DIFFERENT ASPECTS OF DIFFERENTIABILITY
DISSERTATIONES MATHEMATICAE 340 (1995)

ON ISOMORPHISMS OF SPACES OF FUNCTIONAL R-SHIFTS FOR RIGHT INVERTIBLE OPERATORS

ZBIGNIEW BINDERMAN

Academy of Agriculture, Nowoursynowska 166
02-766 Warszawa, Poland

Abstract. This paper provides isomorphisms of a space of analytic functions onto spaces of functional R-shifts introduced by the author in [13]. The space contains all functions analytic in a ring $\{h \in \mathbb{C} : 0 < |h| < \rho\}$, $0 < \rho \leq +\infty$, do not having an essential singularity at the origin. The results obtained include and extend some of those in the author's works [5], [7].

0. Let X be a linear space over the field $\mathbb{C}$ of the complex numbers. Denote by $L(X)$ the set of all linear operators with domains and ranges in X and by $L_0(X)$ the set of those operators from $L(X)$ which are defined on the whole space X. An operator $D \in L(X)$ is said to be *right invertible* if there exists an operator $R \in L(X)$ such that $DR = I$. The set of all right invertible operators belonging to $L(X)$ will be denoted by $R(X)$. For an $D \in R(X)$ we denote by $\mathcal{R}_D$ the set of all its right inverses. In the sequel we shall assume that $\dim\ker D > 0$, i.e. D is right invertible but not invertible, and that the set $\mathcal{R}_D \subset L_0(X)$. An operator $F \in L_0(X)$ is said to be an *initial* operator for D corresponding to an $R \in \mathcal{R}_D$ if

$$F^2 = F, \quad FX = \ker D \quad \text{and} \quad FR = 0.$$

This definition implies that F is an initial operator for D if and only if there is an operator $R \in \mathcal{R}_D$ such that $F = I - RD$ on $\operatorname{dom} D$. The set of all initial operators for a given $D \in R(X)$ is denoted by $\mathcal{F}_D$. One can prove that any projection onto $\ker D$ is an initial operator for D. If we know at least one right inverse R, we can determine the set $\mathcal{R}_D$ of all right inverses and the set $\mathcal{F}_D$ of all initial operators for a given $D \in R(X)$. The theory of right invertible operators and its applications is presented by D. Przeworska-Rolewicz in the book [18].

1991 *Mathematics Subject Classification*: 47B99,47E05,47G99.

Key words and phrases: right invertible operators, functional R-shifts, functional shifts, isomorphisms of analytic functions.

The paper is in final form and no version of it will be published elsewhere.

Here and in the sequel we admit that $0^0 := 1$. We also write $\mathbb{N}$ for the set of all positive integers, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and $H(G)$ for the class of all functions analytic on a set $G \subset \mathbb{C}$.

For a given operator $D \in R(X)$ we shall write (cf. [18]):

$$S := \bigcup_{i=1}^{\infty} \ker D^i. \tag{0.1}$$

If $R \in \mathcal{R}_D$ then the set S is the linear span $P(R)$ of all D-monomials, i.e.

$$S = P(R) := \operatorname{lin}\{R^k z : z \in \ker D,\ k \in \mathbb{N}_0\}. \tag{0.2}$$

Evidently, the set $P(R)$ is independent of the choice of the right inverse R.

1. In this section, Ω will stand for a ring $K_\rho := \{h \in \mathbb{C} : 0 < |h| < \rho\}$, $0 < \rho \leq +\infty$. A non-empty set $K \subseteq \Omega$ is arbitrarily fixed. Write:

$$H_{\Leftarrow}(\Omega) := \Big\{ f \in H(\Omega) : f(h) = \sum_{k=-n}^{\infty} a_k h^k \text{ for all } h \in \Omega \Big\}, \quad n \in \mathbb{N}, \tag{1.1}$$

i.e. if $f \in H_{\Leftarrow}(\Omega)$ then f not does have an essential singularity at the origin.

Suppose that a function $f \in H_{\Leftarrow}(\Omega)$ has the following expansion:

$$f(h) = \sum_{k=-n}^{\infty} a_k h^k \quad \text{for all } h \in \Omega, \tag{1.2}$$

where $n \in \mathbb{N}_0$.

Definition 1.1. Suppose that $D \in R(X), \dim \ker D > 0$ and $R \in \mathcal{R}_D$ is arbitrarily fixed. A family $T_K = \{T_h\}_{h \in K} \subset L_0(X)$ is said to be a family of *functional R-shifts* for the operator D induced by a function $f \in H_{\Leftarrow}(\Omega)$ and R if

$$T_h x := \sum_{k=0}^{\infty} a_k h^k D^k x + \sum_{k=1}^{n} a_{-k} h^{-k} R^k x \quad \text{for all } h \in K,\ x \in S, \tag{1.3}$$

where S, f, are determined by Formulas (0.1), (1.2), respectively.

We should point out that by definition of the set S, the last sum has only a finite number of members different than zero.

Some fundamental properties of functional R-shifts for right invertible operators are given in the author's work [13]. Functional R-shifts which are induced by functions analytic on the set $\Omega \cup \{0\}$ have been called *functional shifts* (cf. [4], [5]). The theory of functional and sequential shifts induced by a right invertible operator is presented in detail in the author's works [1]–[12]. Evidently , the definition of functional shifts for $D \in R(X)$ is independent on $R \in \mathcal{R}_D$. Shifts induced by the function e^h for right invertible operators have been investigated by D. Przeworska-Rolewicz: [17]–[22]. Note, that properties of functional R-shifts induced by functions analytic in a ring having an isolated essential singularity at the center are recently studied by the author [14].

