If Y is the space obtained from (X, τ') by identifying all points which have identical closures, then Y is a developable T_1-space and the natural quotient mapping defines a $\mathscr{U}$-mapping from (X, τ) onto Y. Consequently $X = (X, \tau)$ is D-paracompact. ∎

Clearly every developable space and every fully normal space (Tukey [1940]) is D-paracompact, hence every paracompact Hausdorff space. Every D-paracompact space is D-normal (compare Theorems 5.4 and 4.8). It was known for some time that every perfectly metacompact space ([21]) is subparacompact (Hodel [1970]). Quite recently J. Chaber has proved that, in fact, such a space is already D-paracompact [1983a]. Following (Brandenburg [1985]) we will now show how Chaber's theorem can be derived from Theorem 5.4.

5.5. COROLLARY (Chaber [1983a]). *Every perfectly metacompact space is D-paracompact.*

Proof. Consider a point-finite open cover $\mathscr{U} = (U_i)_{i \in I}$ of a perfectly metacompact space X. By virtue of 5.4 it suffices to show that $\mathscr{U}$ is dissectable. Since $X(n) = \{x \in X \mid \operatorname{ord}(x, \mathscr{U}) \geqslant n\}$ is open, there exists a sequence $(X(n, k))_{k<\omega}$ of closed subsets of X such that $X(n) = \bigcup \{X(n, k) \mid k < \omega\}$ whenever $0 < n < \omega$. Set $\mathscr{U}(n, k) = \mathscr{U} \cup \{X \backslash X(n, k)\}$ and $A(x, n, k) = \bigcap \{X \backslash U \mid U \in \mathscr{U}(n, k),\ x \in X \backslash U\}$. If $m \mapsto (n(m), k(m))$ is a bijection from ω onto $(\omega \backslash \{0\}) \times \omega$ and

$$D(i, m) = \bigcup \{A(x, n(m), k(m)) \mid x \in U_i \cap X(n(m), k(m)),\ \operatorname{ord}(x, \mathscr{U}) = n(m)\}$$

for each pair $(i, m) \in \mathrm{I} \times \omega$, then it is easily seen that $(i, m) \mapsto D(i, m)$ defines a dissection of $\mathscr{U}$. ∎

Concerning the relationships between metacompactness and D-paracompactness in normal T_1-spaces C. M. Pareek has asked whether every normal D-paracompact T_1-space is metacompact, and whether every metacompact

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([21]) A topological space is said to be *perfectly metacompact*, if it is perfect (2.1 (iv)) and metacompact.

normal T_1-space is D-paracompact (Pareek [1972], Problems 6.2 and 6.3). Both questions have negative answers. In fact, R. H. Bing's example F [1951] can easily be seen to be a counterexample to the first question, while in [1974] D. K. Burke has given an example of a metacompact normal T_1-space which is not even subparacompact.

In view of Corollary 5.5 one might wonder whether every perfectly subparacompact space (4.15) is D-paracompact. However, this possibility is ruled out by the following example from (Brandenburg [1985]).

5.6. EXAMPLE. A completely regular perfectly subparacompact T_1-space which is not D-paracompact. Let S be the Sorgenfrey line, i.e. the reals supplied with the topology generated by all half-open intervals $[x_0, x_1)$, $x_0 < x_1$. It is known that $S \times S$ is perfect (Heath and Michael [1971]) and subparacompact (Lutzer [1972]). Originally S was introduced as the first example of a paracompact Hausdorff space whose square is not paracompact (Sorgenfrey [1947]). We will now show that $S \times S$ is not even D-paracompact.

To this end let $U(x) = [x, x+1) \times [-x, -x+1)$ for each $x \in \boldsymbol{R}$ and $U = \boldsymbol{R}^2 \setminus \{(x, -x) \mid x \in \boldsymbol{R}\}$. Then $\mathscr{U} = \{U(x) \mid x \in \boldsymbol{R}\} \cup \{U\}$ is an open cover of $S \times S$. Suppose that $S \times S$ is D-paracompact. Then there exists a $\mathscr{U}$-mapping f from $S \times S$ onto a developable T_1-space Y. Let $(\mathscr{V}_n)_{n<\omega}$ be a development of Y and define $\mathscr{U}_n = \{f^{-1}[V] \mid V \in \mathscr{V}_n\}$. Moreover set $\beta = (\mathscr{U}_n)_{n<\omega}$ and $H = \{(x, y) \in \boldsymbol{R}^2 \mid (x, y) \in \operatorname{int}_\beta U(x)\}$. Then the following holds:

A. H is of the second category with respect to the Euclidean topology of $\boldsymbol{R}^2$.

For suppose that H is of the first category in $\boldsymbol{R}^2$. Then, by a theorem of K. Kuratowski and S. Ulam (e.g. see Kuratowski [1966], p. 247) there exists an $x_0 \in \boldsymbol{R}$ such that $H(x_0) = \{y \in \boldsymbol{R} \mid (x_0, y) \in H\}$ is of the first category in $\boldsymbol{R}$. Since f is a $\mathscr{U}$-mapping, $(x_0, -x_0) \in \operatorname{int}_\beta U(x_0)$. Hence there exists an $r > 0$ such that $[x_0, x_0+r) \times [-x_0, -x_0+r)$ is contained in $\operatorname{int}_\beta U(x_0)$, for $\operatorname{int}_\beta U(x_0)$ is open in $S \times S$. It follows that $[-x_0, -x_0+r)$ is a subset of $H(x_0)$ which is of the second category in $\boldsymbol{R}$. However, since every subset of $H(x_0)$ must be of the first category in $\boldsymbol{R}$, we arrive at a contradiction which proves Claim A.

Now $H = \bigcup \{H_n \mid n < \omega\}$, where $H_n = \{(x, y) \in H \mid \operatorname{St}((x, y), \mathscr{U}_n) \subset U(x)\}$ for each $n < \omega$. Hence there exist an $n < \omega$, a point $(x, y) \in \boldsymbol{R}^2$, and an $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \times (y-\varepsilon, y+\varepsilon)$ is contained in the Euclidean closure of H_n. Moreover, there exists a point $(u, v) \in H_n$ and a $\delta > 0$ such that

$$[u, u+\delta) \times [v, v+\delta) \subset (x-\varepsilon, x+\varepsilon) \times (y-\varepsilon, y+\varepsilon) \cap \operatorname{St}((u, v), \mathscr{U}_n).$$

If $(s, t) \in (u, u+\delta) \times (v, v+\delta) \cap H_n$, then $\operatorname{St}((s, t), \mathscr{U}_n) \subset U(s)$. Since $(u, v) \in \operatorname{St}((s, t), \mathscr{U}_n)$, it follows that $s \leqslant u$, contradicting the fact $u < s$. Hence $S \times S$ cannot be D-paracompact. ∎

Perfectly subparacompact spaces were characterized internally by H. Junnila [1980] and J. Chaber and P. Zenor [1977]. A slight modification of their argument yields the following characterization of perfectly D-paracompact spaces. ([22])

5.7. THEOREM. *For a topological space $X = (X, \tau)$ the following conditions are equivalent:*

(i) *X is perfectly D-paracompact.*

(ii) *For every open cover $\mathscr{U} = (U_\alpha)_{\alpha<\varkappa}$ of X there exists a countable kernel-normal collection β of open covers of X such that $\mathscr{U} \in \beta$ and $\min\{\alpha < \varkappa \mid x \in \operatorname{int}_\beta U_\alpha\} = \min\{\alpha < \varkappa \mid x \in U_\alpha\}$.*

(iii) *For every open cover $(U_\alpha)_{\alpha<\varkappa}$ of X there exists a dissectable open cover $(V_\alpha)_{\alpha<\varkappa}$ of X such that $U_\alpha \setminus \bigcup\{U_\gamma \mid \gamma < \alpha\} \subset V_\alpha \subset U_\alpha$ for each $\alpha < \varkappa$.*

Proof. (i) implies (ii): Consider an arbitrary open cover $\mathscr{U} = (U_\alpha)_{\alpha<\varkappa}$ of X. Assuming (i) there exists, for each $\alpha < \varkappa$, a sequence $(A(\alpha, n))_{0<n<\omega}$ of closed sets such that $\bigcup\{A(\alpha, n) \mid 0 < n < \omega\} = \bigcup\{U_\gamma \mid \gamma < \alpha\}$. Moreover, inductively we can find, for each $(n_0, \ldots, n_{k-1}) \in S$ ([23]), an open cover $\mathscr{U}(n_0, \ldots, n_{k-1})$ of X such that the following conditions are satisfied:

(a) $\mathscr{U}(0) = \mathscr{U}$ and $\mathscr{U}(n)$ refines $\mathscr{U}(0)$ for each $n < \omega$;

(b) $\beta(0) = (\mathscr{U}(n))_{n<\omega}$ and $\beta(n_0, \ldots, n_{k-1}) = \{\mathscr{U}(n_0, \ldots, n_{k-1})\} \cup \{\mathscr{U}(n_0, \ldots, n_{k-1}, n) \mid 0 < n < \omega\}$ are kernel-normal whenever $(n_0, \ldots, n_{k-1}, n) \in S$, $0 < n < \omega$;

(c) $\mathscr{U}(n_0, \ldots, n_{k-1})$ refines $\mathscr{U}(n_0, \ldots, n_{k-2})$ and

$\big(U_\alpha \setminus \big(A(\alpha, n_{k-1}) \cup \operatorname{cl}\{z \mid \operatorname{St}(z, \mathscr{U}(n_0, \ldots, n_{k-2})) \subset \bigcup\{U_\gamma \mid \gamma < \alpha\}\}\big)\big)_{\alpha<\varkappa}$ ([24])

whenever $(n_0, \ldots, n_{k-1}) \in S$ and $k > 1$.

Then $\beta = \bigcup\{\beta(n_0, \ldots, n_{k-1}) \mid (n_0, \ldots, n_{k-1}) \in S\}$ is a countable kernel-normal collection of open covers of X containing $\mathscr{U}$. Suppose that there exists an $x \in X$ such that

$$(*) \qquad \alpha_0 = \min\{\alpha < \varkappa \mid x \in \operatorname{int}_\beta U_\alpha\} > \min\{\alpha < \varkappa \mid x \in U_\alpha\}.$$

Then there is an $(n_0, \ldots, n_{k-1}) \in S$ such that $\operatorname{St}(x, \mathscr{U}(n_0, \ldots, n_{k-1}))$ is contained in U_{α_0}. Because of (a) we may assume that $(n_0, \ldots, n_{k-1}) \neq (0)$. It follows that

$$(**) \quad x \in \operatorname{cl}\{z \mid \operatorname{St}(z, \mathscr{U}(n_0, \ldots, n_{k-1})) \subset \bigcup\{U_\gamma \mid \gamma < \alpha\}\} \text{ whenever } \alpha > \alpha_0.$$

Moreover, from $(*)$ we conclude that $x \in A(\alpha_0, n(x))$ for some $0 < n(x) < \omega$. Since (b) implies that $\operatorname{int}_{\beta(n_0, \ldots, n_{k-1})} \mathscr{U}(n_0, \ldots, n_{k-1}, n(x))$ covers X there are a $U \in \mathscr{U}(n_0, \ldots, n_{k-1}, n(x))$ and a $k(x) < \omega$ such that

$$(***) \qquad \operatorname{St}(x, \mathscr{U}(n_0, \ldots, n_{k-1}, k(x))) \subset U.$$

([22]) A topological space is called *perfectly D-paracompact*, if it is perfect and D-paracompact.

([23]) For the definition of S see § 1.

([24]) Note that this is an open cover of X.

By virtue of (c) there is an $\alpha < \varkappa$ such that

$$U \subset U_\alpha \backslash (A(\alpha, n(x)) \cup \mathrm{cl}\{z \mid \mathrm{St}(z, \mathscr{U}(n_0, \ldots, n_{k-1})) \subset \bigcup \{U_\gamma \mid \gamma < \alpha\}\}).$$

Now (***) and (**) imply that $\alpha \leqslant \alpha_0$. Since $x \in A(\alpha_0, n(x))$, $\alpha = \alpha_0$ is impossible. Therefore $\alpha < \alpha_0$ which, however, contradicts the definition of α_0, for (***) implies that $x \in \mathrm{int}_\beta U_\alpha$. Consequently $\min\{\alpha < \varkappa \mid x \in \mathrm{int}_\beta U_\alpha\} = \min\{\alpha < \varkappa \mid x \in U_\alpha\}$ for each $x \in X$.

That (ii) implies (iii) follows from Lemmas 2.11 and 5.3. Moreover, 5.4 shows that (iii) implies (i), which completes the proof. ∎

In [1972] C. M. Pareek claimed that the following property of a T_1-space X characterizes S-paracompact spaces, where **S** is the class of semi-metrizable spaces (2.1 (ii)).

(†) For every open cover $\mathscr{U}$ of X there exists a sequence $(\mathscr{U}_m)_{m<\omega}$ of open covers of X satisfying the following conditions:

(i) $\mathscr{U} = \mathscr{U}_0$ and $\mathscr{U}_{m+1}$ refines $\mathscr{U}_m$ for each $m < \omega$;

(ii) for each $x \in X$ there exists an $m(x) < \omega$ such that $\mathrm{ord}(x, \mathscr{U}_m) = 1$ whenever $m \geqslant m(x)$;

(iii) $\bigcap\{\mathrm{St}(x, \mathscr{U}_m) \mid m < \omega\}$ is closed for each $x \in X$.

However, as J. E. Mack [1974] already pointed out in his review of Pareek's paper, this characterization is of doubtful validity because its proof is supported by dubious lemmas. In fact, following (Brandenburg [1985]) we will now show that a certain space X considered by D. K. Burke [1970a] for another purpose satisfies (†) without being S-paracompact.

5.8. Example. The underlying set of X is $\omega_1 \times \omega$. The topology of X is defined by specifying a basic system of neighborhoods for each point $(\alpha, n) \in \omega_1 \times \omega$ as follows. If $\alpha > 0$ and $n > 0$, then (α, n) is isolated. Additionally the point $(0, 0)$ is isolated. For each $n > 0$ let $H_n = \{(\alpha, n) \mid 0 < \alpha < \omega_1\}$, and for each $\alpha > 0$ define $V_\alpha = \{(\alpha, n) \mid 0 < n < \omega\}$. Basic neighborhoods of a point $(0, n)$, $n > 0$, are of the form $U(0, n, E) = \{(0, n)\} \cup (H_n \backslash E)$, where E is a finite subset of H_n. A basic neighborhood of a point $(\alpha, 0)$, $\alpha > 0$, is of the form $U(\alpha, 0, F) = \{(\alpha, 0)\} \cup (V_\alpha \backslash F)$, where F is a finite subset of V_α. The resulting space X is locally compact and Hausdorff.

In order to show that X satisfies (†) consider an open cover $\mathscr{U}$ of X. Without loss of generality we may assume that $\mathscr{U}$ is of the form

$$\mathscr{U} = \{(0, 0)\} \cup \{\{(\alpha, n)\} \mid 0 < \alpha < \omega_1,\ 0 < n < \omega\} \\ \cup \{U(0, n, E_n) \mid 0 < n < \omega\} \cup \{U(\alpha, 0, F_\alpha) \mid 0 < \alpha < \omega_1\},$$

where $E_n \subset H_n$ and $F_\alpha \subset V_\alpha$ are finite. Now if

$$\mathscr{U}_m = \{(0, 0)\} \cup \{U(0, n, E_n) \mid 0 < n < \omega\} \cup \\ \{\{(\alpha, n)\} \mid (\alpha, n) \in E_n,\ 0 < n < \omega\} \cup \{U(\alpha, 0, F_\alpha) \backslash \{(\alpha, n) \mid 0 < n \leqslant m\} \mid 0 < \alpha < \omega_1\}$$

for each $m \geqslant 1$ and $\mathscr{U}_0 = \mathscr{U}$, then it is easily verified that $(\mathscr{U}_m)_{m<\omega}$ satisfies conditions (i)–(iii) of (†).

To prove that X is not S-paracompact note first that every S-paracompact space is D-normal (e.g. use Theorem 4.2). Hence it suffices to show that X is not D-normal. To this end consider the disjoint closed subsets $A = \{(0, n) \mid 0 < n < \omega\}$ and $B = \{(\alpha, 0) \mid 0 < \alpha < \omega_1\}$ of X. Suppose that there exists a closed G_δ-set F in X such that $A \subset F$ and $F \cap B = \emptyset$. Then $F = \bigcap \{F(k) \mid 0 < k < \omega\}$, where every $F(k)$ is open in X. For each $n > 0$, $k > 0$ there exists a finite subset $E(n, k)$ of H_n such that $U(0, n, E(n, k)) \subset F(k)$. If $\alpha(n, k) = \max\{\alpha \mid (\alpha, n) \in E(n, k)\}$ and $\alpha_0 = \sup\{\alpha(n, k) \mid n > 0,\ k > 0\}$, then $\alpha_0 + 1 < \omega_1$. Since every neighborhood of $(\alpha_0 + 1, 0)$ intersects F, it follows that $(\alpha_0 + 1, 0)$ is contained in $\operatorname{cl} F = F$, contradicting the fact that $F \cap B = \emptyset$. Consequently, X cannot be D-normal. ∎

In view of the preceding example it remains an open problem to find internal characterizations of S-paracompact spaces. We conjecture that every S-paracompact space is D-paracompact, i.e. that the two notions coincide. This would be the case if the following question could be answered affirmatively.

