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**Limit theorems
for sums of dependent random vectors in $\boldsymbol{R}^d$**

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Introduction

The main purpose of the classical theory of probability was to investigate the limit distributions of an increasing number of independent random vectors. It is fairly obvious that without additional assumptions about summands any probability measure can appear as the limit law. If we restrict our considerations to infinitesimal systems, i.e., systems such that the influence of individual components on the sum decreases uniformly to zero, then the class of possible limit distributions is equal to the class of infinitely divisible distributions. Fundamental information concerning this class is given in Chapter I. Each infinitely divisible probability measure can be described by three simple objects: a vector, a matrix and a measure. The class of all infinitely divisible laws is closed under weak convergence; moreover, there can be given necessary and sufficient conditions for this convergence in terms of the above-mentioned characterisation. Using the fact that the characteristic function of a sum of independent random vectors can be approximated by a suitably constructed infinitely divisible characteristic function, one can obtain criteria for convergence in law to an arbitrarily chosen infinitely divisible probability measure. It is worth mentioning that the possibility of such approximation is guaranteed by our postulate (2.4), which is weaker than the usual a priori assumption of the infinitesimality of the systems. Unfortunately, this assumption has a less intuitive meaning; moreover, it is not clear to what extent it is more general than the infinitesimality. Chapter II contains a brief survey of all classical limit theorems for independent random vectors with the emphasis on the role of condition (2.4). In view of the complete solution of the problem in the case of independence, it is natural to ask when the weak convergence of dependent quantities holds. Dvoretzky has proposed in [2] that sufficient conditions for such convergence can be obtained by a generalization of known results in the following way: we replace all mean values by conditional mean values with respect to suitably chosen σ-fields. Moreover, we replace convergence of numerical sequences by convergence in probability of sequences of random vectors obtained in this way. In [2] Dvoretzky has communicated examples for such a procedure for classical theorems which use the Lévy–Khintchine and the Kolmogorov representations for limit distributions. In [3] he gave a slightly complicated proof of one of those theorems, namely, the generalization of the Lindeberg–Feller theorem on asymptotic normality. Simultaneously Brown and Eagleson proved in [1] the generalization of Kolmogorov theorem to systems of random vectors with finite variances. Using variants of their simple and elegant method, we shall show in Chapters III–V that every fundamental limit theorem can be generalized in the way proposed by Dvoretzky. Observe that the above-mentioned condition (2.4) has its analogue in this theory. Moreover, we can give necessary and sufficient conditions for weak convergence under some assumptions about random vectors. Chapter VI contains corollaries to the simplest laws: degenerate, normal and Poissonian. It is an open question with respect to what σ-fields we can employ the conditional quantities to guarantee the weak convergence of distributions. In this paper we consider almost exclusively conditioning of the martingale type. Chapter VII is devoted to other kind of conditioning, proposed by Dvoretzky. In view of the great number of limit theorems for dependent random variables, we are not able here to study systematically the degree of generality of the results presented. Let us remark only that Dvoretzky in [3] has shown that many known results concerning asymptotic normality follow from his conditional Lindeberg–Feller criterion.

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I. Infinitely divisible probability measures on $\boldsymbol{R}^d$

Let $\boldsymbol{N}, \boldsymbol{R}, \boldsymbol{C}$ denote the sets of natural, real and complex numbers, respectively. By $\boldsymbol{R}^d$ we denote the d-dimensional real Euclidean space. Let $\mathscr{B}^d$ be the σ-field of all Borel sets in $\boldsymbol{R}^d$, i.e., the smallest σ-field containing all open subsets in $\boldsymbol{R}^d$ in the topology generated by the norm $\|\cdot\|$, where $\|\vec{x}\| := \sqrt{x_1^2 + \ldots + x_d^2}$ for $\vec{x} = (x_1, \ldots, x_d) \in \boldsymbol{R}^d$.

Every countably additive set function $P\colon \mathscr{B}^d \to [0, 1]$ such that $P(\boldsymbol{R}^d) = 1$ is said to be a *probability measure* on $\boldsymbol{R}^d$. A set $E \in \mathscr{B}^d$ is called a *continuity set for the measure* P if and only if $P(\partial E) = 0$, where ∂E is a boundary of the set E. The field of all continuity sets for P will be denoted by $\operatorname{Cont} P$.

Let $\mathscr{P}(\boldsymbol{R}^d)$ denote the set of all probability measures on $\boldsymbol{R}^d$ equipped with the weak topology, i.e., the sequence $P_n \in \mathscr{P}(\boldsymbol{R}^d)$, $n \in \boldsymbol{N}$, converges in this topology to $P \in \mathscr{P}(R^d)$ if and only if $\lim_{n\to\infty} \int_{\boldsymbol{R}^d} f dP_n = \int_{\boldsymbol{R}^d} f dP$ for every bounded continuous real function f, or equivalently, $\lim_{n\to\infty} P_n(E) = P(E)$

for every $E \in \mathrm{Cont}P$. In such a case we shall say that the *sequence* P_n, $n \in \boldsymbol{N}$, *converges weakly* to P or $P_n \Rightarrow P$ in symbols.

For any $P_1, P_2 \in \mathscr{P}(\boldsymbol{R}^d)$ *the convolution* $P_1 * P_2 \in \mathscr{P}(\boldsymbol{R}^d)$ is defined as follows:

$$(P_1 * P_2)(E) := \int_{\boldsymbol{R}^d} P_1(E - \vec{x})\, dP_2(\vec{x}), \qquad E \in \mathscr{B}^d,$$

where $E - \vec{x} := \{\vec{y} - \vec{x};\ \vec{y} \in E\}$.

If $P \in \mathscr{P}(\boldsymbol{R}^d)$, its *characteristic function* φ_p: $\boldsymbol{R}^d \to \boldsymbol{C}$ is defined by the formula

$$\varphi_p(\vec{t}) := \int_{\boldsymbol{R}^d} e^{i(\vec{t}, \vec{x})}\, dP(\vec{x}), \qquad \vec{t} \in \boldsymbol{R}^d,$$

where $(\vec{x}, \vec{y}) := x_1 y_1 + \ldots + x_d y_d$, $\vec{x}, \vec{y} \in \boldsymbol{R}^d$, is the inner product in $\boldsymbol{R}^d$. The set of all characteristic functions with the topology of almost uniform convergence will be denoted by $\mathscr{C}h(\boldsymbol{R}^d)$. The mapping Φ: $\mathscr{P}(\boldsymbol{R}^d) \to \mathscr{C}h(\boldsymbol{R}^d)$, $\Phi(P) := \varphi_p$, is one-to-one, bicontinuous and $\Phi(P * Q) = \Phi(P) \cdot \Phi(Q)$ for $P, Q \in \mathscr{P}(\boldsymbol{R}^d)$. Moreover, if the sequence $\varphi_n \in \mathscr{C}h(\boldsymbol{R}^d)$, $n \in \boldsymbol{N}$, converges pointwise to some function φ which is continuous at $\vec{O} \in \boldsymbol{R}^d$, then $\varphi \in \mathscr{C}h(\boldsymbol{R}^d)$ and $P_{\varphi_n} \Rightarrow P_{\varphi}$.

A probability measure $P \in \mathscr{P}(\boldsymbol{R}^d)$ is said to be *infinitely divisible* if for every $n \in \boldsymbol{N}$ there exists a $P_n \in \mathscr{P}(\boldsymbol{R}^d)$ such that $P = \underbrace{P_n * \ldots * P_n}_{n \text{ times}} := (P_n)^{*n}$

A characteristic function $\varphi \in \mathscr{C}h(\boldsymbol{R}^d)$ is said to be *infinitely divisible* if for every $n \in \boldsymbol{N}$ there exists a $\varphi_n \in \mathscr{C}h(\boldsymbol{R}^d)$ such that $\varphi = (\varphi_n)^n$.

Obviously, $P \in \mathscr{P}(\boldsymbol{R}^d)$ is infinitely divisible if and only if so is its characteristic function φ_p. Also the limit of weakly convergent sequence of infinitely divisible probability measures is infinitely divisible.

There exist some canonical decompositions for infinitely divisible characteristic functions in which the latter can be represented by some triples of objects of a simple nature.