PROPOSITION 1.1 (cf. [13]). *Suppose that* $D \in R(X)$ *and* $R \in \mathcal{R}_D$. *Let* $T_{f,K} = \{T_{f,h}\}_{h\in K}$ *be a family of functional* R*-shifts for the operator* D *induced by a function* $f \in H_{\Leftarrow}(\Omega)$ *and the operator* R. *Then*

(i) *The operators* $T_{f,h}$ $(h \in K)$ *are uniquely determined on the set* S.

(ii) *If* X *is a complete linear metric space,* $\overline{S} = X$ *and* $T_{f,h}$ *are continuous for* $h \in K$. *Then* $T_{f,h}$ *are uniquely determined on the whole space.*

(iii) *For all* $h \in K$ *the operators* $T_{f,h}$ *commute on the set* S *with the operator* D.

THEOREM 1.1 (cf. [13]). *Suppose that* $D \in R(X)$ *and* $\dim\ker D > 0$, F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$ *and a family* $T_K = \{T_h\}_{h\in K} \subset L_0(X)$ *is given. Then following two conditions are equivalent*:

a) T_K *is a family of functional* R*-shifts for the operator* D *induced by the function* f *and* $R \in \mathcal{R}_D$,
b) $T_h R^k F = \sum_{j=0}^{k+n} a_{k-j} h^{k-j} R^j F$ *for all* $h \in K$, $k \in \mathbb{N}_0$.

EXAMPLE 1.1 (cf. [13]). Let $X = H(U)$ and $K = U\backslash\{0\}$, where U is the unit disk. If $T_K = \{T_h\}_{h\in K}$ is a family of functional R-shifts for the *Pommiez operator* $D \in L_0(X)$:

$$(Dx)(t) = \frac{x(t) - x(0)}{t} \quad \text{for } x \in X,\ t \in U,$$

where

$$\left.\frac{x(t) - x(0)}{t}\right|_{t=0} := x'(0),$$

induced by the function $f(h) = 1/h(1-h) \in H_{\Leftarrow}(K)$ and the operator $R \in \mathcal{R}_D$:

$$(Rx)(t) = tx(t) \quad \text{for } x \in X,\ t \in U.$$

Then the operators T_h $(h \in K)$ are uniquely determined on X by the formula

$$(T_h x)(t) = \begin{cases} \frac{t^2x(t)-h^2x(h)}{h(t-h)} & \text{for } t \neq h, \\ \frac{d}{dt}[t^2x(t)]\big|_{t=h} = 2hx(h) + h^2x'(h) & \text{for } t = h, \end{cases}$$

where $t \in U$. The operators $T_h (h \in K)$ are continuous.

Let T_K be the set of all families of functional R-shifts for an operator $D \in R(X)$ induced by an $R \in \mathcal{R}_D$ and by the members of the set $H_{\Leftarrow}(\Omega)$, i.e.

$$T_K := \{T_{g,K} : g \in H_{\Leftarrow}(\Omega)\}. \tag{1.4}$$

Let $T_{f,K}, T_{g,K} \in T_K$, where $f, g \in H_{\Leftarrow}(\Omega)$ have the following expansions:

$$f(h) = \sum_{k=-n}^{\infty} a_k h^k, \quad g(h) = \sum_{k=-m}^{\infty} b_k h^k \quad \text{for all } h \in \Omega,\ n, m \in \mathbb{N}_0. \tag{1.5}$$

If $n = m = 0$ then on the set S (cf. [5])

$$T_{f,h}T_{g,h} = T_{g,h}T_{f,h} = T_{fg,h} \quad \text{for all } h \in K. \tag{1.6}$$

In general, the equalities (1.6) do not hold. For an example, let $\Omega = K = K_1$ and let

$$f(h) = h^2 + h^{-1}, \quad g(h) = h + 1 + h^{-2}, \quad h \in K.$$

Then by the definition we have on the set S

$$\begin{aligned} T_{f,h}T_{g,h} &= (h^2D^2 + h^{-1}R)(hD + I + h^{-2}R^2) \\ &= h^3D^3 + h^2D^2 + I + RD + h^{-1}R + h^{-3}R^3, \\ T_{g,h}T_{f,h} &= h^3D^3 + h^2D^2 + I + R^2D^2 + h^{-1}R + h^{-3}R^3, \\ T_{fg,h} &= h^3D^3 + h^2D^2 + 2I + h^{-1}R + h^{-3}R^3 \quad \text{for all } h \in K. \end{aligned}$$

This shows that for all $h \in K$ we have on S

$$T_{f,h}T_{g,h} \neq T_{g,h}T_{f,h}, \quad T_{f,h}T_{g,h} \neq T_{fg,h}, \quad T_{g,h}T_{f,h} \neq T_{fg,h}.$$

LEMMA 1.1. *Suppose that $D \in R(X)$ and an $R \in \mathcal{R}_D$ is arbitrarily fixed. Let $T_{f,K}, T_{g,K} \in T_K$, where $f, g \in H_{\Leftarrow}(\Omega)$ have the expansions* (1.5). *Define the following operation*

$$T_{f,h} \circ T_{g,h} := D^{m+n}\, T_{f,h}T_{g,h}R^{m+n} \quad \text{for } h \in K. \tag{1.7}$$