PROBLEM I. Is every semi-metrizable space, or, more generally, every semi-stratifiable T_1-space D-paracompact?

We consider Problem I. to be the most important of all research problems mentioned in this paper. Note, for example, that an affirmative answer would also yield an affirmative answer to Problem F, for it is easily seen that every D-paracompact space with G_δ-diagonal is subdevelopable. There are two recent partial results concerning Problem I:

5.9. THEOREM (Chaber [1984]). *Every semi-stratifiable meta-Lindelöf space* ([25]) *is D-paracompact.* ∎

5.10. THEOREM. *Every semi-stratifiable orthocompact space is D-paracompact.* ([26]) ∎

Let us mention another interesting problem concerning D-paracompactness.

PROBLEM J. Let X be a completely regular T_1-space such that $X \times \beta X$ is D-normal. Is X necessarily D-paracompact?

Our next theorem characterizes those T_1-spaces which belong to the paracompactness hull of the class of second countable developable T_1-spaces.

([25]) A topological space is *meta-Lindelöf*, if every open cover has a point-countable open refinement.

([26]) For the definition of orthocompactness see footnote (5). That every semi-metrizable orthocompact space is D-paracompact was first shown by R. W. Heath [1984]. The present generalization was suggested by the referee who pointed out that Theorem 4.8 of (Junnila [1978]) can be used to show that every interior-preserving open cover of a semi-stratifiable space is dissectable.

5.11. THEOREM. *For a T_1-space $X = (X, \tau)$ the following conditions are equivalent*:

(i) *For every open cover $\mathscr{U}$ of X there exists a $\mathscr{U}$-mapping from X onto some second countable developable T_1-space.*

(ii) *X is a D-normal space with the property that each γ-sequence $(x_\xi)_{\xi<\gamma}$ in X, $\gamma \geqslant \omega$ an ordinal, which satisfies* (*) *has a cluster point*:

(*) *for every countable kernel-normal open cover $\mathscr{U}$ of X there exist a cofinal subset A of γ and a $U \in \mathscr{U}$ such that $x_\xi \in U$ whenever $\xi \in A$.*

(iii) *X is a (weakly) D-completely regular Lindelöf space.*

Proof. Obviously, (i) implies that X is D-paracompact and therefore D-normal. Let $(x_\xi)_{\xi<\gamma}$, $\gamma \geqslant \omega$, be a γ-sequence in X without cluster point. Then every $x \in X$ has an open neighborhood V_x such that $x_\xi \in X \backslash V_x$ whenever $\xi > \xi(x)$, where for each $x \in X$ $\xi(x)$ is an ordinal such that $\xi(x)+1 < \gamma$. By (i) $\mathscr{V} = \{V_x \mid x \in X\}$ has a countable kernel-normal open refinement $\mathscr{U}$. Since $\{x_\xi \mid \xi \in A\} \not\subset U$ whenever $A \subset \gamma$ is cofinal and $U \in \mathscr{U}$, it follows that $(x_\xi)_{\xi<\gamma}$ does not have property (*).

(ii) implies (iii): ([27]) In 4.3 (ii) we have shown that every D-normal T_1-space is D-completely regular. Suppose that X is not Lindelöf. If $\varkappa = \min\{|\mathscr{U}| \mid \mathscr{U}$ is an open cover of X without countable subcover$\}$, then $\omega < \varkappa$. Let $(U_\alpha)_{\alpha<\varkappa}$ be an open cover of X without countable subcover. Note that $F_\alpha = X \backslash \bigcup\{U_\beta \mid \beta \leqslant \alpha\} \neq \emptyset$ for each $\alpha < \varkappa$. Let $<_\alpha$ be a well-order of F_α and consider the set $F = \{(F_\alpha, x) \mid \alpha < \varkappa,\ x \in F_\alpha\}$. Since F is well-ordered by the relation

$$\text{“}(F_\alpha, x) < (F_\beta, y) \text{ if either } \alpha < \beta \text{ or } \alpha = \beta \text{ and } x <_\alpha y\text{”},$$

there exists an ordinal γ and an isomorphism $f\colon \gamma \to (F, <)$. Observe that $\gamma > \omega$. Let $g\colon F \to X$ be defined by $(F_\alpha, x) \in F \mapsto x = g((F_\alpha, x))$ and set $x_\xi = g \circ f(\xi)$ for each $\xi < \gamma$. Then the following holds:

A. The γ-sequence $(x_\xi)_{\xi<\gamma}$ has no cluster point in X.

For suppose that $x \in X$ is a cluster point of $(x_\xi)_{\xi<\gamma}$. Then $x \in U_{\alpha(x)}$ for some $\alpha(x) < \varkappa$. Moreover, there exists a cofinal subset A of γ such that $\{x_\xi \mid \xi \in A\} \subset U_{\alpha(x)}$. In particular, there is a $\xi \in A$ such that $f(\xi) > (F_{\alpha(x)}, y)$, where $y \in F_{\alpha(x)}$ is an arbitrary point. It follows that $x_\xi = g(f(\xi)) \in F_\alpha$ for some $\alpha \geqslant \alpha(x)$. On the other hand $x_\xi \in U_{\alpha(x)} \subset \bigcup\{U_\beta \mid \beta \leqslant \alpha\} = X \backslash F_\alpha$ — a contradiction!

Assuming (ii) we conclude from A that there exists a countable kernel-normal open cover $\mathscr{V}$ of X such that $\{x_\xi \mid \xi \in B\} \not\subset V$ whenever $B \subset \gamma$ is cofinal and $V \in \mathscr{V}$ Moreover, using Lemma 2.11 it is easily seen that there is a developable topology $\tau' \subset \tau$ such that $w(X, \tau') \leqslant \omega$ and $\{\operatorname{int}_{\tau'} V \mid V \in \mathscr{V}\}$ covers X. We claim that the following holds:

B. $\bigcup\{X \backslash \operatorname{cl}_\tau F_\alpha \mid \alpha < \varkappa\} = X$.

([27]) This proof is a modification of an argument due to N. R. Howes [1980].

Suppose that B is false, i.e. that there exists an $x \in \bigcap \{\mathrm{cl}_{\tau} F_{\alpha} \mid \alpha < \varkappa\}$. Then $x \in \mathrm{int}_{\tau'} V$ for some $V \in \mathscr{V}$ We obtain a contradiction by showing that $B = \{\xi < \gamma \mid x_{\xi} \in \mathrm{int}_{\tau'} V\}$ is cofinal in γ. To this end consider an arbitrary $\eta < \gamma$. If $f(\eta) = (F_{\alpha}, y)$, choose a β such that $\alpha < \beta < \varkappa$. There exists a $z \in \mathrm{int}_{\tau'} V \cap F_{\beta}$. If $\xi < \gamma$ such that $f(\xi) = (F_{\beta}, z)$, then $x_{\xi} = g \circ f(\xi) = z \in \mathrm{int}_{\tau'} V$ and $f(\eta) < f(\xi)$, i.e. $\xi \in B$ and $\eta < \xi$, which completes the argument. Since (X, τ') is a Lindelöf space, it follows from B that there is a countable subset C of $\varkappa$ such that $\bigcup \{X \backslash \mathrm{cl}_{\tau'} F_{\alpha} \mid \alpha \in C\} = X$. For $\alpha \in C$ let $(F(\alpha, n))_{n<\omega}$ be a sequence of closed subsets of X such that $\bigcup \{F(\alpha, n) \mid n < \omega\} = X \backslash \mathrm{cl}_{\tau'} F_{\alpha}$. By the definition of $\varkappa$ there exists, for each pair $(\alpha, n) \in C \times \omega$, a countable subset $C(\alpha, n)$ of α such that $F(\alpha, n) \subset \bigcup \{U_{\beta} \mid \beta \in C(\alpha, n)\}$. But then $\{U_{\beta} \mid \beta \in \bigcup \{C(\alpha, n) \mid (\alpha, n) \in C \times \omega\}\}$ is a countable subcover of $(U_{\alpha})_{\alpha<\varkappa}$, which contradicts our choice of $(U_{\alpha})_{\alpha<\varkappa}$. Consequently X must be a Lindelöf space.

(iii) implies (i): Observe first that if X is a weakly D-completely regular Lindelöf space, then X is D-paracompact. For every open cover of X has a countable open F_{σ}-refinement i.e a σ-dissectable open refinement. Therefore, if $\mathscr{U}$ is an arbitrary open cover of X there exists a countable kernel-normal open refinement $\mathscr{V}$ of $\mathscr{U}$ (5.4). As in the proof of the previous implication it follows that there is a second countable developable topology $\tau' \subset \tau$ such that $\bigcup \{\mathrm{int}_{\tau'} V \mid V \in \mathscr{V}\} = X$. If Y is the space obtained by identifying points which have identical closures in (X, τ'), then the natural quotient mapping from (X, τ') onto Y defines a $\mathscr{U}$-mapping from (X, τ) onto Y. Since Y is a developable T_1-space of weight ω, the proof is complete. ■

Problem K. Let X be a D-normal T_1-space such that every γ-sequence $(x_{\xi})_{\xi<\gamma}$ in X, $\gamma \geqslant \omega$ an ordinal, with property (□) has a cluster point:

(□) for every kernel-normal open cover $\mathscr{U}$ of X there exists a cofinal subset A of γ and a $U \in \mathscr{U}$ such that $x_{\xi} \in U$ whenever $\xi \in A$.

Is X necessarily D-paracompact?

5.12. Remarks. (a) C. M. Pareek [1972] has shown that a topological space *X is D-paracompact if and only if for every open cover $\mathscr{U}$ of X there exists a sequence $(\mathscr{U}_n)_{n<\omega}$ of open covers of X satisfying*:

(i) *$\mathscr{U} = \mathscr{U}_0$ and $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$ for each $n < \omega$;*

(ii) *for each $n < \omega$ and for each $x \in X$ there exist a $k < \omega$ and a $U \in \mathscr{U}_n$ such that* $\mathrm{St}(x, \mathscr{U}_k) \subset U$;

(iii) *for each $n < \omega$ and for each $x \in X$ there exists a $k < \omega$ such that for every $y \in \mathrm{St}(x, \mathscr{U}_k)$ there is a $k(y) < \omega$ such that* $\mathrm{St}(y, \mathscr{U}_{k(y)}) \subset \mathrm{St}(x, \mathscr{U}_n)$.

This characterization can be easily deduced from Theorem 5.4.

(b) It is worth mentioning that a topological space X is D-expandable if and only if every weak $\bar{\theta}$-cover of X is kernel-normal.

(c) In [1984] J. Chaber has introduced the notion of a *collectionwise D-normal space* and shown that *a topological space is D-paracompact if and only if it is collectionwise D-normal and θ-refinable.* He has also obtained a characterization of D-paracompact meta-Lindelöf spaces.

(d) There is an interesting relationship between D-paracompact spaces and para-uniform nearness spaces (2.10(ii)): *A topological nearness space* (Herrlich [1974]) *is para-uniform if and only if its induced topology is D-paracompact* (Brandenburg [1988]).

(e) An example of a D-paracompact space which is neither developable nor paracompact can be obtained as follows. Take a T_1-space X in which points are G_δ-sets but which is not first countable. Then the Pixley–Roy hyperspace PR(X) (e.g. see van Douwen [1977]) of X is perfectly metacompact, hence D-paracompact (5.5). But PR(X) is not developable. If in addition X does not have an antisymmetric neighbornet in the sense of (Junnila [1978]), then PR(X) is not paracompact (e.g. see Tanaka [1982]).

(f) Let us call a topological space X *countably D-paracompact* if for every open cover $(U_n)_{n<\omega}$ of X there exists a dissectable open cover $(V_n)_{n<\omega}$ of X such that $V_n \subset U_n$ for each $n < \omega$. Using slight modifications of previous arguments it can be shown that each of the following conditions is equivalent to the countable D-paracompactness of X:

(i) *X is countably subparacompact*([28]) *and D-normal.*
(ii) *X is countably metacompact and D-normal.*
(iii) *Every countable open cover of X is kernel-normal.*
(iv) *Every countable open cover of X has a countable open F_σ-refinement.*
(v) *$X \times Y$ is D-normal for every compact metric space Y.*
(vi) *$X \times [0, 1]$ is D-normal.*
(vii) *$X \times (\omega+1)$ is D-normal.*

(g) D-paracompactness is additive, but neither hereditary nor productive. However, normally situated subspaces (4.18(e)) of D-paracompact spaces are always D-paracompact (Brandenburg [1985]). Clearly, quotients of D-paracompact spaces need not be D-paracompact. ∎

In [1984] J. Chaber has shown that even perfect preimages of developable spaces need not be D-paracompact, thereby disproving a conjecture of C. M. Pareek [1972]. But the following problem is still open.

PROBLEM L. Are perfect images of D-paracompact spaces always D-paracompact?

Furthermore, nothing is known concerning the final problem in this section.

PROBLEM M. Characterize Q-paracompact spaces, where **Q** is the class of quasi-metrizable T_1-spaces.

§ 6. Some characterizations of developable spaces

During the last twenty-five years a great variety of characterizations of developable spaces involving many different concepts appeared in the literature. These results are too numerous to cite here. In particular, for regular or

([28]) A topological space is *countably subparacompact*, if every countable open cover has a σ-discrete closed refinement.

completely regular T_1-spaces there are several interesting factorizations of developability in terms of G_δ-diagonal concepts, (generalizations of) p-spaces, θ-refinability, (generalizations of) semi-stratifiability, quasi-developability or symmetrizability. Probably the most interesting among the internal characterizations of arbitrary developable spaces is the basic theorem of H. H. Wicke and J. M. Worrell, Jr. [1965] which expresses developability essentially as a combination of a base property (i.e. base of countable order) and a weak covering property (i.e. θ-refinability; see footnote (20)).

Every characterization of developable spaces yields a (not always significant) metrization theorem by simply adding collectionwise normality (Bing [1951]). On the other hand one may ask whether some basic metrization theorems have counterparts in the theory of developable spaces. The above mentioned theorem of Wicke–Worrell, Jr., for instance, generalizes a metrization theorem of A. Arkhangel'skii [1963] which itself is an improvement of the Aleksandrov–Urysohn metrization theorem [1923]. Similarly, the following result may be viewed as an analogue, for developable spaces, of the Nagata–Smirnov metrization theorem.

6.1. THEOREM (Brandenburg [1978], [1980]). *A topological space is developable if and only if it has a σ-dissectable open base.*

Proof. It follows from Lemma 5.3 that every open cover of a developable space is dissectable. Hence every development of a developable space forms a σ-dissectable open base. For the proof of the reverse implication consider a topological space X with a σ-dissectable open base $\mathscr{B} = \bigcup_{n<\omega}\mathscr{B}_n$, where $\mathscr{B}_n = (B(i))_{i\in I(n)}$ for each $n<\omega$. Let D_n: $I(n)\times\omega\to\mathscr{P}(X)$ be a dissection of $\mathscr{B}_n$. For each pair $(n, k)\in\omega\times\omega$ we construct an open cover $\mathscr{U}(n, k)$ of X as follows. If $x\in X(n, k) = \bigcup\{D_n(i, k)\mid i\in I(n)\}$, there exists an open set $U(x, n, k)$ containing x such that

$$U(x, n, k)\subset\bigcap\{B(i)\mid i\in I(n),\ x\in D_n(i, k)\}$$

and

$$U(x, n, k)\subset X\backslash\bigcup\{D_n(i, k)\mid i\in I(n), x\notin D_n(i, k)\}.$$

We define

$$\mathscr{U}(n, k) = \{X\backslash X(n, k)\}\cup\{U(x, n, k)\mid x\in X(n, k)\}$$

and claim that $(\mathscr{U}(n, k))_{(n,k)\in\omega\times\omega}$ is a development of X.

