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ASSOCIATE FORMS, JOINS, MULTIPLICITIES AND AN INTRINSIC ELIMINATION THEORY

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The first time a mathematician hears about "multiplicity" $m_r \in \mathbf{Z}^+$ refers to an m_r-ple root of a polynomial $f(x)$ or binary form $\phi(x_0, x_1)$:

$$(0.0) \quad f(x) = a_0 \prod_{r \in \mathbf{P}_1(\mathbf{C})} (x-r)^{m_r}, \quad \phi(x_0, x_1) = \prod_{r \in \mathbf{P}_1(\mathbf{C})} \begin{vmatrix} x_0 & x_1 \\ r_0 & r_1 \end{vmatrix}^{m_r}, \quad m_r \in \mathbf{Z},\ m_r \geqslant 0$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

$x-r$ (or $\begin{vmatrix} x_0 & x_1 \\ r_0 & r_1 \end{vmatrix}$) is the trivial associate form (a.f.) (cf. Def. 4.6) of the point r (with affine or projective coordinates $x(x_0, x_1)$ in the complex affine (or projective) line). It is natural to ask whether or not this *exponent* is also the natural *intersection multiplicity* of an irreducible component I in the proper intersection $V \cap W$ (V, W irreducible a.v. in $\mathbf{P}_n(\mathbf{C}) = \mathbf{P}(E)$). An affirmative answer is found in [vdW1] only for two irreducible plane curves. This idea of the exponent intersection multiplicity is developed in this paper in the general case by showing that the form

$$F_{V \cdot W} = \prod F_I^{m_I} \quad (I \text{ proper irreducible component of } V \cap W)$$

can be computed by *restriction* of the F_J (associated to the join $J(V \times W) \subset \mathbf{P}(E \oplus E)$, see Def. 1.1, cf. [G1]) to the diagonal subspace (see § 1, 3) $\Delta \subset \mathbf{P}(E \oplus E)$. The method extends naturally to h ($\geqslant 2$) a.v. $V^{(j)} \subset \mathbf{P}_n$, $j = 1, \ldots, h$ (cf. § 2) provided $c = \sum c_j \leqslant n$ ($c_j = \operatorname{codim} V^{(j)}$ in $\mathbf{P}_n$). The geometric interpretation of F_V in terms of the complex $\mathfrak{C}(V^c) = \{\mathbf{P}_{c-1} \subset \mathbf{P}_n | \mathbf{P}_{c-1} \cap V^c \neq \emptyset\}$ leads naturally to an equivalence of the exponent multiplicity with van der Waerden's theory (cf. § 10), [vdW1], [vdW-ZAG].

Since $c = \operatorname{codim} J$ in $\mathbf{P}(E \oplus \ldots \oplus E)$ (h copies of E) a natural discussion arises also in the case $c > n$. Then the old elimination theory (too much discredited because of its heavy dependence on coordinates) can be replaced by intrinsic constructions, cf. § 13, 14. Natural applications are made to Bézout's theorem as well to possible future relations with the "length multiplicity" (cf. Vogel's report here), [vdW-ZAG], [Grö 1, 2].

0. Introduction

Most of the algebraic varieties needed in this paper will be embedded in a fixed complex projective space $\mathbf{P}_n(\mathbf{C}) = \mathbf{P}(E_{n+1}) = E - \{0\}/\mathbf{C}^\times$, with $E = E_{n+1}$ $(n+1)$-dimensional $\mathbf{C}$-vector space. The projection P: $E - \{0\} \to \mathbf{P}_n(\mathbf{C})$ will be denoted also by $\mathbf{P}$ although for a given $v \in E - \{0\}$ we write simply $\mathbf{P}(v) = (v) = (\lambda v)$ ($\forall \lambda \in \mathbf{C}^\times$).

Let V, W be two irreducible algebraic varieties of $\mathbf{P}_n(\mathbf{C})$ meeting properly. Let

$$(0.1) \qquad F_{V \cdot W} = \prod_{C = C_d \subset V \cap W} F_C^{i(V, W; C)}$$

be the associate form (a.f.) of the intersection cycle

$$(0.2) \qquad V \cdot W = \sum_{C = C_d \subset V \cap W} i(V, W; C) C_d$$

where F_C is the (irreducible) a.f. to the irreducible component C_d of $V \cap W$. *The intersection multiplicities $i(V, W; C)$ are uniquely determined as the exponents* *in the prime factor decomposition of* $F_{V\cdot W}$; this remark is useless if there is no way of computing intrinsically $F_{V\cdot W}$ in terms of V and W (i.e. F_V and F_W). This paper shows that *actually* $F_{V\cdot W}$ *is uniquely and intrinsically determined in a natural way by restriction to the diagonal space* $\Delta \subset \mathbf{P}(E\oplus E)$ of the F_J associated to the join $J = J(V\times W) \subset \mathbf{P}(E\oplus E)$ of V and W (§ 1), Def. 1.1. More precisely we have

$$F_{V\cdot W} = \delta^{-1}(F_J|\Delta) \tag{0.3}$$

(cf. § 5) where δ: $\mathbf{P}_n \subsetneq \mathbf{P}(E\oplus E)$ is defined by

$$\delta(x) = ((x, x)) \tag{0.4}$$

for any $x\in E-\{0\}$, $(x) = \mathbf{P}(x)\in\mathbf{P}(E)$, $((x, x))\in\mathbf{P}(E\oplus E)$; δ is the *diagonal injection* and $\Delta = \delta(\mathbf{P}_n)$ is the *diagonal space* (cf. § 3). The a.f. F_J of $J(V\times W)$ *is actually intrinsically determined by standard methods* (cf. § 6).

The construction can be extended in several ways:

(a) If $V\cap W$ is improper, (0.3) is meaningless since $V\cdot W$ is not defined as a cycle, so $F_{V\cdot W}$ is not defined. However the right hand side of (0.1) is always defined and we have

$$\delta^{-1}(F_J|\Delta) = 0, \quad (J = J(V\times W)) \tag{0.3$'$}$$

iff $V\cap W$ *is improper*. Notice that

$$V\cap W \text{ improper} \Leftrightarrow J(V\times W)\cap\Delta \text{ improper}.$$

(b) The construction is valid also for finitely many irreducible varieties $V^{(j)}$ denoted sometimes also by

$$V_{d_j} = V^{c_j} \subset \mathbf{P}_n, \quad d_j+c_j = n,\ j = 1, 2, \ldots, h, \tag{0.5}$$

where we use a double notation $V = V_d = V^c$ for an irreducible $V\subset \mathbf{P}_n$ if there is no ambiguity, where the subscript d indicates the dimension and the superscript c the codimension of V in $\mathbf{P}_n$ $(d+c = n)$.

The join

$$J = J(V^{c_1}\times V^{c_2}\times \ldots \times V^{c_n}) \subset \mathbf{P}(E\oplus\ldots\oplus E), \quad (h \text{ copies of } E) \tag{0.6}$$

(cf. § 1, Def. 1.1) is also irreducible of codimension c in $\mathbf{P}(E\oplus\ldots\oplus E)$. This ambient space of J can be identified with $\mathbf{P}(E\otimes\mathbf{C}^h)$ where the jth direct summand $(0, \ldots, E, \ldots)$ is identified with $E\otimes e_j$, where $e_j = (0, \ldots, 1, \ldots, 0)$, 1 on jth place.

The set-theoretic intersection

$$\bigcap_{j=1}^{h} V^{c_j}$$

does always exist provided

$$c = c_1+c_2+\ldots+c_h \leqslant n. \tag{0.7}$$

(We shall assume (0.7) in the first part of Chapter III.). Then we have

$$\text{codim} \bigcap_{j=1}^{h} V^{c_j} \leqslant n \tag{0.8}$$

and this *intersection is proper* (i.e. its codimension equals c) *iff* $J \cap \Delta$ *is proper in* $\mathbf{P}(E \otimes \mathbf{C}^h)$, because if C runs through the set of irreducible components of the intersection, $\delta(C)$ runs through the set of all irreducible components of $J \cap \Delta$ and $\dim C = \dim \delta(C)$. Then (0.3), (0.3)$'$ can be extended to an arbitrary $h \geqslant 2$ as indicated by the following:

THEOREM 1. *We have* $\delta^{-1}(F_J|\Delta) = 0$ *iff the set-theoretic intersection* $\bigcap_{j=1}^{h} V^{c_j}$ *is improper. Otherwise the intersection cycle* $I = V^{c_1} \cdot V^{c_2} \cdot \ldots \cdot V^{c_h}$ *is well defined in* $\mathbf{P}_n$ *and we have*

$$F_I = \delta^{-1}(F_J|\Delta), \quad \Delta = \delta(P_n). \tag{0.9}$$

The associate form F_J *can be determined by* $V^{c_1}, V^{c_2}, \ldots, V^{c_h}$ *(i.e. by* $F_{V^{(j)}}$, $c_j, j = 1, 2, \ldots, h$*) in the standard way (cf. § 6) for any* h *as well as in the case* $h = 2$.

There are several versions of the associate forms attached to a given pure cycle $V^c \subset \mathbf{P}_n$ (and for each one the restriction symbol $F|\Delta$ appearing in (0.9) has a natural meaning); on the other hand all of them lead to the same intersection-multiplicities. But we shall use only the following three versions of the a.f. (cf. § 5,6):

(0.10) $(x_1, x_2, \ldots, x_c) \mapsto S(x_1, x_2, \ldots, x_c)$ (Cayley–Severi), [C], [P], [S],

(0.11) $(u_1, u_2, \ldots, u_{d+1}) \mapsto Y(u_1, u_2, \ldots, u_{d+1})$
(van der Waerden–Chow), [Ch-vdW],

(0.12) $(v_1, v_2, \ldots, v_{d+2}; x) \mapsto N(v_1, v_2, \ldots, v_{d+2}; x)$
(Barsotti–Weil–Siegel), [Ba], [W], [Si]

where $x_i \in E$, $u_j \in \check{E} = \text{Hom}_{\mathbf{C}}(E, \mathbf{C})$ (cf. § 6) and they are defined up to a proportionality factor $\lambda \in \mathbf{C}^\times$. It suffices to define them first for an irreducible V^c and then to extend to the general Γ^c by "prime factor decomposition". All of them can be defined in terms of a complex defined as follows.

DEFINITION 0.1. The complex $\mathfrak{C}(V^c)$ of $(c-1)$*-dimensional projective subspaces attached to an irreducible* V^c is given by

$$\mathfrak{C}(V^c) = \{\mathbf{P}_{c-1} \subset \mathbf{P}_n | \mathbf{P}_{c-1} \cap V^c \neq \varnothing\} \quad \text{(cf. § 4, 5).} \tag{0.13}$$

In fact $\mathfrak{C}(V^c)$ is represented by an irreducible subvariety of codimension one in the Grassmannian $\mathscr{G}(c-1; n)$. Furthermore V^c is recovered from $\mathfrak{C}(V^c)$ as the locus of singular points of $\mathfrak{C}(V^c)$ (cf. § 6).

The proof of our Theorem I is a consequence of the following fact: if $\bigcap_{j=1}^{h} V^{c_j}$ is improper then the restriction of the complex $\mathfrak{C}(J)$ attached to J

$$\mathfrak{C}(J(V^{c_1} \times \ldots \times V^{c_h})) = \{\mathbf{P}_{c-1} \subset \mathbf{P}(E \otimes \mathbf{C}^h) | \mathbf{P}_{c-1} \cap J \neq \varnothing\}$$

to the diagonal space Δ:

(0.14) $$\mathfrak{C}(J)|\Delta = \{\mathbf{P}_{c-1} \subset \Delta | \mathbf{P}_{c-1} \in \mathfrak{C}(J)\}$$

is the full Grassmannian $\mathscr{G}(c-1; \Delta)$ because if $C^{c'}$ ($c' < c$) is an excedentary irreducible component of the intersection $\bigcap_{j=1}^{h} V^{(j)}$ then every subspace $\mathbf{P}_{c-1}$ of Δ meets the diagonal image $\delta(C^{c'})$.

If the previous intersection is proper the restriction $\mathfrak{C}(J)|\Delta$ is a proper complex of $\mathscr{G}(c-1; \Delta)$ and $\delta^{-1}(\mathfrak{C}(J)|\Delta)$ is a positive divisor $\mathfrak{C}(I)$ of the Grassmannian $\mathscr{G}(c-1; \mathbf{P}_n)$ attached to $I = V^{c_1} \cdot V^{c_2} \cdot \ldots \cdot V^{c_n}$ in a natural way:

$$\mathfrak{C}(I) = \sum i_c \mathfrak{C}(C^c) \Leftrightarrow I = \sum i_c C^c$$

where the sums are taken over $C^c \subset \bigcap_{j=1}^{h} V^{c_j}$ and $i_c = i(\bigcap_{j=1}^{h} V^{c_j}; C^c)$.

The intersection multiplicities i_c equal the exponents of the corresponding F_C's. In fact we recall in § 4 that all the F_V in (0.10), (0.11), (0.12) are defined in terms of $\mathfrak{C}(V)$ by means of conjugation conditions (cf. Def. 5.1, 5.2). It suffices to assume first V irreducible. Namely: S, the Cayley–Severi form of V^c ($= V_d$) is the *conjugation condition with respect to* $\mathfrak{C}(V^c)$ *of* c *points* (x_j), $j = 1, 2, \ldots, c$. Y, the original *zugeordnete Form*, now usually called *Chow form of* V_d (cf. [Ch-vdW]) is the conjugation condition of $d+1$ hyperplanes and the Barsotti–Weil–Siegel form N (cf. [Si]), *Siegel's Normalgleichung of* V_d, is the conjugation condition of $d+2$ hyperplanes $(v_j) \in \mathbf{P}(\check{E})$, $j = 0, 1, \ldots, d+1$, and one point (x) with respect to $\mathfrak{C}(V^c)$. In terms of an exterior algebra: S, Y, N vanish if the products

$$\bigwedge_{j=1}^{c} x_j, \quad \bigwedge_{j=1}^{d+1} u_j, \quad x \lrcorner \bigwedge_{j=0}^{d+1} v_j$$

vanish, cf. [Bou]. If this is not the case any nonzero of these products represents (in the well-known way) a projective subspace $\mathbf{P}_{c-1} \subset \mathbf{P}_n$. Then $S = 0$ (resp. $Y = 0$, $N = 0$) iff such $\mathbf{P}_{c-1} \in \mathfrak{C}(V)$ (cf. § 4 for further details).

If $V = \sum m_I I > 0$ then F_V is defined by $F_V = \prod F_I^{m_I}$ ($F = S, Y, N$ and I is irreducible of dimension d). In any case F_V is well defined up to a factor $\lambda \in \mathbf{C}^{\times}$.

In any case the restrictions $S_J|\Delta$, $Y_J|\Delta$, $N_J|\Delta$ are well defined taking in (0.9) $(x_j) \in \Delta$, $u_j|\Delta$, $j = 1, 2, \ldots, h$.

The condition $c \leqslant n$ of (0.7) — essential to define the previous restrictions to the diagonal space — is not necessary in order to define the join $J = J(V^{(1)} \times \ldots \times V^{(h)})$ of h irreducible varieties $V^{(j)} \subset \mathbf{P}(E)$, $j = 1, 2, \ldots, h$. In the case $c > n$ the given varieties — in general position — do not meet but when $c_1 = c_2 = \ldots = c_h = 1$ the existence and discussion of a non epty intersection

(0.15) $$\bigcap_{j=1}^{h} V^{(j)} \neq \emptyset$$

is precisely the goal of the old elimination theory! Accordingly we devote § 12 to such a problem also with arbitrary c_j's — but $c > 0$. Under this hypothesis the compatibility condition (0.14), equivalent to $J \cap \Delta \neq \emptyset$ can be expressed by the following one:

THEOREM II. *The given h irreducible varieties* $V^{(j)} \subset \mathbf{P}_n, j = 1, 2, \ldots, h$ *with* $c = \sum_{j=1}^{n} c_j > n$ *meet iff the diagonal space* Δ *is singular for the complex* $\mathfrak{C}(J)$ *attached to* $J = J(V^{(1)} \times \ldots \times V^{(h)})$ ($\Leftrightarrow$ *every* $\mathbf{P}_{c-1}$ *satisfying* $\Delta \subset \mathbf{P}_{c-1}$ *belongs to* $\mathfrak{C}(J)$).

