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PROJECTIVE REPRESENTATIONS OF THE GENERALIZED SYMMETRIC GROUPS

J. F. HUMPHREYS

Department of Pure Mathematics, University of Liverpool
Liverpool, England

1. Definitions

The *generalized symmetric groups* are the wreath products $C_m \wr S_n$, where C_m denotes a cyclic group of order m and S_n is the symmetric group on $\{1, \ldots, n\}$. The case $m = 2$ is of special interest since the groups arising in that case are the finite reflection groups of type B_n.

A (complex) *projective representation of degree d* of a group G is a map $P\colon G \to \mathrm{GL}(d, C)$ such that

(i) $P(1_G) = I_d$; and

(ii) for all x, y in G, there is an element $\alpha(x, y)$ in $\mathbf{C}\setminus\{0\}$ such that

$$P(x)P(y) = \alpha(x, y)P(xy).$$

If $\alpha(x, y) = 1$ for all x, y, we say that P is a linear representation of G.

The map $\alpha\colon G \times G \to \mathbf{C}\setminus\{0\}$ satisfies the conditions

(i) $\alpha(1, g) = 1 = \alpha(g, 1)$ for all g in G, and

(ii) $\alpha(x, yz)\alpha(y, z) = \alpha(x, y)\alpha(xy, z)$ for all x, y, z in G, so that α is a 2-cocycle. There is an equivalence relation on 2-cocycles: $\alpha \sim \beta$ if and only if there exists a map $\delta\colon G \to \mathbf{C}\setminus\{0\}$ such that for all x, y in G

$$\alpha(x, y) = \delta(x)\delta(y)\delta(xy)^{-1}\beta(x, y).$$

The equivalence classes of 2-cocycles form a group $M(G)$ under multiplication of values. This group is the Schur multiplier and is finite if G is finite.

In 1904, Schur [4] established the existence of a representation group H for any finite group G. This is a group with the properties:

(i) H has a subgroup A with $A \leqslant Z(H) \cap [H, H]$;

(ii) $H/A \cong G$; and

(iii) $|A| = |M(G)|$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Schur also showed that any projective representation of G can be lifted to a linear representation of H.

The Schur multiplier of $C_m \wr S_n$ has been calculated by Davies and Morris [2]. It is an elementary abelian 2-group of rank $k(m, n)$ where

$$k(m,n) = \begin{cases} 3 & \text{if } m \text{ is even and } n \geq 4, \\ n-1 & \text{if } m \text{ is even and } n \leq 3, \\ 1 & \text{if } m \text{ is odd and } n \geq 4, \\ 0 & \text{if } m \text{ is odd and } n \leq 3. \end{cases}$$

Thus the projective representations of $C_m \wr S_n$ are determined by the linear representations of various "double covers" G of $C_m \wr S_n$

$$1 \to C_2 \to G \to C_m \wr S_n \to 1.$$

In an irreducible representation of degree d of such a group G, by Schur's lemma, the generator of C_2 is represented either as I_d or $-I_d$. In the latter case, the representation is said to be negative. The irreducible nonlinear projective representations of $C_m \wr S_n$ therefore correspond to irreducible negative representations of various double covers of $C_m \wr S_n$.

2. A general construction for double covers

Let $\mathcal{G}$ be the category whose objects are triples (G, z, σ) where G is a finite group, z is a central involution in G and $\sigma: G \to \mathbf{Z}/2$ is a homomorphism with $\sigma(z) = 0$. A morphism between (G_1, z_1, σ_1) and (G_2, z_2, σ_2) is a group homomorphism $\theta: G_1 \to G_2$ such that $\theta(z_1) = z_2$ and with $\sigma_2(\theta(g_1)) = \sigma_1(g_1)$ for all g_1 in G_1. Thus G is a double cover for $\bar{G} = G/\{1, z\}$.

Given N, H in $\mathcal{G}$ and a homomorphism $\theta: H \to \mathrm{Aut}_{\mathcal{G}} N$, such that $\theta(z)$ is the identity. Let $N \hat{\times}_\theta H$ be the Cartesian product $N \times H$ with multiplication

$$(n, h)(n_1, h_1) = (n\theta(h)(n_1), z^{\sigma(h)\sigma(n_1)} hh_1).$$

It may be checked that $N \hat{\times}_\theta H$ is then a group with $\{(1, 1), (1, z), (z, 1), (z, z)\}$ a central subgroup. Let Z be the central subgroup $\{(1, 1), (z, z)\}$ and put

$$N \hat{\curlyvee}_\theta H = N \hat{\times}_\theta H/Z.$$

This is given the structure of an object in $\mathcal{G}$ by setting $z = (1, z)Z$ and

$$\sigma(n, h) = \sigma(n) + \sigma(h).$$

If $\bar{\theta}$ is the homomorphism $\bar{H} \to \mathrm{Aut}(\bar{N})$ produced in the obvious way from θ, then $N \hat{\curlyvee}_\theta H$ is a double cover of the semi-direct product $\bar{N} \times_{\bar{\theta}} \bar{H}$.