To prove this assertion consider an arbitrary point x in X and a neighborhood V of x. Since $\mathscr{B}$ is a base of X, there exists an $n<\omega$ and an $i_0\in I(n)$ such that $x\in B(i_0)\subset V$. Furthermore there is a $k<\omega$ such that $x\in D_n(i_0, k)$. It remains to prove that $\mathrm{St}(x, \mathscr{U}(n, k))$ is contained in V. Since

$$\mathrm{St}(x, \mathscr{U}(n, k)) = \bigcup\{U(y, n, k)\mid y\in X(n, k),\ x\in U(y, n, k)\},$$

it suffices to show that for each $y \in X(n, k)$ with $x \in U(y, n, k)$ the set $U(y, n, k)$ is contained in $B(i_0)$. To this end consider a fixed $y \in X(n, k)$ such that $x \in U(y, n, k)$. For every $j \in I(n)$ such that $x \in D_n(j, k)$ we have $y \in D_n(j, k)$, for otherwise we would have

$$x \in D_n(j, k) \subset \bigcup \{D_n(i, k) \mid i \in I(n),\ y \notin D_n(i, k)\} \subset X \setminus U(y, n, k),$$

which is impossible. Hence

$$\begin{aligned} U(y, n, k) &\subset \bigcap \{B(i) \mid i \in I(n),\ y \in D_n(i, k)\} \\ &\subset \bigcap \{B(i) \mid i \in I(n),\ x \in D_n(i, k)\} \subset B(i_0), \end{aligned}$$

which completes the proof. ■

6.2. COROLLARY (Brandenburg [1980]). *Every topological space with a locally countable base consisting of open F_σ-sets is developable.*

Proof. If a space has a locally countable base consisting of open F_σ-sets, then it has a σ-locally finite base consisting of open F_σ-sets (Charlesworth [1976]), i.e. a σ-dissectable open base. ■

Our second characterization of developable spaces is motivated by Nagata's so-called "double sequence metrization theorem" (Nagata [1957]).

6.3. THEOREM (Brandenburg [1980]). *A topological space X is developable if and only if for every point x in X there exist two sequences $(U(x, n))_{n<\omega}$ and $(V(x, n, k))_{(n,k)\in\omega\times\omega}$ of neighborhoods of x such that*

(i) *$(U(x, n))_{n<\omega}$ is a neighborhood base of x, and for each $n < \omega$ $U(x, n)$ has a decomposition $U(x, n) = \bigcup \{U(x, n, k) \mid k < \omega\}$ such that*

(ii) *$y \notin U(x, n, k)$ implies $V(y, n, k) \cap U(x, n, k) = \emptyset$, and*

(iii) *$y \in U(x, n, k)$ implies $V(y, n, k) \subset U(x, n)$.*

Proof. Let $(\mathscr{V}_n)_{n<\omega}$ be a development of X, where $\mathscr{V}_n = (V(i))_{i\in I(n)}$. For each $n < \omega$ there exists a dissection $D_n\colon I(n) \times \omega \to \mathscr{P}(X)$ of $\mathscr{V}_n$ (5.3). For each $x \in X$ and for each $n < \omega$ choose a fixed $i(x, n) \in I(n)$ such that $x \in V(i(x, n))$. We define $U(x, n) = V(i(x, n))$ and $U(x, n, k) = D_n(i(x, n), k)$ for each $n, k < \omega$. Moreover we put

$$V(x, n, k) = \bigcap \{V(i) \mid i \in I(n),\ x \in D_n(i, k)\} \cap \big(X \setminus \bigcup \{D_n(i, k) \mid i \in I(n),\ x \notin D_n(i, k)\}\big).$$

Evidently, the so-defined sequences of neighborhoods of x satisfy (i)–(iii).

For the proof of the reverse implication define

$$W(x, n, k) = U(x, n) \cap V(x, n, k) \quad \text{and} \quad \mathscr{W}(n, k) = (W(x, n, k))_{x\in X}$$

for each pair $(n, k) \in \omega \times \omega$. Since it is easy to see that $(\mathscr{W}(n, k))_{(n,k)\in\omega\times\omega}$ is a development of X, the proof is complete. ■

For a topological space X let $\exp X$ be the set of all nonempty closed subsets of X supplied with the Vietoris topology, i.e. with the topology which has as a base all sets of the form $\langle U_0, \ldots, U_{k-1}\rangle$, where $U_0, \ldots, U_{k-1}$ are open subsets of X and

$$\langle U_0, \ldots, U_{k-1}\rangle = \{A \in \exp X \mid A \subset U_0 \cup \ldots \cup U_{k-1},\ A \cap U_i \neq \emptyset \text{ for each } i < k\}.$$

Following A. N. Dranishnikov [1978] a mapping φ from $X \times \exp X$ into the nonnegative reals is called an *annihilator for the closed sets of* X, if $A = \{x \in X \mid \varphi(x, A) = 0\}$ for each $A \in \exp X$. It is called *monotone*, if $A, B \in \exp X$ and $A \subset B$ implies $\varphi(x, B) \leqslant \varphi(x, A)$ for each $x \in X$. In [1976] P. Zenor has shown that a T_1-space is metrizable if and only if it has a continuous monotone annihilator for the closed sets. Our third characterization of developable spaces is obtained by weakening the continuity condition in Zenor's metrization theorem.

6.4. THEOREM (Brandenburg [1986]). *A T_1-space is developable if and only if it has a monotone upper semicontinuous annihilator for the closed sets.*

Proof. Let $(\mathcal{U}_n)_{n<\omega}$ be a development of a T_1-space X such that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for each $n < \omega$. If φ: $X \times \exp X \to \boldsymbol{R}$ is defined by

$$\varphi(x, A) = \begin{cases} 0 & \text{if } x \in A, \\ 1/2^{n(x,A)} & \text{if } x \in X \backslash A, \quad \text{where } n(x, A) \\ & = \min\{n < \omega \mid \mathrm{St}(x, \mathcal{U}_n) \subset X \backslash A\}, \end{cases}$$

then it is easily seen that φ is a monotone annihilator for the closed sets of X. We claim that φ is upper semicontinuous. In order to verify this claim let c be a real number such that $0 < c \leqslant 1$([29]) and consider a pair $(x, A) \in X \times \exp X$ such that $\varphi(x, A) < c$. If $n_0 = \max\{n < \omega \mid c \leqslant 1/2^n\}$, then $x \in \mathrm{St}(A, \mathcal{U}_{n_0})$, for otherwise we would have $\mathrm{St}(x, \mathcal{U}_{n_0}) \subset X \backslash A$ and therefore $1/2^{n_0} \leqslant \varphi(x, A) < c \leqslant 1/2^{n_0}$. Consequently, there exists a $U \in \mathcal{U}_{n_0}$ containing x such that $U \cap A \neq \emptyset$. Since $U \times \langle \mathrm{St}(A, \mathcal{U}_{n_0}), U\rangle$ is an open neighborhood of (x, A) in $X \times \exp X$, it suffices to show that $\varphi(y, B) < c$ whenever $(y, B) \in U \times \langle \mathrm{St}(A, \mathcal{U}_{n_0}), U\rangle$. To this end consider an arbitrary $(y, B) \in U \times \langle \mathrm{St}(A, \mathcal{U}_{n_0}), U\rangle$. Since $U \cap B \neq \emptyset$, it follows that $y \in \mathrm{St}(B, \mathcal{U}_{n_0})$ and hence (no matter whether $y \in B$ or not) that $\varphi(y, B) < 1/2^{n_0}$. Therefore, by the definition of n_0, $\varphi(y, B) < c$.

To prove the reverse implication assume now that X is a T_1-space with a monotone upper semicontinuous annihilator φ for the closed sets. If d: $X \times X \to \boldsymbol{R}$ is defined by $d(x, y) = \varphi(x, \{y\}) + \varphi(y, \{x\})$, then d is a symmetric distance function such that $d(x, y) = 0$ if and only if $x = y$, i.e. d is a semi-metric on X. We show that d is compatible with the topology of X, i.e. that for each subset A of X, $x \in \mathrm{cl} A$ if and only if $d(x, A) = 0$ (see 2.1 (ii)).

To this end suppose first that $d(x, A) = 0$ but $x \notin \mathrm{cl} A$. Then $\varphi(x, \mathrm{cl} A) > 0$. Hence there exists an $a \in A$ such that $d(x, a) < \varphi(x, \mathrm{cl} A)$. On the other hand

$$\varphi(x, \mathrm{cl} A) \leqslant \varphi(x, \{a\}) \leqslant \varphi(x, \{a\}) + \varphi(a, \{x\}) = d(x, a)$$

by the monotonicity of φ — a contradiction! Therefore $d(x, A) = 0$ implies $x \in \mathrm{cl} A$. Conversely, if $x \in \mathrm{cl} A$ and $\varepsilon > 0$, choose an n such that $0 < n < \omega$ and

([29]) Evidently, $\varphi^{-1}(\leftarrow, c)$ is open if $c \leqslant 0$ or $1 < c$.

$1/n < \varepsilon/2$. Since $\varphi(x, \{x\}) = 0$, there exists an open neighborhood U of x such that $\varphi(y, B) < 1/n$ whenever $(y, B) \in U \times \langle U \rangle$. Moreover, there exists an $a \in U \cap A$. Since $(x, \{a\}) \in U \times \langle U \rangle$ and $(a, \{x\}) \in U \times \langle U \rangle$, it follows that

$$d(x, a) = \varphi(x, \{a\}) + \varphi(a, \{x\}) < 1/n + 1/n < \varepsilon,$$

which implies that $d(x, A) = 0$.

So far we have shown that X is semi-metrizable. By a folklore result in developability theory it follows that X is developable provided that d has the following property:

(*) Whenever $x \in X$ and (x_n), (y_n) are sequences in X such that $\lim(d(x_n, x)) = \lim(d(y_n, x)) = 0$ it follows that $\lim(d(x_n, y_n)) = 0$.

In order to show that d has property (*) consider two sequences (x_n), (y_n) in X such that $\lim(d(x_n, x)) = \lim(d(y_n, x)) = 0$ for some point $x \in X$. Suppose that the sequence $(d(x_n, y_n) = (\varphi(x_n, \{y_n\}) + \varphi(y_n, \{x_n\}))$ does not converge to $0 \in \mathbf{R}$. Then, without loss of generality, we may assume that $\varphi(x_n, \{y_n\})$ does not converge to $0 \in \mathbf{R}$. Hence there exists a $c > 0$ and a cofinal subset N of ω such that $\varphi(x_n, \{y_n\}) > c$ for each $n \in N$. On the other hand, since $\varphi(x, \{x\}) = 0$, there exists an open neighborhood U of x such that $\varphi(y, B) < c$ whenever $(y, B) \in U \times \langle U \rangle$. Moreover, there exists an $n_0 < \omega$ such that $(x_n, \{y_n\}) \in U \times \langle U \rangle$ for each $n > n_0$. Therefore, if $n \in N$ and $n > n_0$, then $\varphi(x_n, \{y_n\}) < c$, contradicting the fact that $\varphi(x_n, \{y_n\}) > c$. It follows that $\lim(d(x_n, y_n)) = 0$. ■

By virtue of Lemma 5.2 we can reformulate Theorem 6.1 so that we get another characterization of developable spaces in terms of upper semicontinuous real-valued mappings.

6.5. THEOREM (Brandenburg [1986]). *A topological space X is developable if and only if there exists a compatible*(30) *σ-equi-upper semicontinuous*(31) *family $\mathscr{F} = \bigcup_{n<\omega} \mathscr{F}_n$ of mappings from X into* $[0, 1]$ *which has the following property* (D):

(D) *Whenever $x \in X$, $\mathscr{F}' \subset \mathscr{F}_n$, $n < \omega$, such that* $\mathscr{F}'[X] = \{f(x) \mid f \in \mathscr{F}'\} \subset (a, 1]$, $a > 0$, *there exists a neighborhood V of x such that* $\mathscr{F}'[V] = \{f(y) \mid f \in \mathscr{F}', y \in V\} \subset (0, 1]$. ■

Note that this theorem is the analogue, for developable spaces, of a nice metrization theorem due to J.-I. Nagata [1957] and J. A. Guthrie and M. Henry [1977]. Let us call a family $\mathscr{F}$ of upper semicontinuous mappings from a topological space X into $[0,1]$ *sup-complete* if for every subfamily $\mathscr{F}'$ of $\mathscr{F}$ the mapping $\sup\{f \mid f \in \mathscr{F}'\}$ is upper semicontinuous. Since it is easily seen that every equi-upper semicontinuous family of mappings into $[0, 1]$ is sup-complete, we ask:

(30) Compatibility means that $\{f^{-1}(0, 1] \mid f \in \mathscr{F}\}$ is a base for the open sets of X.

(31) This means that every $\mathscr{F}_n$ is equi-upper semicontinuous (5.2).

PROBLEM N. Let X be a topological space which admits a compatible family $\mathscr{F} = \bigcup_{n<\omega}\mathscr{F}_n$ of upper semicontinuous mappings into [0, 1] such that every $\mathscr{F}_n$ is sup-complete and $\mathscr{F}$ satisfies condition (D) in Theorem 6.5. Is X necessarily developable?

That the somewhat odd looking condition (D) in Theorem 6.5 cannot be dropped follows from our next theorem which shows that otherwise a characterization of σ-spaces (2.1 (i)) is obtained.

6.6. THEOREM (Brandenburg [1986]). *For a topological space $X = (X, \tau)$ the following conditions are equivalent:*

(i) *X is a σ-space.*

(ii) *There exists an equi-upper semicontinuous family $\mathscr{F} = (f_U\colon X \to [0, 1])_{U\in\tau}$ of mappings such that $f_U^{-1}[\{0\}] = X\backslash U$ for each $U\in\tau$.*

(iii) *There exists a compatible σ-equi-upper semicontinuous family $\mathscr{F} = \bigcup_{n<\omega}\mathscr{F}_n$ of mappings from X into [0, 1].*

Proof. (i) implies (ii): Let $\mathscr{B} = \bigcup_{n<\omega}\mathscr{B}_n$ be a σ-closure preserving closed network of X. For each $U\in\tau$ define a mapping $f_U\colon X\to[0, 1]$ by

$$f_U(x) = \begin{cases} 0 & \text{if } x\in X\backslash U, \\ 1/2^{n(U,x)} & \text{if } x\in U, \quad \text{where } n(U, x) \\ & = \min\{n < \omega \mid x\in\bigcup\{B\in\mathscr{B}_0\cup\ldots\cup\mathscr{B}_n \mid B\subset U\}\}. \end{cases}$$

Then $\mathscr{F} = (f_U\colon X\to[0, 1])_{U\in\tau}$ is equi-upper semicontinuous and $f_U^{-1}[\{0\}] = X\backslash U$ for each $U\in\tau$.

Obviously, (ii) implies (iii). If $\mathscr{F} = \bigcup_{n<\omega}\mathscr{F}_n$ is a compatible σ-equi-upper semicontinuous family of mappings from X into [0, 1], then $\bigcup\{\{f^{-1}[[1/2^n, 1]] \mid f\in\mathscr{F}_n\} \mid n < \omega\}$ is a σ-closure preserving closed network of X which shows that (iii) implies (i). ∎

Interestingly, a characterization of orthocompact (see footnote (5)) developable spaces is obtained if simultaneously condition (D) in Theorem 6.5 is replaced by a more natural (but stronger) condition and the equi-upper semicontinuity is weakened to sup-completeness.

6.7. THEOREM (Brandenburg [1986]). *For a T_1-space X the following conditions are equivalent:*

(i) *X is developable and orthocompact.*

(ii) *There exists a compatible family $\mathscr{F} = \bigcup_{n<\omega}\mathscr{F}_n$ of upper semicontinuous mappings from X into [0, 1] such that every $\mathscr{F}_n$ is sup-complete and the following condition* (I) *is satisfied:*

(I) *Whenever $x\in X$, $\mathscr{F}'\subset\mathscr{F}_n$, $n<\omega$, such that $\mathscr{F}'[X]\subset(0, 1]$ there exists a neighborhood V of x such that $\mathscr{F}'[V]\subset(0, 1]$.*

Proof. (i) implies (ii): If X is developable and orthocompact, there exists a development $(\mathscr{U}_n)_{n<\omega}$ of X consisting of interior-preserving open covers. By virtue of Lemma 5.3 every $\mathcal{U}_n$ is dissectable. Hence every $\mathcal{U}_n$ induces a family $\mathcal{F}_n$ of upper semicontinuous mappings as in the proof of Lemma 5.2. It is easily seen that $\mathcal{F} = \bigcup_{n<\omega}\mathcal{F}_n$ has the desired properties.