A. The Lévy-Khintchine canonical representation.

THEOREM 1.1 ([7]). *Let* $\mathbf{A} = [a_{ij}]$ *be a nonnegative definite* $d \times d$-*matrix* (i.e., $(\vec{t}, \mathbf{A}\vec{t}) \geqslant 0$ *for* $\vec{t} \in \boldsymbol{R}^d$). *Next, let* μ *be a finite measure on* $\boldsymbol{R}^d$ *such that* $\mu(\{\vec{O}\}) = 0$. *Finally, let us fix a vector* $\vec{a} \in \boldsymbol{R}^d$. *Then the function* φ: $\boldsymbol{R}^d \to \boldsymbol{C}$ *defined by the formula*

$$\varphi(\vec{t}) := \exp\Big\{ i(\vec{t}, \vec{a}) - \tfrac{1}{2}(\vec{t}, \mathbf{A}\vec{t}) + \int_{\boldsymbol{R}^d} \Big(e^{i(\vec{t}, \vec{x})} - 1 - \frac{i(\vec{t}, \vec{x})}{1 + \|\vec{x}\|^2} \Big) \frac{1 + \|\vec{x}\|^2}{\|\vec{x}\|^2}\, d\mu \Big\}, \qquad t \in \boldsymbol{R}^d, \tag{1.1}$$

belongs to $\mathscr{C}h(\boldsymbol{R}^d)$ *and is infinitely divisible.*

Conversely, if $\varphi \in \mathscr{Ch}(\boldsymbol{R}^d)$ is infinitely divisible, then it has a unique decomposition in form (1.1). ■ (■ denotes the end of the proof)

We shall denote P_φ by $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$.

THEOREM 1.2 ([7]). *If $\mathscr{L}(\vec{a}_n, \mathbf{A}_n, \mu_n) \Rightarrow P$, then there exist a vector $\vec{a}$, a matrix $\mathbf{A}$ and a measure μ, with the properties described above, such that $P = \mathscr{L}(\vec{a}, \mathbf{A}, \mu)$ and*

$$\lim_{n\to\infty} \vec{a}_n = \vec{a}; \tag{1.2}$$

$$\lim_{n\to\infty} (\mathbf{A}_n + \mathbf{T}_n) = \mathbf{A} + \mathbf{T}, \tag{1.3}$$

$$\lim_{n\to\infty} \mu_n(E) = \mu(E), \qquad E \in \operatorname{Cont}\mu, \ \vec{O} \notin \bar{E}, \tag{1.4}$$

where $\mathbf{T}_n = [t_{ij}^{(n)}]$, $\mathbf{T} = [t_{ij}]$ are $d\times d$-matrices whose entries are defined by

$$t_{ij}^{(n)} := \int_{\boldsymbol{R}^d} \frac{\Pi_i \vec{x} \, \Pi_j \vec{x}}{\|\vec{x}\|^2} \, d\mu_n(\vec{x}) \tag{1.5}$$

$$t_{ij} := \int_{\boldsymbol{R}^d} \frac{\Pi_i \vec{x} \, \Pi_j \vec{x}}{\|\vec{x}\|^2} \, d\mu(\vec{x}), \qquad 1 \leqslant i, j \leqslant d, \ n \in \boldsymbol{N}. \tag{1.6}$$

Conversely, if (1.2)–(1.4) *hold, then $\mathscr{L}(\vec{a}_n, \mathbf{A}_n, \mu_n) \Rightarrow \mathscr{L}(\vec{a}, \mathbf{A}, \mu)$.* ($\Pi_i$: $\boldsymbol{R}^d \to \boldsymbol{R}$, $1 \leqslant i \leqslant d$, denotes the ith coordinate projection. In the sequel we shall write $x_i := \Pi_i \vec{x}$, $x_{ni} := \Pi_i \vec{x}_n$, $x_{nki} := \Pi_i \vec{x}_{nk}$, etc. for $1 \leqslant i \leqslant d$). ■

B. The Lévy canonical representation. Let μ be an arbitrary finite measure on $\boldsymbol{R}^d$ such that $\mu(\{\vec{O}\} = 0$. We define the σ-finite measure ν on $\boldsymbol{R}^d$ by the equalities

$$\nu(E) := \int_E \frac{1 + \|\vec{x}\|^2}{\|\vec{x}\|^2} \, d\mu, \qquad E \in \mathscr{B}^d, \ \vec{O} \notin E, \tag{1.7}$$

$$\nu(\{\vec{O}\}) := 0. \tag{1.8}$$

It is easy to check that

$$\int_{\|\vec{x}\| \leqslant 1} \|\vec{x}\|^2 d\nu = \int_{\|\vec{x}\| \leqslant 1} (1 + \|\vec{x}\|^2) \, d\mu < +\infty, \tag{1.9}$$

$$\int_{\|\vec{x}\| > 1} d\nu = \int_{\|\vec{x}\| > 1} \frac{1 + \|\vec{x}\|^2}{\|\vec{x}\|^2} \, d\mu < +\infty. \tag{1.10}$$

Conversely, if a measure ν on $\boldsymbol{R}^d$ fulfils (1.8)–(1.10), then the measure μ defined by

$$\mu(E) := \int_E \frac{\|\vec{x}\|^2}{1+\|\vec{x}\|^2}\, d\nu, \qquad E \in \mathscr{B}^d, \tag{1.11}$$

is finite and $\mu(\{\vec{O}\}) = 0$.

These remarks can be summarized in the following

THEOREM 1.3 ([9]). *Every infinitely divisible $\varphi \in \mathscr{C}h(\boldsymbol{R}^d)$ can be uniquely decomposed in the form*

(1.12)

$$\varphi(\vec{t}) = \exp\left\{i(\vec{t}, \vec{a}) - \tfrac{1}{2}(\vec{t}, \mathbf{A}\vec{t}) + \int_{\boldsymbol{R}^d} \left(e^{i(\vec{t}, \vec{x})} - 1 - \frac{i(\vec{t}, \vec{x})}{1+\|\vec{x}\|^2}\right) d\nu\right\}, \qquad \vec{t} \in \boldsymbol{R}^d,$$

where $\vec{a} \in \boldsymbol{R}^d$, $\mathbf{A}$ is a nonnegative definite $d \times d$-matrix and ν is a σ-finite measure on $\boldsymbol{R}^d$ satisfying (1.8)–(1.10). ■

If φ is given by (1.12), then the infinitely divisible measure $P_\varphi \in \mathscr{P}(\boldsymbol{R}_d)$ will be denoted by $l(\vec{a}, \mathbf{A}, \nu)$.

THEOREM 1.4 ([9]). *If $l(\vec{a}_n, \mathbf{A}_n, \nu_n) \Rightarrow P$, then there exist a vector $\vec{a}$, a matrix $\mathbf{A}$ and a measure ν, with the above properties, such that $P = l(\vec{a}, \mathbf{A}, \nu)$; moreover,* (1.2), (1.3) *and*

$$\lim_{n\to\infty} \nu_n(E) = \nu(E), \qquad E \in \mathrm{Cont}\,\nu, \ \vec{O} \notin \bar{E}, \tag{1.13}$$

are satisfied. Conversely, (1.2), (1.3) *and* (1.13) *imply $l(\vec{a}_n, \mathbf{A}_n, \nu_n) \Rightarrow l(\vec{a}, \mathbf{A}, \nu)$.* ■

Let us mention, for future reference, that in this case

$$t_{ij}^{(n)} = \int_{\boldsymbol{R}^d} \frac{x_i x_j}{1+\|\vec{x}\|^2}\, d\nu_n, \tag{1.14}$$

$$t_{ij} = \int_{\boldsymbol{R}^d} \frac{x_i x_j}{1+\|\vec{x}\|^2}\, d\nu, \qquad 1 \leqslant i, j \leqslant d, \ n \in \boldsymbol{N}. \tag{1.15}$$

Before we start to describe other representations, we shall make some remarks about the existence of moments of infinitely divisible probability measures.

We shall say that $P \in \mathscr{P}(\boldsymbol{R}^d)$ has a *finite absolute moment of order $a \geqslant 0$* if and only if $\int_{\boldsymbol{R}^d} \|\vec{x}\|^a dP < +\infty$.

THEOREM 1.5 $P = l(\vec{a}, \mathbf{A}, \nu) \in \mathscr{P}(\boldsymbol{R}^d)$ *has a finite absolute moment of order* $a \geqslant 0$ *if and only if*

$$\int\limits_{\boldsymbol{R}^d \setminus V} \|\vec{x}\|^a d\nu < +\infty, \qquad V \in O. \tag{1.16}$$

(By O we denote the family of bounded sets $E \in \mathscr{B}^d$ such that $\vec{O} \in \operatorname{Int} E$.)

Proof. In the case $d = 1$ this theorem was proved by Ramachandran in [6] (theorem 8). Assume that $d \geqslant 2$. The measure ν being finite outside every $E \in O$, it suffices to prove this theorem for the cube $V := \{\vec{x} \in \boldsymbol{R}^d; |x_i| \leqslant 1, 1 \leqslant i \leqslant d\}$.