In particular for $c = n+1$ we have:

The h varieties $V^{(j)}$ *meet iff the diagonal space* Δ *belongs to the complex* $\mathfrak{C}(J)$.

This condition implies a single equation in the coefficients of F_J *reducing to* $R = 0$ *where* $R = R(f_1, f_2, \ldots, f_{n+1})$ *is the resultant of the* $n+1$ *hypersurfaces* $H_1, H_2, \ldots, H_{n+1}$ *if* $c_1 = c_2 = \ldots = c_{n+1} = 1$.

In the case $c > n+1$ the singularity condition of Δ can be expressed by the identical vanishing of a covariant in agreement with Gram's theorem of invariant theory, [We], § 12.

In the last part of the paper I review some results of the Author (cf. [G2] [G3]) regarding a replacement of the usual Kronecker elimination procedure by the explicit computation of the Cayley–Severi forms $S_I(x_1, x_2, \ldots, x_c)$ attached to an irreducible component $I = I^c$ of codimension c of the Zariski-closed set represented by an arbitrary system

$$(0.16) \qquad f_1 = 0 \quad f_2 = 0 \qquad f_r = 0$$

of homogeneous polynomial equations in the homogeneous coordinates $x_0, x_1, \ldots, x_n$ in $\mathbf{P}_n$. The method rests on the fact that the "elimination of the variables" $x_0, x_1, \ldots, x_i$ represents geometrically the projection of a variety from a certain space of the projective coordinate frame to the opposite face. If we replace these – indeed very particular projections – (essentially attached to the coordinate frame) by appropriate generic projections we obtain the indicated algorithm. But the easy transition form of S to the N forms gives back the old Kronecker elimination theory with respect to a generic frame "Φ built in" in the formulas, (instead of mentioning it but never written as before). I believe that this shows that the Barsotti–Weil–Siegel forms are the best ones-although the Chow forms seem to be the most famous. This inclusion of Φ is actually accomplished by means of an arbitrary basis $u_0, u_1, \ldots, u_n$ of $\check{E}$ acting as coordinate forms for points in $\mathbf{P}(E)$;

$$x \mapsto (\langle u_0, x\rangle, \langle u_1, x\rangle, \ldots, \langle u_n, x\rangle) \in \mathbf{P}(\mathbf{C}^{n+1})$$

for a fixed projective frame with current coordinates functions $(x_0, x_1, \ldots, x_n)$.

In order to see that it suffices to represent the projection center $\mathbf{P}_{c-2}$ defined in S_c by $x_2 \wedge x_3 \wedge \ldots \wedge x_c$ (with $(x_1) = (x)$ acting as a current variable point of the projecting cone of a V^c from $\mathbf{P}_{c-2}$) with hyperplane coordinates

$u_0, u_1, \ldots, u_{d+1}$ in such a way that $x_2 \wedge \ldots \wedge x_c$ and $x \rfloor u_0 \wedge \ldots \wedge u_{d+1}$ are dual, i.e. they represent the same $\mathbf{P}_{c-2}$.

We try to use standard notations as much as possible.

Acknowledgments. The *ruled join* or *join* was introduced by the Author in [G.1] trying to compare F_V, F_W with $F_{V \times W}$ (or $F_{V \cdot W}$ when the intersection cycle does exist) with the name *prodotto rigato*, but actually similar ideas were frequent in the Italian School also in symmetric squares; for instance the symmetric square of a smooth curve was represented frequently by the variety of chords containing the tangential surface as representative of the diagonal. But it was necessary also to recover lost properties of the "*zweifach projektive Räume*" $\mathbf{P}_{m,n}$ remarking that the "point" $(v, w) \sim (\lambda v, \mu w)$ of $\mathbf{P}_{m,n}$ is essentially the same as the line $\lambda(v, 0)+\mu(0, w)$ but certain natural subspaces, such as Δ do not appear in $\mathbf{P}_{m,n}$. Cf. § 1.2 for more details. This construction was also used by Fulton [F], [F-L] to illustrate his intersection theory and in the study of the topology of algebraic subvarieties of $\mathbf{P}_n$. A few years ago Vogel [V.1], [V.2], [F-V] tried successfully to recover the "length multiplicity" – rejected previously for well-known reasons with a sort of reduction to the diagonal using the double projective space $\mathbf{P}_{m,n}$. Kleimann – in a letter to Vogel [K] recommended him to do precisely what I did in the join construction. As a consequence I am coming back to his old technique. I hope to establish a link of the exponent multiplicity (previously used by van der Waerden's elementary cases of Bézout's theorem by means of resultants) with the length multiplicity. The pleasant atmosphere and the kind invitation of the Banach Center of the Polish Academy of Sciences is certainly a good encouragement in this direction.

The last part (of page 105) is just sketched – although the methods are very similar to those of [G.2], [G.3]. We shall come back to this with full details in [G.3] with an application to the Schottky problem (where the Siegel form appears in [SI]).

I am indebted to the wonderful facilities of the Max-Planck-Institute in Bonn – in particular to the extreme patience of the typist Frau Wolf–Gazo who did a beautiful job with them.

I. Generalities on joins

The reduction to the diagonal (cf. formula (1.1) below) introduced by C. Segre and Severi (fixed points of correspondences) and widely used later in Topology was applied by Weil [W] and others to local intersection multiplicity theories. The global extension to varieties in a projective space has some difficulties due to the fact that the diagonal Σ is not any more a linear space. Σ is a Segre variety:

$$\Sigma = \mathbf{P}\{x \otimes n \mid x \in E - \{0\},\ n \in \mathbf{C}^h - \{0\}\},$$

cf. § 1, 3. We show [G.1], § 1, 2, 3 that a naturally chosen generator $\Delta \subsetneq \Sigma \subset \mathbf{P}(E \otimes \mathbf{C}^h)$,

(1.0) $\Delta = \mathbf{P}\{x \otimes (1, 1, \dots, 1) \mid x \in E - \{0\}\} = \mathbf{P}\{(x, x, \dots, x) \mid x \in E - \{0\}\}$

plays the same role as in the affine case, although it is essential to introduce the space $\mathbf{P}(E \otimes \mathbf{C}^h)$ *instead of the "h-fach projektive Raum"* of [vdW1], [vdW-ZAG], [H-P]. The affine formulas (1.4) lead naturally to the (1.4)′ suggesting the definition of the *join* (cf. Def. 1.1) and the *projective reduction to the diagonal*, cf. formula (1.10) on page 80.

1. The "reduction to the diagonal". A projective version

Let $A_j \neq \emptyset, j = 1, 2, \dots, h$ be h nonempty subsets of an ambient set E. Let $\Pi = E \times E \times \overset{h}{\dots} \times E$ be the hth Cartesian power of E. We have:

$$\delta\Big(\bigcap_{j=1}^{h} A_j\Big) = A_1 \times A_2 \times \dots \times A_h \cap \Delta \tag{1.1}$$

where $\delta \colon E \hookrightarrow \Pi$ is the diagonal injection $\delta(x) = (x, x, \overset{h}{\dots}, x)$, $\forall x \in E$ and $\Delta = \delta(E)$ is the diagonal of Π.

This simple remark has many applications in algebraic geometry and it is regarded as a "reduction" (in spite of the fact that Π seems more complicated than E) because of the following reasons:

(a) If E is an algebraic variety and the A_j are all subvarieties, Π is also an algebraic variety and $A_1 \times \dots \times A_h$ and Δ are algebraic subvarieties of Π with Δ independent of the A_j.

(b) The subvarieties of Π are graphs of algebraic h-correspondences on E, in particular they might be graphs of maps and Δ is the graph of the identity. If we can "move" Π in an algebraic system, it is possible to move the A_j to generic positions $\bar{A}_j, j = 1, 2, \dots, h$ in such a way that we can predict geometric statements on the original A_j's by a subsequent specialization.

(c) In particular: if E is an affine space, Π is another one and Δ is a linear subspace of Π with $\dim E = \dim \Delta$. In this case $\delta(I)$ is an irreducible component of $A_1 \times \dots \times A_n \cap \Delta$ iff I is an irreducible component of $\bigcap_{j=1}^h A_j$. Accordingly $\Pi \cap \Delta$ is proper iff $\bigcap_{j=1}^h A_j$ is proper. Since the definition of the intersection multiplicities looks easier if one of the intersecting varieties is a linear space the diagonal provides a way to define

$$i(A_1 \cdot \ldots \cdot A_h;\ I) = i(\Pi \cdot \Delta;\ \delta(I)) \tag{1.2}$$

i.e.: It suffices to know how to define i for $\Pi \cdot \Delta$ ($h = 2$) and Δ a linear space. Cf. [W], [F]. The affine case is sufficient for all the local theories.

If E is a projective space $\mathbf{P}_n$, Π and Δ are not projective spaces, but Segre varieties, cf. [SE], [H-P]. However, the explicit description of Δ in the affine case leas naturally to the "join construction" (cf. Introduction) as follows: Let us assume $h = 2$. Then Δ is characterized by the system of linear equations

(1.3) $$x_j - y_j = 0, \quad j = 1, 2, \ldots, n,$$

if $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are current affine coordinates in the two copies of E. If $f_i(x) = 0$; and $g_j(x) = 0$ are two systems of equations defining A_1, A_2 the system

(1.4) $$f_j(x) = 0, \quad g_h(y) = 0,$$

defines $A_1 \times A_2$. (1.4) and (1.3) together define $\Pi \cap \Delta$.

In the projective case the x, y can be regarded as absolute coordinates in the **C**-vector space $E = E_{n+1}$ or as homogeneous coordinates in $\mathbf{P}_n = \mathbf{P}(E)$ and (1.3) is replaced by

(1.5) $$\operatorname{rank}\begin{pmatrix} x_0\, x_1 \ldots x_n \\ y_0\, y_1 \ldots y_n \end{pmatrix} = 1 \Rightarrow \begin{vmatrix} x_i\, x_j \\ y_i\, y_j \end{vmatrix} = 0, \quad 0 \leqslant i \leqslant j \leqslant n.$$

Then the (1.4) can be replaced by

(1.4)′ $$f_j(\lambda x) = 0, \quad g_h(\mu y) = 0,$$

where all the f_j and g_h are homogeneous and λ, μ are two independent non zero proportionality factors. Moreover the equations (1.5) define the Segre variety representing $\mathbf{P}(E) \times \mathbf{P}(\mathbf{C}^1)$ ($\Leftrightarrow$ locus of $x \otimes (\lambda, \mu) \in \mathbf{P}(E \oplus E) = \mathbf{P}(E \otimes \mathbf{C}^2)$). Cf. § 3 for further details.

Remarks. (1) We do not need the homogeneous equations $f_i(x) = 0$ $g_j(x) = 0$ anymore to establish (1.1).

(2) A_1, A_2 can be arbitrary empty subset of $\mathbf{P}(E)$.

(3) It suffices to define the two injections i_1, i_2: $\mathbf{P}(E) \hookrightarrow \mathbf{P}(E \oplus E)$ by

(1.6) $$i_1(x) = (x, 0), \quad i_2(x) = (0, x).$$

i_1, i_2 have the following properties:

$$i_1(\mathbf{P}(E)) = \mathbf{P}(E \oplus 0) = \mathbf{P}(E \otimes (1, 0)).$$

(1.7) $$i_2(\mathbf{P}(E)) = \mathbf{P}(0 \oplus E) = \mathbf{P}(E \otimes (0, 1)).$$

(1.8) $$i_1(\mathbf{P}(E)) \cap i_2(\mathbf{P}(E)) = \emptyset.$$

(1.9) $$\Delta = \mathbf{P}(E \otimes (1, 1)).$$

In other words the two copies of $\mathbf{P}(E)$ in $\mathbf{P}(E \oplus E) = \mathbf{P}(E \otimes \mathbf{C}^2)$ do not meet; accordingly any ordered pair $(P, Q) \in \mathbf{P}(E) \times \mathbf{P}(E)$ can be represented by the line joining $i_1(P)$ with $i_2(Q)$ and conversely any line joining one point of $\mathbf{P}(E \oplus 0)$ with another one of $\mathbf{P}(0 \oplus E)$ represents a uniquely defined ordered pair (P, Q).

More generally we have the following formal definition of the join (used already before).

Definition 1.1. (4) The *join* of A_1, A_2, denoted by $J(A_1 \times A_2)$, is the locus of all (always well defined!) lines joining points of $i_1(A_1)$ with points of $i_2(A_2)$. In particular; $J(\mathbf{P}(E) \times \mathbf{P}(E))$ is the subvariety of $\mathbf{P}(E \oplus E)$ consisting of lines joining points of $\mathbf{P}(E \oplus 0)$ and $\mathbf{P}(E \oplus 0)$.

(5) The following natural generalizations are possible

$$(\emptyset \neq) A_1 \subset \mathbf{P}(E) | (\emptyset \neq) A_2 \subset \mathbf{P}(F) \Rightarrow J(A_1 \times A_2) \subset \mathbf{P}(E \oplus F)$$

because i_1: $\mathbf{P}(E) \subsetneq \mathbf{P}(E \oplus F)$, i_2: $\mathbf{P}(F) \subsetneq \mathbf{P}(E \oplus F)$ are still valid.

(6) We can consider any finite number h of nonempty subsets $A_j \subset \mathbf{P}(E_{(j)})$, $j = 1, 2, \ldots, h$.

We shall consider this general set up in § 2 in order to clarify the relationship between the diagonal subspace Δ and the diagonal variety Σ in § 3.

The "reduction to the diagonal" in $\mathbf{P}(E)$ has finally the following expression:

$$\delta(A_1 \times A_2) = J(A_1 \times A_2) \cap \Delta \tag{1.10}$$

where A_1, A_2 are arbitrary nonempty subsets of $\mathbf{P}(E)$, Δ is the diagonal space (cf. (1.9)), and δ: $\mathbf{P}(E) \subsetneq \mathbf{P}(E \oplus E)$ is defined by (0.4):

(1..1) $$\delta(x) = ((x, x) = (x \otimes (1, 1)) \quad \forall x \in E - \{0\}.$$

Remark. We see that in the formula (1.10) one needs the points of $\mathbf{P}(E \oplus E)$, for instance those $((x, x)) \in \Delta$, not just the lines $\lambda(x, 0) + \mu(0, y)$. This justifies our preference for the join construction rather than the use of the two-way projective spaces $\mathbf{P}_{m,n}$; in $\mathbf{P}_{m,n}$ the previous line is the "point" $(x, y) \sim (\lambda x, \mu y)$, $\lambda \neq 0$, $\mu \neq 0$.

2. Recall of the join of h varieties. Relation with the Segre model of the product $V^{(1)} \times V^{(2)} \times \ldots \times V^{(h)}$

Let $\mathbf{P}(E_j) = E_j - \{0\}/\mathbf{C}^\times$, $j = 1, 2, \ldots, h$, be h ($\geqslant 2$) complex projective spaces generated by the corresponding vector spaces E_j. Let $\mathbf{P}(S)$ be the quotient projective space of the direct sum

$$S = E_1 \oplus E_2 \oplus \ldots \oplus E_h. \tag{2.1}$$

Let us call $S_j = (0, \ldots, E_j, \ldots, 0)$, $j = 1, 2, \ldots, h$. $\mathbf{P}(S)$ is the ambient projective space containing copies $\mathbf{P}(S_j) = i_j(\mathbf{P}(E_j))$, $j = 1, 2, \ldots, h$ of the given spaces $\mathbf{P}(E_j)$ satisfying the following properties (already checked for $h = 2$):

(a) For every ordered h-tuple $(x_1, x_2, \ldots, x_h) \in \prod_{j=1}^h \mathbf{P}(E_j)$ the corresponding images $i_j(x_j)$ ($j = 1, \ldots, h$) are linearly independent.