(ii) implies (i): Let $\mathcal{F} = \bigcup_{n<\omega}\mathcal{F}_n$ be a compatible family of upper semicontinuous mappings from X into $[0, 1]$ such that every $\mathcal{F}_n$ is sup-complete and (I) is satisfied. If $\mathcal{B}_n = \{f^{-1}(0, 1] \mid f \in \mathcal{F}_n\}$ for each $n < \omega$, then (I) implies that every $\mathcal{B}_n$ is interior-preserving, i.e. $\mathcal{B} = \bigcup_{n<\omega}\mathcal{B}_n$ is a σ-interior-preserving open base of X. Since every semi-stratifiable T_1-space with a σ-interior-preserving base is developable and orthocompact (e.g. see Fletcher and Lindgren [1982], Chapter 7), it suffices to prove that X is semi-stratifiable (2.1 (iii)). To this end let $n \mapsto (k(n), m(n))$ be a bijection from ω onto $\omega \times \omega$. For each open set U in X and for each $n < \omega$ define

$$U_n = \mathrm{cl}\{f^{-1}(1/2^{k(n)}, 1] \mid f \in \mathcal{F}_{m(n)},\ f^{-1}(1/2^{k(n)+1}, 1] \subset U\}.$$

Evidently, every U_n is closed, and $U_n \subset V_n$ whenever U, V are open and $U \subset V$. In order to verify that $U \subset \bigcup\{U_n \mid n < \omega\}$ consider an arbitrary point $x \in U$. Since $\mathcal{F}$ is compatible, there exists an $m < \omega$ and an $f \in \mathcal{F}_m$ such that $x \in f^{-1}(0, 1] \subset U$. Hence, if $k < \omega$ and $n < \omega$ are chosen in such a way that $1/2^k < f(x)$ and $(k(n), m(n)) = (k, m)$, then $x \in U_n$. Therefore the proof is complete once we have shown that $U_n \subset U$ for each open set U and for each $n < \omega$. But the latter is an immediate consequence of the inclusions

$$\begin{aligned}
&\bigcup\{f^{-1}(1/2^{k(n)}, 1] \mid f \in \mathcal{F}_{m(n)},\ f^{-1}(1/2^{k(n)+1}, 1] \subset U\} \\
&\quad \subset (\sup\{f \in \mathcal{F}_{m(n)} \mid f^{-1}(1/2^{k(n)+1}, 1] \subset U\})^{-1}(1/2^{k(n)}, 1] \\
&\quad \subset (\sup\{f \in \mathcal{F}_{m(n)} \mid f^{-1}(1/2^{k(n)+1}, 1] \subset U\})^{-1}[1/2^{k(n)}, 1] \\
&\quad \subset \bigcup\{f^{-1}(1/2^{k(n)+1}, 1] \mid f \in \mathcal{F}_{m(n)},\ f^{-1}(1/2^{k(n)+1}, 1] \subset U\} \subset U,
\end{aligned}$$

for the sup-completeness of $\mathcal{F}_{m(n)}$ implies that

$$(\sup\{f \in \mathcal{F}_{m(n)} \mid f^{-1}(1/2^{k(n)+1}, 1] \subset U\})^{-1}[1/2^{k(n)}, 1]$$

is closed. ∎

The proof of the preceding theorem and a recent characterization of stratifiable spaces by C. R. Borges and G. Gruenhage [1983] suggest the following question.

PROBLEM O. Is a topological space semi-stratifiable if and only if it has a compatible family $\mathcal{F} = \bigcup_{n<\omega}\mathcal{F}_n$ of upper semicontinuous mappings into $[0, 1]$ such that every $\mathcal{F}_n$ is sup-complete?

Let us close this section by mentioning yet another characterization of developable spaces which actually was the first result of this type we ever proved.

6.8. THEOREM. *A topological space is developable if and only if its topology is induced by a nearness structure which has a countable base.* ∎

For a proof of this theorem we refer to (Brandenburg [1978], 2.2.1) or to (Carlson [1980]), where it was obtained independently.

6.9. Remarks. (a) A characterization of orthocompact developable spaces similar to Theorem 6.1 was recently proved by T. Mizokami [1987].

(b) In [1988] S. Romaguera has characterized quasi-metrizable developable spaces (= strongly quasi-metrizable spaces) in terms of upper semicontinuous mappings.

§ 7. On inverse limits of developable spaces

In [1939] J. Dieudonné has shown that a completely regular T_1-space X is homeomorphic to the limit of an inverse system of metrizable topological spaces if and only if the fine uniformity on X is complete. Since then inverse limits of metrizable spaces are called *topological complete* or *Dieudonné complete spaces.* By combining results of B. A. Pasynkov [1968] and T. Shirota [1952] it follows that inverse limits of second countable metrizable spaces are precisely the *realcompact spaces* as introduced by E. Hewitt [1948]. By Shirota's theorem [1952] a completely regular T_1-space is realcompact if and only if it is a Dieudonné complete space in which every closed discrete subspace is of nonmeasurable cardinality. These classical results motivate our procedure in this section, i.e. we intend to characterize those topological spaces which are limits of inverse systems of (second countable) developable T_1-spaces. In this way we obtain a natural generalization of realcompactness which has not been studied before. In particular, it will turn out that the resulting class is nothing but the epireflective hull of the class of (second countable) developable T_1-spaces in a suitably chosen subcategory of the category of all topological spaces (see 7.19(a)). All results in this section are new and appear here for the first time.

As Theorem 2.13 indicates, para-uniform nearness spaces will have to play the role of uniform spaces. However, while there is a natural concept of completeness for uniform spaces, the situation is essentially different as soon as one considers non-regular nearness spaces (see Bentley and Herrlich [1979]). Both completeness properties for nearness spaces which have been studied most, i.e. Herrlich completeness (= cluster completeness; Herrlich [1974]) and ultrafilter completeness (Carlson [1975]), turn out to be not suitable for our purpose. What we need is the following notion which we attribute to K. Morita [1951].

7.1. Definition. A nearness space (X, μ) is called *Morita complete,* if $\bigcap \{\mathrm{cl} F \mid F \in \mathscr{F}\} \neq \emptyset$ for every strong Cauchy filter $\mathscr{F}$ on X, where a filter $\mathscr{F}$ is called a *strong Cauchy filter* if for every $\mathscr{U} \in \mu$ there exist a $U \in \mathscr{U}$, an $F \in \mathscr{F}$, and a $\mathscr{V} \in \mu$ such that $\mathrm{St}(F, \mathscr{V}) \subset U$.

Before we can formulate the first main theorem of this section we need to introduce some more terminology.

7.2. DEFINITION (Mrówka [1957]). For a subset A of a topological space X the set

$$\mathrm{cl}^\delta A = \{x \in X \mid G \cap A \neq \emptyset \text{ for every } G_\delta\text{-set } G \text{ in } X \text{ containing } x\}$$

is called the *G_δ-closure of A* in X. A is said to be *G_δ-closed* in X if $\mathrm{cl}^\delta A = A$. A is called *G_δ-dense* in X, if $\mathrm{cl}^\delta A = X$.

7.3. DEFINITION. (i) If $\mathscr{B}$ is a collection of subsets of a set X which is closed with respect to finite intersections, a nonempty collection $\mathscr{F} \subset \mathscr{B}$ of nonempty sets is called a *$\mathscr{B}$-filter* if the following conditions are satisfied:

(F.1) $F \cap G \in \mathscr{F}$ whenever $F \in \mathscr{F}$ and $G \in \mathscr{F}$.

(F.2) If $F \in \mathscr{F}$ and $G \in \mathscr{B}$ satisfies $F \subset G$, then $G \in \mathscr{F}$.

A *$\mathscr{B}$-filter* $\mathscr{F}$ is said to be a *$\mathscr{B}$-ultrafilter* if $G \in \mathscr{B}$ and $G \cap F \neq \emptyset$ for each $F \in \mathscr{F}$ implies that $G \in \mathscr{F}$.

(ii) For every topological space X we denote by $\mathscr{D}(X)$ the collection of all D-closed subsets (1.3) of X.

For the definition of (limits of) inverse systems of topological spaces we refer to (Engelking [1977]). We can now prove:

7.4. THEOREM. *For a D-completely regular space $X = (X, \tau)$ the following conditions are equivalent:*

(i) *X is homeomorphic to the limit of an inverse system of developable T_1-spaces.*

(ii) *X is homeomorphic to a G_δ-closed subset of a product of developable T_1-spaces.*

(iii) *X is homeomorphic to a G_δ-closed subset of a product of D-paracompact T_1-spaces.*

(iv) *X has a G_δ-base $\mathscr{B}$ for the closed sets, closed with respect to countable intersections, such that $\{X \backslash F \mid F \in \mathscr{F}\}$ is kernel-normal whenever $\mathscr{F}$ is a $\mathscr{B}$-ultrafilter with* cip[32] *and $\bigcap \{F \mid F \in \mathscr{F}\} = \emptyset$.*

(v) *$\{X \backslash F \mid F \in \mathscr{F}\}$ is kernel-normal whenever $\mathscr{F}$ is a $\mathscr{D}(X)$-ultrafilter with* cip *and $\bigcap \{F \mid F \in \mathscr{F}\} = \emptyset$.*

(vi) *(X, μ_f) is Morita complete* (see 2.13).

Proof. (i) implies (ii): Suppose that there is an inverse system $(f_i^j\colon X_j \to X_i)_{i,j \in I; i \leq j}$ of developable T_1-spaces such that X is homeomorphic to $X' = \varprojlim (f_i^j\colon X_j \to X_i)_{i,j \in I, i \leq j}$. It suffices to show that $X' \subset \prod_{i \in I} X_i$ is G_δ-closed. So consider a point $x \in \prod_{i \in I} X_i \backslash X'$. There exist $i, j \in I$ such that $i \leq j$ and $f_i^j \circ p_j(x) \neq p_i(x)$, where p_i, p_j are the projections from $\prod_{i \in I} X_i$ onto X_i respectively X_j. But then

$$G = (f_i^j \circ p_j)^{-1}[\{f_i^j p_j(x)\}] \cap p_i^{-1}[\{p_i(x)\}]$$

(32) $\mathscr{F}$ has the cip (*countable intersection property*), if $\bigcap \{F \mid F \in \mathscr{F}'\} \neq \emptyset$ for each $\mathscr{F}' \in [\mathscr{F}]^{\leq \omega}$.

is a G_δ-set in $\prod_{i\in I} X_i$ containing x such that $G \cap X' = \emptyset$.

Evidently, (ii) implies (iii).

(iii) implies (iv): For simplicity we assume that X itself is a G_δ-closed subspace of a product $\prod_{i\in I} Y_i$ of D-paracompact T_1-spaces Y_i. For each $i\in I$ let $\mathscr{B}_i$ be a G_δ-base for the closed sets of Y_i. Then the collection $\mathscr{B}$ consisting of all countable intersections of finite unions of sets of the form $(p_i \restriction X)^{-1}[B]$, $B\in\mathscr{B}_i$, $i\in I$, is a G_δ-base for the closed sets of X which is closed with respect to countable intersections. Consider a $\mathscr{B}$-ultrafilter $\mathscr{F}$ on X with cip such that $\bigcap\{F \mid F\in\mathscr{F}\} = \emptyset$. If $\mathscr{G}_i = \{G\in\mathscr{B}_i \mid (p_i\restriction X)^{-1}[G]\in\mathscr{F}\}$ for each $i\in I$, then the following holds:

A. $\bigcap\{G \mid G\in\mathscr{G}_i\} = \emptyset$ for some $i\in I$.

Note that once we have proved Claim A it follows that $\{(p_i\restriction X)^{-1}[Y_i\backslash G] \mid G\in\mathscr{G}_i\}$ is a kernel-normal open cover of X (5.4 (viii)) which refines $\{X\backslash F \mid F\in\mathscr{F}\}$. Hence $\{X\backslash F \mid F\in\mathscr{F}\}$ is kernel-normal itself.

Suppose now that Claim A is false, i.e. that for each $i\in I$ there is a $y_i\in\bigcap\{G \mid G\in\mathscr{G}_i\}$. Then:

B. $\mathscr{G}_i = \{G\in\mathscr{B}_i \mid y_i\in G\}$ for each $i\in I$.

In fact, if $G\in\mathscr{B}_i$ contains y_i, there exists a sequence $(G_n)_{n<\omega}$ in $\mathscr{B}_i$ such that $Y_i\backslash G = \bigcup\{G_n \mid n<\omega\}$ and $(p_i\restriction X)^{-1}[G_n]\notin\mathscr{F}$ for each $n<\omega$. Since $\mathscr{F}$ is a $\mathscr{B}$-ultrafilter, we can find $F_n\in\mathscr{F}$ such that $F_n\cap(p_i\restriction X)^{-1}[G_n] = \emptyset$ for each $n<\omega$. By the cip of $\mathscr{F}$ it follows that $\bigcap\{F_n \mid n<\omega\}\in\mathscr{F}$. Since $\bigcap\{F_n \mid n<\omega\}\subset(p_i\restriction X)^{-1}[G]$, we see that $G\in\mathscr{G}_i$.

Now consider the point $y = (y_i)_{i\in I}$ in $\prod_{i\in I} Y_i$. We claim that $y\in X$. In order to verify this assertion it suffices to show that every G_δ-set in $\prod_{i\in I} Y_i$ containing y meets X, for X is G_δ-closed in $\prod_{i\in I} Y_i$. So let $(U_n)_{n<\omega}$ be a sequence of open subsets of $\prod_{i\in I} Y_i$ such that $y\in U = \bigcap\{U_n \mid n<\omega\}$. For each $n<\omega$ there exist a $J_n\in[I]^{<\omega}$ and open sets $U(j,n)\subset Y_j$ for each $j\in J_n$ such that $y\in\bigcap\{p_j^{-1}[U(j,n)] \mid j\in J_n\}\subset U_n$. For each $n<\omega$ and for each $j\in J_n$ we can find a $G(j,n)\in\mathscr{B}_j$ such that $y_j\in G(j,n)\subset U(j,n)$. By virtue of Claim B, $(p_j\restriction X)^{-1}[G(j,n)]\in\mathscr{F}$ whenever $n<\omega$ and $j\in J_n$. Hence there exists an $x\in\bigcap\{p_j\restriction X)^{-1}[G(j,n)] \mid n<\omega, j\in J_n\}$. In particular, $x\in U\cap X$. It follows that $y\in X$.

Now $\bigcap\{F \mid F\in\mathscr{F}\}$ being empty, there must be an $F\in\mathscr{F}$ such that $y\in X\backslash F$. Therefore there are a $J\in[I]^{<\omega}$ and open sets $V_j\subset Y_j$ for each $j\in J$ such that $y\in\bigcap\{(p_j\restriction X)^{-1}[V_j] \mid j\in J\}\subset X\backslash F$. Moreover, there exist $G_j\in\mathscr{B}_j$ satisfying $y_j\in G_j\subset V_j$ for each $j\in J$. It follows from B that $\bigcap\{(p_j\restriction X)^{-1}[G_j] \mid j\in J\}\in\mathscr{F}$, contradicting the fact that $F\in\mathscr{F}$. Therefore Claim A must be true.

(iv) implies (v): Assume that $\mathscr{B}$ is a G_δ-base for the closed sets of X, closed with respect to countable intersections, such that $\{X\backslash G \mid G\in\mathscr{G}\}$ is kernel-normal whenever $\mathscr{G}$ is a $\mathscr{B}$-ultrafilter with cip and $\bigcap\{G \mid G\in\mathscr{G}\} = \emptyset$. Let $\mathscr{F}$ be a $\mathscr{D}(X)$-ultrafilter with cip such that $\bigcap\{F \mid F\in\mathscr{F}\} = \emptyset$. By Zorn's

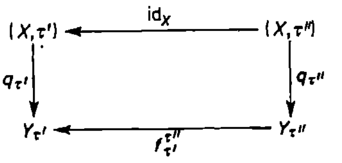
Lemma there exists a $\mathscr{B}$-ultrafilter $\mathscr{G}$ containing $\{G\in\mathscr{B}\mid F\subset G$ for some $F\in\mathscr{F}\}$. Suppose that there is a $\mathscr{G}'\in[\mathscr{G}]^{\leqslant\omega}$ such that $\bigcap\{G\mid G\in\mathscr{G}'\}=\emptyset$. For each $G\in\mathscr{G}'$ choose a $\mathscr{B}_G\in[\mathscr{B}]^{\leqslant\omega}$ such that $X\backslash G=\bigcup\{B\mid B\in\mathscr{B}_G\}$. Then $X=\bigcup\{B\mid B\in\bigcup\{\mathscr{B}_G\mid G\in\mathscr{G}'\}\}$. Moreover, there exist a $G_0\in\mathscr{G}'$ and a $B_0\in\mathscr{B}_{G_0}$ such that $B_0\in\mathscr{F}$, for otherwise we could find $F_B\in\mathscr{F}$ such that $F_B\cap B=\emptyset$ for every $B\in\bigcup\{\mathscr{B}_G\mid G\in\mathscr{G}'\}$. Since $\bigcap\{F_B\mid B\in\bigcup\{\mathscr{B}_G\mid G\in\mathscr{G}'\}\}=\emptyset$, this would contradict the fact that $\mathscr{F}$ has the cip. It follows that both B_0 and G_0 belong to $\mathscr{F}$ – a contradiction, since $B_0\cap G_0=\emptyset$. Therefore $\mathscr{G}$ has the cip. Since it is easily seen that $\mathscr{G}\subset\mathscr{F}$ and $\bigcap\{G\mid G\in\mathscr{G}\}=\emptyset$, $\{X\backslash G\mid G\in\mathscr{G}\}$ is a kernel-normal open cover of X which refines $\{X\backslash F\mid F\in\mathscr{F}\}$. Hence $\{X\backslash F\mid F\in\mathscr{F}\}$ is kernel-normal itself.