For every $n \in N$ there exists a $P_n \in \mathscr{P}(\boldsymbol{R}^d)$ such that $P = (P_n)^{*n}$. Let us denote by $P^{(i)}$, $P_n^{(i)}$, $1 \leqslant i \leqslant d$, the one-dimensional marginal distributions of P and P_n, respectively. It is obvious that $P^{(i)}$, $1 \leqslant i \leqslant d$, are infinitely divisible and $P^{(i)} = (P_n^{(i)})^{*n}$, $1 \leqslant i \leqslant d$, $n \in N$. Let ν_i denote the Lévy measure of $P^{(i)}$ and let $\nu^{(i)}$ denote the one-dimensional marginal measure of ν, $1 \leqslant i \leqslant d$. If follows from Theorem 4.2 below that

$$\lim_{n\to\infty} nP_n(E) = \nu(E), \qquad E \in \operatorname{Cont}\nu,\ \vec{O} \notin \bar{E}, \tag{1.17}$$

$$\lim_{n\to\infty} nP_n^{(i)}(A) = \nu_i(A), \qquad A \in \operatorname{Cont}\nu_i,\ O \notin \bar{A}. \tag{1.18}$$

Observe that $A \in \operatorname{Cont}\nu^{(i)}$ if and only if $\Pi_i^{-1}A \in \operatorname{Cont}\nu$. It follows from (1.17) that

$$\lim_{n\to\infty} nP_n^{(i)}(A) = \lim_{n\to\infty} nP_n(\Pi_i^{-1}A) = \nu(\Pi_i^{-1}A) = \nu^{(i)}(A)$$

for $A \in \operatorname{Cont}\nu^{(i)}$, $O \notin \bar{A}$, and then by (1.18) we have $\nu^{(i)}(A) = \nu_i(A)$ for $A \in \operatorname{Cont}\nu^{(i)} \cap \operatorname{Cont}\nu_i$, $O \notin \bar{A}$.

Thus for $1 \leqslant i \leqslant d$ the condition

$$\int\limits_{|x_i|>1} |x_i|^a d\nu_i < +\infty \tag{1.19}$$

is equivalent to

$$\int\limits_{|x_i|>1} |x_i|^a d\nu^{(i)} < +\infty \tag{1.20}$$

for $a \geqslant 0$. Now, if (1.16) is satisfied, then in virtue of the inequalities

$$\int\limits_{|x_i|>1} |x_i|^a d\nu^{(i)}(x_i) \leqslant \int\limits_{\boldsymbol{R}^d \setminus V} \|\vec{x}\|^a d\nu(\vec{x}), \tag{1.21}$$

$$\int\limits_{\boldsymbol{R}^d \setminus V} \|\vec{x}\|^a dP(\vec{x}) \leqslant d^a \int\limits_{\boldsymbol{R}^d \setminus V} \max_{1\leqslant i\leqslant d} |\Pi_i \vec{x}|^a dP(\vec{x}) \tag{1.22}$$

$$\leqslant d^a \sum_{i=1}^{d} \int\limits_{\{\vec{x};\, |\Pi_i \vec{x}|>1\}} |\Pi_i \vec{x}|^a dP(\vec{x}) = d^a \sum_{i=1}^{d} \int\limits_{|x_i|>1} |x_i|\, dP^{(i)}(x_i)$$

by the theorem of Ramachandran for the one-dimensional case $P^{(i)}$'s and next P have a finite absolute moment of order a. Similarly we prove the converse part of this theorem. ■

C. The canonical representation for infinitely divisible probability measures with finite expectations. The expectation of the probability measure $P \in \mathscr{P}(\boldsymbol{R}^d)$ is defined as the Lebesgue integral $\int_{\boldsymbol{R}^d} \vec{x}\, dP(\vec{x})$, if the latter exists. Theorems 1.1, 1.3, 1.5 imply at once

THEOREM 1.6. *Let $P \in \mathscr{P}(\boldsymbol{R}^d)$ be infinitely divisible. P has finite expectation $\vec{m}$ if and only if its characteristic function φ_p can be uniquely decomposed in the following equivalent forms:*

(1.23)
$$\varphi_p(\vec{t}) = \exp\Big\{i(\vec{t}, \vec{m}) - \tfrac{1}{2}(\vec{t}, \mathbf{A}\vec{t}) + \int_{\boldsymbol{R}^d} \big(e^{i(\vec{t}, \vec{x})} - 1 - i(\vec{t}, \vec{x})\big)\, d\nu\Big\}, \quad \vec{t} \in \boldsymbol{R}^d,$$

(1.24)
$$\varphi_p(\vec{t}) = \exp\Big\{i(\vec{t}, \vec{m}) - \tfrac{1}{2}(\vec{t}, \mathbf{A}\vec{t}) + \int_{\boldsymbol{R}^d} \big(e^{i(\vec{t}, \vec{x})} - 1 - i(\vec{t}, \vec{x})\big) \frac{1 + \|\vec{x}\|^2}{\|\vec{x}\|^2}\, d\mu\Big\}, \quad \vec{t} \in \boldsymbol{R}^d,$$

(1.25)
$$\varphi_p(\vec{t}) = \exp\Big\{i(\vec{t}, \vec{m}) - \tfrac{1}{2}(\vec{t}, \mathbf{A}\vec{t}) + \int_{\boldsymbol{R}^d} \big(e^{i(\vec{t}, \vec{x})} - 1 - i(\vec{t}, \vec{x})\big) \frac{d\varkappa}{\|\vec{x}\|^2}\Big\}, \quad \vec{t} \in \boldsymbol{R}^d,$$

where $\mathbf{A}$ *is a nonnegative definite* $d \times d$*-matrix, the* σ*-finite measure* ν *on* $\boldsymbol{R}^d$ *satisfies* (1.8), (1.9) *and*

(1.26)
$$\int_{\boldsymbol{R}^d \setminus V} \|\vec{x}\|\, d\nu < +\infty, \quad V \in O,$$

the finite measure μ *satisfies*

(1.27)
$$\int_{\boldsymbol{R}^d} \|\vec{x}\|\, d\mu < +\infty,$$

the σ*-finite measure* $\varkappa$ *fulfils*

(1.28)
$$\int_{\boldsymbol{R}^d \setminus V} \frac{d\varkappa}{\|\vec{x}\|} < +\infty, \quad V \in O,$$

(1.29)
$$\varkappa(V) < +\infty, \quad V \in O. \;■$$

Let us remark that the measures μ and ν, occurring in (1.23) and (1.24) are the same as those in (1.12) and (1.1), respectively. Moreover, the relations between μ, ν and $\varkappa$ are given by

$$\varkappa(E) = \int_E (1+\|\vec{x}\|^2)\,d\mu, \qquad E \in \mathscr{B}^d, \tag{1.30}$$

and

$$\varkappa(E) = \int_E \|\vec{x}\|^2\,d\nu, \qquad E \in \mathscr{B}^d. \tag{1.31}$$

Observe that

$$\vec{m} = \vec{a} + \int_{R^d} \vec{x}\,d\mu = \vec{a} + \int_{R^d} \frac{\vec{x}\,\|\vec{x}\|^2}{1+\|\vec{x}\|^2}\,d\nu. \tag{1.32}$$

We shall denote by $k(\vec{m}, \mathbf{A}, \varkappa)$ the probability measure whose characteristic function is given by (1.25).

D. The Kolmogorov canonical representation for infinitely divisible probability measure with finite variance. The variance of $P \in \mathscr{P}(\boldsymbol{R}^d)$ is defined by $v(P) := \int_{\boldsymbol{R}^d} \|\vec{x}-\vec{m}\|^2 dP$ and its covariance matrix $\mathbf{C}(P) = [c_{ij}]$ by $c_{ij} := \int_{\boldsymbol{R}^d} (x_i - m_i)(x_j - m_j)\,dP(\vec{x})$, $1 \leqslant i, j \leqslant d$ ($\vec{m}$ denotes the expectation of P), where all the c_{ij} are finite if and only if $v(P) < +\infty$, or equivalently, the second absolute moment of P is finite.

Let $P = \mathscr{L}(\vec{a}, \mathbf{A}, \mu) = l(\vec{a}, \mathbf{A}, \nu) \in \mathscr{P}(\boldsymbol{R}^d)$ be infinitely divisible. By Theorem 1.5, P has finite variance if and only if

$$\int_{R^d} \|\vec{x}\|^2\,d\nu < +\infty \tag{1.33}$$

or equivalently,

$$\int_{\boldsymbol{R}^d} \|\vec{x}\|^2\,d\mu < +\infty. \tag{1.34}$$

Then the measure $\varkappa$ in (1.25) is finite. Conversely, if φ_p is given in form (1.25) with finite $\varkappa$, then one can easily obtain its decomposition in the Lévy form or in the equivalent Lévy-Khinthine form, where (1.33) and (1.34) are satisfied because of the finiteness of $\varkappa$.

Therefore we have

THEOREM 1.7. *An infinitely divisible $P \in \mathscr{P}(\boldsymbol{R}^d)$ has finite variance if and only if φ_p can be represented in form* (1.25) *with the finite measure $\varkappa$.* ∎

In this case we shall use the notation $\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa)$ for P. It can be checked by computing the partial derivatives of φ_p that the expectation of $\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa)$ is equal to $\vec{m}$ and its covariance matrix is $\mathbf{A}+\mathbf{S}$, where

$d \times d$-matrix $\mathbf{S} = [S_{ij}]$ is defined by

$$S_{ij} := \int_{\mathbf{R}^d} \frac{x_i x_j}{\|\vec{x}\|^2} d\varkappa, \quad 1 \leqslant i, j \leqslant d. \tag{1.35}$$

In particular, we get $v(\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa)) = \mathrm{Tr}(\mathbf{A}+\mathbf{S}) = \mathrm{Tr}\,\mathbf{A}+\varkappa(\mathbf{R}^d)$.