Then on the set S

$$T_{f,h} \circ T_{g,h} = T_{g,h} \circ T_{f,h} \quad \text{for all } h \in K. \tag{1.8}$$

$$T_{f,h} \circ T_{g,h} = T_{fg,h} \quad \text{for all } h \in K. \tag{1.9}$$

P r o o f. Let $h \in K$ be arbitrarily fixed. Our assumptions and Theorem 1.1 together imply that the operators $D, R, T_{f,h}, T_{g,h} \in L_0(S)$. We have on the set S

$$\begin{aligned} &T_{f,h} \circ T_{g,h} - T_{g,h} \circ T_{f,h} \\ &\quad = D^{m+n}\Big(\sum_{k=0}^{\infty} a_k h^k D^k + \sum_{k=1}^{n} a_{-k} h^{-k} R^k\Big) \\ &\qquad \Big(\sum_{k=0}^{\infty} b_k h^k D^k + \sum_{k=1}^{m} b_{-k} h^{-k} R^k\Big) R^{m+n} \\ &\qquad - D^{m+n}\Big(\sum_{k=0}^{\infty} b_k h^k D^k + \sum_{k=1}^{m} b_{-k} h^{-k} R^k\Big) \\ &\qquad \Big(\sum_{k=0}^{\infty} a_k h^k D^k + \sum_{k=1}^{n} a_{-k} h^{-k} R^k\Big) R^{m+n} \\ &\quad = D^{m+n}\Big\{\Big(\sum_{k=0}^{\infty} a_k h^k D^k \sum_{k=0}^{\infty} b_k h^k D^k\Big) \end{aligned}$$

$$
\begin{aligned}
&- \Big(\sum_{k=0}^{\infty} b_k h^k D^k \sum_{k=0}^{\infty} a_k h^k D^k\Big)\Big\} R^{m+n} \\
&+ D^{m+n}\Big\{\Big(\sum_{k=1}^{n} a_{-k} h^{-k} R^k \sum_{k=1}^{m} b_{-k} h^{-k} R^k\Big) \\
&- \Big(\sum_{k=1}^{m} b_{-k} h^{-k} R^k \sum_{k=1}^{n} a_{-k} h^{-k} R^k\Big)\Big\} R^{m+n} \\
&+ D^{m+n}\Big(\sum_{k=0}^{\infty} a_k h^k D^k \sum_{k=1}^{m} b_{-k} h^{-k} R^k\Big) R^{m+n} \\
&+ D^{m+n}\Big(\sum_{k=1}^{n} a_{-k} h^{-k} R^k \sum_{k=0}^{\infty} b_k h^k D^k\Big) R^{m+n} \\
&- D^{m+n}\Big(\sum_{k=0}^{\infty} b_k h^k D^k \sum_{k=1}^{n} a_{-k} h^{-k} R^k\Big) R^{m+n} \\
&- D^{m+n}\Big(\sum_{k=1}^{m} b_{-k} h^{-k} R^k \sum_{k=0}^{\infty} a_k h^k D^k\Big) R^{m+n} \\
= &\sum_{j=1}^{m}\Big(\sum_{k=j}^{m} a_{k-j} b_{-k}\Big) h^{-j} R^j \\
&+ \sum_{j=0}^{\infty}\Big(\sum_{k=1}^{m} a_{j+k} b_{-k}\Big) h^j D^j + \sum_{j=1}^{n}\Big(\sum_{k=j}^{n} a_{-k} b_{k-j}\Big) h^{-j} R^j \\
&+ \sum_{j=0}^{\infty}\Big(\sum_{k=1}^{n} a_{-k} b_{j+k}\Big) h^j D^j \\
&- \sum_{j=1}^{n}\Big(\sum_{k=j}^{n} b_{k-j} a_{-k}\Big) h^{-j} R^j - \sum_{j=0}^{\infty}\Big(\sum_{k=1}^{n} b_{j+k} a_{-k}\Big) h^j D^j \\
&- \sum_{j=1}^{m}\Big(\sum_{k=j}^{m} b_{-k} a_{k-j}\Big) h^{-j} R^j - \sum_{j=0}^{\infty}\Big(\sum_{k=1}^{m} b_{-k} a_{j+k}\Big) h^j D^j = 0.
\end{aligned}
$$