(b) The $S_{h-1} = S_{h-1}(x_1, \ldots, x_h)$ space spanned by the x_j meet $\mathbf{P}(S_j)$ precisely in the point x_j:

$$S_{h-1} \cap \mathbf{P}(S_j) = x_j \quad j = 1, 2, \ldots, h.$$

As a consequence we have:

(c) There is a bijection of $\prod_{j=1}^{h} \mathbf{P}(E_j)$ with the subset $\mathscr{J}(\mathbf{P}(E_1)\times \ldots \times \mathbf{P}(E_j)) \subset \mathscr{G}(h-1; \mathbf{P}(E_1 \oplus \ldots \oplus E_h))$ of the shown Grassmannian of $(h-1)$-spaces defined by:

$$(2.2) \quad \mathscr{J} = \mathscr{J}(\mathbf{P}(E_1)\times \ldots \times \mathbf{P}(E_j)) = \{\mathbf{P}_{h-1} \subset \mathbf{P}(E_1 \oplus \ldots \oplus E_h) \mid \mathbf{P}_{h-1} \cap i_j(\mathbf{P}(E_j)) = (x_j),\ j = 1, 2, \ldots, h\}$$

for $j = 1, 2, \ldots, h$. $\mathscr{J}$ is closely related to J by

DEFINITION 2.1. $J = J(\mathbf{P}(E_1)\times \ldots \times \mathbf{P}(E_h))$ is defined in terms of $\mathscr{J}$ (cf. (2.2)) by

$$(2.3) \quad J = J(\mathbf{P}(E_1)\times \ldots \times \mathbf{P}(E_h)) = \{\mathbf{P}_{h-1} \subset \mathbf{P}(E_1 \oplus \ldots \oplus E_h) \mid \mathbf{P}_{h-1} \in \mathscr{J}\}$$

is called the *ruled join* (or just *join*) of the given spaces $\mathbf{P}(E_1), \mathbf{P}(E_2), \ldots, \mathbf{P}(E_h)$. Def. 2.1 is the extension of Def. 1.1, page 80 for any $h \geqslant 2$.

A vector of S is regarded as an ordered h-tuple $(v_1, v_2, \ldots, v_h)$ with $v_j \in E_j$, $j = 1, 2, \ldots, h$. Let i_j: $E_j \hookrightarrow S$ be the natural injection defined by

$$(2.4) \qquad i_j(v) = (0, 0, \ldots, \overset{j}{\breve{v}} \ldots, 0), \quad v \in E_j$$

where $i_j(\mathbf{P}(E_j)) = \mathbf{P}(S_j) = \mathbf{P}(0, \ldots, E_j, \ldots, 0)$. We shall use the same symbol i_j for the corresponding maps between projective spaces.

$$(2.5) \qquad i_j\colon \mathbf{P}(E_j) \hookrightarrow \mathbf{P}(S), \quad i_j(\mathbf{P}(E_j)) = \mathbf{P}(S_j), \quad j = 1, 2, \ldots, h.$$

It is easy to check both conditions (a), (b) for the h copies $\mathbf{P}(S_1), \mathbf{P}(S_2), \ldots, \mathbf{P}(S_h)$ of given projective spaces $\mathbf{P}(E_j)$. In fact any ordered h-tuple $(x_1 \times x_2 \times \ldots \times x_h) \in \prod_{j=1}^{h} \mathbf{P}(E_j)$ $((E_j - \{0\})$ defines an h-tuple of linearly independent vectors $i_j(x_j) \in S_j = 1, 2, \ldots, h$ $(\Leftrightarrow \bigwedge_{j=1}^{h} i_j(x_j) \neq 0)$. They define a subspace $S(x_1, x_2, \ldots, x_h)$ of dimension $h-1$ in $\mathbf{P}(S)$ – the projection in $\mathbf{P}(S)$ of the h-dimensional vector space locus of points of the type

$$(2.6) \qquad (\lambda_1(v_1, 0, \ldots, 0) + \lambda_2(0, v_2, \ldots, 0) + \ldots + \lambda_h(0, 0, \ldots, v_h))$$

in such a way that

$$(2.7) \qquad S(x_1, \ldots, x_h) \cap \mathbf{P}(S_j) = (x_j), \quad j = 1, 2, \ldots, h$$

and conversely.

Another $(y_1)\times \ldots \times (y_h) \in \prod_{j=1}^{h} \mathbf{P}(E_j)$ $(E_j - \{0\})$ defines the same h-tuple of points in $\mathbf{P}(S_1)\times \ldots \times \mathbf{P}(S_h)$ and also the same S_{h-1} iff $y_j = \lambda_j x_j$ $\lambda_j \in \mathbf{C}^{\times}$ $j = 1, 2, \ldots, h$, (i.e. iff $(x_1, \ldots, x_h) \sim (y_1, \ldots, y_h)$ as points of the h-way projective space of $\mathbf{P}_{n,n,\ldots,n}$ (cf. Introduction, [H-P], [vdW1], [vdW-ZAG]); in other words:

$$S(x_1, \ldots, x_h) = S(y_1, \ldots, y_h) \Leftrightarrow y_j = \lambda_j x_j, \quad j = 1, 2, \ldots, h.$$

This construction leads to two modifications of Def. 2.1 obtained taking into account rather than the $\mathbf{P}_{h-1}$ of $\mathscr{G}$ some set of points in $\mathbf{P}(E_1 \oplus \ldots \oplus E_h)$.

DEFINITION 2.1′. The *pre-join*

$$J_p = J_p(\mathbf{P}(E_1)\times\ldots\times\mathbf{P}(E_n))$$
$$= \{((x_1, \ldots, x_h))\in\mathbf{P}(E_1\oplus\ldots\oplus E_h) \mid x_j\neq 0,\ j = 1, 2, \ldots, h\}$$

DEFINITION 2.2′. The *full join* J is the Zariski closure of J_p:

$$J = \bar{J}_p = \bigcup_{\mathbf{P}_{h-1}\in\mathscr{J}} \mathbf{P}_{h-1}. \tag{2.3)'}$$

However, in spite of the differences between $\mathscr{J}$, J_p, J the context will indicate without confusion which one we need, and we prefer the simplest notation J.

Remark. The name *ruled join* (*"prodotto rigato"*) is clear since an h-tuple of $\mathbf{P}(E_1)\times\ldots\times\mathbf{P}(E_h)$ is not represented by a point of another space but by a $\mathbf{P}_{h-1}$, i.e. by a line for $h = 2$ (cf. Introduction).

The product $\mathbf{P}(E_1)\times\ldots\times\mathbf{P}(E_h)$ is represented also by the quotient set

$$J_p/\sim\ = J_p/\mathbf{C}^\times\times\ldots\times\mathbf{C}^\times = \prod_{j=1}^{h}(E_j-\{0\})/\mathbf{C}^\times\times\ldots\times\mathbf{C}^\times \tag{2.7}$$

usually called the *r-way projective space* $\mathbf{P}_{n_1,n_2,\ldots,n_h}$ where $n_j = \dim\mathbf{P}(E_j)$ by [vdW], [vdW-ZAG]; see also [H-P].

Remarks. (1) Since there is a bijection between "points" $(v_1, \ldots, v_h)$ of $\mathbf{P}_{n_1,n_2,\ldots,n_h}$ and $(h-1)$-dimensional subspaces of type $S(v_1, v_2, \ldots, v_h)$, the relation between J_p, $J_p/\sim\ = \mathbf{P}_{n_1,n_2,\ldots,n_h}$ and J is very close (cf. Def. 2.1). The reason of our preference of J over $\mathbf{P}_{n_1,n_2,\ldots,n_h}$ is due to the fact that in the interpretation of the reduction to the diagonal (cf. § 1) we need J (rather than J_p or $\mathscr{J}$) and the subset $\Delta\subset\mathbf{P}(E\oplus\ldots\oplus E)$ (which do not belong to $\mathbf{P}_{n_1,\ldots,n_h}$). In other words the equivalence relation defining $\mathbf{P}_{n_1,\ldots,n_h}$ looses the points $\mathbf{P}(E\oplus\ldots.\oplus E)$ needed essentially in the reduction to the diagonal.

EXAMPLE. The product $\mathbf{P}_1\times\mathbf{P}_1 = \mathbf{P}(E)$ ($\dim E = 2$) is represented by the set (*line congruence*) J of lines joining pairs of points of $\mathbf{P}(S_1) = \mathbf{P}(E\oplus 0)$ and

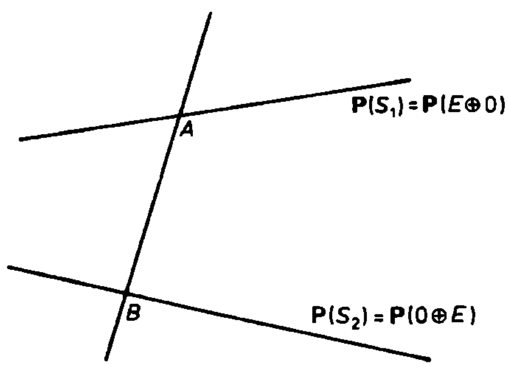

Fig. 1

$\mathbf{P}(S_2) = \mathbf{P}(0 \oplus E_2)$. The two lines $\mathbf{P}(S_1)$, $\mathbf{P}(S_2)$ do not meet and conversely *any line of this congruence determines uniquely the pair of points* (A, B).

The relation of the ruled model $J(\mathbf{P}(E_1) \times \ldots \times \mathbf{P}(E_h))$ with the usual Segre model $\Sigma_{n_1, \ldots, n_h}$ is very simple. It suffices to show it for $h = 2$:

Let $J(m, n) = J(\mathbf{P}_m \times \mathbf{P}_n)$ be the join and let $\Sigma_{m,n} \subset \mathbf{P}(E_1 \otimes E_2)$ be the Segre model; let us recall that $\Sigma_{m,n}$ is the image of the set of ($\neq 0$) monomial elements $x \otimes y$ ($x \in E_1$, $y \in E_2$) in the tensor product $E_1 \otimes E_2$ by the canonical projection $E_1 \otimes E_2 \to \mathbf{P}(E_1 \otimes E_2)$ in such a way that the pair $(x) \times (y) \in \mathbf{P}(E_1) \times \mathbf{P}(E_2)$ is represented by $(x \otimes y) \in \mathbf{P}(E_1 \otimes E_2)$. The Grassmann coordinates of the line joning $(x, 0)$ with $(0, y)$ are the two-minors of the matrix

$$\begin{pmatrix} x^0\, x^1 \ldots x^m\, 0 & 0 & \ldots 0 \\ 0 & 0 & \ldots 0 & y^{0'}\, y^{1'} \ldots y^{n'} \end{pmatrix} \tag{2.8}$$

where we choose a couple of bases in E_1, E_2 labelling the coordinates with the indices $0, 1, \ldots, m, 0', 1', \ldots, n'$; we have $p^{ij} = p^{i'j'} = 0$ but

$$p^{ij'} = x^i y^j = \text{coordinates of } x \otimes y. \tag{2.9}$$

In other words: *the products $x^i y^j$ representing the coordinates of $x \otimes y$ in a canonical basis represent also the essential Grassmann coordinates of the line joining $(i_1(x))$ with $(i_2(y))$.*

Intrinsically: we can identify $x \otimes y$ with $i_1(x) \wedge i_2(y)$ inside $E_1 \oplus E_2$; similarly we have for any $h \geqslant 2$

$$x_1 \otimes x_2 \otimes \ldots \otimes x_h \Leftrightarrow i_1(x_1) \wedge i_2(x_2) \wedge \ldots \wedge i_h(x_h) \tag{2.10}$$

in $E_1 \oplus E_2 \oplus \ldots \oplus E_h$. Cf. [SG], [B], [H-P].

The join of h irreducible subvarieties $V^{(j)} \subset \mathbf{P}(E_j)$ is naturally defined by restriction as follows:

Definition 2.2. Let $i_j(V^{(j)}) \subset \mathbf{P}(S_j)$ be the corresponding copies of the h given subvarieties. The join $J(V^{(1)} \times \ldots \times V^{(h)})$ of $V^{(1)}, V^{(2)}, \ldots, V^{(h)}$ is the restriction of $J = J(\mathbf{P}(E_1) \times \ldots \times \mathbf{P}(E_{h-1}))$ to the $\mathbf{P}_{h-1}$ subspaces of J joining points of the $i_j(V^{(j)})$, $j = 1, 2, \ldots, h$.

$$J(V^{(1)} \times \ldots \times V^{(h)}) = \{\mathbf{P}_{h-1} \in J(\mathbf{P}(E_1) \times \ldots \times \mathbf{P}(E_h)) | \mathbf{P}_{h-1} \cap S_j = V^{(j)},\ j = 1, 2, \ldots, h\}.$$

We shall use the following properties of $J(V^{(1)} \times \ldots \times V^{(h)})$:

(1) $J(V^{(1)} \times V^{(2)} \times \ldots \times V^{(h)})$ *is irreducible if* $V^{(j)}$ *is irreducible* (for $j = 1, 2, \ldots, h$). Moreover:

$$\dim J(V^{(1)} \times V^{(2)} \times \ldots \times V^{(h)}) = d_1 + d_2 + \ldots + d_h + h - 1 \tag{2.11}$$

where $d_j = \dim V^{(j)}$, $j = 1, 2, \ldots, h$.