It is not difficult to reformulate Theorems 1.2 and 1.4 in terms of other representations we have considered. Let us give a simple result of such a type:

THEOREM 1.8. *If* $k(\vec{m}_n, \mathbf{A}_n, \varkappa_n) \Rightarrow k(\vec{m}, \mathbf{A}, \varkappa)$, *then the following conditions are satisfied*:

$$\lim_{n\to\infty} \left((\vec{m}_n - \int_{\mathbf{R}^d} \frac{\vec{x}}{1+\|\vec{x}\|^2} d\varkappa_n \right) = \vec{m} - \int_{\mathbf{R}^d} \frac{\vec{x}}{1+\|\vec{x}\|^2} d\varkappa, \tag{1.36}$$

$$\lim_{n\to\infty} (\mathbf{A}_n + \mathbf{T}_n) = \mathbf{A}+\mathbf{T}, \tag{1.37}$$

where $\mathbf{T}_n$, $\mathbf{T}$ *are the* $d\times d$*-matrices defined in Theorem* 1.2, *whose entries are expressed in the terms of* $\varkappa$:

$$t_{ij}^{(n)} = \int_{\mathbf{R}^d} \frac{x_i x_j}{(1+\|\vec{x}\|^2)\|\vec{x}\|^2} d\varkappa_n, \quad n \in \mathbf{N},\ 1 \leqslant i, j \leqslant d,$$

$$t_{ij} = \int_{\mathbf{R}^d} \frac{x_i x_j}{(1+\|\vec{x}\|^2)\|\vec{x}\|^2} d\varkappa, \quad 1 \leqslant i, j \leqslant d,$$

and

$$\lim_{n\to\infty} \int_E \frac{d\varkappa_n}{\|\vec{x}\|^2} = \int_E \frac{d\varkappa}{\|\vec{x}\|^2}, \quad E \in \mathrm{Cont}\,\varkappa,\ \vec{O} \notin \bar{E} \tag{1.38}$$

or, equivalently,

$$\lim_{n\to\infty} \int_E \frac{d\varkappa_n}{1+\|\vec{x}\|^2} = \int_E \frac{d\varkappa}{1+\|\vec{x}\|^2}, \quad E \in \mathrm{Cont}\,\varkappa,\ \vec{O} \notin \bar{E}. \tag{1.39}$$

Conversely, if a sequence $k(\vec{m}_n, \mathbf{A}_n, \varkappa_n)$, $n \in \mathbf{N}$, *satisfies* (1.36), (1.37) *and* (1.38) *or* (1.39) *with some* σ*-finite* $\varkappa$, *then it converges weakly to* $k(\vec{m}, \mathbf{A}, \varkappa)$. *If* $\varkappa$ *is finite, we obtain the weak convergence to* $\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa)$. ∎

It should be remarked that in Theorem 1.8 nothing has been assumed about the variances $v(k(\vec{m}_n, \mathbf{A}_n, \varkappa_n))$ (if they exist), but in such a general case the expectations $\vec{m}_n$ may not converge to the expectation $\vec{m}$ of the limit measure.

THEOREM 1.9 ([8]). *Let* $\mathcal{K}(\vec{m}_n, \mathbf{A}_n, \varkappa_n)$, $n \in N$, *be infinitely divisible probability measures on* $\mathbf{R}^d$ *such that* $v(\mathcal{K}(\vec{m}_n, \mathbf{A}_n, \varkappa_n))$ *are commonly bounded. Then* $\mathcal{K}(\vec{m}_n, \mathbf{A}_n, \varkappa_n) \Rightarrow \mathcal{K}(\vec{m}, \mathbf{A}, \varkappa)$ *if and only if the following properties hold:*

$$\lim_{n\to\infty} \vec{m}_n = \vec{m}, \tag{1.40}$$

$$\lim_{n\to\infty} (\mathbf{A}_n + \mathbf{T}_n^V) = \mathbf{A} + \mathbf{T}^V, \quad V \in \operatorname{Cont} \varkappa \cap O, \tag{1.41}$$

where $\mathbf{T}_n^V = [t_{ijn}^V]$, $\mathbf{T}^V = [t_{ij}^V]$, *are* $d \times d$*-matrices whose entries are defined by*

$$t_{ijn}^V := \int_V \frac{x_i x_j}{\|\vec{x}\|^2} d\varkappa_n, \quad 1 \leqslant i, j \leqslant d, \ n \in N, \tag{1.42}$$

$$t_{ij}^V := \int_V \frac{x_i x_j}{\|\vec{x}\|^2} d\varkappa, \quad 1 \leqslant i, j \leqslant d, \tag{1.43}$$

$$\lim_{n\to\infty} \varkappa_n(E) = \varkappa(E) \quad \text{for every bounded } E \in \operatorname{Cont} \varkappa, \ \vec{O} \notin \bar{E}. \tag{1.44}$$

Assume in addition that

$$\lim_{n\to\infty} v(\mathcal{K}(\vec{m}_n, \mathbf{A}_n, \varkappa_n)) = v(\mathcal{K}(\vec{m}, \mathbf{A}, \varkappa)).$$

Then $\mathcal{K}(\vec{m}_n, \mathbf{A}_n, \varkappa_n) \Rightarrow \mathcal{K}(\vec{m}, \mathbf{A}, \varkappa)$ *if and only if the following conditions are satisfied:*

$$\lim_{n\to\infty} \vec{m}_n = \vec{m}, \quad \lim_{n\to\infty} (\mathbf{A}_n + \mathbf{S}_n) = \mathbf{A} + \mathbf{S}, \tag{1.45}$$

where $\mathbf{S}_n, \mathbf{S}$ *are defined by* (1.35),

$$\lim_{n\to\infty} \varkappa_n(E) = \varkappa(E), \quad E \in \operatorname{Cont} \varkappa, \ \vec{O} \notin \bar{E}. \tag{1.46}$$

II. The classical limit theorems for sums of independent random vectors

In this section we give a survey of known results concerning the convergence of distributions of sums of independent random quantities.

Let us consider a system of d-dimensional random vectors arranged in a double array:

$$(2.1)\qquad \begin{array}{l} \vec{X}_{11}, \vec{X}_{12}, \ldots \vec{X}_{1,k_1}, \\ \vec{X}_{21}, \vec{X}_{22}, \ldots \vec{X}_{2,k_2}, \\ \cdots\cdots\cdots\cdots \\ \vec{X}_{n,1}, \vec{X}_{n,2}, \ldots, \vec{X}_{n,k_n}, \\ \cdots\cdots\cdots\cdots \end{array}$$

all these vectors being defined on the probability space $(\Omega, \mathscr{F}, P)$. We define the sequence of row sums for (2.1):

$$\vec{S}_n := \sum_{k=1}^{k_n} \vec{X}_{nk}, \qquad n \in \boldsymbol{N},$$

and the sequence of its probability distributions:

$$\sigma_n(E) := P(\vec{S}_n^{-1}E), \qquad E \in \mathscr{B}^d, \qquad n \in \boldsymbol{N}.$$

If the sequence σ_n, $n \in \boldsymbol{N}$, converges weakly to a probability measure $P \in \mathscr{P}(\boldsymbol{R}^d)$, then the system of random vectors (2.1) is said to *converge in law to* P.

The random vectors (2.1) determine in a natural way a double array of their d-dimensional probability distributions:

$$(2.2)\qquad P_{nk}(E) := P(\vec{X}_{nk}^{-1}E), \qquad E \in \mathscr{B}^d,\ 1 \leqslant k \leqslant k_n,\ n \in \boldsymbol{N}.$$

Assume that system (2.1) is *row-wise independent*, i.e., for every $n \in \boldsymbol{N}$ the vectors $\vec{X}_{n1}, \ldots, \vec{X}_{nk_n}$ are independent. Then

$$(2.3)\qquad \sigma_n = P_{n_1} * \ldots * P_{n,k_n}, \qquad n \in \boldsymbol{N},$$

and hence in this case the main problem is to give necessary and sufficient conditions for the weak convergence of convolutions of probabilities on $\boldsymbol{R}^d$.

For arbitrary $V \in O$ we define the vectors

$$\vec{a}_{nk}^V = \mathrm{E}[\vec{X}_{nk} I(\vec{X}_{nk} \in V)], \qquad 1 \leqslant k \leqslant k_n,\ n \in \boldsymbol{N}.$$

(We write $\mathrm{E}\vec{X} := \int_\Omega \vec{X}\,dP$, $I(A)$ is the indicator function of the set A.)