We assume that $m > n \geq 2$. Then

$$
\begin{aligned}
T_{f,h} \circ T_{g,h} = &\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} a_j b_{k-j}\Big) h^k D^k + \sum_{k=2}^{n}\Big(\sum_{j=1}^{k-1} a_{-j} b_{j-k}\Big) h^{-k} R^k \\
&+ \sum_{k=n+1}^{m}\Big(\sum_{j=1}^{n} a_{-j} b_{j-k}\Big) h^{-k} R^k \\
&+ \sum_{k=m+1}^{m+n}\Big(\sum_{j=k-m}^{n} a_{-j} b_{j-k}\Big) h^{-k} R^k
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{m}\Big(\sum_{j=k}^{m} a_{j-k}b_{-j}\Big)h^{-k}R^k+\sum_{k=0}^{\infty}\Big(\sum_{j=1}^{m} a_{j+k}b_{-j}\Big)h^{k}D^k\\
&+\sum_{k=1}^{n}\Big(\sum_{j=k}^{n} a_{-j}b_{j-k}\Big)h^{-k}R^k+\sum_{k=0}^{\infty}\Big(\sum_{j=1}^{n} a_{-j}b_{j+k}\Big)h^{k}D^k\\
=&\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} a_j b_{k-j}+\sum_{j=k}^{m} a_{j+k}b_{-j}+\sum_{j=1}^{n} a_{-j}b_{j+k}\Big)h^kD^k\\
&+\sum_{k=1}^{m}\Big(\sum_{j=k}^{m} a_{j-k}b_{-j}\Big)h^{-k}R^k+\sum_{k=1}^{n}\Big(\sum_{j=k}^{n} a_{-j}b_{j-k}\Big)h^{-k}R^k\\
&+\sum_{k=2}^{n}\Big(\sum_{j=1}^{k-1} a_{-j}b_{j-k}\Big)h^{-k}R^k\\
&+\sum_{k=n+1}^{m}\Big(\sum_{j=1}^{n} a_{-j}b_{j-k}\Big)h^{-k}R^k+\sum_{k=m+1}^{m+n}\Big(\sum_{j=k-m}^{n} a_{-j}b_{j-k}\Big)h^{-k}R^k\\
=&\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} a_j b_{k-j}+\sum_{j=1}^{m} a_{j+k}b_{-j}+\sum_{j=1}^{n} a_{-j}b_{j+k}\Big)h^kD^k\\
&+\Big(\sum_{j=1}^{m} a_{j-1}b_{-j}+\sum_{j=1}^{n} a_{-j}b_{j-1}\Big)h^{-1}R\\
&+\sum_{k=2}^{n}\Big(\sum_{j=k}^{m} a_{j-k}b_{-j}+\sum_{j=1}^{n} a_{-j}b_{j-k}\Big)h^{-k}R^k\\
&+\sum_{k=n+1}^{m}\Big(\sum_{j=k}^{m} a_{j-k}b_{-j}+\sum_{j=1}^{n} a_{-j}b_{j-k}\Big)h^{-k}R^k\\
&+\sum_{k=m+1}^{m+n}\Big(\sum_{j=k-m}^{n} a_{-j}b_{j-k}\Big)h^{-k}R^k\\
=&\sum_{k=0}^{\infty}\Big(\sum_{j=-n}^{-1} a_j b_{k-j}+\sum_{j=0}^{k} a_j b_{k-j}+\sum_{j=k+1}^{m+k} a_j b_{k-j}\Big)h^kD^k\\
&+\sum_{k=1}^{m+n}\Big(\sum_{j=-n}^{m-k} a_j b_{-k-j}\Big)h^{-k}R^k\\
=&\sum_{k=0}^{\infty} c_k h^k D^k+\sum_{k=1}^{m+n} c_{-k}h^{-k}R^k,
\end{aligned}
$$

where

$$c_k := \sum_{j=-n}^{m+k} a_j b_{k-j} \quad \text{for } k \geq -(m+n). \tag{1.10}$$

One can prove that $fg \in H_{\Leftarrow}(\Omega)$ and has the following expansion:

$$f(t)g(t) = \sum_{k=-(m+n)}^{\infty} c_k t^k \quad \text{for } t \in \Omega, \tag{1.11}$$

where c_k $[-(m+n) \leq k < +\infty]$ are determined by Formula (1.10). Clearly, for the coefficients c_k we have

$$c_k = \sum_{j=-\infty}^{\infty} a_j b_{k-j} \quad (k \geq -(m+n)).$$

This implies that $T_{f,h} \circ T_{g,h} = T_{fg,h}$. Similar proofs can be given for the other cases. ■

It is easy to observe that the set $H_{\Leftarrow}(\Omega)$ is a commutative linear ring with the following algebraic operations:

$$f(h) + g(h) = (f+g)(h), \quad \alpha f(h) = (\alpha f)(h), \quad f(h)g(h) = (fg)(h),$$

where $f, g \in H_{\Leftarrow}(\Omega)$, $\alpha \in \mathbb{C}$, $h \in \Omega$.

Lemma 1.1. implies

Proposition 1.1. *The set T_K of all families of functional R-shifts for $D \in R(X)$ induced by the set $H_{\Leftarrow}(\Omega)$ and an operator $R \in \mathcal{R}_D$ defined on the set S is a commutative linear ring with the operations*

$$T_{f,K} + T_{g,K} := T_{f+g,K}, \quad \alpha T_{f,K} := T_{\alpha f,K}, \quad T_{f,K} \circ T_{g,K} := T_{fg,K}, \tag{1.12}$$

where $f, g \in H_{\Leftarrow}(\Omega)$, $\alpha \in \mathbb{C}$.

The neutral elements and units of the commutative linear rings $H_{\Leftarrow}(\Omega)$ and T_K are

$$\mathbf{0}(h) \equiv 0, \quad \mathbf{1}(h) \equiv 1 \quad \text{on } \Omega,$$
$$OR^k z = 0, \quad IR^k z = R^k z \quad \text{for all } k \in \mathbb{N}_0,\ z \in \ker D,\ R \in \mathcal{R}_D,$$

respectively.

Observe that

$$T_{\mathbf{0},h} = O, \quad T_{-f,h} = -T_{f,h}, \quad T_{\mathbf{1},h} = I, \quad h \in K,\ f \in H_{\Leftarrow}(\Omega).$$

Moreover, if $f \in H_{\Leftarrow}(\Omega)$ then f and $1/f$ are meromorphic in the set $\Omega \cup \{0\}$ and the function $1/f$ has singular points in Ω only at zeroes of f. These singular points are poles for $1/f$. So that, if $f(h) \neq 0$ on Ω then $1/f \in H_{\Leftarrow}(\Omega)$ and by Lemma 1.1,

$$I = T_{\mathbf{1},h} = T_{f[1/f],h} = T_{f,h} \circ T_{[1/f],h}, \quad h \in K.$$

This implies that

$$[T_{f,h}]^{-1} = T_{[1/f],h} \quad \text{for all } h \in K.$$

By the definition, if $f, g \in H_{\Leftarrow}(\Omega)$ and $T_{f,h} = T_{g,h}$ for all $h \in K$ on the set S then $f \equiv g$ on K.