3. An application of the construction

From now on we take m to be even. Let A be the direct product $C_m \times \{1, z\}$. Regard A as an object in $\mathscr{G}$ by choosing a homomorphism σ: $A \to Z/2$ with $\sigma(z) = 0$. Now iterate the construction of Section 2 on A n times to produce a group N which is a double cover of C_m^n. (The maps θ are all taken to be the trival map $h \to \mathrm{id}_N$).

Next let H be the group $\tilde{S}_n(\varepsilon)$ generated by $z, t_1, \ldots, t_{n-1}$ subject to the relations

$$\begin{aligned} z^2 &= 1, & zt_i &= t_i z, & & 1 \leqslant i \leqslant n-1, \\ t_i^2 &= 1, & & & & 1 \leqslant i \leqslant n-1, \\ (t_i t_{i+1})^3 &= 1, & & & & 1 \leqslant i \leqslant n-2, \\ t_i t_j &= z^\varepsilon t_j t_i, & |i-j| &\geqslant 2 & \text{and} \quad & 1 \leqslant i, j \leqslant n-1. \end{aligned}$$

Thus $S_n(0)$ is the direct product $S_n \times \{1, z\}$ while $\tilde{S}_n(1)$ is one of the representation groups of S_n constructed by Schur if $n \geqslant 4$. In either case, H is regarded as an object in $\mathscr{G}$ by taking $\sigma(z)$ to be 0 and $\sigma(t_i) = \varepsilon$.

One further homomorphism α: $A \to Z/2$ is used to specify the map θ: $H \to \mathrm{Aut}_{\mathscr{G}}(N)$. Thus θ maps t_i to τ_i where

$$\tau_i(g_1, \ldots, g_n) = z^{n_i}(g_1, \ldots, g_{i-1}, g_{i+1}, g_i, \ldots, g_n)$$

with

$$n_i = \sigma(g_i)\sigma(g_{i+1}) + \sum_{k \neq i, i+1} \alpha(g_k).$$

The construction of Section 2 is then applied to yield a group $Y_n(\alpha, \sigma, \varepsilon)$. There are therefore eight groups arising from the possible choices of α, σ and ε and these are precisely the eight double covers of $C_m \wr S_n$.

When $\alpha = 0$, we have the important "Young" property that $Y_k(0, \sigma, \varepsilon)\, \hat{\mathbf{Y}}_{\theta=1}\, Y_l(0, \sigma, \varepsilon)$ is a subgroup of $Y_{k+l}(0, \sigma, \varepsilon)$.

4. Representation theory

For G in $\mathscr{G}$, let $T^1(G)$ be the Grothendieck group generated by the finite dimensional negative representations. We also consider $Z/2$-graded negative representations of G. These are pairs $\{V_0, V_1\}$ of finite-dimensional vector spaces such that $V_0 \oplus V_1$ is a negative representation of G and also that $gV_i \subseteq V_{i+\sigma(g)}$. Let $T^*(G)$ be the $Z/2$-graded group $T^0(G) \otimes T^1(G)$.

Now define L to be the ring $Z[\lambda]/(\lambda^3 - 2\lambda)$. A $Z/2$-grading is determined on L by requiring that $\lambda \in L^1$. As abelian groups, L^1 has basis λ and L^0 has basis $\{1, \varrho\}$ where $\varrho = \lambda^2 - 1$. In fact $T^*(G)$ is a $Z/2$-graded L-module. The following result was proved in [3].

THEOREM. *Given A, B in $\mathcal{G}$, there is an isomorphism*

$$T^*(A) \otimes_L T^*(B) \to T^*(A \hat{Y} B).$$

As a consequence of this, the graded ring $\bigoplus_{n \geq 1} T^*(Y_n(0, \sigma, \varepsilon))$ has a multiplication (induction product). It also has a comultiplication (arising from restrictions) together with a positivity property (elements corresponding to irreducible representations) and a self-adjointness property (from inner products). We therefore have the following result.

THEOREM. *The algebra $\bigoplus_{n \geq 1} T^*(Y_n(0, \sigma, \varepsilon))$ is an L-PSH algebra.*

[The notion of an L-PSH algebra was introduced in [1]. It is a natural generalization of the concept of Z-PSH algebra as introduced by Zelevinsky [5]. The standard example of a Z-PSH algebra is the graded algebra of Grothendieck groups of linear representations of S_n].

In [1], Bean and Hoffman have given a classification of L-PSH algebras analogous to Zelevinsky's classification of Z-PSH algebras. In the case of L-PSH algebras, each is a tensor product of "atoms". There are four types of atom, each of which is realized in one of our four algebras. Two of our four algebras are atomic PSH-algebras, and the other two are both tensor products of two atoms. In each case the algebra structure is known and information such as the rank of each algebra in dimension n may easily be calculated.

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