Let us define by φ_{nk} the characteristic function of the measure P_{nk}. Moreover, for arbitrary $V \in O$ we define

$$\varphi_{nk}^V(\vec{t}) := \exp\{i(\vec{t}, \vec{a}_{nk}^V) + \gamma_{nk}^V(\vec{t})\}, \qquad t \in \boldsymbol{R}^d,$$

where

$$\gamma_{nk}^V(\vec{t}) := \mathrm{E}[e^{i(\vec{t}, \vec{X}_{nk} - \vec{a}_{nk}^V)} - 1], \qquad \vec{t} \in \boldsymbol{R}^d,\ 1 \leqslant k \leqslant k_n,\ n \in \boldsymbol{N}.$$

It can easily be verified that $\varphi_{nk}^V \in \mathscr{C}h(\boldsymbol{R}^d)$ are infinitely divisible. Finally, let us define

$$\varphi_n := \varphi_{n1}\cdot\varphi_{n2}\cdot \ldots \cdot\varphi_{n,k_n} = \varphi_{\sigma_n}, \qquad n\in\boldsymbol{N},$$
$$\varphi_n^V := \varphi_{n1}^V\cdot\varphi_{n2}^V\cdot \ldots \cdot\varphi_{nk_n}^V, \qquad n\in\boldsymbol{N}.$$

Obviously, the functions $\varphi_n^V\in\mathscr{C}h(\boldsymbol{R}^d)$ are infinitely divisible: furthermore, for every $\vec{t}\in\boldsymbol{R}^d$ and $n\in\boldsymbol{N}$,

$$\left|\varphi_n(\vec{t})-\varphi_n^V(\vec{t})\right| \leqslant \sum_{k=1}^{k_n}|\varphi_{nk}(\vec{t})-\varphi_{nk}^V(\vec{t})| = \sum_{k=1}^{k_n}|e^{i(\vec{t},\vec{a}_{nk})}[1+\gamma_{nk}^V(\vec{t})-e^{\gamma_{nk}^V(\vec{t})}]|$$
$$\leqslant \tfrac{1}{2}\sum_{k=1}^{k_n}|\gamma_{nk}^V(\vec{t})|^2\exp|\gamma_{nk}^V(\vec{t})| \leqslant 5\sum_{k=1}^{k_n}|\gamma_{nk}^V(\vec{t})|^2.$$

Thus, if the random vectors (2.1) fulfil the condition

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}|\gamma_{nk}^V(\vec{t})|^2 = 0, \qquad \vec{t}\in\boldsymbol{R}^d, \tag{2.4}$$

for some $V\in O$, then $\lim_{n\to\infty}\varphi_n = \varphi$ if and only if $\lim_{n\to\infty}\varphi_n^V = \varphi$ and in this case the limit function $\varphi\in\mathscr{C}h(\boldsymbol{R}^d)$ must be infinitely divisible.

THEOREM 2.1. *If the row-wise independent system* (2.1) *satisfies* (2.4) *for some* $V\in O$, *then it converges in law to* $\mathscr{L}(\vec{a}, \mathrm{A}, \mu)$ *if and only if the following properties hold* (*for the above-mentioned* V):

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\left\{\vec{a}_{nk}^V+\mathrm{E}\left(\frac{\vec{X}_{nk}-\vec{a}_{nk}^V}{1+\|\vec{X}_{nk}-\vec{a}_{nk}^V\|^2}\right)\right\} = \vec{a}, \tag{2.5}$$

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\mathrm{E}\left(\frac{(X_{nki}-a_{nki}^V)(X_{nkj}-a_{nkj}^V)}{1+\|\vec{X}_{nk}-\vec{a}_{nk}^V\|^2}\right) = a_{ij}+\int_{\boldsymbol{R}^d}\frac{x_ix_j}{\|\vec{x}\|^2}\,d\mu, \qquad 1\leqslant i,\ j\leqslant d, \tag{2.6}$$

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\mathrm{E}\left(\frac{\|\vec{X}_{nk}-\vec{a}_{nk}^V\|^2}{1+\|\vec{X}_{nk}-\vec{a}_{nk}^V\|^2}\,I(\vec{X}_{nk}-\vec{a}_{nk}^V\in E)\right) = \mu(E), \qquad E\in\mathrm{Cont}\,\mu,\ \vec{O}\notin\bar{E}. \tag{2.7}$$

Proof. Observe that for every $\vec{t}\in\boldsymbol{R}^d$ and $n\in\boldsymbol{N}$

$$\varphi_n^V(\vec{t}) = \exp\sum_{k=1}^{k_n}\{i(\vec{t},\vec{a}_{nk}^V)+\mathrm{E}\,[e^{i(\vec{t},\vec{X}_{nk}-\vec{a}_{nk}^V)}-1]\}$$
$$= \exp\left\{i(\vec{t},\vec{a}_n^V)+\int_{\boldsymbol{R}^d}\left(e^{i(\vec{t},\vec{x})}-1-\frac{i(\vec{t},\vec{x})}{1+\|\vec{x}\|^2}\right)\frac{1+\|\vec{x}\|^2}{\|\vec{x}\|^2}\,d\mu_n(\vec{x})\right\},$$

where

$$\vec{a}_n^V := \sum_{k=1}^{k_n} \left\{ \vec{a}_{nk}^V + \mathrm{E}\left(\frac{\vec{X}_{nk} - \vec{a}_{nk}^V}{1 + \|\vec{X}_{nk} - \vec{a}_{nk}^V\|^2} \right) \right\}, \qquad n \in N,$$

$$\mu_n(E) := \sum_{k=1}^{k_n} \int_E \frac{\|\vec{x}\|_2}{1 + \|\vec{x}\|^2} \, dP_{nk}(\vec{x} + \vec{a}_{nk}^V)$$

$$= \sum_{k=1}^{k_n} \mathrm{E}\left(\frac{\|\vec{X}_{nk} - \vec{a}_{nk}^V\|^2}{1 + \|\vec{X}_{nk} - \vec{a}_{nk}\|^2} I(\vec{X}_{nk} - \vec{a}_{nk}^V \in E) \right), \qquad E \in \mathscr{B}^d, \ n \in N.$$

Hence by Theorem 1.2 the sequence φ_n^V, $n \in N$, is pointwise convergent to the characteristic function of $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$ if and only if (2.5)–(2.7) hold. By (2.4) the same is true for φ_n, $n \in N$. ∎

THEOREM 2.2. *Under the assumptions of Theorem* 2.1, *system* (2.1) *converges in law to* $l(\vec{a}, \mathbf{A}, \nu)$ *if and only if for the above-mentioned* V *the following properties hold*:

(2.5) $$\lim_{n\to\infty} \sum_{k=1}^{k_n} \left\{ \vec{a}_{nk}^V + \mathrm{E}\left(\frac{\|\vec{X}_{nk} - \vec{a}_{nk}^V\|^2}{1 + \|\vec{X}_{nk} - \vec{a}_{nk}^V\|^2} \right) \right\} = \vec{a},$$

(2.8) $$\lim_{n\to\infty} \sum_{k=1}^{k_n} E - \left(\frac{(X_{nki} - a_{nki}^V)(X_{nkj} - a_{nkj}^V)}{1 + \|\vec{X}_{nk} - \vec{a}_{nk}^V\|^2} \right) = a_{ij} + \int_{R^d} \frac{x_i x_j}{1 + \|\vec{x}\|^2} \, d\nu,$$
$$1 \leqslant i, \ j \leqslant d,$$

(2.9) $$\lim_{n\to\infty} \sum_{k=1}^{k_n} P(\vec{X}_{nk} - \vec{a}_{nk}^V \in E) = \nu(E), \qquad E \in \mathrm{Cont}\,\nu, \ \vec{O} \notin \bar{E}.$$ ∎

System (2.1) will be called *infinitesimal* if

(2.10) $$\lim_{n\to\infty} \max_{1 \leqslant k \leqslant k_n} P(\vec{X}_{nk} \notin V) = 0, \qquad V \in O,$$

or, equivalently,

(2.11) $$\lim_{n\to\infty} \max_{1 \leqslant k \leqslant k_n} \mathrm{E}\left(\frac{\|\vec{X}_{nk}\|^2}{1 + \|\vec{X}_{nk}\|^2} \right) = 0.$$

If (2.1) is infinitesimal, then for each $V \in O$ we have

(2.12) $$\lim_{n\to\infty} \max_{1 \leqslant k \leqslant k_n} \|\vec{a}_{nk}^V\| = 0$$

and

(2.13) $$\lim_{n\to\infty} \max_{1 \leqslant k \leqslant k_n} \mathrm{E}\left(\frac{\|\vec{X}_{nk} - \vec{a}_{nk}^V\|^2}{1 + \|\vec{X}_{nk} - \vec{a}_{nk}^V\|^2} \right) = 0.$$

This implies that the random vectors $\vec{X}_{nk}-\vec{a}_{nk}^{V}$, $1 \leqslant k \leqslant k_n$, $n \in \boldsymbol{N}$, form an infinitesimal system, but the converse of course is not true.