It is well-known that if $f, g \in H_{\Leftarrow}(\Omega)$, $f(h_n) = g(h_n)$ for $n \in \mathbb{N}$ and the sequence $\{h_n\}$ has a limit point in Ω, then $f(h) = g(h)$ in Ω. Thus, if $f, g \in H_{\Leftarrow}(\Omega)$, $K \subseteq \Omega$ is an open set and $T_{f,h} = T_{g,h}$ for all $h \in K$ on S then $f \equiv g$ on the set Ω. This implies

THEOREM 1.2. *Suppose that $D \in R(X)$, an $R \in \mathcal{R}_D$ is arbitrarily fixed, $K \subseteq \Omega$ is an open set and T_K is the set of all families of functional R-shifts for D defined on the set S induced by the set $H_{\Leftarrow}(\Omega)$ and the operator R. Then the rings $H(\Omega)$ and T_K are isomorphic. The mapping $T : f \Rightarrow T_{f,K}$ is an isomorphism of $H_{\Leftarrow}(\Omega)$ onto T_K.*

Clearly, Theorem 1.2 implies

COROLLARY 1.1 (cf. [5]). *Suppose that all assumptions of Theorem 1.2 are satisfied and T'_K denotes the set of all families of functional shifts for D defined on S induced by the set $H(\Omega \cup \{0\})$. Then*

(i) *The set T'_K is a commutative linear ring with the operations*

$$T'_{f,K} + T'_{g,K} := T'_{f+g,K}, \quad \alpha T'_{f,K} := T'_{\alpha f,K}, \quad T'_{f,K} T'_{g,K} := T'_{fg,K},$$

where $f, g \in H(\Omega \cup \{0\})$, $\alpha \in \mathbb{C}$.

(ii) *The rings $H(\Omega \cup \{0\})$ and T'_K are isomorphic. The mapping $T' : f \Rightarrow T'_{f,K}$ is a ring isomorphism of $H(\Omega \cup \{0\})$ onto T'_K.*

2. In this section we assume that X is a linear topological space. As before, Ω is a ring $K_\rho (0 < \rho \leq +\infty)$, an open set $K \subseteq \Omega$ is arbitrarily fixed, the function $f \in H_{\Leftarrow}(\Omega)$ has the expansion (1.2).

Let $D \in R(X)$ and F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Write (cf. [7], [18], [23]):

$$D_\infty := \bigcap_{k \in \mathbb{N}_0} D_k, \quad \text{where } D_0 := X, D_k := \operatorname{dom} D^k (k \in \mathbb{N}), \tag{2.1}$$

$$E_\lambda := \ker(D - \lambda I), \quad \lambda \in \mathbb{C}, \tag{2.2}$$

$$E := \bigcup_{\lambda \in \mathbb{C}} E_\lambda. \tag{2.3}$$

$$E^\lambda(R) := \ker(I - \lambda R), \quad \lambda \in \mathbb{C}, \tag{2.4}$$

$$S_f^{(n)}(D) := \Big\{x \in D_\infty : \sum_{k=0}^{\infty} a_k h^k D^{k+n} x \text{ is convergent for all } h \in K\Big\}, \ n \in \mathbb{N}_0,$$

$$S_f(D) := S_f^{(0)}(D), \tag{2.5}$$

$$S_f^{\infty}(D) := \bigcap_{n\in\mathbb{N}_0} S_f^{(n)}(D). \tag{2.6}$$

It is obvious that $S \subset D_\infty \subset S_f^{\infty}(D) \subset S_f(D)$, where the set S is defined by Formula (0.1).

In a similar way as in Section 1, we make

Definition 2.1. Let X be a linear topological space and let $f \in H_{\Leftarrow}(\Omega)$ have the expansion (1.2). Suppose that $D \in R(X), \dim\ker D > 0$ and $R \in \mathcal{R}_D$. A family $T_{f,K} = \{T_{f,h}\}_{h\in K} \subset L_0(X)$ is said to be a family of *R-functional shifts* for the operator D induced by the function f and R if Formula (1.3) holds for all $h \in K$, $x \in S_f(D)$, where the set $S_f(D)$ is defined by Formula (2.5).

This definition immediately implies

Proposition 2.1. *Suppose that $D \in R(X)$, $R \in \mathcal{R}_D$ and $T_{f,K} = \{T_{f,h}\}_{h\in K}$ is a family of R-functional shifts induced by a function $f \in H_{\Leftarrow}(\Omega)$ and R. Then*

(i) *The operator $T_{f,h}$ of the functional variable $f \in H_{\Leftarrow}(\Omega)$ and the complex variable $h \in K$ as an operator acting in the space $L_0(S_f(D))$ is linear*, *i.e.*

$$T_{\lambda f+\mu g,h} = \lambda T_{f,h} + \mu T_{g,h} \quad \text{for all } f, g \in H_{\Leftarrow}(\Omega);\ \lambda, \mu \in \mathbb{C}.$$

(ii) *For all $h \in K$ the operators $T_{f,h}$ are uniquely determined on the set $S_f(D)$.*

(iii) *If $\overline{S_f(D)} = X$ and $T_{f,h}$ are continuous for $h \in K$ then $T_{f,h}$ are uniquely determined on the whole space X.*

Theorem 2.1 (cf. [13]). *Suppose that $D \in R(X), F$ is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $T_{f,K} = \{T_{f,h}\}_{h\in K}$ is a family of R-functional shifts induced by a function $f \in H_{\Leftarrow}(\Omega)$ and R. Let $E_\lambda \neq \{0\}$, let $\lambda K := \{\lambda h : h \in K\} \subset \Omega$, where $\lambda \in \mathbb{C}$ and let E_λ be defined by Formula* (2.2). *Then*

(i) *$E_\lambda \subset S_f(D)$.*

(ii) *For all $h \in K$ and $x \in E_\lambda$,*

$$T_{f,h}x = f(\lambda h)x - \sum_{k=0}^{n-1}\Big(\sum_{j=-n}^{-k-1} a_j(\lambda h)^j\Big)\lambda^k R^k F x. \tag{2.7}$$

In a similar way as in the author's work [7] (see also [14]) we prove

Proposition 2.2. *Suppose that $D \in R(X)$ and $E_\lambda = \ker(D - \lambda I) \neq \{0\}$ for $\lambda \in \mathbb{C}$ with $\lambda K \subset \Omega$. Let $T'_K|_{E_\lambda}$ be the set of all families of functional shifts for D induced by the set $H(\Omega \cup \{0\})$ defined on E_λ. Then*

(i) *The set* $T'_K|_{E_\lambda}$ *is a commutative linear ring with the operations defined by Formula* (1.13).