(2) $J(V^{(1)}\times\ldots\times V^{(h)})$

$$= \bigcap_{j=1}^{h} J\big(\mathbf{P}(E_1)\times\ldots\times\mathbf{P}(E_{j-1})\times V^{(j)}\times\mathbf{P}(E_{j+1})\times\ldots\times\mathbf{P}(E_h)\big).$$

(3) The codimension c of $J(V^{(1)}\times\ldots\times V^{(h)})$ in $\mathbf{P}(E_1\oplus\ldots\oplus E_h)$ is equal to the sum of the codimensions $c_j = n-d_j$, $j = 1, 2, \ldots, h$

$$c = c_1+c_2+\ldots+c_h. \tag{2.12}$$

3. Case $n_1 = n_2 = \ldots = n_h = n$. The diagonals Σ, Δ

The case $E_1 = E_2 = \ldots = E_h = E$, $S = E\oplus E\oplus\ldots\oplus E$, $\dim E = n+1$ is particularly important in the intersection problems, because then we need to consider the representation of the abstract diagonal

$$D = \{P_1\times P_2\times\ldots\times P_n\in\mathbf{P}(E)\times\mathbf{P}(E)\times\ldots\times\mathbf{P}(E)|P_1 = P_2 = \ldots = P_h\}$$

in the abstract product. D is represented in the Segre model $\Sigma_{n_1,n_2,\ldots,n_h}$ by a Veronese variety $V(D)$ (cf. [B])

$$V(D) = \{\lambda(x_1\otimes x_2\otimes\ldots\otimes x_h)\in\Sigma_{n,n,\ldots,n}|x_1 = x_2 = \ldots = x_n \neq 0\} \tag{3.1}$$

In the join the image of $(\lambda_1 x)\times(\lambda_2 x)\times\ldots\times(\lambda_n x)$, $\lambda_j \neq 0$, $j = 1, 2, \ldots,$ is the subspace $S(x, x, \ldots, x)$, thus the image of D is

$$\Sigma_D = U\{S(x_1, x_2, \ldots, x_h)|x_1 = x_2 = \ldots = x_h\}. \tag{3.2}$$

$\Sigma = \Sigma_D$ is a Segre variety model of $\mathbf{P}(E)\times\mathbf{P}(\mathbf{C}^h)$. In order to see that it is convenient to introduce the following identifications:

$$S = E\otimes\mathbf{C}^h,\qquad S_j = E\otimes u_j,\qquad S = \bigoplus_{j=1}^{h} S_j \tag{3.3}$$

where $u_j = (0, 0, \ldots, \overset{j}{1}, \ldots, 0)$, $j = 1, 2, \ldots, h$.

$$(x_1, x_2, \ldots, x_h)\subset J \Leftrightarrow (x_1\otimes u_1, x_2\otimes u_2, \ldots, x_h\otimes u_h) \tag{3.4}$$

(3.4) implies in the diagonal case $x_1 = x_2 = \ldots = x_h = x \neq 0$

$$(\lambda_1 x, \lambda_2 x, \ldots, \lambda_h x)\Leftrightarrow x\otimes(\lambda_1, \lambda_2, \ldots, \lambda_h). \tag{3.5}$$

The generating spaces $\mathbf{P}(E)\otimes(\lambda_1, \lambda_2, \ldots, \lambda_h)$, $((\lambda_1, \ldots, \lambda_h))\in\mathbf{P}(\mathbf{C}^h)$ and $(x)\otimes\mathbf{P}(\mathbf{C}^h)$ are represented by

$$\mathbf{P}\{x\otimes(\lambda_1, \ldots, \lambda_h)|x\in E\} \quad\text{and}\quad \mathbf{P}(x\otimes(\lambda_1, \ldots, \lambda_h)|(\lambda_1, \ldots, \lambda_h)\in\mathbf{C}^h)$$

respectively. The latter is the image of the abstract diagonal point $(x)\times(x)\times\ldots\times(x)$, i.e. by the span of the h copies of (x) in $\mathbf{P}(S_j)$, $j = 1, 2, \ldots, h$. The former is a copy of $\mathbf{P}(E)$, the copy maps being

$$(x)\mapsto(x)\otimes(\lambda_1, \ldots, \lambda_h).$$

In particular we have the following distinguished copies

$$\mathbf{P}(S_j) = \mathbf{P}(E \otimes u_j) \subset \Sigma_D, \quad j = 1, 2, \ldots, h,$$

$$\Delta = \mathbf{P}(E \otimes (1, 1, \ldots, 1)) = \mathbf{P}\{(x, x, \ldots, x) | x \in E\} \subset \Sigma_D \subset \Gamma(E \otimes \mathbf{C}^h).$$

Δ is the diagonal space (cf. Introduction) not to be confused with Σ_D.

The reduction to the diagonal for h arbitrary nonempty subsets $A_1, \ldots, A_h$ of $\mathbf{P}(E)$ has the final form:

$$\delta\Big(\bigcap_{j=1}^{h} A_j\Big) = J(A_1 \times A_2 \times \ldots \times A_h) \cap \Delta \tag{3.6}$$

where J is the full join: $J(A_1 \times \ldots \times A_h) = \{\mathbf{P}_{h-1} | \mathbf{P}_{h-1} \in \mathscr{J}(A_1 \times A_2 \times \ldots \times A_h)\}$.

Let us come back to our interesting case $A_j = V^{(j)}$ irreducible algebraic subvariety of $\mathbf{P}(E)$ of dimension d_j and codimension c_j. We know (cf. formula (2.12)) that $\operatorname{cod} J$ in $\mathbf{P}(E \otimes \mathbf{C}^h)$ is equal to $c = c_1 + c_2 + \ldots + c_h$. Then our discussions lead naturally to the two cases $c \leqslant n$ and $c > n$.

If $c \leqslant n$ then always $\bigcap V^{(j)} \neq \emptyset \Leftrightarrow J(V^{(1)} \times \ldots \times V^{(h)}) \cap \Delta \neq \emptyset$.

If $c > n$, $\bigcap V^{(j)} = \emptyset$ for the $V^{(j)}$ in generic position $\Leftrightarrow$ the diagonal space Δ does not meet the join:

$$J(V^{(1)} \times \ldots \times V^{(h)}) \cap \Delta = \emptyset \Leftrightarrow \bigcap_{j=1}^{h} V^{(j)} = \emptyset. \tag{3.7}$$

4. Joins and h-collineations

The h-way projective space $\mathbf{P}_{n_1, n_2, \ldots, n_h} = \prod_{j=1}^{h} (E_j - \{0\}) / \mathbf{C}^\times \times \overset{h}{\ldots} \times \mathbf{C}^\times$ where $\dim E_j = n_j + 1$ was introduced by van der Waerden [Ch-vdW] to study the correspondences in $\mathbf{P}_{n_1} \times \mathbf{P}_{n_2} \times \ldots \times \mathbf{P}_{n_h}$ (cf. also [H-P], Vol. I, Chapter V, § 10 and specifically Vol. II, Ch. XI). An irreducible correspondence in

$$\Pi_{n_1, n_2, \ldots, n_h} = \mathbf{P}_{n_1} \times \ldots \times \mathbf{P}_{n_h}$$

is an irreducible subvariety of this product. The natural way to study them is to introduce the systems of homogeneous polynomial equations; a polynomial $f \in \mathbf{C}[x^{(1)}, x^{(2)}, \ldots, x^{(h)}]$ (where $x^{(j)} = (x_0^{(j)}, x_1^{(j)}, \ldots, x_{n_j}^{(j)})$, $j = 1, 2, \ldots, h$) is called *homogeneous of degree* $(m_1, m_2, \ldots, m_h)$ iff

$$f(\lambda_1 x^{(1)}, \lambda_2 x^{(2)}, \ldots, \lambda_h x^{(h)}) = \lambda_1^{m_1} \lambda_2^{m_2} \ldots \lambda_h^{m_h} f(x^{(1)}, \ldots, x^{(h)}). \tag{4.1}$$

In the interpretation of the points of $\mathbf{P}_{n_1, n_2, \ldots, n_h}$ as $(h-1)$-subspaces of $J(\mathbf{P}_{n_1} \times \ldots \times \mathbf{P}_{n_h})$ any subvariety of $\mathbf{P}_{n_1, \ldots, n_h}$ might be regarded as a Grasmannian subvariety:

$$\mathscr{S} \subset \mathscr{J}(\mathbf{P}_{n_1} \times \ldots \times \mathbf{P}_{n_h}) \subset \mathscr{G}(h-1; \mathbf{P}(E_i \oplus \ldots \oplus E_h)).$$

The transition of $\mathscr{J}$ to J originates *a ruled variety*

(4.2) $$S = \bigcup_{\mathbf{P}_{h-1} \in \mathscr{S}} \mathbf{P}_{h-1}.$$

We shall omit the easy transition of the language developed in [H-P] for $\mathbf{P}_{n_1, n_2, \ldots, n_h}$ to our "join"-interpretation with the exception of the *h-collineations* among h copies of $\mathbf{P}(E) = \mathbf{P}_h$: *they have some special properties closely related to the subspaces of* $\mathbf{P}(E \otimes \mathbf{C}^h)$ *which will enable us in* § 10 *to show the equivalence of the exponent multiplicity with van der Waerden's.*

Let us recall the following ones:

(4) Let $P = (v_1, v_2, \ldots, v_h) \in \mathbf{P}(E \otimes \mathbf{C}^h)$ be one point of $\mathbf{P}(J_p)$ ($\Leftrightarrow v_j \neq 0$, $j = 1, \ldots, h$). Then there is one and only one $\mathbf{P}_{h-1} \in \mathscr{J}(\mathbf{P}(E) \times \ldots \times \mathbf{P}(E))$ containing P.

Let $U \subset \mathbf{P}(J_p)$ be a *J-unisecant variety* $\Leftrightarrow U$ does not contain two different points belonging to the same $\mathbf{P}_{h-1} \in \mathscr{J}(\mathbf{P}_n \times \overset{h}{\ldots} \times \mathbf{P}_n)$. *Then U represents in a natural way the same h-correspondence that the ruled variety R locus of* $\mathbf{P}_{h-1} \in \mathscr{J}(\mathbf{P}(E) \times \ldots \times \mathbf{P}(E))$ *meeting* U:

$$R = \bigcup_{\mathbf{P}_{h-1} \in \mathscr{J}(\mathbf{P}(E) \times \ldots \times \mathbf{P}(E)) \mid \mathbf{P}_{h-1} \cap U \neq \emptyset} \mathbf{P}_{h-1}.$$

Let $\mathscr{D}$ be the collineation group of $\mathbf{P}(E \otimes \mathbf{C}^h)$ in itself represented by homogeneous diagonal matrices: $\operatorname{diag}(\lambda_1 \ldots \overset{n+1}{\ldots} \lambda_1; \lambda_2 \ldots \overset{n+1}{\ldots} \lambda_2, \ldots, \lambda_h \ldots \overset{n+1}{\ldots} \lambda_h)$ with h nonzero scalars $\lambda_j, j = 1, 2, \ldots, n$.

Then U and γU represent the same correspondence for any

$$\gamma = D_{\lambda_1 \lambda_2 \ldots \lambda_h} \in \mathscr{D}.$$

EXAMPLE. The diagonal space Δ has the two properties we want: $\Delta \in \mathbf{P}(J_p)$ and Δ does not contain two different points of the same $\mathbf{P}_{h-1}$ of $J(\mathbf{P}(E) \times \ldots \times \mathbf{P}(E))$. In this case $R_\Delta = \Sigma \cdot \Delta$ and Σ represent both the *diagonal* ($\Leftrightarrow$ "*identity*") in the abstract product $\mathbf{P}(E) \times \ldots \times \mathbf{P}(E)$.

However there are other linear spaces $\mathbf{P}(E \otimes \mathbf{C}^h)$ having this property, for instance those (replacing Δ) obtained "moving" the h identifications) i_j: $\mathbf{P}(E) \to \mathbf{P}(S_j)$. Let us replace them by h arbitrary nondegenerate collineations γ_j: $\mathbf{P}(E) \to \mathbf{P}(S_j), j = 1, 2, \ldots, h$. Then we have: The correspondence γ, locus of $(\gamma_1(P), \gamma_2(P), \ldots, \gamma_h(P))$, $P \subset \mathbf{P}(E)$ will be called a *nondegenerate h-collineation.* It is represented by a Segre variety Σ_γ (reducing to Σ for $\gamma = i_j, j = 1, 2, \ldots, h$) whose vertical $(h-1)$-spaces belong to $J(\mathbf{P}(E) \times \ldots \times \mathbf{P}(E))$. Any horizontal one $H \neq \mathbf{P}(S_1), \mathbf{P}(S_2), \ldots, \mathbf{P}(S_h)$ represents γ, i.e. $H \subset (J_p)$: H has no two different points in the same $\mathbf{P}_{h-1}$ of the join and $R_H = \Sigma_\gamma$.

In the case $h = 2$, $\gamma_2 \gamma_1^{-1} (\gamma_1 \gamma_2^{-2})$ represents a collineation $\mathbf{P}(S_1) \to \mathbf{P}(S_2)$ (or its inverse $\mathbf{P}(S_2) \to \mathbf{P}(S_1)$).

Let us see these properties more closely using a basis:

Let $B_j(u_0^{(j)}, u_1^{(j)}, \ldots, u_n^{(j)})$, $j = 1, 2, \ldots, h$ be a basis of E ($\Leftrightarrow \bigwedge_{i=0} u_i^{(j)} \neq 0$ $j = 1, 2, \ldots, h$). Then we have:

Such h bases define a nondegenerate h-collineation where $(x_1, \ldots, x_n)$ correspond if x_j has the same homogeneous coordinates in B_j for $j = 1, 2, \ldots, h$. But the h vectors

$$(u_j^{(1)}, u_j^{(2)}, \ldots, u_j^{(h)}) \in J$$

are linearly independent and they define an $S_n \subset \mathbf{P}(E \otimes \mathbf{C}^h)$. The h bases $(\lambda B_1, \lambda B_2, \ldots, \lambda B_h)$ define the same S for any $\lambda \neq 0$. Let $\lambda_1, \lambda_2, \ldots, \lambda_h$ be h different nonzero scalars. Then $(\lambda_1 B_1, \lambda_2 B_2, \ldots, \lambda_h B_h)$ define a different $S'_n = DS_n$ where $D = D_{\lambda_1 \lambda_2 \ldots \lambda_h}$. But S_n and S'_n define the same h-collineation.

EXAMPLE. For $h = 2$ we have: If $(\lambda, \mu) \neq (0, 0)$, $(\lambda B_1, \lambda B_2)$ define a subspace S_{n-1} representing the nondegenerate collineation (B_1, B_2). (λ', μ') defines the same S_{n-1} iff $(\lambda', \mu') = \nu(\lambda, \mu)$.

(B_1, B_2) and (B'_1, B'_2) define the same collineation iff $B'_i = B_1 T$, $B'_2 = B_2 T$ where T is a $(n+1) \times (n+1)$ matrix with $\det T \neq 0$.

Then we can see that $R_S = R_{S'}$ is a Segre variety.

Let us introduce back coordinate systems $(x_0, x_1, \ldots, x_n)$ in E as well as $(x_0^{(j)}, \ldots, x_n^{(j)})$ in S_j interpreted as homogeneous coordinates when needed. Then for $\gamma_1, \gamma_2, \ldots, \gamma_n$ nondegenerate we can assign to any set of h nonsingular matrices $G_1, G_2, \ldots, G_h$ the n-subspaces S_n of $\mathbf{P}(E \otimes \mathbf{C}^h)$ generated by the $n+1$ rows of $G_1 G_2 \ldots G_h$. The nonsingularity condition $\det G_j \neq 0$ is equivalent to the fact that $S_n \cap (S_j) = \emptyset$ where

$$S_j = E \ldots \overset{j}{0} \ldots E.$$

Thus $S_n \subset \mathbf{P}(J_p) \Leftrightarrow \det G_j \neq 0$ for $j = 1, 2, \ldots, n$.

$$(G_1 G_2 \ldots G_h) \quad \text{and} \quad (G_1 T, G_2 T, \ldots, G_h T)$$

are two different bases of S_n if $\det T \neq 0$, and we can assume either one $G_j = \mathbf{P}_n$. If $\lambda_j \neq 0$ for $j = 1, 2, \ldots, n$, $(\lambda_1 G_1, \lambda_2 G_2, \ldots, \lambda_h G_h)$ defines $D_{\lambda_1 \lambda_2 \ldots \lambda_h} S_n$ with $D_{\lambda_1 \ldots \lambda_h} \in \mathscr{D}$.

Let us forget now the condition $\det G_j \neq 0$ for some (or all j) but keeping the fact that rank $(G_1, G_2, \ldots, G_h) = n+1$. Then the condition $S_n \subset \mathbf{P}(J_p)$ fails $\Leftrightarrow$ S_n meets some $\mathbf{P}(S_j)$. However we can assign to S_n a correspondence $\Gamma(S_n)$ where $(x_1, x_2, \ldots, x_h) \in \Gamma$ iff the $\mathbf{P}_{h-1}$ space $\{\lambda_1 x_1 + \mu_2 x_2 + \ldots + \mu_h x_h\}$ meets S_n (we cannot insure anymore that it meets in a single point).

EXAMPLE. Let S_α, S_β be two subspaces of $\mathbf{P}_n$. Then $J(S_\alpha \times S_\beta)$ is a subspace of dimension $\alpha + \beta + 1$ of $\mathbf{P}(E \oplus E)$, but $J(S_\alpha \times S_\beta) \cap \mathbf{P}(S_1) = i_1(S_\alpha)$, $J(S_\alpha \times S_\beta) \cap \mathbf{P}(S_2) = i_2(S_\beta)$; if $(x) \in S_\alpha$, $(y) \in S_\beta$ the whole line $\lambda(x, 0) + \mu(0, y)$ is contained in $J(S_\alpha \times S_\beta)$.