(v) implies (vi): Let $\mathscr{G}$ be a strong Cauchy filter on X with respect to (X,μ_f). By Zorn's Lemma there exists a $\mathscr{D}(X)$-ultrafilter $\mathscr{F}$ on X containing $\{F\in\mathscr{D}(X)\mid G\subset F$ for some $G\in\mathscr{G}\}$. Now the key observation is:

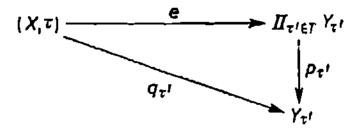
C. There is no $\mathscr{F}'\subset\mathscr{F}$ such that $\{X\backslash F\mid F\in\mathscr{F}'\}$ is a kernel-normal open cover of X.

For suppose that $\{X\backslash F\mid F\in\mathscr{F}'\}\in\mu_f$ for some $\mathscr{F}'\subset\mathscr{F}$. Then there exist an $F\in\mathscr{F}'$, a $G\in\mathscr{G}$, and a $\mathscr{U}\in\mu_f$ such that $\mathrm{St}(G,\mathscr{U})\subset X\backslash F$. Without loss of generality we may assume that $\mathscr{U}$ is a kernel-normal open cover of X. Let $\tau'\subset\tau$ be a developable topology such that $\mathrm{int}_{\tau'}\mathscr{U}=\{\mathrm{int}_{\tau'}U\mid U\in\mathscr{U}\}$ covers X (2.11). Then $H=X\backslash\mathrm{St}(F,\mathrm{int}_{\tau'}\mathscr{U})$ is a D-closed set in X such that $G\subset H$. It follows that $H\in\mathscr{F}$, hence $H\cap F\neq\emptyset$ – a contradiction! Since every countable cover of X consisting of D-open sets is kernel-normal, it follows from C that $\mathscr{F}$ has the cip. But then (v) and C imply that $\bigcap\{F\mid F\in\mathscr{F}\}\neq\emptyset$. Hence $\bigcap\{\mathrm{cl}\,G\mid G\in\mathscr{G}\}\neq\emptyset$ which proves the implication.

(vi) implies (i): Obviously, $T=\{\tau'\subset\tau\mid(X,\tau')$ is developable$\}$ is directed by set inclusion, i.e. $\tau'\leqslant\tau''$ if and only if $\tau'\subset\tau''$. For each $\tau'\in T$ let $Y_{\tau'}$ be the space obtained by identifying points in (X,τ') which have identical closures and denote by $q_{\tau'}$ the corresponding quotient mapping from (X,τ') onto $Y_{\tau'}$. Whenever $\tau',\tau''\in T$ and $\tau'\leqslant\tau''$ let $f_{\tau'}^{\tau''}\colon Y_{\tau''}\to Y_{\tau'}$ be the unique continuous mapping for which the following diagram commutes:



Then $(f_{\tau'}^{\tau''}\colon Y_{\tau''}\to Y_{\tau'})_{\tau',\tau''\in T;\,\tau'\leqslant\tau''}$ is an inverse system of developable T_1-spaces. We claim that X is homeomorphic to $Y=\varprojlim(f_{\tau'}^{\tau''}\colon Y_{\tau''}\to Y_{\tau'})_{\tau',\tau''\in T;\,\tau'\leqslant\tau''}$. Since the family $(q_{\tau'}\colon(X,\tau')\to Y_{\tau'})_{\tau'\in T}$ of continuous mappings separates points from closed sets, the unique mapping $e\colon(X,\tau)\to\prod_{\tau'\in T}Y_{\tau'}$ for which the diagram



commutes for all $\tau'\in T$ is an embedding, where $p_{\tau'}$ is the natural projection. Thus, in order to verify our claim it suffices to prove that $e[X]=Y$.

If $x\in X$ and $\tau',\tau''\in T$ such that $\tau'\leqslant\tau''$, then

$$f_{\tau'}^{\tau''}\circ p_{\tau''}(e(x)) = f_{\tau'}^{\tau''}(q_{\tau''}(x)) = q_{\tau'}(x) = p_{\tau'}(e(x)),$$

which shows that $e[X]\subset Y$.

For the proof of the converse inclusion consider an arbitrary $y\in Y$. We show that

$$\mathscr{F} = \{q_{\tau'}^{-1}[F]\,|\,\tau'\in T,\ p_{\tau'}(y)\in F\subset Y_{\tau'}\}$$

is a strong Cauchy filter with respect to μ_f. In fact, $\mathscr{F}$ is a nonempty collection of nonempty subsets of X, and every subset of X containing an element of $\mathscr{F}$ belongs to $\mathscr{F}$. If $q_{\tau'}^{-1}[F]\in\mathscr{F}$ and $q_{\tau''}^{-1}[G]\in\mathscr{F}$, there exists a $\tau'''\in T$ such that $\tau'\leqslant\tau'''$ and $\tau''\leqslant\tau'''$. Since $y\in Y$, $p_{\tau'''}(y)\in H=(f_{\tau'}^{\tau'''})^{-1}[F]\cap(f_{\tau''}^{\tau'''})^{-1}[G]$, i.e. $q_{\tau'''}^{-1}[H]\in\mathscr{F}$. But $q_{\tau'''}^{-1}[H]=q_{\tau'}^{-1}[F]\cap q_{\tau''}^{-1}[G]$, which proves that $\mathscr{F}$ is a filter on X. Furthermore, if $\mathscr{U}\in\mu_f$, there exists a $\tau'\in T$ such that $\mathrm{int}_{\tau'}\mathscr{U}=\{\mathrm{int}_{\tau'}U\,|\,U\in\mathscr{U}\}$ covers X (2.11). In particular, $p_{\tau'}(y)\in\mathrm{int}_{\tau'}U$ for some $U\in\mathscr{U}$. Hence, if $(\mathscr{U}_n)_{n<\omega}$ is a development of (X,τ'), then $\mathrm{St}(p_{\tau'}(y),\mathscr{U}_n)\in\mathrm{int}_{\tau'}U$ for some $n<\omega$. Since $\mathscr{U}_n\in\mu_f$, this shows that $\mathscr{F}$ is strongly Cauchy with respect to μ_f. Assuming (vi) it follows that there is an $x\in X$ which is contained in every element of $\mathscr{F}$. Suppose that $e(x)\neq y$. Then $q_{\tau'}(x)=p_{\tau'}(e(x))\neq p_{\tau'}(y)$ for some $\tau'\in T$, i.e. $x\notin q_{\tau'}^{-1}[\{p_{\tau'}(y)\}]$ although $q_{\tau'}^{-1}[\{p_{\tau'}(y)\}]\in\mathscr{F}$ – a contradiction! Consequently, $y=e(x)\in e[X]$, which completes the proof. ∎

Of course, every Dieudonné complete space satisfies the conditions of the preceding theorem. Moreover, it follows from 5.4(viii) and 7.4(v) that every D-paracompact T_1-space is an inverse limit of developable T_1-spaces, in particular every perfectly metacompact T_1-space (5.5). Clearly, none of these implications can be reversed.

Our next aim is to characterize those topological spaces which are homeomorphic to the limit of an inverse system of second countable developable T_1-spaces. For this purpose we need some preparations.

7.5. LEMMA. *Every Lindelöf subspace of a D-completely regular space X is G_δ-closed in X.*

Proof. Let $A\subset X$ be a Lindelöf subspace and consider a point $x\in\mathrm{cl}^\delta A$. Then $\bigcap\{G\cap A\,|\,G\in\mathscr{D}(X),\ x\in G\}\neq\varnothing$, for otherwise the open cover $\{A\backslash(G\cap A)\,|\,G\in\mathscr{D}(X),\ x\in G\}$ of A would have a countable subcover

$\{A\setminus(G_n\cap A) \mid G_n\in\mathscr{D}(X),\ x\in G_n,\ n<\omega\}$. Then $\bigcap\{G_n \mid n<\omega\}$ would be a G_δ set in X containing x which would not meet A, contrary to the fact that $x\in \mathrm{cl}^\delta A$. Since X is D-completely regular,

$$\bigcap\{G\cap A \mid G\in\mathscr{D}(X),\ x\in G\} = A\cap\bigcap\{G\in\mathscr{D}(X) \mid x\in G\} = A\cap\{x\}.$$

Therefore $x\in A$, which shows that $\mathrm{cl}^\delta A = A$. ■

Let $\mathscr{B}$ be a σ-algebra on a set X. A measure(33) $\nu\colon \mathscr{B}\to[0,\to)$ is called *2-valued* if $\nu(X)=1$ and $\nu(B)\in\{0,1\}=2$ for each $B\in\mathscr{B}$. It is called a *Dirac measure* if it is 2-valued and there is a point $x\in X$ such that $\nu(B)=1$ if and only if $x\in B$. A cardinal $\varkappa$ is said to be *Ulam-measurable* if there exists a 2-valued measure $\nu\colon \mathscr{P}(\varkappa)\to 2$ such that $\nu(\{\alpha\})=0$ for each $\alpha<\varkappa$. If $X=(X,\tau)$ is a D-completely regular space, $\mathscr{B}_D(X)$ is the σ-algebra generated by $\mathscr{D}(X)$, while $\mathscr{B}(X)$ denotes the σ-algebra of all Borel subsets of X. Obviously, $\mathscr{B}_D(X)\subset\mathscr{B}(X)$. A measure defined on $\mathscr{B}_D(X)$ will be called a *$\mathscr{B}_D(X)$-measure* on X, whereas a measure with domain $\mathscr{B}(X)$ is called a *Borel measure* on X. A Borel measure ν on X is said to be *regular* if

$$\nu(B)=\inf\{\nu(U) \mid B\subset U,\ U \text{ open}\} = \sup\{\nu(A) \mid A\subset B,\ A \text{ closed}\}$$

for each Borel set B in X. It is called *τ-additive* if

$$\nu(U)=\sup\{\nu(V) \mid V\in\mathscr{V}\}$$

whenever $U\in\tau$ and $\mathscr{V}$ is an upwards directed(34) collection of open sets such that $U=\bigcup\{V \mid V\in\mathscr{V}\}$. In analogy, a $\mathscr{B}_D(X)$-measure ν on X will be called τ-additive if

$$\nu(U)=\sup\{\nu(V) \mid V\in\mathscr{V}\}$$

whenever $U\subset X$ is D-open (1.3) and $\mathscr{V}$ is an upwards directed collection of D-open sets such that $U=\bigcup\{V \mid V\in\mathscr{V}\}$.

Finally, the *support* of a $\mathscr{B}_D(X)$-measure ν on X is the set $\mathrm{supp}(\nu)=\{x\in X \mid \nu(U)>0$ for each D-open subset U of X containing $x\}$. We can now prove:

7.6. THEOREM. *For a nonempty D-completely regular space $X=(X,\tau)$ the following conditions are equivalent:*

(i) *X is homeomorphic to the limit of an inverse system of second countable developable T_1-spaces.*

(ii) *X is homeomorphic to the limit of an inverse system of D-completely regular Lindelöf spaces.*

(iii) *X is homeomorphic to a G_δ-closed subset of a product of D-completely regular Lindelöf spaces.*

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(33) For basic facts concerning measure theory we refer to (Cohn [1980]).

(34) A collection $\mathscr{V}$ of subsets of X is called *upwards directed*, if for every $\{V_0, V_1\}\in[\mathscr{V}]^2$ there is a $V_2\in\mathscr{V}$ such that $V_0\cup V_1\subset V_2$.

(iv) *X is homeomorphic to a G_δ-closed subset of ${}^{\varkappa}\boldsymbol{D}_1$ for some cardinal $\varkappa$.*

(v) *There exists a G_δ-base $\mathscr{B}$ for the closed sets of X, closed with respect to countable intersections, such that every $\mathscr{B}$-ultrafilter $\mathscr{F}$ on X with* cip *is fixed, i.e. $\bigcap\{F \mid F \in \mathscr{F}\} \neq \emptyset$.*

(vi) *Every 2-valued $\mathscr{B}_D(X)$-measure on X is τ-additive.*

(vii) *For every 2-valued $\mathscr{B}_D(X)$-measure ν on X there exists a unique τ-additive Borel measure ν' on X such that $\nu = \nu' \restriction \mathscr{B}_D(X)$.*

(viii) *X is closed-complete*([35]) *and every 2-valued $\mathscr{B}_D(X)$-measure on X has a regular Borel extension.*

(ix) *Every 2-valued $\mathscr{B}_D(X)$-measure on X has a nonempty support.*

(x) *Every 2-valued $\mathscr{B}_D(X)$-measure on X is a Dirac measure.*

(xi) *Every $\mathscr{D}(X)$-ultrafilter $\mathscr{F}$ on X with* cip *is fixed.*

(xii) *X has no closed discrete subspace of Ulam-measurable cardinality and $\{X \backslash F \mid F \in \mathscr{F}\}$ is kernel-normal whenever $\mathscr{F}$ is a $\mathscr{D}(X)$-ultrafilter with* cip *and $\bigcap\{F \mid F \in \mathscr{F}\} = \emptyset$.*

(xiii) *(X, μ_ω) is Morita complete* (see 2.13).

Proof. Evidently (i) implies (ii). That (ii) implies (iii) can be shown as in the proof of Theorem 7.4 "(i) implies (ii)".

(iii) implies (iv): Let X be homeomorphic to a G_δ-closed subspace of a product $\prod_{i \in I} Y_i$ of D-completely regular Lindelöf spaces Y_i. For each $i \in I$ there exists a cardinal $\varkappa(i)$ such that Y_i is homeomorphic to a subspace Y_i' of ${}^{\varkappa(i)}\boldsymbol{D}_1$ (2.2). By virtue of Lemma 7.5 every Y_i' is G_δ-closed in ${}^{\varkappa(i)}\boldsymbol{D}_1$. Hence X is homeomorphic to a G_δ-closed subspace of ${}^{\varkappa}\boldsymbol{D}_1$, where $\varkappa = \sum_{i \in I} \varkappa(i)$.

(iv) implies (v): For simplicity we assume that X itself is a G_δ-closed subspace of ${}^{\varkappa}\boldsymbol{D}_1$ for some cardinal $\varkappa$. For each $\alpha < \varkappa$ denote by p_α the natural projection from ${}^{\varkappa}\boldsymbol{D}_1$ onto $\boldsymbol{D}_1$. Let $\mathscr{C}$ be a G_δ-base for the closed sets of $\boldsymbol{D}_1$. If $\mathscr{B}$ consists of all countable intersections of finite unions of subsets of X of the form $(p_\alpha \restriction X)^{-1}[C]$, $\alpha < \varkappa$, $C \in \mathscr{C}$, then $\mathscr{B}$ is a G_δ-base for the closed sets of X which is closed with respect to countable intersections. That every $\mathscr{B}$-ultrafilter on X with cip is fixed can be shown by some obvious modifications of the argument used to prove that (iii) implies (iv) in Theorem 7.4.

(v) implies (vi): Assume that there is a G_δ-base $\mathscr{B}$ for the closed sets of X, closed with respect to countable intersections, such that every $\mathscr{B}$-ultrafilter with cip is fixed: Let ν be a 2-valued $\mathscr{B}_D(X)$-measure on X. A straightforward application of Zorn's Lemma yields a $\mathscr{B}$-ultrafilter $\mathscr{F}$ on X such that

$$\{F \in \mathscr{B} \mid A \subset F \text{ for some } A \in \mathscr{D}(X) \text{ with } \nu(A) = 1\} \subset \mathscr{F}$$

Assume for a moment that the following claim is true.

A. $\mathscr{F}$ has the cip.

([35]) A topological space X is called *closed-complete* (Blair [1977]) or *a-realcompact* (Dykes [1970]) if every closed ultrafilter on X with cip is fixed.

Then we may argue as follows. Suppose that there is a $\mathscr{D}$-open subset U of X and an upwards directed collection $\mathscr{V}$ of D-open sets such that $U = \bigcup\{V \mid V \in \mathscr{V}\}$ and $\nu(U) \neq \sup\{\nu(V) \mid V \in \mathscr{V}\}$. Then $\nu(U) = 1$ and $\nu(V) = 0$ for each $V \in \mathscr{V}$ If $(A_n)_{n<\omega}$ is a sequence in $\mathscr{D}(X)$ such that $U = \bigcup\{A_n \mid n < \omega\}$, there exists an $n_0 < \omega$ such that $\nu(A_{n_0}) = 1$. Since $\mathscr{B}$ is a base for the closed sets, there is a subcollection $\mathscr{B}'$ of $\mathscr{B}$ such that $A_{n_0} = \bigcap\{B \mid B \in \mathscr{B}'\}$. Hence

$$\bigcap\{F \mid F \in \mathscr{F}\} \subset \bigcap\{B \mid B \in \mathscr{B}'\} = A_{n_0} \subset U.$$

By virtue of A there is an $x \in \bigcap\{F \mid F \in \mathscr{F}\} \subset U$. Moreover, there exist a $B_x \in \mathscr{B}$ and a $V_x \in \mathscr{V}$ such that $x \in X \backslash B_x \subset V_x$. Since $\nu(X \backslash B_x) = 0$, it follows that $B_x \in \mathscr{F}$, hence $x \in B_x$ — a contradiction which proves that (v) implies (vi).