(ii) *The rings* $H(\Omega \cup \{0\})$ *and* $T'_K|_{E_\lambda}$ *are isomorphic. The mapping* $T' : f \mapsto T'_{f,K}$ *is a ring isomorphism of* $H(\Omega \cup \{0\})$ *onto* $T'_K|_{E_\lambda}$.

Proposition 2.2 implies (cf. [7])

COROLLARY 2.2. *Suppose that* $D \in R(X)$, $\Omega \cup \{0\} = K \cup \{0\} = \mathbb{C}$ *and* $T'_K|_E$ *denotes the set of all families of functional shifts for* D *defined on* E *induced by the set* $H(\mathbb{C})$. *Then*

(i) *The set* $T'_K|_E$ *is a commutative linear ring with the operations defined by Formula* (1.13).

(ii) *If* $E \neq \ker D$, *then the rings* $H(\mathbb{C})$ *and* $T'_K|_E$ *are isomorphic. The mapping* $T' : f \mapsto T'_{f,K}$ *is a ring isomorphism of* $H(\mathbb{C})$ *onto* $T'_K|_E$.

Suppose that all assumptions of Theorem 2.1 are satisfied. Then

$$E^\lambda(R) = \ker(I - \lambda R) \subset E_\lambda$$

and

$$T_{f,h}x = f(\lambda h)x \quad \text{for all } h \in K,\ x \in E^\lambda(R)$$

(cf. [13]). This implies that, in a similar way as in [7], we can prove

PROPOSITION 2.3. *Suppose that all assumptions of Theorem* 2.1 *are satisfied and* $E^\lambda(R) \neq \{0\}$. *Let* $T_K|_{E^\lambda(R)}$ *be the set of all families of functional* R*-shifts for* D *defined on* $E^\lambda(R)$ *induced by the set* $H_{\Leftarrow}(\Omega)$ *and* R. *Then*

(i) *The set* $T_K|_{E^\lambda(R)}$ *is a commutative linear ring with the operations defined by Formula* (1.13).

(ii) *The rings* $H_{\Leftarrow}(\Omega)$ *and* $T_K|_{E^\lambda(R)}$ *are isomorphic. The mapping* $T : f \mapsto T_{f,K}$ *is a ring isomorphism of* $H_{\Leftarrow}(\Omega)$ *onto* $T_K|_{E^\lambda(R)}$.

We need the following

LEMMA 2.1. *Suppose that all assumptions of Theorem* 2.1 *are satisfied. Then*

$$R^p(E_\lambda) \subset S_f(D), \qquad p \in \mathbb{N},$$

and

$$(2.8) \quad T_{f,h}R^p x = \lambda^{-p}\Big[f(\lambda h)x - \sum_{k=0}^{n-1}\Big(\sum_{j=-n}^{-k-1}(\lambda h)^j a_j\Big)\lambda^k R^k F x - \sum_{k=0}^{p-1}\lambda^k \sum_{j=0}^{k+n} a_{k-j}h^{k-j}R^j F x\Big]$$

for all $h \in K$, $x \in E_\lambda$, $p \in \mathbb{N}$.

Proof. Let $h \in K$, $x \in E_\lambda$ be arbitrarily fixed. Observe that

$$\lambda^p R^p x = x - \sum_{k=0}^{p-1} \lambda^k R^k F x \quad \text{for } p \in \mathbb{N}. \tag{2.9}$$

Indeed (by induction), if $p = 1$ then by the definition

$$\lambda R x = R(\lambda x) = RDx = x - Fx.$$

Suppose Formula (2.9) to be true for an arbitrarily fixed $p \geq 1$. Then

$$\begin{aligned}\lambda^{p+1} R^{p+1} x &= \lambda R(\lambda^p R^p) x = \lambda R\Big(x - \sum_{k=0}^{p-1} \lambda^k R^k F x\Big)\\ &= \lambda R x - \sum_{k=0}^{p-1} \lambda^{k+1} R^{k+1} F x = x - Fx - \sum_{k=1}^{p} \lambda^k R^k F x\\ &= x - \sum_{k=0}^{p} \lambda^k R^k F x.\end{aligned}$$

This proves Formula (2.9) for all positive integers.

Formula (2.9), Theorem 1.1 and Theorem 2.1 together imply

$$\begin{aligned}T_{f,h} R^p x &= T_{f,h}\Big(\lambda^{-p} x - \lambda^{-p} \sum_{k=0}^{p-1} \lambda^k R^k F x\Big) = \lambda^{-p}\Big(T_{f,h} x - \sum_{k=0}^{p-1} \lambda^k T_{f,h} R^k F x\Big)\\ &= \lambda^{-p}\Big[f(\lambda h) x - \sum_{k=0}^{n-1}\Big(\sum_{j=-n}^{-k-1} (\lambda h)^j a_j\Big)\lambda^k R^k F x\\ &\quad - \sum_{k=0}^{p-1} \lambda^k \sum_{j=0}^{k+n} a_{k-j} h^{k-j} R^j F x\Big] \quad \text{for all } p \in \mathbb{N}. \ \blacksquare\end{aligned}$$

PROPOSITION 2.4. *Suppose that $D \in R(X)$ and $R \in \mathcal{R}_D$. Let $E_\lambda = \ker(D - \lambda I) \neq \{0\}$ and let $\lambda K \subset \Omega$, where $\lambda \in \mathbb{C}$. Then the set $T_K|_{E_\lambda}$ of all families of functional R-shifts defined on E_λ induced by the set $H_{\Leftarrow}(\Omega)$ and R is a commutative linear ring with the operations defined by Formula* (1.12).