II. Generalities on the complex $\mathfrak{C}(V)$ attached to a $V \subset \mathbf{P}_n$

We shall complete with appropriate references some of the information already given in the Introduction. It is well known that not every complex in $\mathscr{G}(c-1; n)$ is attached to a V. Such particular complexes are indeed very special; they will be called *nucleated* with nucleus V^c. The characteristic nuclearity conditions for a $\mathfrak{C} \subset \mathscr{G}(c-1; n)$ can be expressed by a system of homogeneous polynomials equations — the so called Chow equations (cf. [Ch-vdW]) they are use; to prove that the set of positive cycles of codimension c in $\mathbf{P}_n$ is Zariski closed.

5. The complex $\mathfrak{C}(V^c)$ of $\mathbf{P}_{c-1}$. Recall on Zugeordnete Formen

The word *complex of subspaces* $\mathbf{P}_d$ in $\mathbf{P}_n$ $(0 \leqslant d \leqslant n)$ is used here in the XIX-th century sense—namely as a synonym of *Grassmann divisor* (in $\mathscr{G}(d; \mathbf{P}_n)$). We identify $\mathfrak{C}$ with its image in the Grassmann embedding

$$(5.1) \qquad \mathscr{G} = \mathscr{G}(d;\ n) \subsetneq \mathbf{P}(\bigwedge^{d+1} E_{n+1}), \qquad \mathbf{P}_n = \mathbf{P}(E_{n+1}).$$

A $\mathbf{P}_d(\subset \mathbf{P}_n)$ can be determined uniquely by $d+1$ linearly independent points in $\mathbf{P}_n$ or by $n-d$ l.i. hyperplanes meeting in $\mathbf{P}_d$. Accordingly we define the *conjugation condition* with respect to a complex $\mathfrak{C}$ of d-spaces in $\mathbf{P}_n$ as follows:

Definition 5.1. $d+1$ linearly independent points $P_1, P_2, \ldots, P_{d+1}$ of $\mathbf{P}_n$ are called *conjugate with respect to* $\mathfrak{C}$ iff the unique $S_d \ni P_j$ $(j = 1, 2, \ldots, d+1)$ belongs to $\mathfrak{C}$.

Definition 5.2. $n-d$ linearly independent hyperplanes $H_1, H_2, \ldots$ $\ldots, H_{n-d}(\subset \mathbf{P}_n)$ are called *conjugate with respect to* $\mathfrak{C}$ iff the unique $S_d = H_1 \cap H_2 \cap \ldots \cap H_{n-d}$ belongs to $\mathfrak{C}$.

The conjugation condition of $d+1$ points with respect to an irreducible $\mathfrak{C}$ $(\subset \mathscr{G}(d; n))$ (cf. Def. 5.1) can be determined by a single irreducible equation

$$(5.2) \qquad F(x_1, x_2, \ldots, x_{d+1}) = 0$$

where F is a polynomial homogeneous of the same degree g with respect to each one of the $d+1$ variable vectors $x_j \in E_{n+1}$ representing the points P_j; $j = 1, 2, \ldots, h$.

Similarly we have another plurihomogeneous form G (with the same g for the $n-d$ variables $u^j \in \check{E}$ (dual of E_{n+1}), such that

$$(5.3) \qquad G(u^1, u^2, \ldots, u^{n-d}) = 0$$

characterizes the conjugation condition of the H_j $(= \mathbf{P}(u^j))$, $j = 1, 2, \ldots, n-d$, of Def. 4.2. F and G can be written uniquely as C-linear combinations of standard monomials $p(S)$, $q(\Sigma)$ of degree g (cf. [H-P] vol. II, Ch. XIV, page 377) in the Grassmann coordinates of $\mathbf{P}_d$ $(\mathbf{P}_d^{\perp})\, p^{i_1 i_2 \ldots i_{d+1}}$, $(q_{j_1 j_2 \ldots j_{n-d}})$

(5.4) $$F = \sum \lambda_S p(S), \qquad G = \sum_{\Sigma} \mu_\Sigma q(\Sigma).$$

F and G are *uniquely determined by* $\mathfrak{C}$ (*up to a* $\mathbf{C}^{\times}$*-factor*). Accordingly $(\lambda_1, \lambda_2, \ldots)$ or $(\mu_1, \mu_2, \ldots)$ are well defined homogeneous coordinates representing $\mathfrak{C}$. The procedure is extended to arbitrary positive Grassmann divisors by prime factor decomposition $F = \prod F_j^{m_j}$, $G = \prod G_j^{m_j}$. Both expressions (4.4) are not essentially different because of the well known identities between the p and q.

When $\mathfrak{C} = \mathfrak{C}(V)$ $(d = c-1)$ (cf. Introduction, page 75 and page 73) these conjugation conditions (5.2), (5.4) define the *Cayley–Severi form* (or the *Chow form* respectively) of $V = V^c = V_d$. We emphasize that the number of vectors ($\Leftrightarrow$ belonging to E) in (5.2) is equal to the codimension c of V^c, thus it gives back the equation of a V^1 (i.e. of a hypersurface), for $c = 1$. The Chow forms of V contain a number of covectors (belonging to $\check{E}$) equal to $\dim V + 1$. Then (5.2), resp. (5.3) represent the characteristic condition for a S to meet V (here S_{c-1} is uniquely determined by c points, resp. as intersection of $d+1$ hyperplanes).

In order to introduce the formal definition for nucleated complexes (Def. 5.5) we shall need to consider certain exceptional behaviour of points and S-spaces $(m > d)$ with respect to a complex of d-spaces.

Definition 5.3. Let P be a point of $\mathbf{P}_n$. P is called *singular with respect to* $\mathfrak{C}$ iff every $S_d \ni P$ belongs to $\mathfrak{C}$.

Definition 5.4. The subspace S_m $(m < d)$ of $\mathbf{P}_m$ is called *singular with respect to* $\mathfrak{C}$ iff every $S_d \subset S_m$ belongs to $\mathfrak{C}$. Cf. [S].

We shall introduce now formally the complex $\mathfrak{C}(V^c)$ attached to an irreducible subvariety V^c of codimension c in $\mathbf{P}_n$. It necessary to check first the following property:

The set

(5.5) $$\mathfrak{C}(V^c) = \{\mathbf{P}_{c-1} \subset \mathbf{P}_n \mid \mathbf{P}_{c-1} \cap V^c \neq \emptyset\} \subset \mathscr{G}(c-1;\, \mathbf{P}_n)$$

is an irreducible complex in $\mathscr{G}(c-1;\, \mathbf{P}_n)$. The variety V^c is the locus of singular points of $\mathfrak{C}(V^c)$ (cf. [S] [H-P], vol. I, II); i.e. $\mathfrak{C}(V^c)$ is nucleated with locus V^c.

Definition 5.5. The set $\mathfrak{C}(V^c)$ defined by (5.6) is called the *complex attached to* V^c.

Examples. (1) For $c = 1$, $\mathfrak{C}(V^1)$ is just the set of points of the irreducible $V^1 \subset \mathbf{P}_n$.

(2) For $c = 2$, $\mathfrak{C}(V^2)$ is the set of lines meeting V^2. For instance, if $V^2 = \Gamma_1$ is an irreducible curve of $\mathbf{P}_3$, $\mathfrak{C}(\Gamma)$ is the complex of lines of $\mathbf{P}_3$ meeting Γ.

DEFINITION 5.6. The conjugation conditions of points (or hyperplanes) with respect to $\mathfrak{C}(V^c)$ are called the *Cayley–Severi form* (or *Chow form*) of V^c.

$$S(x_1, \ldots, x_c) = 0. \tag{5.6}$$

$$Y(u^1, u^2, \ldots, u^{d+1}) = 0. \tag{5.7}$$

Actually $Y = 0$ is the first systematic "*zugeordnete Form*" (cf. [vdW-ZAG]) or *associated form*. In the case of an irreducible plane curve Γ the left hand side of (5.7) is the resultant $R(f; u, v)$ where $f = 0$ represents Γ and u, v are linear forms.

In the introduction we mentioned also the *characteristic form* (Weil) (or *Normalgleichung*) (Siegel) (valid also for a non nucleated $\mathfrak{C}$) containing $\dim V+2$ covectors and a single vector; it was also introduced by Barsotti [Ba]. In the general case we have this "mixed" equation:

$$N(u_1, u_2, \ldots, u_{d+2}; x) = 0. \tag{5.8}$$

Remark. Severi pointed out in [S] that $S = 0$ is the real generalization of the equation of an irreducible hypersurface V^1, since the number of vector variables equals the codimension. But $S = 0$ was described by Cayley (as early as 1860) for conics in $\mathbf{P}_3$ of [C₁], [C₂]. If we keep fixed $c-1$ linearly independent variables $a_1 \ldots a_{c-1}$ in $S = 0$ in such a way that $(a_1 \wedge \ldots \wedge a_{c-1})$ does not meet V^c then

$$S(a_1, \ldots, a_{c-1}; x) = 0 \tag{5.9}$$

represents the projecting cone of V^c from $\mathbf{P}_{c-2}$; accordingly V^c is recovered from $S = 0$ as the intersection of all the projecting cones of V^c from the $\mathbf{P}_{c-2}(\bigcap V^c = \emptyset)$. If we replace $a_1 \wedge \ldots \wedge a_{c-1} \in \bigwedge^{c-1} E$ by the corresponding $u_1 \wedge \ldots \wedge u_{d+2} \in \bigwedge^{d+2} \check{E}$ we have the Barsotti–Weil–Siegel equation (5.8)

$$N(u_1, u_2, \ldots, u_{d+2}; x) = 0 \tag{5.10}$$

representing $V^c = V_d$ as the intersection of all the projecting cones from generic spaces $(u_1 \wedge \ldots \wedge u_{d+2})$ not meeting V^c.

EXAMPLES (for an irreducible curve Γ in $\mathbf{P}_3$).

$$S(x_1, x_2) = 0, \qquad Y(u, v) = 0, \qquad N(u_0, u_1, u_2; x) = 0$$

represent Γ via the complex $\mathfrak{C}(\Gamma)$ (cf. Fig. 2), where a line $l \in \mathfrak{C}(\Gamma)$ is defined by a couple of points (or of planes $(c = 2,\ d = 1)$). Γ appears as intersection of all its projecting cones from outside points $P = (a) = (u_0 \wedge u_1 \wedge u_2)$ given by a single $a \in E$ or as intersection of three linearly independent planes (u_0), (u_1), (u_2).

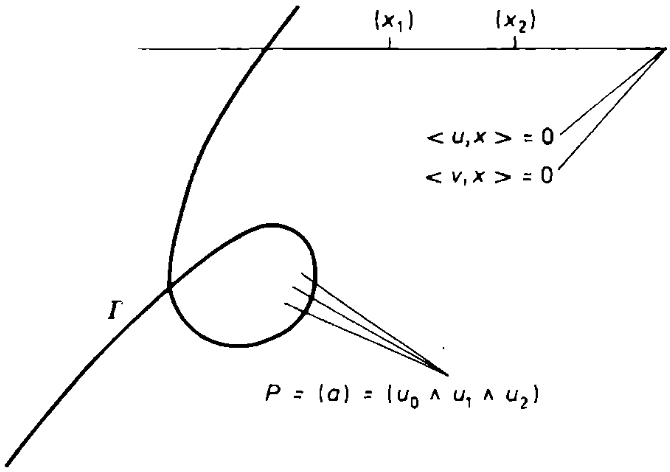

Fig. 2

6. Review on associate forms

We shall recall here the main properties of the a.f. needed subsequently referring — for further details to the original papers [vdW-Ch], the Einführung [vdW 1] (with the 2-nd historical appendix), the Zag book, [H-P II], Ch. X, § 6, 7, 8 and Severi's comments in his paper on Grassmannians [S]. First of all there are uniquely defined linear combinations of the standard power products of Grassmann coordinates $p^{i_1 i_2 \ldots i_c}$ (or $^{\cdot}p_{j_0 j_1 \ldots j_d}$) of $x_1 \wedge x_2 \wedge \ldots \wedge x_c$ (or $u_0 \wedge u_1 \wedge \ldots \wedge u_d$) in $\bigwedge^c E$ (or $\bigwedge^{d+1} \check{E}$) such that

$$S(x_1, x_2, \ldots, x_c) = \mathbf{S}(x_1 \wedge \ldots \wedge x_c) \tag{6.1}$$

$$Y(u_0, u_1, \ldots, u_d) = \mathbf{Y}(u_0 \wedge u_1 \wedge \ldots \wedge u_d) \tag{6.2}$$

and the transition between the right hand sides of (6.1), (6.2) is given by the well-known formulas of type

$$S_b(x_1, x_2, \ldots, x_{c-k}) = S(x_1, x_2, \ldots, x_{c-k} b_{c-k+1} \ldots b_c)$$

where S_b is the Cayley–Severi form of the projecting cone of V from the S_{k-1}-subspace represented by $(b_{c-k+1} \wedge \ldots \wedge b_c)$ with $S_{k-1} \cap V = \emptyset$. The identical vanishing takes place iff $S_{k-1} \cap V \neq \emptyset$.

In particular for $k = c-1$ we obtain back the original Cayley's idea of representing V as intersection of all its projecting cones from S_{c-2} projecting centers not meeting V.

The fact that V is the locus of singular points of the complex $\mathfrak{C}(V)$ gives rise to a canonical system of equations of V expressing the fact that for a point $(a) \in V$ the equation of the projecting cone from (a) vanishes identically $\Leftrightarrow$ (a) is a singular point of $\mathfrak{C}(V)$.

In order to get the properties of the Barsotti–Weil–Siegel form (*characteristic form* = *Normalgleichung*) it is convenient to represent the projection center S_{c-2} ($\cap V = \emptyset$) as a complete intersection of $d+2$ hyperplanes

$$(u_0), (u_1), \ldots, (u_{d+1}) (\in \mathbf{P}(\check{E})).$$

This will give us an identity of type:

(6.4) $$S(x;\, x_2, \ldots, x_c) = N(u_0, u_1, \ldots, u_{d+1};\, x).$$

We shall give a more explicit expression of (6.4) using the fact that we can write:

(6.5) $$v_0 \wedge v_1 \wedge \ldots \wedge v_d = x \rfloor u_0 \wedge u_1 \wedge \ldots \wedge u_{d+1}.$$

(cf. [Bou]) if we normalize conveniently x, where the point (x) belongs to the intersection of the $d+2$ linearly independent hyperplanes $\langle u_j, x\rangle = 0$, $j = 0, 1, \ldots, d+1$, namely

(6.4)′ $$S(x;\, x_2, \ldots, x_c) = N(u_0, u_1, \ldots, u_{d+1};\, x) = \mathbf{N}(\xi_0, \xi_1, \ldots, \xi_{d+1}).$$

(6.6) $$\xi_j = \langle u_j, x\rangle = \sum_{l=1}^{n} u_{jl} x_l, \quad j = 0, 1, \ldots, d+1.$$

The form $\mathbf{N}$ contains coefficients depending on $u_0, u_1, \ldots, u_{d+1}$ that can be determined explicitly. (6.5) has the following remarkable geometric interpretation:

Let $\mathbf{P}(E/E_{c-1})$ be the quotient projective space of E with respect to the subspace E_{c-1} represented by $x_2 \wedge \ldots \wedge x_c \in \bigwedge^{c-2} E$ (which is also represented by $(v_0 \wedge \ldots \wedge v_{d+1})$ $(\in \bigwedge^{d+2} E)$).

(6.7) $$\dim E/E_{c-1} = d+2 \Leftrightarrow \dim \mathbf{P}(E/E_{c-1}) = d+1.$$

The $d+2$ forms u_j linearly independent of $\check{E}$ can be regarded also as forms in E/E_{c-1} because E_{c-1} is defined by $\langle u_j, x\rangle = 0$ for, $j = 0, \ldots, d+1$: $\langle u_j, x\rangle = \langle u_j, x+y\rangle$ $\forall y \in E_{c-1}$.