It remains to verify Claim A. To this end assume that there is an $\mathscr{F}' \in [\mathscr{F}]^{\leq\omega}$ such that $\bigcap\{F \mid F \in \mathscr{F}'\} = \emptyset$. Since $\mathscr{B}$ is a G_δ-base, we can find, for each $F \in \mathscr{F}'$, a $\mathscr{B}_F \in [\mathscr{B}]^{\leq\omega}$ such that $X \backslash F = \bigcup\{B \mid B \in \mathscr{B}_F\}$. It follows that

$$X = \bigcup\{X \backslash F \mid F \in \mathscr{F}'\} = \bigcup\{B \mid B \in \bigcup\{\mathscr{B}_F \mid F \in \mathscr{F}'\}\}.$$

Since $\bigcup\{\mathscr{B}_F \mid F \in \mathscr{F}'\}$ is countable, there exists an $F_0 \in \mathscr{F}'$ and a $B_0 \in \mathscr{B}_{F_0}$ such that $\nu(B_0) = 1$. But then $B_0 \in \mathscr{F}$ such that $B_0 \cap F_0 = \emptyset$, which is impossible. Hence Claim A must be true.

That (vi) implies (vii) follows from a general measure extension theorem due to K. P. Dalgas ([1982], Theorem 3.7), and that (vii) implies (viii) becomes obvious in view of the following facts.

B. Every τ-additive 2-valued Borel measure on X is regular.

C. X is closed-complete if and only if every regular 2-valued Borel measure on X is τ-additive.

D. If ν and ν' are regular 2-valued Borel measures on X such that $\nu \upharpoonright \mathscr{B}_D(X) = \nu' \upharpoonright \mathscr{B}_D(X)$ and ν' is τ-additive, then $\nu = \nu'$.

We omit the routine verifications of B and D and refer to (Dalgas [1978], 2.2.6) for a proof of C.([36])

(viii) implies (ix): Let ν be a 2-valued $\mathscr{B}_D(X)$-measure on X. If $\mathscr{N} = \{U \subset X \mid U \ D\text{-open},\ \nu(U) = 0\}$, then

$$\mathscr{V} = \{V \mid \text{there exists an } \mathscr{N}' \in [\mathscr{N}]^{<\omega} \text{ such that } V = \bigcup\{U \mid U \in \mathscr{N}'\}\}$$

is an upwards directed collection of open sets such that $X \backslash \mathrm{supp}(\nu) = \bigcup\{V \mid V \in \mathscr{V}\}$. Assuming (viii) there exists a τ-additive 2-valued Borel measure ν' on X such that $\nu' \upharpoonright \mathscr{B}_D(X) = \nu$. Since $\nu'(X \backslash \mathrm{supp}(\nu)) = 0$, it follows that $\mathrm{supp}(\nu) \neq \emptyset$.

(x) and (xi) are equivalent by a general result due to W. Adamski ([1976], Theorem 2.1). Therefore we proceed by showing that (ix) implies (xi). To this end consider a $\mathscr{D}(X)$-ultrafilter $\mathscr{F}$ on X with cip. If

([36]) For Hausdorff spaces, C was proved by R. J. Gardner [1975].

$$\nu(B) = \begin{cases} 0 & \text{whenever } F \subset X \backslash B \text{ for some } F \in \mathscr{F}, \\ 1 & \text{whenever } F \subset B \text{ for some } F \in \mathscr{F}, \end{cases}$$

for each $B \in \mathscr{B}_D(X)$, then ν is a 2-valued $\mathscr{B}_D(X)$-measure on X. Assuming (ix) there exists an $x \in \operatorname{supp}(\nu) \subset \bigcap \{F \mid F \in \mathscr{F}\}$.

(xi) implies (xii): Evidently, (xi) implies that every closed discrete subspace of X is realcompact. Hence X has no closed discrete subspace of Ulam-measurable cardinality (Gillman and Jerison [1960], 12.2).

(xii) implies (xiii): Suppose that there is a strong Cauchy filter $\mathscr{G}$ with respect to (X, μ_ω) such that $\bigcap \{\operatorname{cl} G \mid G \in \mathscr{G}\} = \emptyset$. Let $\mathscr{F}$ be a $\mathscr{D}(X)$-ultrafilter containing $\{F \in \mathscr{D}(X) \mid G \subset F$ for some $G \in \mathscr{G}\}$. $\mathscr{F}$ has the cip, for otherwise there would be an $\mathscr{F}' \in [\mathscr{F}]^{\leq \omega}$ such that $\{X \backslash F \mid F \in \mathscr{F}'\} \in \mu_\omega$ and hence a $G \in \mathscr{G}$, a $\mathscr{V} \in \mu_\omega$, and an $F \in \mathscr{F}'$ such that $\operatorname{St}(G, \mathscr{V}) \subset X \backslash F$. By virtue of Lemma 2.11 we could find a developable topology $\tau(\mathscr{V}) \subset \tau$ such that $\{\operatorname{int}_{\tau(\mathscr{V})} V \mid V \in \mathscr{V}\}$ is a covering of X. Since $\operatorname{cl}_{\tau(\mathscr{V})} G \subset \operatorname{St}(G, \mathscr{V})$, it follows that $\operatorname{cl}_{\tau(\mathscr{V})} G \in \mathscr{F}$ and $\operatorname{wcl}_{\tau(\mathscr{V})} G \cap F = \emptyset$, which is impossible.

Assuming (xii) it follows that $\{X \backslash F \mid F \in \mathscr{F}\}$ is kernel-normal. Applying Lemma 2.11 there exist a developable topology $\tau' \subset \tau$ and a τ'-open cover $\mathscr{U} = (U_F)_{F \in \mathscr{F}}$ such that $U_F \subset X \backslash F$ for each $F \in \mathscr{F}$. Moreover, since (X, τ') is subparacompact, there is a sequence $(\mathscr{U}_n)_{n<\omega}$ of τ'-open refinements of $\mathscr{U}$ satisfying

(*) for each $x \in X$ there is an $n(x) < \omega$ such that $\operatorname{ord}(x, \mathscr{U}_{n(x)}) = 1$

(Burke [1970]). For each $n < \omega$ define $\mathscr{U}'_n = \{U \in \mathscr{U}_n \mid A(U) \neq \emptyset\}$, where $A(U) = X \backslash \bigcup \{V \in \mathscr{U}_n \mid V \neq U\}$. Then it is easily seen that each $\{A(U) \mid U \in \mathscr{U}'_n\}$ is closed and discrete with repect to τ'. Hence $X(n) = \bigcup \{A(U) \mid U \in \mathscr{U}'_n\} \in \mathscr{D}(X)$ for each $n < \omega$. We show that $N = \{n < \omega \mid X(n) \in \mathscr{F}\}$ is nonempty. Suppose that $X(n) \notin \mathscr{F}$ for each $n < \omega$. Then there are $F_n \in \mathscr{F}$ such that $X(n) \cap F_n = \emptyset$ for each $n < \omega$. Since $\mathscr{F}$ has the cip, $\bigcap \{F_n \mid n < \omega\} \in \mathscr{F}$ But this is impossible, because property (*) implies that

$$\bigcap \{F_n \mid n < \omega\} \subset \bigcap \{X \backslash X(n) \mid n < \omega\} = X \backslash \bigcup \{X(n) \mid n < \omega\} = \emptyset.$$

For each $U \in \bigcup \{\mathscr{U}'_n \mid n < \omega\}$ choose an $x_U \in A(U)$. Then every $A(n) = \{x_U \mid U \in \mathscr{U}'_n\}$, $n < \omega$, is a closed discrete subspace of X. By Zorn's Lemma there exists, for each $n \in N$, an ultrafilter $\mathscr{G}_n$ on $A(n)$ such that $\{G(F, n) \mid F \in \mathscr{F}\} \subset \mathscr{G}_n$, where $G(F, n) = \{x_U \in A(n) \mid F \cap A(U) \neq \emptyset\}$.

E. $\bigcap \{G \mid G \in \mathscr{G}_n\} = \emptyset$ for each $n \in N$.

For suppose that there is an $n \in N$ and an $x_U \in A(n)$ such that $x_U \in \bigcap \{G \mid G \in \mathscr{G}_n\}$. It follows that $X \backslash U \notin \mathscr{F}$, for otherwise $x_U \in G(X \backslash U, n) \in \mathscr{G}_n$, hence $(X \backslash U) \cap A(U) \neq \emptyset$, which is impossible. Since $\mathscr{F}$ is a $\mathscr{D}(X)$-ultrafilter, $(X \backslash U) \cap F_0 = \emptyset$, for some $F_0 \in \mathscr{F}$, i.e. $F_0 \subset U$. On the other hand $\mathscr{U}_n$ refines $\{X \backslash F \mid F \in \mathscr{F}\}$. Hence $F_0 \subset U \subset X \backslash F_1$ for some $F_1 \in \mathscr{F}$, contradicting the fact that $\mathscr{F}$ is a $\mathscr{D}(X)$-filter.

By virtue of (xii) and (Gillman and Jerison [1960], 12.2), every subspace $A(n)$, $n \in N$, is realcompact. Therefore E implies that for each $n \in N$ there is a $\mathscr{G}'_n \in [\mathscr{G}_n]^{\leqslant \omega}$ such that $\bigcap \{G \mid G \in \mathscr{G}'_n\} = \emptyset$. Moreover, the following holds.

F. For each $n \in N$ and for each $G \in \mathscr{G}'_n$ there is an $F(G, n) \in \mathscr{F}$ such that $F(G, n) \subset \bigcup \{U \in \mathscr{U}'_n \mid x_U \in G\}$.

To prove this suppose that there exist an $n \in N$ and a $G \in \mathscr{G}'_n$ such that $F \cap (X \backslash \bigcup \{U \in \mathscr{U}'_n \mid x_U \in G\}) \neq \emptyset$ for each $F \in \mathscr{F}$. Then $X \backslash \bigcup \{U \in \mathscr{U}'_n \mid x_U \in G\} \in \mathscr{F}$ and therefore $G(X \backslash \bigcup \{U \in \mathscr{U}'_n \mid x_U \in G\}, n) \in \mathscr{G}_n$, contradicting the fact that $G \cap (X \backslash \bigcup \{U \in \mathscr{U}'_n \mid x_U \in G\}) = \emptyset$.

Now, for each $m \in \omega \backslash N$ let $F(m)$ be an element of $\mathscr{F}$ such that $F(m) \cap X(m) = \emptyset$. Since $\mathscr{F}$ has the cip, there exists an

$$x \in \bigcap \{F(G, n) \mid n \in N,\ G \in \mathscr{G}'_n\} \cap \bigcap \{F(m) \mid m \in \omega \backslash N\}.$$

By virtue of property (*) there is an $n(x) < \omega$ such that $x \in X(n(x))$. It follows that $n(x) \in N$. If $U(x)$ is the only member of $\mathscr{U}'_{n(x)}$ such that $x \in A(U(x))$, then $x_{U(x)} \notin G_x$ for some $G_x \in \mathscr{G}'_{n(x)}$. Consequently, $x \notin \bigcup \{U \in \mathscr{U}'_{n(x)} \mid x_U \in G_x\}$. On the other hand, F implies that $x \in F(G_x, n(x)) \subset \bigcup \{U \in \mathscr{U}'_{n(x)} \mid x_U \in G_x\}$ – a contradiction which completes the argument.

Finally, that (xiii) implies (i) can be shown as in the proof of the corresponding implication in Theorem 7.4. ■

Let us call a topological space X *D-complete* if it is D-completely regular and satisfies one – and therefore all – of the conditions in the preceding theorem. Evidently, every D-completely regular Lindelöf space is D-complete. Moreover, it follows from Theorem 7.6 that the following implications hold:

$$\text{realcompact} \Rightarrow D\text{-complete} \Rightarrow \text{closed-complete}.$$

Clearly, D_1 is D-complete, but not realcompact. The well-known Isbell–Mrówka space Ψ (see Gillman and Jerison [1960], 5.I) is an example of a completely regular D-complete space, in fact a Moore space, which is not realcompact. However, for normal spaces, D-completeness and realcompactness coincide.

7.7. COROLLARY. *A normal T_1-space is realcompact if and only if it is D-complete.*

Proof. It is easily seen that in a normal T_1-space the collection of D-closed subsets coincides with the collection of zero-sets. Therefore, for normal T_1-spaces condition (xi) in Theorem 7.6 is a characterization of realcompactness. ■

Observe that ω_1 is a normal T_1-space which is not D-complete. As P. Simon [1971] has shown, the Dowker space constructed by M. E. Rudin [1971] is an example of a normal T_1-space which is closed-complete but not realcompact, hence not D-complete. By virtue of Theorem 4.4 the collection of all closed subsets of a topological space X coincides with $\mathscr{D}(X)$ if and only if X is perfect. Hence a perfect T_1-space is D-complete if and only if it is closed-complete. But this observation can be slightly improved.

7.8. COROLLARY. *Let X be a D-completely regular space satisfying*

(Δ) *Whenever $(A_n)_{n<\omega}$ is a sequence of closed subsets of X such that $A_{n+1} \subset A_n$ for each $n < \omega$ and $\bigcap\{A_n \mid n < \omega\} = \emptyset$, there exists a sequence $(F_n)_{n<\omega}$ in $\mathscr{D}(X)$ such that $A_n \subset F_n$ for each $n < \omega$ and $\bigcap\{F_n \mid n < \omega\} = \emptyset$.*

Then the following conditions are equivalent:

(i) *X is D-complete.*

(ii) *X is closed-complete.*

Proof. It suffices to show that (ii) implies (i). To this end consider a $\mathscr{D}(X)$-ultrafilter $\mathscr{F}$ such that $\bigcap\{F \mid F \in \mathscr{F}\} = \emptyset$. According to condition (xi) in Theorem 7.6 we have to show that $\mathscr{F}$ does not have the cip. By Zorn's Lemma there exists a closed ultrafilter $\mathscr{G}$ on X containing $\mathscr{F}$ Since $\bigcap\{G \mid G \in \mathscr{G}\} = \emptyset$, there exists a sequence $(G_n)_{n<\omega}$ in $\mathscr{G}$ such that $G_{n+1} \subset G_n$ for each $n < \omega$ and $\bigcap\{G_n \mid n < \omega\} = \emptyset$. By property ($\Delta$) we can find a sequence $(F_n)_{n<\omega}$ in $\mathscr{D}(X)$ such that $G_n \subset F_n$ for each $n < \omega$ and $\bigcap\{F_n \mid n < \omega\} = \emptyset$. Since $F_n \in \mathscr{F}$ for each $n < \omega$, the proof is complete. ∎

7.9. COROLLARY. *Let X be a T_1-space which has no closed discrete subspace of Ulam-measurable cardinality. Each of the following conditions implies that X is D-complete:*

(i) *X is developable.*

(ii) *X is a σ-space.*

(iii) *X is semi-metrizable.*

(iv) *X is semi-stratifiable.*

(v) *X is D-paracompact.*

(vi) *X is perfectly subparacompact.*

(vii) *X is D-normal and θ-refinable.*

Proof. If X is θ-refinable, then our assumption on X implies that X is closed-complete (Blair [1977]). If in addition X is assumed to be D-normal, then X is countably D-paracompact (5.12(f)). Since every countably D-paracompact space has property (Δ) of the preceding corollary, we see that (vii) implies that X is D-complete. But (vii) is the weakest of the above properties. ∎

Our next theorem which is motivated by recent work of A. V. Arkhangel'skii [1983] and V. V. Uspenskii [1983], shows that an entirely different condition on a D-completely regular space also implies its D-completeness.

7.10. THEOREM. *Let X be a D-completely regular space satisfying*

(∇) *A mapping φ: $C_p(X, \boldsymbol{D}_1) \to \boldsymbol{D}_1$ is continuous*[37] *if and only if for each*

[37] $C_p(X, \boldsymbol{D}_1)$ is the space obtained by supplying the set $C(X, \boldsymbol{D}_1)$ of all continuous mappings from X into $\boldsymbol{D}_1$ with the topology of pointwise convergence.

$F \in [C(X, D_1)]^{\leq \omega}$ there is a continuous mapping φ_F: $C_p(X, D_1) \to D_1$ such that $\varphi_F \upharpoonright F = \varphi$.

Then X is D-complete.

Proof. Denote by C the set of all continuous mappings from $C_p(X, D_1)$ into D_1. By virtue of conditions (iii) or (iv) in Theorem 7.6 it suffices to verify the following claims.

A. X is homeomorphic to a G_δ-closed subspace of C.

B. C is a G_δ-closed subspace of the product space ${}^{C(X,D_1)}D_1$.

For the proof of Claim A note first that the correspondence $x \in X \mapsto e(x) \in C$ defines an embedding of X into C, where $e(x)(f) = f(x)$ for each $x \in X$ and for each $f \in C_p(X, D_1)$. Thus, we only have to prove that $e[X]$ is G_δ-closed in C.