Proof. Let $T_{f,K}, T_{g,K} \in T_K|_{E_\lambda}$ where $f, g \in H_{\Leftarrow}(\Omega)$ have the expansion (1.5). Clearly, it is enough to show that for $x \in E_\lambda$

$$T_{f,h} \circ T_{g,h} x = D^{m+n} T_{f,h} T_{g,h} R^{m+n} x = T_{g,h} \circ T_{f,h} x \quad \text{for } h \in K$$

and

$$T_{f,h} \circ T_{g,h} x = T_{fg,h} x \quad \text{for } h \in K.$$

Let $h \in K$, $x \in E_\lambda$ be arbitrarily fixed. By F we denote an initial operator for D

corresponding to R. Lemma 2.1, Formula (2.7) and Theorem 1.1 together imply

$$
\begin{aligned}
&(T_{g,h}\circ T_{f,h})(x) = (D^{m+n}T_{g,h}T_{f,h})(R^{m+n}x)\\
&\quad = D^{m+n}T_{g,h}\Big\{\lambda^{-(m+n)}\Big[f(\lambda h)x - \sum_{k=0}^{n-1}\Big(\sum_{j=-n}^{-k-1}(\lambda h)^j a_j\Big)\lambda^k R^k Fx\\
&\qquad - \sum_{k=0}^{m+n-1}\lambda^k\sum_{j=0}^{k+n} a_{k-j}h^{k-j}R^jFx\Big]\Big\}\\
&\quad = \lambda^{-(m+n)}D^{m+n}\Big[f(\lambda h)T_{g,h}x - \sum_{k=0}^{n-1}\Big(\sum_{j=-n}^{-k-1}(\lambda h)^j a_j\Big)\lambda^k T_{g,h}R^kFx\\
&\qquad - \sum_{k=0}^{m+n-1}\lambda^k\sum_{j=0}^{k+n} a_{k-j}h^{k-j}T_{g,h}R^jFx\Big]\\
&\quad = \lambda^{-(m+n)}D^{m+n}\Big\{f(\lambda h)\Big[g(\lambda h)x - \sum_{k=0}^{m-1}\Big(\sum_{j=-m}^{-k-1}(\lambda h)^j b_j\Big)\lambda^k R^kFx\Big]\\
&\qquad - \sum_{k=0}^{n-1}\Big(\sum_{j=-n}^{-k-1}(\lambda h)^j a_j\Big)\lambda^k\sum_{p=0}^{k+m} b_{k-p}h^{k-p}R^pFx\\
&\qquad - \sum_{k=0}^{m+n-1}\lambda^k\sum_{j=0}^{k+n} a_{k-j}h^{k-j}\sum_{p=0}^{j+m} b_{j-p}h^{j-p}R^pFx\Big\}\\
&\quad = \lambda^{-(m+n)}\Big\{f(\lambda h)g(\lambda h)D^{m+n}x - f(\lambda h)\sum_{k=0}^{m-1}\Big(\sum_{j=-m}^{-k-1}(\lambda h)^j b_j\Big)\lambda^k D^{m+n}R^kFx\\
&\qquad - \sum_{k=0}^{n-1}\Big(\sum_{j=-n}^{-k-1}(\lambda h)^j a_j\Big)\lambda^k\sum_{p=0}^{k+m} b_{k-p}h^{k-p}D^{m+n}R^pFx\\
&\qquad - \sum_{k=0}^{m+n-1}\lambda^k\sum_{j=0}^{k+n} a_{k-j}h^{k-j}\sum_{p=0}^{j+m} b_{j-p}h^{j-p}D^{m+n}R^pFx\Big\}\\
&\quad = \lambda^{-(m+n)}\Big\{f(\lambda h)g(\lambda h)\lambda^{m+n}x\\
&\qquad - \sum_{k=0}^{m+n-1}\lambda^k\sum_{j=n}^{k+n} a_{k-j}h^{k-j}\sum_{p=m+n}^{j+m} b_{j-p}h^{j-p}R^{p-m-n}Fx\Big\}\\
&\quad = f(\lambda h)g(\lambda h)x - \sum_{k=0}^{m+n-1}\sum_{j=n}^{k+n}\sum_{p=m+n}^{j+m}\lambda^{k-m-n}h^{k-p}a_{k-j}b_{j-p}R^{p-m-n}Fx\\
&\quad = f(\lambda h)g(\lambda h)x - \sum_{k=0}^{m+n-1}\sum_{j=0}^{k}\sum_{p=0}^{j}\lambda^{k-m-n}h^{k-m-n-p}a_{k-n-j}b_{j-p-m}R^pFx
\end{aligned}
$$

$$= f(\lambda h)g(\lambda h)x - \sum_{k=0}^{m+n-1}\sum_{p=0}^{k}\sum_{j=p}^{k}\lambda^{k-m-n}h^{k-m-n-p}a_{k-n-j}b_{j-p-m}R^pFx$$

$$= f(\lambda h)g(\lambda h)x - \sum_{p=0}^{m+n-1}\Big(\sum_{k=p}^{m+n-1}\sum_{j=p}^{k}(\lambda h)^{k-m-n-p}a_{k-n-j}b_{j-p-m}\Big)\lambda^pR^pFx$$