As a consequence: *the* $d+2$ *forms* ξ_j $(j = 0, 1, \ldots, d+1)$ *are homogeneous coordinates in the quotient projective space* $\mathbf{P}_{d+1} = \mathbf{P}(E/E_{c-1})$.

The equation:

(6.8) $$\mathbf{N}(\xi_0, \xi_2, \ldots, \xi_{d+1}) = 0$$

cf. (6.4)′ represents a hypersurface model of V_d lying in $\mathbf{P}_{d+1} = \mathbf{P}(E/E_{c-1})$ whose points are naturally mapped to the generators of the projecting cone of V from $\mathbf{P}(E_{c-1})$.

Since E_{c-1} can be any vector subspace of E such that $\mathbf{P}(E_{c-1}) \cap V = \emptyset$ we have a refinement of Cayley's idea in the sense that given one of those Cayley's projection centers $\mathbf{P}(E_{c-1})$ $(\cap V = \emptyset)$ (6.8) defines an ordinary equation of a hypersurface $H_{E_{c-1}}$ model of V, for every choice of forms u_j $(j = 0, 1, \ldots, d+1)$ defining E_{c-1}.

The points of this hypersurface correspond bijectively with the $\mathbf{P}_{c-1}$ generators of the Cayley cone of center $\mathbf{P}(E_{c-1})$. For a generic choice of $\mathbf{P}(E_{c-1})$ the generic generator of this projecting cone contains just one point of V, the exceptional ones correspond bijectively with the singular points of $H_{E_{c-1}}$.

In particular if $d = \dim V = n-1$ the $n+1$ linearly independent forms $u_0, u_1, \ldots, u_n$ in $\check{E}$ define a coordinate system in $\check{E} \Rightarrow$ a projective system in $\mathbf{P}_n$; thus in this case

$$(6.9) \qquad \mathbf{N}(\langle u_0, x\rangle, \langle u_1, x\rangle, \ldots, \langle u_n, x\rangle) = 0$$

with $\langle u_j, x\rangle = \sum_{k=0}^{n} u_{jk} x^k$ defines the equation of the hypersurface V in this coordinate system, or in the language of invariants:

(6.9) represents all the possible equations of the hypersurface V for all the choices with $u_0 \wedge \ldots \wedge u_n \neq 0$.

Examples. In the case of Fig. 2, page 91, any triple of linearly independent linear forms u_0, u_1, u_2 define a projective coordinate system with (u_j) $(j = 0, 1, 2)$ as coordinate planes and $(u_0+u_1+u_2)$ as the unit "line" in the abstract plane $\mathbf{P}(E_4/E_1)$, where $P = \mathbf{P}(E_1)$ is any point of $\mathbf{P}_3 = \mathbf{P}(E_4)$ outside V intersection of the three planes $\langle u_j, x\rangle = 0$. The equations

$$\mathbf{N}(\xi_0, \xi_1, \xi_2) = 0, \qquad N(\langle u_0, x\rangle, \langle u_1, x\rangle, \langle u_2, x\rangle) = 0$$

represent a model of V in $\mathbf{P}(E_4/E_1)$ or the projecting cone of V with vertex P.

Remark. The explicit computation of $\mathbf{N}$ in terms of $\mathbf{S}$ can be achieved expressing (6.5) in coordinates, replacing $p^{i_0 i_1 \ldots i_d}$ by

$$(6.15)' \qquad p^{i_0 i_1 \ldots i_d} = \sum x^i q^{i i_0 i_1 \ldots i_d}$$

where $q^{j_0 j_1 \ldots j_{d+1}}$ are the coordinates of $x \lrcorner u_0 \wedge u_1 \wedge \ldots \wedge u_{d+1}$ leading to

$$(6.10) \qquad p^{i_0 i_1 \ldots i_d} = \begin{vmatrix} \langle u_0, x\rangle & u_{0i_0} & u_{0i_1} & & u_{0i_d} \\ \langle u_1, x\rangle & u_{1i_0} & u_{1i_1} & & u_{1i_d} \\ \vdots & \vdots & \vdots & & \vdots \\ \langle u_{d+1}, x\rangle & u_{d+1,i_0} & u_{d+1,i_1} & \cdots & u_{d+1,i_d} \end{vmatrix}.$$

III. Applications

The construction of $J = J(V^{(1)} \times \ldots \times V^{(h)})$, $V^{(j)} \subset \mathbf{P}_n$, $j = 1, 2, \ldots, h$, has two natural applications depending on the codimension of J in $\mathbf{P}(E \otimes \mathbf{C}^h)$. Cf. (2.12), page 84; if $c \leqslant n$ the h given varieties $V^{(j)}$ always meet in $\mathbf{P}_n$ ($\Leftrightarrow \Delta$ always meet J in $\mathbf{P}(E \otimes \mathbf{C}^h)$). If $c > n$ the given varieties do not meet ($\Leftrightarrow \Delta \cap J = \emptyset$) if they are in generic position, but the discussion of their meeting gives a new form to the old "elimination theory" which can be made intrinsic. We divided the paper in two parts according to both possibilities:

In Part I, page 94, § 7, 8, 9, 10 we deal with the case $c \leqslant n$. If $S \in \mathfrak{C}(J)$, $J = J(V^{(1)} \times \ldots \times V^{(h)})$, since $\dim S = c-1 < n$, it makes sense to introduce the restriction to the diagonal space Δ in $\mathbf{P}(E \otimes \mathbf{C}^h)$ (cf. § 3, page 84). Such a restriction is trivial ($\Leftrightarrow \mathfrak{C}(J) | \Delta = \mathscr{G}(c-1; \Delta)$) iff the intersection $\bigcap_{j=1}^{h} V^{(j)}$ is improper. Otherwise, there is a well defined complex $\mathfrak{C}(J) | \Delta$ whose pull-back to $\mathbf{P}(E)$ by δ^{-1} gives the natural definition of $\mathfrak{C}(I)$, where $I = V^{(1)} \cdot V^{(2)} \cdot V^{(h)}$ is the intersection cycle. The prime factor decomposition gives the intersection multiplicity as the exponent of either one of S_C, Y_C or N_C (any two of them agree) for any irreducible component C of I. See our main definition Def. 7.1, page 95. In particular, we can prove Bézout's theorem since $\deg \dot{J} = \prod_{j=1}^{h} \deg V^{(j)}$ can be proved with a rigorous degeneration method using the characteristic transversality condition for multiplicity one.

The announced equivalence of the exponent intersection multiplicity with the original one of van der Waerden follows easily from the interpretation of the S_n subspaces of $\mathbf{P}(E \otimes \mathbf{C}^h)$ as representatives of h-collineations cf. § 4.

First part
The exponent intersection multiplicity

7. Restriction to the diagonal of $\mathfrak{C}(J(V^{(1)} \times V^{(2)} \times \ldots \times V^{(h)}))$, $c \leqslant n$

Let us come back to the constructions of § 0, 1. Let $J = J(V^{(1)} \times \ldots \times V^{(h)})$ be the join of the h shown irreducible varieties of codimensions c_j in $\mathbf{P}_n$ satisfying (0.7). Let us consider the complex $\mathfrak{C}(J)$ of $(c-1)$-dimensional subspaces of $\mathbf{P}(E \otimes \mathbf{C}^h) = \mathbf{P}_{h(n+1)-1}$ attached to J:

(7.1) $$\mathfrak{C}(J) = \{\mathbf{P}_{c-1} \subset \mathbf{P}(E \otimes \mathbf{C}^h) \mid \mathbf{P}_{c-1} \cap J \neq \emptyset\} \subset \mathscr{G}(c-1; h(n+1)-1)$$

then, since $c \leqslant n$ the restriction to the n-dimensional diagonal space Δ:

(7.2) $$\mathfrak{C}(J) \mid \mathscr{G}(c-1, \Delta)$$

makes sense. We shall distinguish two cases:

(1) If $\bigcap_{j=1}^{h} V^{(j)}$ is improper $\Leftrightarrow$ if there is at least one excedentary irreducible component $X \Leftrightarrow \operatorname{cod} X < n$ then

$$\delta(X) \cap \mathbf{P}_{c-1} \neq \emptyset, \quad \forall \mathbf{P}_{c-1} \subset \Delta.$$

(2) On the contrary: If $\bigcap_{j=1}^{h} V^{(j)}$ is proper we can construct some subspace $\mathbf{P}_{c-1} \subset \Delta$ satisfying

$$\delta(\bigcap V^{(j)}) \cap \mathbf{P}_{c-1} = \emptyset.$$

It suffices to take the diagonal image of a $\mathbf{P}_{c-1}$ of $\mathbf{P}_n$ not meeting $\bigcap_{j=1}^{h} V^{(j)}$.

In other words, we have proved the following:

LEMMA. *The diagonal space $\dot{\Delta}$ of $P_{h(n+1)-1}$ (cf. § 3) is a singular space of $\mathfrak{C}(J)$ iff $\bigcap_{j=1}^{h} V^{(j)}$ is improper. Otherwise the restriction of $\mathfrak{C}(J)$ to Δ defines the complex* (7.4) *below which will be attached to the intersection cycle I by the formula*

(7.4) $$\mathfrak{C}(I) = \delta^{-1}(C(J) \mid \Delta)$$

where $I = V^{(1)} \cdot V^{(2)} \cdot \ldots \cdot V^{(h)}$.

MAIN DEFINITION

DEFINITION 7.1. The complex of S_{c-1} subspaces of $\mathbf{P}_n$ defined by (7.4) is called *the complex attached to the* (well-defined) *intersection cycle* $I = V^{(1)} \cdot V^{(2)} \cdot \ldots \cdot V^{(h)}$ of the h given properly intersecting varieties.

Remark. The effective restriction $\mathfrak{C}(J) = \delta^{-1}(\mathfrak{C}(J) | \Delta)$ can be achieved by means of either one of the associated forms discussed in § 6, namely:

We know that the Severi form S_J attached to J contains c covariant vector variables $x_1, x_2, \ldots, x_c$. It suffices to take $x_j \in \Delta$ for $j = 1, 2, \ldots, c$ to get the desired restriction. For Y_J there are $d+1$ hyperplane variables representing a $\mathbf{P}_{c-1} \in \mathfrak{C}(J)$ where $d = h(n+1)-1-c$. The corresponding number of a $\mathbf{P}_{c-1} \subset \Delta$ is $n-c+1$. The difference $(n+1)-(h-1)$ equals the number of equations of type

$$x_j^{(r)} - x_j^{(1)} = 0, \quad r = 2, \ldots, h;\ j = 0, 1, \ldots, n \tag{7.5}$$

defining Δ. Thus, we shall define a $\mathbf{P}_{c-1} \subset \Delta$ with forms containing the (7.4). The rest define the same $\mathbf{P}_{c-1}$ as a subspace of Δ.

Similarly, the Barsotti–Weil–Siegel form suffices to restrict the generic projection center of dimension $c-2$ — in the ambient space of J — to the diagonal subspace Δ.

In the three cases we have prime factor decompositions of $S | \Delta$, $Y | \Delta$, $N | \Delta$ with prime factors S_C, Y_C, N_C attached bijectively to all the proper irreducible components C of $\bigcap_{j=1}^h V^{(j)}$ and equal exponents i_C:

$$S_I = \prod S_C^{i_C}, \quad Y_I = \prod Y_C^{i_C}, \quad N_I = \prod N_C^{i_C}. \tag{7.6}$$

Such equality is indeed a consequence of the transformation formulas between S_I, Y_I, N_I studied in § 6.

MAIN DEFINITION 2:

DEFINITION 7.2. The positive integer i_C well defined by either one of the (7.5) in an intrinsic way is called *the exponent intersection multiplicity of* C *in* I (cf. (7.4)).

8. Computation of F_J. Bézout's theorem

The computation of the Chow form Y_V on any irreducible $V \subset \mathbf{P}_n$ is based on the theory of the u-resultant (cf. [H-P], I). It can be applied to $J = J(V \times W)$ when we give any two systems of equations in (x), (y) to represent V and W. From Y_V we can construct S_V and N_V. A direct computation of any S_V with $\operatorname{cod} V = c$ can be obtained by

$$S_V(x_1, x_2, \ldots, x_c) = \text{h.c.d.}(\ldots, R_k, \ldots) \tag{8.1}$$

where the R_k are resultant forms with respect to $\lambda_1, \ldots, \lambda_c$ in the equations

$$f_k\left(\sum_{j=1}^{c} \lambda_j x_j\right) = 0$$

obtained by the specialization $x \mapsto \sum_{j=1}^{c} \lambda_j x_j$ in the equations $\ldots f_k(x) = 0 \ldots$ representing V.

Remark. It is remarkable, very simple and essentially "new" (since the N-form is not widely used in the literature) that the equation

$$N\left(\sum_{k=0}^{n} u_{jk} x_k\right) = 0, \quad j = 0, 1, \ldots, d+1, \tag{8.2}$$

can be obtained immediately observing that the transcendence degree of the projecting cone $\Gamma(V)$ of V is equal to $d+1$. Accordingly: The $d+2$ restrictions

$$\left(\sum_{k=0}^{n} u_{jk} x_k\right) \mid \Gamma(V),$$

$j = 0, \ldots, d+1$, are algebraically dependent.

For instance, let V be an irreducible algebraic curve in $\mathbf{P}_n$. Then we write immediately an irreducible equation

$$F(\langle u_1, x\rangle, \langle u_2, x\rangle, \langle u_3, x\rangle) = 0 \tag{8.3}$$

representing V as intersection of all the projecting cones from $\mathbf{P}_{n-3}$-subspaces, complete intersections of the three hyperplanes

$$\langle u_j, x\rangle = 0, \quad j = 1, 2, 3, \tag{8.4}$$

where $\langle u_i, x\rangle = \sum_{k=0}^{n} u_{jk} x_k = 0$.

In particular, if V is a canonical curve — non hyperelliptic — of genus g in $\mathbf{P}_{g-1}$ we can take three generic holomorphic differentials to define the Barsotti–Weil–Siegel form. We shall apply elsewhere this remark to the Schottky problem, cf. [G.4].

Remark. The following natural question arises; let F_j be associate forms (of the same kind S, Y, N) corresponding to h algebraic irreducible $V^{(j)} \subset \mathbf{P}_n$. Can we compute F_J in terms of the F_j? (where $J = J(V^{(1)} \times \ldots \times V^{(h)})$). If the $V^{(j)}$ are all hypersurfaces: $c_j = 1$ and $c = h \leqslant n$, the answer is positive, because $F_J =$ Resultant form with respect to $\lambda_1, \lambda_2, \ldots, \lambda_c$ of the c equations

$$F_j(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_c x_c) = 0, \quad j = 1, 2, \ldots, c. \tag{8.5}$$

If $h = c = 2$ a good improvement can be made remarking that then the resultant of the two binary forms in (λ_1, λ_2) has the explicit well-known Sylvester form. Since the S and N a.f. represent a given irreducible V as complete intersection of projecting cones we can try to reduce the computation of F_J with $J = J(V \times W)$ in terms of F_V and F_W to the previous case as follows: Let $\mathbf{P}_{c_1-2}$ and $\mathbf{P}_{c_2-2}$ be two generic projection centers for V^{c_1}, W^{c_2} lying in $\mathbf{P}(E, 0)$ and $\mathbf{P}(0, E)$, respectively. Let $S_{c-1} = J(S_{c_1-1} \times S_{c_2-1})$ be their join with $\dim S_{c-1} = c-1$, $c = c_1 + c_2$ with $S_{c-1} \supset \mathbf{P}_{c_1-1}$, $S_{c-1} \supset \mathbf{P}_{c_2-2}$ and $S_{c-1} \supset J(S_{c_1-2} \times S_{c_2-2}) = \Sigma_{c-3}$. Then we can compute the equation of the projecting cone of $J(V\times W)$ from any $\mathbf{P}_{c-2}$ joining Σ_{c-3} with any point (x) by means of a Sylvester determinant D:

$$(8.6)\qquad D = \begin{vmatrix} a_0 & a_1 & \dots\dots & a_f & & & \\ 0 & a_0 & \dots\dots\dots & & a_f & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & a_0 & \dots\dots\dots\dots & & & a_f \\ b_0 & b_1 & \dots\dots\dots\dots & b_g & & & \\ & b_0 & b_1 & \dots\dots\dots\dots & b_g & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & b_0 & b_1 \dots & b_g \end{vmatrix}$$

where $f = \deg V$, $g = \deg W$, and $F(\lambda x+\mu y) = \sum_{j=0}^{f} a_j \lambda^j \mu^{f-j}$, $G(\lambda x+\mu y) = \sum_{j=0}^{g} b_j \lambda^j \mu^{f-j}$ where $F(x) = 0$, $G(x) = 0$ are the equations of the projecting cones of $V(W)$ from $\mathbf{P}_{c-2}(\mathbf{P}_{c_2-2})$, respectively. An immediate consequence of this property is the following:

The degree of $J(V\times W)$ is equal to the product of degrees of $J(V)$, $J(W)$

$$(8.7)\qquad \deg J(V\times W) = \deg V\cdot \deg W.$$

The intersection $J\cap\Delta$ has the same degree; accordingly we have:

BÉZOUT'S THEOREM. *Let $V\cdot M$ be the intersection cycle of two irreducible algebraic varieties V, W_{c_n} meeting properly. We have*

$$(8.8)\qquad \deg V\cdot W = \deg V\cdot \deg W$$

as a consequence of (8.7).