Suppose there is a $\varphi \in C$ such that $\varphi \in \mathrm{cl}_C^\delta e[X] \backslash e[X]$. Let a, b be two distinct points in D_1. If c_a, $c_b \in C_p(X, D_1)$ are the constant mappings which map X onto $\{a\}$ respectively $\{b\}$, then the following holds:

A.1 $\varphi(c_a) = a$.

A.2 Every open neighborhood G of c_a in $C_p(X, D_1)$ contains an f_G such that $\varphi(f_G) = b$.

Once we have proved these claims we obtain a contradiction as follows. There exists an open neighborhood V of $\varphi(c_a) = a$ in D_1 such that $b \in D_1 \backslash V$. By the continuity of φ and A.2 there must be an $f_{\varphi^{-1}[V]} \in \varphi^{-1}[V]$ such that $\varphi(f_{\varphi^{-1}[V]}) = b$, contradicting the fact that $b \in D_1 \backslash V$.

In order to verify Claim A.1 suppose that $\varphi(c_a) \neq a$. Then there is an open neighborhood U of $\varphi(c_a)$ such that $a \notin U$. Since $\varphi \in \mathrm{cl}_C^\delta e[X]$ and $\Psi = \{\psi \in C \mid \psi(c_a) \in U\}$ is a G_δ-set in C containing φ, there exists an $x \in X$ such that $e(x) \in \Psi$. It follows that $a = c_a(x) = e(x)(c_a) \in U$, a contradiction which proves Claim A.1.

For the proof of Claim A.2 consider an arbitrary open neighborhood G of c_a in $C_p(X, D_1)$. There exist an $A \in [X]^{<\omega}$ and an open subset $U_x \subset D_1$ for each $x \in A$ such that

$$c_a \in F = \{f \in C_p(X, D_1) \mid f(x) \in U_x \text{ for each } x \in A\} \subset G.$$

Since $\varphi \notin e[X]$, we can find a D-closed subset Γ of C such that $\varphi \in \Gamma$ and $\Gamma \cap e[A] = \emptyset$. If $B = \{x \in X \mid e(x) \in \Gamma\}$, then B is a nonempty closed subset of X such that $A \cap B = \emptyset$. By virtue of 2.2 there exist disjoint D-closed subsets A', B' of X such that $A \subset A'$ and $B \subset B'$, for A is finite. Using 1.6 we can find an $f \in C_p(X, D_1)$ such that $f[A'] \subset \{a\}$ and $f[B'] \subset \{b\}$. Since $f(x) = a = c_a(x) \in U_x$ for each $x \in A$, it follows that $f \in F \subset G$. We claim that $\varphi(f) = b$. For otherwise there would be an open subset W of D_1 such that $\varphi(f) \in W$ and $b \notin W$. Then $\Gamma' = \Gamma \cap \{\psi \in C \mid \psi(f) \in W\}$ would be a G_δ-subset of C containing φ. Hence we could find a $z \in X$ such that $e(z) \in \Gamma' \subset \Gamma$. But this is impossible, for

$z \in B$ and therefore $b = f(z) = e(z)(f) \in W$, contradicting the fact that $b \notin W$. This contradiction completes the proof of Claim A.[38]

For the proof of Claim B consider an arbitrary $\psi_0 \in {}^{C(X,\boldsymbol{D}_1)}\boldsymbol{D}_1 \setminus C$. By virtue of property (∇) there exists an $F \in [C(X, \boldsymbol{D}_1)]^{\leq\omega}$ such that $\psi_0 \restriction F \neq \psi \restriction F$ for each $\psi \in C$. If h: ${}^{C(X,\boldsymbol{D}_1)}\boldsymbol{D}_1 \to {}^F\boldsymbol{D}_1$ is defined by

$$\psi \in {}^{C(X,\boldsymbol{D}_1)}\boldsymbol{D}_1 \mapsto h(\psi) = \psi \restriction F \in {}^F\boldsymbol{D}_1,$$

then h is continuous and $h(\psi_0) \notin h[C]$. Since ${}^F\boldsymbol{D}_1$ is developable, it follows that $h^{-1}[\{h(\psi_0)\}]$ is a G_δ-set in ${}^{C(X,\boldsymbol{D}_1)}\boldsymbol{D}_1$ containing ψ_0 such that $h^{-1}[\{h(\psi_0)\}] \cap C = \emptyset$, i.e. $\psi_0 \notin \mathrm{cl}^\delta C$. ■

PROBLEM P. Characterize internally those D-completely regular spaces which have property (∇) of Theorem 7.10. Does every D-complete space have this property?

In the remainder of this section we will show that there are analogues, for D-completely regular spaces, of the Hewitt realcompactification respectively the Dieudonné completion of a completely regular T_1-space. For this purpose we need the following extension theorem.

7.11. PROPOSITION. *Let f be a continuous mapping from a G_δ-dense subspace A of an arbitrary topological space X into $\boldsymbol{D}_1$. If $\mathrm{cl}_X f^{-1}[F] \cap \mathrm{cl}_X f^{-1}[G] = \emptyset$ for each pair F, G of disjoint closed subsets of $\boldsymbol{D}_1$, then there is a continuous mapping g from X into $\boldsymbol{D}_1$ such that $g \restriction A = f$.*

Proof.[39] Let π: $\boldsymbol{D} \to \boldsymbol{D}_1$ be the T_0-reflection and define a mapping E from S into the closed subsets of X by

$$E(n_0, \ldots, n_{k-1}) = \mathrm{cl}_X f^{-1}[\pi[A(n_0, \ldots, n_{k-1})]]$$

for each $(n_0, \ldots, n_{k-1})$ in S, where $\mathscr{A} = \{A(n_0, \ldots, n_{k-1}) \mid (n_0, \ldots, n_{k-1}) \in S\}$ is the canonical subbase for the closed sets of $\boldsymbol{D}$. Evidently, E has property (E.2) of Lemma 1.2. In order to prove that E has property (E.1), consider an arbitrary $(n_0, \ldots, n_{k-1}) \in S \setminus \{(0)\}$. We have to show that

$$(*) \qquad X \setminus (E(0) \cup E(n_0, \ldots, n_{k-1})) = \bigcup \{E(n_0, \ldots, n_{k-1}, n) \mid 0 < n < \omega\}.$$

Suppose that there is an $x \in X \setminus (E(0) \cup E(n_0, \ldots, n_{k-1}))$ such that $x \notin \bigcup \{E(n_0, \ldots, n_{k-1}, n) \mid 0 < n < \omega\}$. Then

$$H = X \setminus (E(0) \cup E(n_0, \ldots, n_{k-1})) \cap \bigcap \{X \setminus E(n_0, \ldots, n_{k-1}, n) \mid 0 < n < \omega\}$$

is a G_δ-set in X containing x such that $H \cap A = \emptyset$, contradicting our

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(38) Note that we have not yet used property (∇), i.e. Claim A holds for every D-completely regular space.

(39) See § 1 for the relevant definitions.

assumption that A is G_δ-dense in X. Consequently, in (*) the inclusion $\subset$ holds. For the proof of the converse inclusion note that

$$\pi[A(n_0, \dots, n_{k-1}, n)] \cap (\pi[A(0)] \cup \pi[A(n_0, \dots, n_{k-1})]) = \emptyset$$

whenever $0 < n < \omega$. By the property of f it follows that

$$E(n_0, \dots, n_{k-1}, n) \subset X \setminus (E(0) \cup E(n_0, \dots, n_{k-1}))$$

for each n such that $0 < n < \omega$, i.e. that (*) holds. Since the remaining case $((n_0, \dots, n_{k-1}) = (0))$ can be treated in the same way, we conclude from Lemma 1.2 that there is a continuous mapping $f_E\colon X \to \boldsymbol{D}$ such that $f_E^{-1}[A(n_0, \dots, n_{k-1})] = E(n_0, \dots, n_{k-1})$ for each $(n_0, \dots, n_{k-1})$ in S. Then $g = \pi \circ f_E$ is a continuous mapping from X into $\boldsymbol{D}_1$ such that $g \restriction A = f$. ■

7.12. COROLLARY. *For a G_δ-dense subspace A of a D-completely regular space X the following conditions are equivalent:*

(i) *A is D-embedded in X.*

(ii) *For every continuous mapping f from A into a D-complete space Y there exists a continuous mapping g from X into Y such that $g \restriction A = f$.*

(iii) $\mathrm{cl}_X \bigcap \{F_n \mid n < \omega\} = \bigcap \{\mathrm{cl}_X F_n \mid n < \omega\}$ *for each sequence* $(F_n)_{n<\omega}$ *in* $\mathscr{D}(A)$.

(iv) $\bigcap \{\mathrm{cl}_X F_n \mid n < \omega\} = \emptyset$ *for each sequence* $(F_n)_{n<\omega}$ *in* $\mathscr{D}(A)$ *with* $\bigcap \{F_n \mid n < \omega\} = \emptyset$.

(v) $\mathrm{cl}_X F \cap \mathrm{cl}_X G = \emptyset$ *whenever F, G are disjoint D-closed subsets of A.*

Proof. (i) implies (ii): Let f be a continuous mapping from A into a D-complete space Y. By virtue of Theorem 7.6(iv) we may assume that Y is a G_δ-closed subset of ${}^\varkappa\boldsymbol{D}_1$ for some cardinal $\varkappa$. Assuming (i) there exists, for each $\alpha < \varkappa$, a continuous mapping g_α from X into $\boldsymbol{D}_1$ such that $g_\alpha \restriction A = (p_\alpha \restriction Y) \circ f$, where $p_\alpha\colon {}^\varkappa\boldsymbol{D}_1 \to \boldsymbol{D}_1$ is the natural projection. If $g(x) = (g_\alpha(x))_{\alpha<\varkappa}$ for each $x \in X$, then g is a continuous mapping from X into ${}^\varkappa\boldsymbol{D}_1$ such that $g \restriction A = f$. The observation that $g[X] = g[\mathrm{cl}^\delta A] \subset \mathrm{cl}^\delta g[A] = \mathrm{cl}^\delta f[A] \subset \mathrm{cl}^\delta Y = Y$ completes the argument.

(ii) implies (iii): Let $(F_n)_{n<\omega}$ be a sequence in $\mathscr{D}(A)$. By virtue of Proposition 1.4 there exist a point $b \in \boldsymbol{D}_1$ and continuous mappings f_n from A into $\boldsymbol{D}_1$ such that $F_n = f_n^{-1}[\{b\}]$ for each $n < \omega$. Another application of the same proposition yields a continuous mapping h from ${}^\omega\boldsymbol{D}_1$ into $\boldsymbol{D}_1$ such that $h^{-1}[\{b\}] = \{(b_n)_{n<\omega}\}$, where $b_n = b$ for each $n < \omega$. Let $f\colon A \to \boldsymbol{D}_1$ be defined by $x \mapsto f(x) = h((f_n(x))_{n<\omega})$. Assuming (ii) we can find continuous mappings g_n from X into $\boldsymbol{D}_1$ such that $g_n \restriction A = f_n$ for each $n < \omega$. Then $x \mapsto g(x) = h((g_n(x))_{n<\omega})$ defines a continuous mapping $g\colon X \to \boldsymbol{D}_1$ such that $g \restriction A = f$. Since it is easily seen(40) that

(40) Use the fact that $\varphi^{-1}[B] = \mathrm{cl}_X(\varphi \restriction A)^{-1}[B]$ for each continuous mapping φ from X into $\boldsymbol{D}_1$ and for each closed set B in $\boldsymbol{D}_1$.

$$\mathrm{cl}_X \bigcap \{F_n \mid n < \omega\} = \mathrm{cl}_X f^{-1}[\{b\}] = g^{-1}[\{b\}]$$
$$= \bigcap \{g_n^{-1}[\{b\}] \mid n < \omega\} = \bigcap \{\mathrm{cl}_X F_n \mid n < \omega\},$$

the implication is proved.

Obviously, (iii) implies (iv), (iv) implies (v), and (v) implies (i) according to the preceding proposition. ■

Based on these preparations we can now prove:

7.13. THEOREM. *For every D-completely regular space X there exists a D-complete space δX and a continuous mapping i_X: $X \to \delta X$ such that the following conditions are satisfied:*

(i) *X is homeomorphic to $i_X[X]$.*

(ii) *$i_X[X]$ is G_δ-dense in δX.*

(iii) *Whenever f is a continuous mapping from X into a D-complete space Y, there exists a unique continuous mapping δf from δX into Y such that $\delta f \circ i_X = f$.*

(iv) *δX and i_X are unique, in the following sense: Whenever j_X is a continuous mapping from X into a D-complete space ηX such that the analogues of* (i) *–*(iii) *are satisfied, there exists a homeomorphism h: $\delta X \to \eta X$ such that $h \circ i_X = j_X$.*

Proof.(41) Let X^* be the Wallman–Frink compactification of X with respect to $\mathscr{D}(X)$, i.e. X^* is the set of all $\mathscr{D}(X)$-ultrafilters supplied with the topology which has the collection $\{F^* \mid F \in \mathscr{D}(X)\}$ as a base for the closed sets, where $F^* = \{\mathscr{F} \in X^* \mid F \in \mathscr{F}\}$. It is wellknown that X^* is a compact T_1-space and that the correspondence $x \mapsto i_X(x) = \{F \in \mathscr{D}(X) \mid x \in F\}$ defines an embedding of X into X^* (e.g. see Steiner [1968]). We claim that $\delta X = \{\mathscr{F} \in X^* \mid \mathscr{F}$ has the cip$\}$ has the desired properties. In order to verify this we prove first:

A. $\delta X = \mathrm{cl}^{\delta}_{X^*} i_X[X]$.

In fact, if $\mathscr{F} \in \delta X$ and $(G_n)_{n<\omega}$ is a sequence in $\mathscr{D}(X)$ such that $\mathscr{F} \in \bigcap \{X^* \backslash G_n^* \mid n < \omega\}$, then there exist $F_n \in \mathscr{F}$ such that $F_n \cap G_n = \emptyset$ for each $n < \omega$. Since $\mathscr{F}$ has the cip, we can find an $x \in \bigcap \{F_n \mid n < \omega\}$. But then $i_X(x) \in \bigcap \{X^* \backslash G_n^* \mid n < \omega\}$, which proves that $\mathscr{F} \in \mathrm{cl}^{\delta}_{X^*} i_X[X]$. Conversely, if $\mathscr{F} \in X^* \backslash \delta X$, there is an $\mathscr{F}' \in [\mathscr{F}]^{\leqslant \omega}$ such that $\bigcap \{F \mid F \in \mathscr{F}'\} = \emptyset$. For each $F \in \mathscr{F}'$ there exists a $\mathscr{G}_F \in [\mathscr{D}(X)]^{\leqslant \omega}$ such that $X \backslash F = \bigcup \{G \mid G \in \mathscr{G}_F\}$. It follows that $\bigcap \{X^* \backslash G^* \mid G \in \bigcup \{\mathscr{G}_F \mid F \in \mathscr{F}'\}\}$ is a G_δ-set in X^* containing $\mathscr{F}$ which has an empty intersection with $i_X[X]$, thus proving that $\mathscr{F} \in X^* \backslash \mathrm{cl}^{\delta}_{X^*} i_X[X]$.

If $F, G \in \mathscr{D}(X)$ are disjoint, then $F^* \cap G^* = \emptyset$. Therefore $i_X[X]$ is D-embedded in δX (7.12), a fact which immediately implies (iii). (For it is easily seen that for a given continuous mapping f from X into a D-complete space Y there is at most one continuous mapping δf from X into Y satisfying $\delta f \circ i_X = f$.) Since it is a matter of routine to verify (iv), it only remains to prove that δX is D-complete. We do this by showing that $\mathscr{B} = \{F^* \cap \delta X \mid F \in \mathscr{D}(X)\}$ is

(41) The easiest way to prove this theorem would be to consider the G_δ-closure of the image of X under the natural embedding of X into ${}^{C(X,D_1)}D_1$. But we prefer another approach which provides more insight into the structure of δX.

a G_δ-base for the closed sets of δX with the property that every $\mathscr{B}$-ultrafilter with cip is fixed (7.6(v); evidently $\mathscr{B}$ is closed with respect to countable intersections).

To this end consider an arbitrary $F \in \mathscr{D}(X)$. There exists a sequence $(F_n)_{n<\omega}$ in $\mathscr{D}(X)$ such that $X \backslash F = \bigcup \{F_n \mid n < \omega\}$. Obviously, $\bigcup \{F_n^* \cap \delta X \mid n < \omega\} \subset \delta X \backslash F^*$. Conversely, if $\mathscr{F} \in \delta X \backslash F^*$, there is an $n < \omega$ such that $\mathscr{F} \in F_n^* \cap \delta X$, for otherwise there would exist a $G \in \mathscr{F}$ and $G_n \in \mathscr{F}$ such that $G \cap F = \emptyset$ and $G_n \cap F_n = \emptyset$ for each $n < \omega$. But then $G \cap \bigcap \{G_n \mid n < \omega\} = \emptyset$, contradicting the fact that $\mathscr{F}$ has the cip. It follows that $\delta X \backslash F^* = \bigcup \{F_n^* \cap \delta X \mid n < \omega\}$, which proves that $\mathscr{B}$ is a G_δ-base.