THEOREM 2.3. *Let system* (2.1) *of d-dimensional random vectors be row-wise independent and infinitesimal. Then it converges in law to* $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$ *if and only if it satisfies conditions* (2.5)–(2.7) *for each* $V \in O$. *System* (2.1) *converges in law to* $l(\vec{a}, A, \nu)$ *if and only if it fulfils the conditions* (2.5), (2.8) *and* (2.9) *for all* $V \in O$.

Proof. Observe that for each $V \in O$ and every $\vec{t} \in \boldsymbol{R}^d$ there exists a constant $\eta(V, \vec{t})$, $0 < \eta(V, \vec{t}) < +\infty$, such that

$$(2.14)\qquad |\gamma_{nk}^{V}(\vec{t})| \leqslant \eta(V, \vec{t})\,\mathrm{E}\left(\frac{\|\vec{X}_{nk}-\vec{a}_{nk}^{V}\|^2}{1+\|\vec{X}_{nk}-\vec{a}_{nk}^{V}\|^2}\right), \qquad 1 \leqslant k \leqslant k_n,\ n \in \boldsymbol{N}$$

(we shall prove this inequality subsequently in a more general case; cf. the proof of Theorem 3.5). Then if for $V \in O$ there exists a constant $0 < C_V < +\infty$ such that

$$(2.15)\qquad \sum_{k=1}^{k_n} \mathrm{E}\left(\frac{\|\vec{X}_{nk}-\vec{a}_{nk}^{V}\|^2}{1+\|\vec{X}_{nk}-\vec{a}_{nk}^{V}\|^2}\right) < C_V, \qquad n \in \boldsymbol{N},$$

then (2.4) holds. Now, if we assume (2.6) or (2.8), then we have (2.15) and next (2.4). Hence by Theorems 2.1 and 2.2 we obtain the sufficiency part of this theorem. It is fairly obvious that without any additional restrictions the infinitesimal double array (2.1) does not necessarily fulfil (2.15). Takano has proved in [7] (Lemmas 9.2 and 9.3), using an analogous result of Gnedenko in the one-dimensional case, that if we assume the convergence in law of system (2.1), then (2.15) is satisfied for every $V \in O$. This gives the necessity of conditions (2.5)–(2.9) for every $V \in O$. ∎

Takano has proved in [9] the following multidimensional central convergence criterion for infinitesimal systems:

THEOREM 2.4. *Under the assumptions of Theorem* 2.3 *system* (2.1) *converges in law to* $l(a, \mathbf{A}, \nu)$ *if and only if the following properties hold*:

$$(2.16)\qquad \lim_{n\to\infty} \mathrm{E}\big(\vec{X}_{nk} I(\vec{X}_{nk} \in V)\big) = \vec{a} + \int_{V} \frac{\vec{x}\,\|\vec{x}\|^2}{1+\|x\|^2}\,d\nu - \int_{\boldsymbol{R}^d\setminus V} \frac{\vec{x}}{1+\|\vec{x}\|^2}\,d\nu, \qquad V \in O \cap \mathrm{Cont}\,\nu,$$

$$(2.17)\qquad \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \sum_{k=1}^{k_n} \big\{\mathrm{E}\big(X_{nki} X_{nkj}\, I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big) - \mathrm{E}\big(X_{nki}\, I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big)\,\mathrm{E}\big(X_{nkj}\, I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big)\big\} = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \ldots = a_{ij}, \qquad 1 \leqslant i, j \leqslant d,$$

$$\lim_{n\to\infty}\sum_{k=1}^{k_n} P(\vec{X}_{nk}\in E) = \nu(E), \qquad E\in \mathrm{Cont}\,\nu,\ \vec{O}\notin \bar{E}. \ \blacksquare \tag{2.18}$$

(A proof of this theorem in a more general situation will be given in Chapter IV.)

Now let us consider a row-wise independent double array (2.1) of d-dimensional random vectors such that each $\vec{X}_{nk}$ has finite expectation $\vec{a}_{nk} := E\vec{X}_{nk}$. Assume that this system satisfies

$$\lim_{n\to\infty}\sum_{k=1}^{k_n} |\gamma_{nk}(\vec{t})|^2 = 0, \qquad \vec{t}\in \boldsymbol{R}^d, \tag{2.19}$$

where $\gamma_{nk}(\vec{t}) := \mathrm{E}[e^{i(\vec{t},\, X_{nk}-\vec{a}_{nk})}-1]$, $1\leqslant k\leqslant k_n$, $n\in \boldsymbol{N}$, $\vec{t}\in \boldsymbol{R}^d$. Then it converges in law to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$ if and only if conditions (2.5)–(2.7) are satisfied with $\vec{a}_{nk}^{V}$ replaced by $\vec{a}_{nk}$; it converges in law to $l(\vec{a}, \mathbf{A}, \nu)$ if and only if conditions (2.5), (2.8), (2.9) hold with $\vec{a}_{nk}^{V}$ replaced by $\vec{a}_{nk}$.

Even if the random vectors $\vec{X}_{nk}-\vec{a}_{nk}$, $1\leqslant k\leqslant k_n$, $n\in \boldsymbol{N}$, form an infinitesimal system, they do not necessarily satisfy (2.19). But in this case

$$\lim_{n\to\infty}\max_{1\leqslant k\leqslant k_n} |\gamma_{nk}(\vec{t})| = 0, \qquad \vec{t}\in \boldsymbol{R}^d, \tag{2.20}$$

and therefore for (2.19) to hold it suffices to assume that

$$\sum_{k=1}^{k_n} |\gamma_{nk}(\vec{t})| \leqslant C_0 = C_0(\vec{t}) < +\infty, \qquad n\in \boldsymbol{N},\ \vec{t}\in \boldsymbol{R}^d. \tag{2.21}$$

A slightly stronger assumption is that

$$\sum_{k=1}^{k_n} |(\vec{t}, \vec{X}_{nk}-\vec{a}_{nk})| \leqslant C_1 = C_1(\vec{t}) \leqslant +\infty, \qquad n\in \boldsymbol{N},\ \vec{t}\in \boldsymbol{R}^d,$$

or, equivalently,

$$\sum_{k=1}^{k_n} \mathrm{E}\,\|\vec{X}_{nk}-\vec{a}_{nk}\| \leqslant C, \qquad n\in \boldsymbol{N}, \tag{2.22}$$

for some $C > 0$.

Assume in addition that all random vectors $\vec{X}_{nk}$ have finite variances $v(\vec{X}_{nk}) := \mathrm{E}\,\|\vec{X}_{nk}-\vec{a}_{nk}\|^2$. In this case φ_n can be compared with the infinitely divisible characteristic function

$$\tilde{\varphi}_n(\vec{t}) := \exp\left\{i(\vec{t}, \vec{m}_n) + \int_{R^d} \left(e^{i(\vec{t},\vec{x})}-1-i(\vec{t}, \vec{x})\right)\frac{d\varkappa_n}{\|\vec{x}\|^2}\right\}, \qquad t\in \boldsymbol{R}^d,$$

where $\vec{m}_n := \sum_{k=1}^{k_n} \vec{a}_{nk}$, $\varkappa_n(E) := \sum_{k=1}^{k_n} \mathrm{E}\big(\|\vec{X}_{nk} - \vec{a}_{nk}\|^2 I(\vec{X}_{nk} - \vec{a}_{nk} \in E)\big)$, $n \in \boldsymbol{N}$, $E \in \mathscr{B}^d$. Using Theorem 1.8, we can formulate the neccessary and sufficient conditions for the convergence in law of system (2.1) to $\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa)$ or $k(\vec{m}, \mathbf{A}, \varkappa)$. Such results differ from Theorems 2.1 and 2.2 only in that $\vec{a}_{nk}^V$ are replaced by $\vec{a}_{nk}$; moreover, the measures μ, ν and the vector $\vec{a}$ are expressed in terms of $\varkappa$ and $\vec{m}$.