9. On the proof of the Theorems

In the expository part of the introduction and in the exposition of the adaptation of the reduction to the diagonal in the projective case (§ 1, 2, 3) we gave already all the necessary ingredients to prove Theorem I but – since there we lacked some technical tools, for instance the relation between the diagonal space Δ and the diagonal Σ (cf. (9.1), below), the more precise recall on associate forms, etc.:

$$(9.1)\qquad \Sigma = \text{Segre variety} = \mathbf{P}\{x\otimes(\lambda_1, \dots, \lambda_h) \mid x\in E,\ (\lambda_1, \dots, \lambda_h)\in \mathbf{C}^h\}$$

with $\Delta = \mathbf{P}(E\otimes(1, 1, \dots, 1))$ – it is convenient for the reader to have now a complete version of the proof. On the other hand with the same procedure we shall prove also Theorem II, although we shall come back to it, in § 12.

Proof of Theorem I. If the set-theoretic intersection $\bigcap_{j=1}^{h} V^{c_j}$ with $c = \sum_{j=1}^{h} c_j \leqslant n$ is improper there is at least one excedentary irreducible component X of codimension less than c; as a consequence every $\mathbf{P}_{c-1}\subset \mathbf{P}_n$

meets I; this is equivalent to the fact that every $\mathbf{P}_{c-1} \subset \Delta$ meets $\delta(I)$; i.e. Δ is a singular space for $\mathfrak{C}(J(C))$, where

$$\mathscr{J}(C) \leqslant \mathscr{G}(c-1; \mathbf{P}(E \otimes \mathbf{C}^h)).$$

cf. Definition 5.4, page 89.

On the other hand, iff $\bigcap_{j=1}^{h} V^{c_j}$ is proper it is always possible to find a $\mathbf{P}_{c-1} \subset \mathbf{P}_n$ such that $\mathbf{P}_{c-1} \cap (\bigcap_{j=1}^{h} V^{(j)}) = \emptyset$. This implies that there exists some $\mathbf{P}_{c-1} \subset \Delta$ which do not meet $J(V^{(1)} \times \ldots \times V^{(h)})$; in other words, we have a proper restriction to Δ of the complex $\mathfrak{C}(J)$ attached to J iff $\cap$ is proper; such restriction can be effectively computed by restriction to Δ of either one of the equivalent form S_J, Y_J, N_J. The prime factor decompositions determine uniquely the exponents of the irreducible factors C (cf. § 5, 6); each irreducible factor has the form S_C, Y_C on N_C appearing with the same well defined exponent

$$i_I = i(V^{(1)}, V^{(2)}, \ldots, V^{(h)}; I).$$

10. Equivalence of the exponent multiplicity with van der Waerden's theory

The exponent multiplicity theory enables an easy transition (in both directions) between the so-called *static* and the *dynamic* multiplicity theories, (cf. [F]) roughly speaking it is equivalent *to move the intersecting varieties V, W or to move the diagonal space Δ*. But Δ belongs to $\mathscr{G}(n; 2n+1)$ and such *motion* is quite well understood. On the other hand a generic $S_n \subset \mathbf{P}(E \otimes E)$ represents a nonsingular collineation γ in $\mathbf{P}_n$ (where S_n, S'_n represent the same collineations iff they are equivalent under the group $\mathscr{D}$) of collineations of $\mathbf{P}(E \otimes E)$ in itself (cf. § 4).

The original van der Waerden's multiplicity theory (cf. ZAG-papers, the historical survey [vdW2] and [H-P] (vol. II)) relies precisely in a motion of the pair (V, W) of irreducible subvarieties in $\mathbf{P}_n$ to $(\gamma_1 V, \gamma_2 W)$ by means of generic collineations $\gamma_1, \gamma_2 \leftarrow \mathbf{P}\,\mathrm{GL}(E)$. $(\gamma_1 V, \gamma_2 W)$ gives essentially the same as $(\gamma\gamma_1 V, \gamma\gamma_2 W)$ where $\gamma \in \mathrm{GL}(E)$, thus we can consider also $(V, \gamma_1^{-1}\gamma_2 W)$ or $(\gamma_2^{-1}\gamma_1, V, W)$ (with the inconvenience of a subsequent need of a proof of the symmetry of $i(I; V \cdot W)$ when we permute V and W. Anyway the intersection multiplicity $i(I; V \cdot W)$ (for I irreducible component of $V\,W$) is defined in vdW's theory by specialization when $(\gamma_1 \gamma') \to$ Identity.

The equivalence of the exponent multiplicity with van der Waerden's appears naturally when we replace the motion of γ with the (equivalent) "motion of Δ". We shall make explicit this equivalence:

We recall that a nonsingular collineation γ: $\mathbf{P}_n \to \mathbf{P}_n (\mathbf{P}_n = \mathbf{P}(E))$ is uniquely defined by a pair of bases B, B' where $\gamma = \mathbf{P}(L)$, $L \in \mathrm{GL}(E)$ and L is uniquely defined by $B' = LB$. Let us construct the $n+1$ vectors

$$(b_j, b'_j) \in E \otimes E, \quad j = 0, 1, \ldots, n. \tag{10.1}$$

They form a basis of the subspace $E \otimes E$. The bases $(\lambda_1 B_1, \lambda_2 B_2)$, $\lambda_1, \lambda_2 \in \mathbf{C}^{\times}$ define another subspace $D_{\lambda_1, \lambda_2}\mathbf{L}$ representing the same collineation $\gamma = \mathbf{P}(L)$ as L. $D_{\lambda_1, \lambda_2} L = L$ iff $\lambda_1 = \lambda_2$.

Two points $P = (x)$, $P' = (x')$ correspond in γ iff

$$(10.2) \qquad x = \sum_0^n \lambda_j b_j, \quad x' = \sum_0^n \lambda_j b'_j$$

with $(\lambda_0, \lambda_1, \ldots, \lambda_n) \neq (0, 0, \ldots, 0)$; $(x, x') \in \mathbf{L}$.

Conversely any point $(\neq 0)$. $(x, x') \in \mathbf{L}$ defines a pair of corresponding points (x), (x') in $\mathbf{P}_n$.

The set-theoretic intersection $\mathbf{L} \cap J(V \times W)$ can be interpreted as the set of pairs $(x) \times (x') \in V \times W$ with $(x') = \gamma(x)$. The specialization $\gamma \to \text{id}$, will give back $\Delta \cap J(V \times W)$ leading to the natural definition of the intersection cycle $A \cdot W$.

The precise nature of the equivalence between the exponent multiplicity and the original VdW's can be retraced quoting the following paragraph of the historical survey [vdW] or the Einführung [vdW] (page 276): "*Applying a projective transformation*": ... "*therefore I proposed in* 1928 *to bring V and W into a generic relative position by applying to one of them a projective transformation T with indeterminate coefficients. The transformed variety T intersects W in a finite number of points. If T is specialized to the identity, the points of intersection of TV and W specialize to points of intersection of V and W. If V and W meet in a finite number of points, each of these appears with a certain multiplicity, which may be defined to be the intersection multiplicity* ..."

In order to adapt vdW's words to our procedure with the join in $\mathbf{P}(E \otimes E)$ let us assume now $\dim V \cdot W = 0 \Leftrightarrow c = n$. Then to the projective T let us associate the $S_n \subset \mathbf{P}(E \otimes E)$ defined by the $n+1$ points of the $(n+1) \times 2(n+1)$ matrix:

$$(10.3) \qquad (1_{n+1}\ T)$$

where 1_{n+1} is the $(n+1) \times (n+1)$ unit matrix. The specialization $T \to$ Identity specializes (10.3) to $(1_{n+1}\ 1_{n+1})$ defining the diagonal space.

In the general case for any $c < n$, the intersection of $J(V \times W) \cdot S_c$ reduces the problem to $J(V \cdot S_c \times W \cdot S_c) \subset S_c$ where $V \cdot S_c \subset S_c$, $W \cdot S_c \subset S_c$ and we have again the previous case: $\dim V \cdot W = 0$.

In the discussion with the complex $\mathfrak{C}(J)$, we need to consider a variable S_{c-1}. The same reduction to $V \cdot S_{c-1} \times W \cdot S_{c-1} \subset \mathbf{P}(E_c \otimes E_c)$ $(S_{c-1} = \mathbf{P}(E_c))$ leads to a case discussed in Part II of this paper, because now for the varieties in generic position $V \cdot S_{c-1} \cap W \cdot S_{c-1} = \emptyset$ and $\operatorname{cod} J(V \cdot S_{c-1} \times W \cdot S_{c-1})$ in $\mathbf{P}(E_c \oplus E_c)$ is equal to $c = \dim S_{c-1} + 1$.

11. Bézout's Theorems with a new degeneration method

The original discovery of the property

$$\text{dig}\, F \cdot G = fg, \quad f = \deg F, \quad g = \deg G \tag{11.1}$$

of the intersection cycle $F \cdot G$ of two irreducible algebraic curves in $\mathbf{P}_2$ was obtained in a pure heuristic – nonrigorous way – by degeneration of F, G in generic sets of f (resp. g) different lines–intersecting together in fg simple points. I do not believe that anybody thought of this remark as a proof, but it has been always interesting to know whether this can be transformed indeed in a proof. We shall show here that by means of a certain degeneration (not of F, G, but of a secant space of complementary dimension) we can prove that

$$\deg J(V \times W) = \deg V \cdot \deg W \tag{11.2}$$

where V, W are again two irreducible varieties $V \subset \mathbf{P}(E_1)$, $W \subset \mathbf{P}(E_2)$. In fact, it is well known that we can choose subspaces $L \subset \mathbf{P}(E_1)$, $M \subset \mathbf{P}(E_2)$ such that the intersection cycles consist of different simple points:

$$V \cdot L = P_1 + P_2 + \ldots + P_f, \qquad W \cdot M = Q_1 + Q_2 + \ldots + Q_g, \tag{11.3}$$

$f = \deg V$, $g = \deg W$, $P_i \neq P_j$, $Q_i \neq Q_j$, $i \neq j$.

On the other hand, the join $J(L \times M)$ is a subspace of $\mathbf{P}(E_1 \oplus E_2)$ of dimension equal to cod V + cod W + 1. The set-theoretic intersection consists of fg lines $J(P_i \times Q_j)$:

$$J(V \times W) \cap J(L \times N) = \bigcup J(P_i \times Q_j), \quad i = 1, 2, \ldots, f, \ j = 1, 2, \ldots, g. \tag{11.4}$$

The transversality criterion for multiplicity one in each P_i or Q_j implies the transversality condition for the line $J(P_i \times Q_j)$. As a consequence we have:

$$J(V \times W) \cdot J(L \times N) = \sum J(P_i \times Q_j), \quad i = 1, 2, \ldots, f, \ j = 1, 2, \ldots, g. \tag{11.5}$$

In the same way we can see that we can choose a hyperplane $\sum u_i x_i + \sum v_j y_j = 0$ in $\mathbf{P}(E_1 \oplus E_2)$ transversal to each fixed $J(P_i \times Q_j)$ because the opposite implies

$$\lambda \left(\sum u_k \xi_k^{(i)}\right) + \mu \sum v_i \eta_i^{(j)}) = 0 \tag{11.6}$$

where $i = 1, 2, \ldots, \deg V$, $j = 1, 2, \ldots, \deg W$.

Second part: $c > n$

Let us consider now the case $c > n$. Then if the given irreducible $V^{(j)} \subset \mathbf{P}_n$ are generically located the intersection is empty, i.e. we have

$$\bigcap_{j=1}^{h} V^{(j)} = \emptyset \Leftrightarrow J(V^{(1)} \times \ldots \times V^{(h)}) \cap \Delta = \emptyset. \tag{11.7}$$

The complex $\mathfrak{C}(J)$ consists then of spaces of dimension $c-1 \geqslant n$ and our task is just to express the exceptional behaviour:

$$\text{(11.8)} \qquad \bigcap_{j=1}^{h} V^{(j)} \neq \emptyset \Leftrightarrow J \cap \Delta \neq \emptyset$$

in terms of associate forms.

The extreme case $c = n+1$ appears in our treatment because then a $\mathbf{P}_{c-1}$ is a $\mathbf{P}_n$ and in particular the alternative (11.7) or (11.8) is equivalent to *the diagonal space Δ does not belong to $\mathfrak{C}(J)$ iff the intersection is empty or $\Delta \in \mathfrak{C}(J)$ iff $\bigcap_{j=1}^{h} V^{(j)} \neq \emptyset$.*

For $c > n+1$ the property $\Delta \cap J \neq \emptyset$ implies that every $\mathbf{P}_{c-1}$ containing Δ meets J:

$$\mathbf{P}_{c-1} \supset \Delta \Rightarrow \mathbf{P}_{c-1} \cap J \neq \emptyset$$

but the converse property is true:

If every $\mathbf{P}_{c-1} \supset \Delta$ meets J then Δ meets J (equivalently, if $\Delta \cap J = \emptyset$ it is possible to find a $\mathbf{P}_{c-1} \supset \Delta$ such that $\mathbf{P}_{c-1} \cap J = \emptyset$).

This property leads naturally to express the condition $J \cap \Delta \neq \emptyset$ by the identical vanishing of a covariant, as indicated in the Introduction.

12. A geometrical theory for resultant systems

In the particular case $c_1 = c_2 = \ldots = c_h = 1$, $h = c > n$ we come back to the compatibility conditions of a system of $h = c > n$ homogeneous polynomial equations

$$\text{(12.0)} \qquad F_1 = 0, \quad F_2 = 0, \quad \ldots, \quad F_h = 0$$

of degrees $m_1, m_2, \ldots, m_h$.

It is well known that in the extreme case $c = n+1 = h$ the compatibility conditions are characterized by the vanishing on a single equation

$$\text{(12.1)} \qquad R = 0$$

where R is a polynomial homogeneous of degree m/m_j in the $\binom{n+m_j}{n}$ indeterminate coefficients of a generic form of degree m_j where

$$\text{(12.2)} \qquad m = m_1 m_2 \ldots m_{n+1}$$

$R = 0$ is equivalent to $\Delta \cap J \neq \emptyset$ where $J = J(H_1 \times \ldots \times H_{n+1})$ as before and $F_j = 0$ defines the irreducible hypersurface H_j, $j = 1, 2, \ldots, n+1$.