$$= f(\lambda h)g(\lambda h)x - \sum_{p=0}^{m+n-1}\Big(\sum_{k=p}^{m+n-1}(\lambda h)^{k-m-n-p}\Big(\sum_{j=p}^{k}a_{k-n-j}b_{j-p-m}\Big)\Big)\lambda^pR^pFx$$

$$= f(\lambda h)g(\lambda h)x - \sum_{p=0}^{m+n-1}\Big(\sum_{j=-n-m}^{-p-1}(\lambda h)^{j}\Big(\sum_{k=-n}^{j+m}a_{j-k-n+m}b_{k+n-m}\Big)\Big)\lambda^pR^pFx$$

$$= f(\lambda h)g(\lambda h)x - \sum_{k=0}^{m+n-1}\Big(\sum_{j=-n-m}^{-k-1}(\lambda h)^{j}\Big(\sum_{p=-m}^{j+n}b_pa_{j-p}\Big)\Big)\lambda^kR^kFx.$$

Hence,

$$T_{g,h}\circ T_{f,h}x = f(\lambda h)g(\lambda h)x - \sum_{k=0}^{m+n-1}\Big(\sum_{j=-n-m}^{-k-1}(\lambda h)^{j}c_j\Big)\lambda^kR^kFx, \tag{2.10}$$

where $c_j = \sum_{p=-m}^{j+n} b_pa_{j-p}$ for $j = -1, -2, \ldots, -m-n$. This proves that $T_{g,h}\circ T_{f,h}x = T_{f,h}\circ T_{g,h}x$. Formula (2.10),Theorem 2.1 and Formula (1.11) together imply that $T_{g,h}\circ T_{f,h}x = T_{fg,h}x$. ∎

In a similar way as Theorem 1.1 we prove the following

THEOREM 2.2. *Suppose that all assumptions of Proposition* 2.4 *are satisfied. Then the rings $H_{\Leftarrow}(\Omega)$ and $T_K|_{E_\lambda}$ are isomorphic. The mapping $T : f \mapsto T_{f,K}$ is a ring isomorphism of $H_{\Leftarrow}(\Omega)$ onto $T_K|_{E_\lambda}$.*

Remark 2.1. In the author's work [8]:

— an isomorphism of a ring of analytic functions onto a ring of functional shifts defined on the space of D-analytic elements (cf. [19], [18]) is established,

— applications of rings of functional shifts to obtain summation formulas of the Euler–Maclaurin type (cf. [15], [21]) are given.

References

[1] Z. Binderman, *Complex R-shifts for right invertible operators*, Demonstratio Math. 23 (1990), 1043–1053.

[2] —, *On some properties of complex R-shifts*, ibid. 25 (1992), 207–217.

[3] —, *Some properties of operators of complex differentation and shifts*, Scientific Bulletin of Łódź Technical University, Matematyka, 24 (1993), 5–18.

[4] —, *Cauchy integral formula induced by right invertible operators*, Demonstratio Math. 25 (1992), 671–690.

[5] Z. Binderman, *Functional shifts induced by right invertible operators*, Math. Nachr. 157 (1992), 211–224.

[6] —, *On periodic solutions of equations with right invertible operators induced by functional shifts*, Demonstratio Math. 26 (1993), 535–543.

[7] —, *A unified approach to shifts induced by right invertible operators*, Math. Nachr. 161 (1993), 239–252.

[8] —, *On summation formulas induced by functional shifts of right invertible operators*, Demonstratio Math. 28 (1995), to appear.

[9] —, *Periodic solutions of equations of higher order with a right invertible operator induced by functional shifts*, Ann. Univ. M. Curie-Skłodowska 46 (1992), 9–22.

[10] —, *On some functional shifts induced by operators of complex differentation*, Opuscula Math. 14 (1994), 53–65.

[11] —, *Some remarks on sequential shifts induced by right invertible operators*, Funct. Approx. 22 (1993), 71–84.

[12] —, *Applications of sequential shifts to an interpolation problem*, Collect. Math. 44 (1993), 47–57.

[13] —, *Some fundamental properties of functional R-shifts for right invertible operators*, Comment. Math. 33 (1994), 9–22.

[14] —, *A note on functional R-shifts for right invertible operators*, Discussiones Math., to appear.

[15] J. B. Miller, *The standard summation operator, the Euler–Maclaurin sum formula and the Laplace transformation*, J. Austral. Math. Soc. 39 (1985), 376–390.

[16] Nguyen van Mau, *Boundary value problems and controllability of linear systems with right invertible operators*, Dissertationes Math. 316 (1992).

[17] D. Przeworska-Rolewicz, *Shifts and Periodicity for Right Invertible Operators*, Research Notes in Math. 43, Pitman Adv. Publ. Program, Boston-London-Melbourne, 1980.

[18] —, *Algebraic Analysis*, PWN – Polish Scientific Publishers, and D. Reidel, Warszawa–Dordrecht, 1988.

[19] —, *Spaces of D-paraanalytic elements*, Dissertationes Math. 302 (1990).

[20] —, *True shifts*, J. Math. Anal. Appl. 170 (1992), 27–48.

[21] —, *Generalized Bernoulli operator and Euler–Maclaurin Formula*, in: Advances in Optimization, Proc. 6-th French-German Colloquium on Optimization, Lambrecht, 2–9 June, 1991, Lecture Notes in Econom. and Math. Systems, 382, Springer, Berlin, 1992, 355–368.

[22] —, *The operator* $\exp(hD)$ *and its inverse formula*, Demonstratio Math. 26 (1993), 545–552.

[23] H. von Trotha, *Structure properties of D-R spaces*, Dissertationes Math. 180 (1981).