A more interesting theorem can be obtained by applying Theorem 1.9:

THEOREM 2.5. *Let the row-wise independent system* (2.1) *of d-dimensional random vectors with finite variances satisfy* (2.19) *and*

$$\sum_{k=1}^{k_n} \mathrm{E}\,\|\vec{X}_{nk} - \vec{a}_{nk}\|^2 \leqslant C, \qquad n \in \boldsymbol{N}, \tag{2.23}$$

for some $C > 0$. Then it converges in law to $\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa)$ if and only if the following conditions hold true:

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \vec{a}_{nk} = \vec{m}, \tag{2.24}$$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathrm{E}\big((X_{nki}-a_{nki})(X_{nkj}-a_{nkj}) I(\vec{X}_{nk}-\vec{a}_{nk} \in V)\big) = a_{ij} + \int_V \frac{x_i x_j}{\|\vec{X}\|^2}\, d\varkappa, \tag{2.25}$$
$$V \in O \cap \mathrm{Cont}\,\varkappa,\ 1 \leqslant i,\ j \leqslant d,$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathrm{E}\big(\|\vec{X}_{nk} - \vec{a}_{nk}\|^2 I(\vec{X}_{nk} - \vec{a}_{nk} \in V)\big) = \varkappa(E) \tag{2.26}$$

for every bounded $E \in \mathrm{Cont}\,\varkappa$, $\vec{O} \notin \bar{E}$, or, equivalently,

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} P(\vec{X}_{nk} - \vec{a}_{nk} \in E) = \int_E \frac{d\varkappa}{\|\vec{x}\|^2}, \qquad E \in \mathrm{Cont}\,\varkappa,\ \vec{O} \notin \bar{E}. \tag{2.27}$$

Assume in addition that

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathrm{E}\,\|\vec{X}_{nk} - \vec{a}_{nk}\|^2 = \mathrm{Tr}\,\mathbf{A} + \varkappa(\boldsymbol{R}^d) = v\big(\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa)\big). \tag{2.28}$$

Then system (2.1) *converges in law to $\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa)$ if and only if it satisfies* (2.24) *and*

(2.29)
$$\lim_{n\to\infty}\sum_{k=1}^{k_n} \mathrm{E}\big((X_{nki}-a_{nki})(X_{nkj}-a_{nkj})\big) = a_{ij}+\int_{R^d}\frac{x_i x_j}{\|\vec{x}\|^2}\,d\varkappa, \quad 1\leqslant i,\ j\leqslant d,$$
(2.30)
$$\lim_{n\to\infty}\sum_{k=1}^{k_n} \mathrm{E}\big(\|\vec{X}_{nk}-\vec{a}_{nk}\|^2 I(\vec{X}_{nk}-\vec{a}_{nk}\in E)\big) = \varkappa(E), \quad E\in \mathrm{Cont}\,\varkappa,\ \vec{O}\notin \bar{E}. \ \blacksquare$$

COROLLARY 2.1. *If the random vectors* $\vec{X}_{nk}-\vec{a}_{nk}$, $1\leqslant k\leqslant k_n$, $n\in \boldsymbol{N}$, *form an infinitesimal system and fulfil* (2.23), *then* (2.19) *is satisfied and hence the conclusion of Theorem* 2.5 *holds.* ■

Observe that it is not necessary to assume in the theorems of this chapter that system (2.1) is row-wise independent. The proofs thereof depend only on the weaker property (2.3).

III. Convergence in law to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$ for sums of dependent random vectors

The purpose of this section is to formulate some conditions which guarantee the convergence in law of system (2.1), not necessarily row-wise independent, to the arbitrarily fixed $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)\in \mathscr{P}(\boldsymbol{R}^d)$.

Let $\vec{X}\colon (\Omega, \mathscr{F}, P)\to \boldsymbol{R}^d$, $\vec{X} = (X_1, \ldots, X_d)$, be a random vector with finite expectation and let $\mathscr{F}_0$ be an arbitrary σ-subfield of $\mathscr{F}$. The *conditional expectation* of $\vec{X}$ with respect to $\mathscr{F}_0$ is defined (P-almost surely) by $\mathrm{E}(\vec{X}|\mathscr{F}_0) := \big(\mathrm{E}(X_1|\mathscr{F}_0), \ldots, \mathrm{E}(X_d|\mathscr{F}_0)\big)$. The random vector $\mathrm{E}(\vec{X}|\mathscr{F}_0)$ is $\mathscr{F}_0$-measurable and
$$\int_A \vec{X}\,dP = \int_A \mathrm{E}(\vec{X}|\mathscr{F}_0)\,dP, \quad A\in\mathscr{F}_0.$$
Moreover, $\mathrm{E}(\vec{X}|\mathscr{F}_0)$ is uniquely determined by those properties up to an event of probability zero. (In the sequel all inequalities and equalities between random vectors are considered in the sense "with probability one".)

In the subsequent part of this paper we shall very often use the following property of convergence in probability P:

LEMMA 3.1. *Let there be given a sequence* $\vec{X}_n\colon \Omega\to\boldsymbol{R}^d$, $n\in\boldsymbol{N}$. *Assume that there exists a sequence of reals* $\varepsilon_k \xrightarrow[k\to\infty]{} 0$ *such that for every* $k\in\boldsymbol{N}$ *there exists a sequence* $\vec{X}_n^k\colon \Omega\to\boldsymbol{R}$, $n\in\boldsymbol{N}$, *so that* $\vec{X}_n^k \xrightarrow{P} \vec{\delta}_k\in\boldsymbol{R}^d$, $\|\vec{\delta}_k\|\leqslant\varepsilon_k$ *and* $\|\vec{X}_n\| \leqslant \varepsilon_k+\|\vec{X}_n^k\|$ *for every* $n\in\boldsymbol{N}$. *Then* $\vec{X}_n\xrightarrow{P}\vec{O}$. ■

("$\xrightarrow{P}$" denotes convergence in probability P.)

We shall say that the random vectors (2.1) form a $\mathscr{D}(\vec{a}, \mathbf{A}, \mu)$-*system*, or shortly a $\mathscr{D}$-*system*, if there exists a double array of σ-subfields of $\mathscr{F}$

$$\begin{array}{c} \mathscr{F}_{1,0} \subset \mathscr{F}_{1,1} \subset \ldots \subset \mathscr{F}_{1,k_1}, \\ \mathscr{F}_{2,0} \subset \mathscr{F}_{2,1} \subset \ldots \subset \mathscr{F}_{2,k_2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \mathscr{F}_{n,0} \subset \mathscr{F}_{n,1} \subset \ldots \subset \mathscr{F}_{n,k_n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \tag{3.1}$$

such that every $\vec{X}_{nk}$ is $\mathscr{F}_{n,k}$-measurable and if there is a $V \in O$ such that for

$$\vec{A}_{nk} := E_{n,k-1}\big(\vec{X}_{nk} I(\vec{X}_{nk} \in V)\big), \tag{3.2}$$

$$\vec{Y}_{nk} := \vec{X}_{nk} - \vec{A}_{nk}, \qquad 1 \leqslant k \leqslant k_n,\ n \in N, \tag{3.3}$$

one has

$$\sum_{k=1}^{k_n} \left\{\vec{A}_{nk} + E_{n,k-1}\left(\frac{\vec{Y}_{nk}}{1+\|\vec{Y}_{nk}\|^2}\right)\right\} \xrightarrow{P} \vec{a}, \tag{3.4}$$

$$\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki} Y_{nkj}}{1+\|\vec{Y}_{nk}\|^2}\right) \xrightarrow{P} a_{ij} + \int_{R^d} \frac{x_i x_j}{\|\vec{x}\|^2}\, d\mu, \qquad 1 \leqslant i,\ j \leqslant d, \tag{3.5}$$

$$\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\, I(\vec{Y}_{nk} \in E)\right) \xrightarrow{P} \mu(E), \qquad E \in \operatorname{Cont}\mu,\ \vec{O} \notin E. \tag{3.6}$$

(Here and in the sequel we use the notation

$$E_{n,k-1}\vec{X} := E(\vec{X} \mid \mathscr{F}_{n,k-1}), \qquad P_{n,k-1}(A) := P(A \mid \mathscr{F}_{n,k-1}).)$$

THEOREM 3.1. *Let the $\mathscr{D}$-system* (2.1) *fulfil the condition*

$$\sum_{k=1}^{k_n} E_{n,k-1}\left\{\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right\} \leqslant C, \qquad n \in N, \tag{3.7}$$

for some $C > 0$. Then it converges in law to $\mathscr{L}(a, \mathbf{A}, \mu)$ if and only if it satisfies

$$\lim_{n\to\infty} E\left[\exp\left\{i \sum_{k=1}^{k_n} (\vec{t}, \vec{Y}_{nk}) - \sum_{k=1}^{k_n} E_{n,k-1}\left(e^{i(\vec{t}, \vec{Y}_{nk})} - 1\right)\right\}\right] = 1, \qquad \vec{t} \in \boldsymbol{R}^d. \tag{3.8}$$

Proof. Observe that $\varphi = \varphi_{\mathscr{L}(\vec{a}, \mathbf{A}, \mu)}$ can be written in the form

$$\varphi(\vec{t}) = \exp\left\{i(\vec{t}, \vec{a}) - \tfrac{1}{2}\big(\vec{t}, (A+T)\vec{t}\big) + \int_{R^d} h(\vec{x}, \vec{t})\, d\mu(\vec{x})\right\}, \qquad \vec{t} \in \boldsymbol{R}^d,$$

where

$$h(\vec{x}, \vec{t}) = \begin{cases} \left(e^{i(\vec{t}, \vec{x})} - 1 - \dfrac{i(\vec{t}, \vec{x})^2}{1+\|\vec{x}\|^2} + \dfrac{(\vec{t}, \vec{x})}{2(1+\|\vec{x}\|^2)}\right) \dfrac{1+\|\vec{x}\|^2}{\|\vec{x}\|^2}, & \vec{x} \neq \vec{O},\ \vec{t} \in \boldsymbol{R}^d, \\ 0, & \vec{x} = \vec{O},\ \vec{t} \in \boldsymbol{R}^d, \end{cases}$$

and $\mathbf{T}$ is defined by (1.6).