Thus in the general case $c = n+1$, the characteristic condition $J \cap \Delta \neq \emptyset \Leftrightarrow H_j \in \mathfrak{C}(J)$ is a generalization of the equation $R = 0$.

In the case $c > n+1$ we checked already in the introduction that $\Delta \cap J \neq \emptyset$ is equivalent to the identical vanishing of the Cayley–Severi form S_J (see Def. 5.6, page 90) for $U_0, U_1, \ldots, U_n$, $X_1, X_2, \ldots, X_{c-n-1}$ where

$U_j = (u_j, u_j, \overset{h}{\ldots}, u_j) \in \Delta$, $j = 0, 1, \ldots, n$ and the x_i are arbitrary vectors of $E \otimes \mathbf{C}^h$; $l = 1, 2, \ldots, c-n-1$, i.e.

$$S_J(U_0, U_2, \ldots, U_n; X_1, X_2, \ldots, X_{c-n-1}) \equiv 0 \tag{12.3}$$

iff $\bigcap_{j=1}^{h} V^{(j)} \neq \emptyset \Leftrightarrow J(V^{(1)} \times \ldots V^{(h)}) \cap \Delta \neq \emptyset$.

In particular, in the "elimination case" again $c_1 = c_2 = \ldots = c_h = 1$, $h = c > n+1$ the condition (12.2) is a covariant in the coefficients of the forms F_j of degree m_j containing $c-n-1$ arbitrary series of variables $X_1, X_2, \ldots, X_{c-n-1}$. The coefficients of the power products in these X's give a system of resultant forms. We hope to study in the near future the relation between this invariant-theoretic approach and the classical ones.

An intrinsic elimination theory

13. Historical approach

The elimination theory has been completely "eliminated" from algebraic geometry! I believe that the main reason is that it was not intrinsic enough; as a matter of fact it was always presented in relation with a coordinate system. For instance, the Henzelt–Noether sophistication of the Kronecker elimination method was presented as follows ([H-N]).

Let $\mathfrak{m} \subset K[x_1, \ldots, x_n]$ be a polynomial ideal. We can associate to $\mathfrak{m}$ a "resultant form"

$$R_{\mathfrak{m}} = R^{(1)}(x_1, \ldots, x_n) \ldots R^{(i)}(x_i, \ldots, x_n) \ldots R^{(n)}(x_n) \equiv 0(\mathfrak{m}) \tag{13.1}$$

in such a way that R vanishes for all the solutions of $\mathfrak{m}$ and only for them. If $\mathfrak{n} \supset \mathfrak{m}$ and $R_{\mathfrak{m}} = R_{\mathfrak{n}}$ then $\mathfrak{m} = \mathfrak{n}$.

We can appreciate that the x_i are explicitly used in the statement and in a given order.

The geometrical meaning of the $R^{(i)}$ is clear. $R^{(1)}$ represents the irreducible components of $V = V(\mathfrak{m})$ of dimension equal to one precisely if

$$R^{(1)} = \prod F_{rk}^{m_k} \tag{13.2}$$

the hypersurface $F_{1k} = 0$ is an irreducible hypersurface contained in the solution variety $V = V(\mathfrak{m})$ and conversely any such hypersurface appears as a prime factor of $R^{(1)}$, $R^{(2)}(x_2, \ldots, x_n) = 0$ represents the projection in the hyperplane $x_1 = 0$ of the locus of irreducible components of codimension two, ... and

$$R^{(i)}(x_i, x_{i+2}, \ldots, x_n) = 0$$

appears as the projection in the coordinate space $x_1 = 0, x_2 = 0, \ldots, x_{i-1} = 0$ of the locus of irreducible components of $V = V(\mathfrak{m})$ of codimension equal to i: $i = 1, 2, \ldots, n$. More precisely, if we want to deal again with projective varieties in $\mathbf{P}_n$ we need to introduce the homogeneous coordinates $x_0, x_1, \ldots, x_n$ and to assume that $\mathfrak{m}$ is homogeneous. Besides *it is necessary to assume that the projective frame of reference is generically located with respect to V.* If this is not the case *it is necessary to apply previously a generic projective transformation to achieve this goal.* We emphasize that: *Such generic projective coordinate changes were never written in the notations*; as a consequence they did not appear in the formulas; accordingly the results are wrongly applied when the reference frame is badly located with respect to the variety defined by (12.0) and again the results are misleading.

The fact that the homogeneous $R^{(i)} = R(x_{i-1}, \ldots, x_n)$ appear as a projection from the coordinate space joining the vertices $P_0, P_1, \ldots, P_{i-2}$ (assumed previously as "well-located") suggests naturally the idea of projecting the locus $\Gamma^{(i)}$ of irreducible components of V of codimension equal to i from a generic $\mathbf{P}_{i-2}$. But this is Cayley's idea! As a consequence the Author in the two papers [G2], [G3] replaces the original problem of "elimination" by the following one:

Let

$$F_j = 0, \quad j = 1, 2, \ldots, r \tag{13.3}$$

be an arbitrary basis of the homogeneous ideal $\mathfrak{m}$. *We shall compute the Cayley–Severi forms* $S^{(i)}$ *of* $\Gamma^{(i)}$, $i = 1, 2, \ldots,$

$$S^{(1)}(x_1) = 0, \quad S^{(2)}(x_1, x_2) = 0, \quad \ldots, \quad S^{(i)}(x_1, \ldots, x_i) = 0, \ldots \tag{13.4}$$

following the same steps as the traditional Kronecker elimination method.

The first step is obvious; $S_1 = \operatorname{hcd}(F_1, F_2, \ldots, F_r)$, i.e. the hypersurface component appears in the same way as in the Kronecker method. The elimination of one variable (which one?) depends on the choice of a well-located ($\Leftrightarrow$ not belonging to $\Gamma^{(2)}$) vertex of the projective frame. If we choose a generic projection center (y) we are reduced to the first step again because such a cone has codimension one. This can be achieved in an elementary way writing $F_j = S^{(1)}\, G_j$, then $G_j = G_j(\lambda x + \mu y)$ and a resultant system in (λ, μ):

$$G_{2k}(x, y) = 0, \quad k = 1, 2, \ldots, r_2. \tag{13.5}$$

Then

$$S^{(2)} = \operatorname{hcd}(G_1, G_2, \ldots, G_{r_2}).$$

In such a way – by induction we construct associate systems of equations

$$F_{ck}(x_1, x_2, \ldots, x_c) = 0, \quad k = 1, 2, \ldots, r_k$$

where $F_{1k} = F_k$, $r_1 = r$. Then $S_c = \operatorname{hcd}(F_{ck} \ldots)$.

With this procedure we can attach to any system (13.3) the associate forms to the $\Gamma^{(i)}$, $i = 1, 2, \ldots, n$. The prime factor decomposition of $S_c(x_1, \ldots, x_c)$

gives all the Cayley–Severi forms of the irreducible components of codimension c of V with a certain intrinsic exponent depending only on $\mathfrak{m}$.

We refer to [G2], [G3] for more details. There is a curious paradoxon in this procedure pointed out already in [G2]: instead of decreasing the number of coordinates by successive "elimination" of $x_0, x_1, \ldots, x_n$ we increase by $n+1$ homogeneous coordinates of $(x_1, x_2, \ldots, x_c, \ldots)$ in every step. However, let us recall that there exists an expression

$$S_c = \mathbf{S}_c(\ldots, p^{i_1 i_2 \ldots i_c}, \ldots)$$

unique if we assume that all the power products of the $p^{\cdots}$ are standard. Let us specialize the projection points—coming back to the elimination theory, assuming them to be the vertices of the projective reference frame $P_0, P_{1\ldots}$ (assuming again that they are well-located to avoid identical vanishing ...). Then we have the coordinate matrix

$$\begin{bmatrix} 1 & 0 & 0 & \ldots 0 & 0 & \ldots 0 \\ 0 & 1 & 0 & \ldots 0 & 0 & \ldots 0 \\ \cdot & \cdot & \cdot & \ldots \cdot & \cdot & \ldots \\ 0 & 0 & 0 & \ldots 1 & 0 & \ldots 0 \\ x_0 & x_1 & x_2 & \ldots x_{c-1} & x_c & \ldots x_n \end{bmatrix}$$

and we remark *that then nonzero Grassmann coordinates are* $x_c, x_{c+1}, \ldots, x_n$. *The* $x_0, x_1, \ldots, x_{c-1}$ *are "eliminated" again.*

Since the three types of associate forms can be transformed among them it is not difficult to compute the Chow or the Barsotti–Weil–Siegel forms. We are definitely interested in the latter because we shall prove in [G4] (cf. § 14 for a short Introduction) that *the computation of these forms is equivalent to use the Kronecker elimination method with generic projective coordinate systems explicitly given in the formulas* by means of basis $u_0\ u_1\ \ldots\ u_n$ of the dual vector space $\check{E}$.

The generic coordinates

$$\xi_j = \langle u_j, x\rangle = \sum_{k=0}^{n} u_{jk} x_k, \quad j = 0, 1, \ldots, n,$$

of any vector $x \in E$ can be interpreted as the projective coordinates of the point (x) (whenever $x \neq 0$). The elimination of the generic variables $\xi_0, \xi_1, \ldots, \xi_{i-1}$ leads naturally to forms of the type

$$N(\xi_i, \xi_{i+1}, \ldots, \xi_n)$$

and we know that the ξ'_j, represent actually homogeneous coordinates in the more sophisticated projective space $\mathbf{P}(E/E_{n-i})$ where E_{n-i} is defined by $\langle u_j, x\rangle = 0$ for $j = 0, 1, \ldots, i-1$ cf. § 6. Actually the projection on the coordinate space opposite to E_{n-i} is not needed. The genericity insures that

$$\mathbf{P}(E_{n-i}) \cap V(\mathfrak{m}) - \Gamma^{(1)} - \Gamma^{(2)} - \ldots - \Gamma^{(i-1)} = \emptyset.$$

14. Intrinsic elimination theory using Barsotti–Weil–Siegel forms

Let us replace the Cayley–Severi forms by the corresponding Barsotti–Weil–Siegel ones using formulas of type:

$$(14.1)\qquad \begin{aligned} S_c(x, x_0, \ldots, x_{c-2}) &= N_c(u_0, u_1, \ldots, u_{d+1}; x) \\ &= \mathbf{N}(\langle u_0, x\rangle, \ldots, \langle u_{d+1}, x\rangle) \end{aligned}$$

where $x \in E - \{0\}$ is regarded as variable in the cone $S(x, x_0, \ldots, x_{c-2}) = 0$ of vertex $\mathbf{P}_{c-2}$ spanned by $(x_0), (x_1); \ldots; (x_{c-2})$ ($\Leftrightarrow$ intersection of the $d+2$ hyperplanes (u_j), $j = 0, 1, \ldots, d+1$). The variables $\xi_j = \langle u_j, x\rangle$ are again the homogeneous coordinates in the abstract $(d+1)$-dimensional space $\mathbf{P}(E/E_{c-1})$ where $\mathbf{P}(E_{c-1}) = \mathbf{P}_{c-2}$.

If we take the $n+1$ forms $u_0, u_1, \ldots, u_n \in \check{E}$ dual to $x_0, x_1, \ldots, x_n \in E$, $x_j \neq 0$, $u_j \neq 0$, we have the following sequence of Weil–Siegel forms:

$$\begin{aligned} S_1(x) &= N_1(x; u_0, u_1, \ldots, u_n) = \mathbf{N}_1(\xi_0, \xi_1, \ldots, \xi_n), \\ S_2(x; x_0) &= N_2(x; u_1, u_2, \ldots, u_n) = \mathbf{N}_2(\xi_1, \ldots, \xi_n), \\ (14.2)\qquad S_3(x; x_0, x_1) &= N_3(x; u_2, \ldots, u_n) = \mathbf{N}_3(\xi_2, \ldots, \xi_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ S_c(x; x_0, x_1, \ldots, x_{c-2}) &= N_c(x; u_{c-1}, u_c, \ldots, u_n) = \mathbf{N}_c(\xi_{c-1}, \ldots, \xi_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

We remark that formally, when we read the (14.2) from top to bottom we have:

$$(14.3)\qquad N(u_0, u_1, \ldots, u_n; x) = \mathbf{N}(\xi_0, \xi_1, \ldots, \xi_n) = 0$$

as *the equation of the hypersurface* (or better, *of the divisor attached to* $\Gamma^{(i)}$) written in the projective coordinate system $((u_0), (u_1), \ldots, (u_n); (u_0 + u_1 + \ldots + u_n))$ which can be regarded as "indeterminate": more precisely, *if we write*

$$\langle u_j, x_j\rangle = \sum_{k=0}^{n} u_{jk} x_{jk}$$

we have the $(n+1)\times(n+1)$ *matrix* (u_{jk}) *representing the preventive coordinate "system" cf. Introduction but written in the formula instead of being ignored.*

If we write the system (13.3) in this "invariant way":

$$F_j = F_j(x; u_0, u_1, \ldots, u_n) = \mathbf{F}_j(\langle u_0, x\rangle \ldots \langle u_n, x\rangle)$$

we can perform equally the first step of Kronecker's elimination method:

$$N_j(x; u_0, u_1, \ldots, u_n) = \mathbf{N}_j(\xi_0, \xi_1, \ldots, \xi_n).$$

The next step is to divide each F_j by $\mathbf{N}_1$, $F_j = \mathbf{N}_1 G_j$. Then we can "eliminate ξ_0" (but within the generic projective frame $(u_0), (u_1), \ldots, (u_n)$; $(u_0 + \ldots + u_n)$). The new system

$$G_j(x; u_0, u_1, \ldots, u_n) = G_j(\xi_0, \xi_1, \ldots, \xi_n)$$

represents the variety $V - \Gamma^{(1)}$ of codimension two which is not contained in $\xi_1 = 0$. Let us cut $V - \Gamma^{(1)}$ with this hyperplane; we shall have only the useful "generic" variables $\xi_1, \xi_2, \ldots, \xi_n$, i.e. we have a system of type:

$$G_j(x; u_1, u_2, \ldots, u_n) = \tilde{G}_j(\xi_1, \xi_2, \ldots, \xi_n)$$

representing the projecting cone of $V - \Gamma^{(1)}$ from the intersection point of the n hyperplanes $\langle u_j, x \rangle = 0$, $j = 1, 2, \ldots, n$ (of vertex P_0 in the corresponding generic projective frame). Then hcd of the $G_j(\xi_1, \ldots, \xi_n)$ will give us back the Barsotti–Weil–Siegel form attached to $\Gamma^{(1)}$, i.e. to the cycle of codimension 2 represented by $\mathfrak{m}$.

In other words, we can prove the announced result:

The systematic computation of the Barsotti–Weil–Siegel forms $\mathbf{N}_1(\xi_{c-1}, \xi_c, \ldots, \xi_n)$ for $c = 1, 2, \ldots, n$ is equivalent to the old Kronecker elimination method but with the preventive projective coordinate system built in the formulas.

Remarks. (1). We emphasize the use of the quotient projective spaces $\mathbf{P}(E/E_{c-1})$ corresponding to coordinate spaces $\mathbf{P}(E_{c-1})$ instead of the projection or the face opposite to $\mathbf{P}(E_{c-1})$.

(2). In order to check all the necessary cautions we follow [vdW1] IV Kap. § 31, page 116; as well as the second edition of vdW's *Algebra*.

The first steps are possible because we know, that the coefficient of the highest power of each x_i is $\neq 0$ (because the corresponding projection space never met the projecting variety. The resultant systems of relative prime forms cannot be identically zero). The coefficient of x_2^q for a Weil–Siegel form is equal to

$$\pm Y(u_0, \ldots, u_i, \ldots, u_{j+1}) \neq 0$$

for ($i = 1, 2, \ldots$, etc.), cf. Remark, § 6, page 93, formula (6.10).

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