Now, if $\mathscr{H}^*$ is a $\mathscr{B}$-ultrafilter with cip, then $\mathscr{F} = \{F \in \mathscr{D}(X) \mid F^* \cap \delta X \in \mathscr{H}^*\}$ is a $\mathscr{D}(X)$-ultrafilter with cip such that $F \in \bigcap \{H \mid H \in \mathscr{H}^*\}$, which completes the proof. ∎

If X is a D-completely regular space, we will henceforth not distinguish between X and $i_X[X]$, i.e. we will always assume that X itself is a G_δ-dense subspace of δX. δX will be called the *D-completion* of X. Since for a normal T_1-space the collection of D-closed subsets is identical with the collection of zero-sets (7.7), the proof of the preceding theorem yields:

7.14. COROLLARY. *For a normal T_1-space X, δX is identical with the Hewitt realcompactification vX of X.* ∎

However, the Isbell–Mrówka space Ψ (Gillman and Jerison [1960], 5 I) shows that for a completely regular T_1-space X it may occur that $\delta X \neq vX$. But there is always a continuous mapping from δX into vX leaving the points of X fixed.

Our next theorem clarifies under which circumstances the D-completion is compact.

7.15. THEOREM. *For a D-completely regular space $X = (X, \tau)$ the following conditions are equivalent:*

(i) *δX is D-compact.*

(ii) *Every countable D-open cover of X has a finite subcover.*

(iii) *(X, μ_ω) is totally bounded*(42) (2.13).

(iv) *Every $\mathscr{D}(X)$-ultrafilter has the* cip.

(v) *There exists a D-compact space αX and a continuous mapping $j_X\colon X \to \alpha X$ such that $j_X[X]$ is G_δ-dense in αX, X is homeomorphic to $j_X[X]$, and the following condition is satisfied:*

(α) *Whenever f is a continuous mapping from X into a D-compact space Y, there exists a unique continuous mapping αf from αX into Y such that $\alpha f \circ j_X = f$.*

Proof. (i) implies (ii): Suppose there is a D-open cover $(U_n)_{n<\omega}$ of X without finite subcover. Then $\{\mathrm{cl}_{\delta X}(X \backslash U_n) \mid n < \omega\}$ is a collection of closed

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(42) A nearness space (X, μ) is said to be *totally bounded*, if for each $\mathscr{U} \in \mu$ there is a finite $\mathscr{V} \in \mu$ which refines $\mathscr{U}$.

subsets of δX with the finite intersection property such that $\bigcap\{\mathrm{cl}_{\delta X}(X \setminus U_n) \mid n < \omega\} = \emptyset$ (7.12), contradicting our assumption that δX is compact.

(ii) implies (iii): Consider an arbitrary $\mathscr{U} \in \mu_\omega$. There exists a countable kernel-normal open cover $\mathscr{V}$ of X which refines $\mathscr{U}$. By virtue of Lemma 2.11 we can find a developable topology $\tau' \subset \tau$ such that $\{\mathrm{int}_{\tau'} V \mid V \in \mathscr{V}\}$ is a cover of X. Assuming (ii) there is a $\mathscr{V}' \in [\mathscr{V}]^{<\omega}$ such that $\{\mathrm{int}_{\tau'} V \mid V \in \mathscr{V}'\}$ covers X. Since $\{\mathrm{int}_{\tau'} V \mid V \in \mathscr{V}'\}$ belongs to μ_ω, the argument is complete.

(iii) implies (iv): Let $\mathscr{F}$ be a $\mathscr{D}(X)$-ultrafilter. If there were an $\mathscr{F}' \in [\mathscr{F}]^{\leqslant\omega}$ such that $\bigcap\{F \mid F \in \mathscr{F}'\} = \emptyset$, then $\{X \setminus F \mid F \in \mathscr{F}'\}$ would be a member of μ_ω without finite subcover.

If X satisfies (iv), then the proof of Theorem 7.13 shows that $\delta X = X^*$, where X^* is the Wallman–Frink compactification of X with respect to $\mathscr{D}(X)$. Hence $\alpha X = \delta X$ has all properties mentioned in (v).

(v) implies (i): By virtue of 7.13(iv) it suffices to show that for αX and j_X condition (iii) of Theorem 7.13 is satisfied. To this end consider a continuous mapping f from X into a D-complete space Y. We may assume that Y is a subspace of a D-compact space Y' (2.8). Assuming (v) there exists a unique continuous mapping αf from αX into Y' such that $\alpha f \circ j_X = f$. It follows that

$$\alpha f[\alpha X] = \alpha f[\mathrm{cl}^\delta_{\alpha X} j_X[X]] \subset \mathrm{cl}^\delta_{Y'} \alpha f[j_X[X]] = \mathrm{cl}^\delta_{Y'} f[X] \subset \mathrm{cl}^\delta_{Y'} Y.$$

Since a slight modification of the argument used to prove Lemma 7.5 shows that $\mathrm{cl}^\delta_{Y'} Y = Y$, the proof is complete. ■

We call a D-completely regular space *D-pseudocompact* if it has one — and therefore all — of the properties in the preceding theorem.

7.16. COROLLARY. (i) *A normal T_1-space is D-pseudocompact if and only if it is pseudocompact.*

(ii) *Every countably compact D-completely regular space is D-pseudocompact.*

(iii) *A countably D-paracompact T_1-space* (5.12(f)) *is D-pseudocompact if and only if it is countably compact.*

(iv) *A D-completely regular space is D-compact if and only if it is D-complete and D-pseudocompact.* ■

Let us mention without proof a characterization of those D-completely regular spaces for which the D-completion is Lindelöf.

7.17. THEOREM. *For a D-completely regular space X the following conditions are equivalent:*

(i) *δX is a Lindelöf space.*

(ii) *For every $\mathscr{D}(X)$-filter $\mathscr{F}$ with* cip *there exists a $\mathscr{D}(X)$-ultrafilter $\mathscr{G}$ with* cip *such that $\mathscr{F} \subset \mathscr{G}$.* ■

Our final theorem provides the analogue, for D-completely regular spaces, of the Dieudonné completion.

7.18. THEOREM. *For every D-completely regular space $X = (X, \tau)$ there* *exists a space* εX *and a continuous mapping* i_X: $X \to \varepsilon X$ *such that the following conditions are satisfied*:

(i) εX *is an inverse limit of developable* T_1*-spaces.*

(ii) X *is homeomorphic to* $i_X[X]$.

(iii) $i_X[X]$ *is* G_δ*-dense in* εX.

(iv) *Whenever* f *is a continuous mapping from* X *into an inverse limit* Y *of developable* T_1*-spaces, there exists a unique continuous mapping* εf *from* εX *into* Y *such that* $\varepsilon f \circ i_X = f$.

(v) εX *and* i_X *are unique, in the following sense: Whenever* ηX *is an inverse limit of developable* T_1*-spaces and* j_X: $X \to \eta X$ *is a continuous mapping such that the analogues of* (ii)–(iv) *are satisfied, there exists a homeomorphism* h: $\varepsilon X \to \eta X$ *such that* $h \circ i_X = j_X$.

Proof. As in the proof of Theorem 7.13 we consider the Wallman–Frink compactification X^* of X with respect to $\mathscr{D}(X)$. Our plan is to show that the following subspace of $\delta X = \{\mathscr{F} \in X^* \mid \mathscr{F}$ has the cip$\}$ has the desired properties:

$$\varepsilon X = \{\mathscr{F} \in \delta X \mid \bigcap \{F \mid F \in \mathscr{F}\} \neq \emptyset\} \cup \{\mathscr{F} \in \delta X \mid \bigcap \{F \mid F \in \mathscr{F}\} = \emptyset \text{ and } \{X \backslash F \mid F \in \mathscr{F}\} \text{ is not kernel-normal}\}.$$

Of course, we let i_X: $X \to \varepsilon X$ be the natural embedding of X into X^*, i.e. $i_X(x) = \{F \in \mathscr{D}(X) \mid x \in F\}$ for each $x \in X$. By the proof of Theorem 7.13 it is clear that εX is D-completely regular and that (ii) and (iii) above are satisfied. Since it is a matter of routine to verify (v), it only remains to prove (i) and (iv).

For the proof of (i) recall that $\mathscr{B} = \{F^* \cap \varepsilon X \mid F \in \mathscr{D}(X)\}$ is a G_δ-base for the closed sets of εX which is closed with respect to the formation of countable intersections, where $F^* = \{\mathscr{F} \in X^* \mid F \in \mathscr{F}\}$. Let $\mathscr{H}^*$ be a $\mathscr{B}$-ultrafilter on εX with cip such that $\bigcap \{H^* \mid H^* \in \mathscr{H}^*\} = \emptyset$. By virtue of 7.4(iv) it suffices to show that $\{\varepsilon X \backslash H^* \mid H^* \in \mathscr{H}^*\}$ is kernel-normal. To this end note that $\mathscr{F} = \{F \in \mathscr{D}(X) \mid F^* \cap \varepsilon X \in \mathscr{H}^*\}$ is a $\mathscr{D}(X)$-ultrafilter with cip. Since $\bigcap \{H^* \mid H^* \in \mathscr{H}^*\} = \emptyset$, it follows that $\mathscr{F} \in \delta X \backslash \varepsilon X$. Hence $\{X \backslash F \mid F \in \mathscr{F}\}$ is kernel-normal. By Lemma 2.11 there exists a developable topology $\tau' \subset \tau$ and a τ'-open cover $\{U_F \mid F \in \mathscr{F}\}$ such that $U_F \subset X \backslash F$ for each $F \in \mathscr{F}$ Let π: $(X, \tau') \to X'$ be the T_0-reflection of (X, τ'). There exists a unique continuous mapping $g = \delta(\pi)$ from δX into $\delta X'$ such that $g \circ i_X = i_{X'} \circ \pi$, where $\delta X'$ is the set of all $\mathscr{D}(X')$-ultrafilters with cip, considered as a subspace of the Wallman–Frink compactification of X' with respect to $\mathscr{D}(X')$ and $i_{X'}$: $X' \to \delta X'$ is the natural embedding (7.13(iii)). It is easily seen that

$$g(\mathscr{G}) = \{G \in \mathscr{D}(X') \mid \pi^{-1}[G] \in \mathscr{G}\} \qquad \text{for each } \mathscr{G} \in \delta X.$$

Moreover, the following holds:

A. $g[\varepsilon X] \subset i_{X'}[X']$.

For suppose that there is a $\mathscr{G}\in\varepsilon X$ such that $g(\mathscr{G})\notin i_{X'}[X']$. Then $\{\pi^{-1}[X'\backslash G]\mid G\in\mathscr{D}(X'),\ \pi^{-1}[G]\in\mathscr{G}\}$ is a kernel-normal open cover of X which refines $\{X\backslash G\mid G\in\mathscr{G}\}$, contradicting the fact that $\mathscr{G}\in\varepsilon X$.

Because of A, $\mathscr{U}=\{g^{-1}[i_{X'}\circ\pi[U_F]]\cap\varepsilon X\mid F\in\mathscr{F}\}$ is a kernel-normal open cover of εX. Hence the proof of (i) is complete once we have shown that $\mathscr{U}$ refines $\{\varepsilon X\backslash H^*\mid H^*\in\mathscr{H}^*\}$. For this purpose it suffices to show that $g^{-1}[i_{X'}\circ\pi[U_F]]\cap\varepsilon X\subset\delta X\backslash F^*$ for each $F\in\mathscr{F}$ Suppose there is an $F\in\mathscr{F}$ and a $\mathscr{G}\in g^{-1}[i_{X'}\circ\pi[U_F]]\cap\varepsilon X$ such that $\mathscr{G}\in F^*$. Then there is an $x\in U_F$ such that

$$\{G\in\mathscr{D}(X')\mid\pi^{-1}[G]\in\mathscr{G}\}=\{G\in\mathscr{D}(X')\mid\pi(x)\in G\}.$$

On the other hand, $F\in\mathscr{G}$, $X'\backslash\pi[U_F]\in\mathscr{D}(X')$, and $F\subset X\backslash U_F$ imply that $\pi(x)\in X'\backslash\pi[U_F]$ – a contradiction!

For the proof of (iv) consider a continuous mapping f from X into a space Y which is an inverse limit of developable T_1-spaces. Let $\delta f\colon\delta X\to\delta Y$ be the unique continuous mapping satisfying $\delta f\circ i_X=i_Y\circ f$ (7.13(iii)), where again we assume that δY is the subspace of the Wallman–Frink compactification of Y with respect to $\mathscr{D}(Y)$ consisting of all $\mathscr{D}(Y)$-ultrafilters with cip, and i_Y is the corresponding embedding of Y into δY. Then

$$\delta f(\mathscr{G})=\{A\in\mathscr{D}(Y)\mid f^{-1}[A]\in\mathscr{G}\}\text{ for each }\mathscr{G}\in\delta X.$$

An obvious modification of the argument used to prove Claim A shows that $\delta f[\varepsilon X]\subset i_Y[Y]$. Hence $\varepsilon f=(i_Y)^{-1}\circ\delta f\restriction\varepsilon X$ satisfies $\varepsilon f\circ i_X=f$. Since it is easily seen that there is at most one continuous mapping from εX into Y with this property, the proof is complete. ■

7.19. Remarks. (a) The main results of this section have a nice categorical interpretation. Let us call a topological space X a *G_δ-Hausdorff space* if for every pair x, y of distinct points in X there exists a pair G_x, G_y of disjoint G_δ-sets such that $x\in G_x$ and $y\in G_y$. Moreover, let G_δ-HAUS be the full subcategory of the category TOP of topological spaces consisting of all G_δ-Hausdorff spaces. By an obvious modification of the well-known argument used to characterize the epimorphisms in the category HAUS of Hausdorff spaces it can be shown that a morphism $f\colon X\to Y$ in G_δ-HAUS is an epimorphism if and only if $f[X]$ is G_δ-dense in Y.(44) Therefore Theorem 7.4 is nothing but a characterization of the objects of the epireflective hull of the class of developable T_1-spaces in the category G_δ-HAUS, i.e. the spaces belonging to $\mathrm{EH}_{G_\delta\mathrm{HAUS}}(\mathbf{D})$. And Theorem 7.6 shows that the epireflective hull of the class of second countable developable T_1-spaces in G_δ-HAUS is simply generated by $\boldsymbol{D}_1$ and consists of all D-complete spaces. Moreover, Theorems 7.13 respec-

(44) For background in category theory respectively categorical topology we refer to (Herrlich [1968], [1971], [1983]).

tively 7.18 provide descriptions of the corresponding epireflections. We find it interesting to note that in this context the well-known characterization of realcompact spaces due to Mrówka [1957] shows that the class of realcompact spaces is the epireflective hull of the class of compact Hausdorff spaces in G_δ-HAUS, i.e. we have:

E	$EH_{HAUS}(E)$	$EH_{G_\delta HAUS}(E)$	$EH_{TOP}(E)$
compact Hausdorff spaces	compact Hausdorff spaces (Tikhonov [1930])	realcompact spaces (Mrówka [1957])	completely regular T_1-spaces (Tikhonov [1930])
[0, 1]	compact Hausdorff spaces	realcompact spaces	completely regular T_1-spaces
$\boldsymbol{R}$	realcompact spaces (Hewitt [1948])	realcompact spaces	completely regular T_1-spaces
metrizable spaces	Dieudonné complete spaces (Dieudonné [1939])	Dieudonné complete spaces	completely regular T_1-spaces
developable T_1-spaces	—	Theorem 7.4	D-completely regular spaces (Theorem 2.2)
$\boldsymbol{D}_1$	—	D-complete spaces (Theorem 7.6)	D-completely regular spaces

(b) It is worth mentioning that Proposition 7.11 can be used to prove that every D-compact space of weight $\varkappa$ is a continuous image of some G_δ-closed subspace of ${}^\varkappa 2$.

(c) Note that a D-completely regular space X is an inverse limit of developable T_1-spaces if and only if for each $x \in \delta X \setminus X$ there is a continuous mapping f from X into some developable T_1-space which cannot be continuously extended to $X \cup \{x\}$. ■

We conclude by mentioning three interesting problems.

Problem Q. Characterize internally those D-completely regular spaces X which have the property that $\delta(X \times Y) = \delta X \times \delta Y$ for each D-completely regular space Y, where $\delta(X \times Y) = \delta X \times \delta Y$ means that $X \times Y$ is D-embedded in $\delta X \times \delta Y$.

Problem R. Characterize internally those D-completely regular spaces X which have the property that $\varepsilon(X \times Y) = \varepsilon X \times \varepsilon Y$ for each D-completely regular space Y.

Problem S. Characterize internally those topological spaces which are homeomorphic to the limit of an inverse system of quasi-metrizable spaces.

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