It is easily seen that h is continuous and bounded on each set of the form $\boldsymbol{R}^d \times \{\|\vec{t}\| \leqslant \tau\}$, $\tau > 0$; moreover, $\lim\limits_{\substack{\vec{x}\to\vec{O} \\ \vec{t}\to\vec{t}_0}} h(\vec{x}, \vec{t}) = 0$ for each $\vec{t}_0 \in \boldsymbol{R}^d$.

Let $\vec{t} \in \boldsymbol{R}^d$ be arbitrarily fixed and let $M > 0$ be such that $|h(\vec{x}, \vec{t})| \leqslant M$, $\vec{x} \in \boldsymbol{R}^d$. First we prove that

$$W_n(\vec{t}) \xrightarrow{P} \frac{1}{\varphi(\vec{t})}, \tag{3.9}$$

where

$$W_n(\vec{t}) := \exp\Bigg\{-i \sum_{k=1}^{k_n} \left(\vec{t}, \vec{A}_{nk} + E_{n,k-1}\left(\frac{\vec{Y}_{nk}}{1+\|\vec{Y}_{nk}\|^2}\right)\right) +$$
$$+ \frac{1}{2} \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{(\vec{t}, \vec{Y}_{nk})^2}{1+\|\vec{Y}_{nk}\|^2}\right) - \sum_{k=1}^{k_n} E_{n,k-1}\left(h(\vec{Y}_{nk}, \vec{t})\, \frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right)\Bigg\}, \quad n \in \boldsymbol{N}.$$

By (3.4), (3.5) and the continuity of the function exp it suffices to prove

$$\sum_{k=1}^{k_n} E_{n,k-1}\left(h(\vec{Y}_{nk}, \vec{t})\, \frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right) \xrightarrow{P} \int_{\boldsymbol{R}^d} h(\vec{x}, \vec{t})\, d\mu(\vec{x}). \tag{3.10}$$

Let us fix an arbitrary $\varepsilon > 0$. Given $0 < \delta < 1$, let $S_\delta := K(\vec{O}, 1/\delta) \setminus K(\vec{O}, \delta)$, where $K(\vec{O}, r) := \{\vec{x} \in \boldsymbol{R}^d;\ \|\vec{x}\| < r\}$. Since the measure μ is finite, there are arbitrarily small $\delta > 0$ so that $S_\delta \in \operatorname{Cont}\mu$ and

$$\mu(S_\delta) > \mu(\boldsymbol{R}^d) - \frac{\varepsilon}{4M}. \tag{3.11}$$

Observe that for such δ

$$\Bigg|\sum_{k=1}^{k_n} E_{n,k-1}\left(h(\vec{Y}_{nk}, \vec{t})\, \frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\| > \delta)\right) - \tag{3.12}$$
$$- \sum_{k=1}^{k_n} E_{n,k-1}\left(h(\vec{Y}_{nk}, \vec{t})\, \frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in S_\delta)\right)\Bigg| \leqslant L_n^\varepsilon,$$

where

$$L_n^\varepsilon := M \cdot \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I\left(\|\vec{Y}_{nk}\| > \frac{1}{\delta}\right)\right), \qquad n \in \boldsymbol{N},$$

and by (3.6) and (3.11)

$$L_n^\varepsilon \xrightarrow{P} M \cdot \mu(\|\vec{x}\| > 1/\delta) < \varepsilon/4. \tag{3.13}$$

Now fix δ, $0 < \delta < 1$, so that, in addition to (3.11),

$$\left|\int_{\boldsymbol{R}^d} h(\vec{x}, \vec{t})\, d\mu(\vec{x}) - \int_{S_\delta} h(\vec{x}, \vec{t})\, d\mu(\vec{x})\right| < \varepsilon/4 \tag{3.14}$$

and

$$|h(\vec{x}, \vec{t})| \leqslant \varepsilon/4C, \qquad \|\vec{x}\| \leqslant \delta. \tag{3.15}$$

By (3.15) and (3.7) we obtain

$$\left|\sum_{k=1}^{k_n} E_{n,k-1}\left(h(\vec{Y}_{nk}, \vec{t}) \frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\| \leqslant \delta)\right)\right| \leqslant \frac{\varepsilon}{4C} \cdot C = \frac{\varepsilon}{4}. \tag{3.16}$$

Since the function $h(\cdot, \vec{t})$ is uniformly continuous on the set S_δ, there exists an $\eta_\varepsilon > 0$ such that $|h(\vec{x}, \vec{t}) - h(\vec{y}, \vec{t})| < \varepsilon/4C$ if $\|\vec{x} - \vec{y}\| < \eta_\varepsilon$; $\vec{x}, \vec{y} \in S_\delta$.

Split the set S_δ into disjoint non-empty subsets $I_m \in \operatorname{Cont}\mu$, $m = 1, 2, \ldots, s$, with diameters less than η_ε and choose points $\vec{x}_{(m)} \in I_m$, $m = 1, 2, \ldots, s$, such that

$$\left|\int_{S_\delta} h(\vec{x}, \vec{t})\, d\mu(\vec{x}) - \sum_{m=1}^{s} h(\vec{x}_{(m)}, \vec{t})\, \mu(I_m)\right| < \varepsilon/4. \tag{3.17}$$

Using (3.7), we get

$$\begin{aligned} &\left|\sum_{k=1}^{k_n} \sum_{m=1}^{s} E_{n,k-1}\left(h(\vec{x}_{(m)}, \vec{t}) \frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in I_m)\right) - \right. \\ &\left. - \sum_{k=1}^{k_n} \sum_{m=1}^{s} E_{n,k-1}\left(h(\vec{Y}_{nk}, \vec{t}) \frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in I_m)\right)\right| < \frac{\varepsilon}{4C} \cdot C = \frac{\varepsilon}{4}. \end{aligned} \tag{3.18}$$

Moreover, (3.6) implies

$$\begin{aligned} M_n^\varepsilon := \sum_{k=1}^{k_n} \sum_{m=1}^{s} h(\vec{x}_{(m)}, \vec{t})\, E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in I_m)\right) - \\ - \sum_{m=1}^{s} h(\vec{x}_{(m)}, \vec{t})\, \mu(I_m) \xrightarrow{P} 0. \end{aligned} \tag{3.19}$$

Therefore, by (3.12)-(3.14), (3.16)-(3.19) we have obtained

$$\left|\sum_{k=1}^{k_n} E_{n,k-1}\left(h(\vec{Y}_{nk}, \vec{t})\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right) - \int_{R^d} h(\vec{x}, \vec{t})\,d\mu(\vec{x})\right| < \varepsilon + L_n^\varepsilon + |M_n^\varepsilon|, \qquad n \in N,$$

where $L_n^\varepsilon + |M_n^\varepsilon| \xrightarrow{P} \varepsilon_0 < \varepsilon$. Hence, $\varepsilon > 0$ being arbitrary, (3.10) follows from Lemma 3.1.

We have the inequalities

$$(3.20)\qquad \left|\sum_{k=1}^{k_n} E_{n,k-1}\left(h(\vec{Y}_{nk}, \vec{t})\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right)\right| \leqslant M\cdot \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right),$$

$$(3.21)\qquad \left|\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{(\vec{t}, \vec{Y}_{nk})^2}{1+\|\vec{Y}_{nk}\|^2}\right)\right| \leqslant \|\vec{t}\|^2 \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right),$$

which by (3.7) yield the boundedness of the sequence $W_n(\vec{t})$, $n \in N$ ($\vec{t}$ is fixed!). Then by (3.10)

$$(3.22)\qquad \lim_{n\to\infty} \mathrm{E}\left|W_n(\vec{t}) - \frac{1}{\varphi(\vec{t})|}\right| = 0, \qquad \vec{t} \in R^d.$$

Applying Lemma 3.2 below, we conclude that the sequence φ_{σ_n}, $n \in N$, is pointwise convergent to φ if and only if

$$\lim_{n\to\infty} \mathrm{E}\{W_n(\vec{t})\exp[i(\vec{t}, \vec{S}_n)]\} = 1, \qquad \vec{t} \in R^d,$$

which is equivalent to (3.8). ∎

LEMMA 3.2. *Let $\vec{X}_n$: $\Omega \to R^d$, $n \in N$, be a sequence of random vectors. Assume that for each $\vec{t} \in R^d$ we have a sequence of random variables $Y_n(\vec{t})$: $\Omega \to C$, $n \in N$, such that*

$$\lim_{n\to\infty} \mathrm{E}\left|Y_n(\vec{t}) - \frac{1}{\varphi(\vec{t})}\right| = 0, \qquad \vec{t} \in R^d,$$

for some nonvanishing complex function φ.

Then the characteristic functions $\varphi_n(\vec{t}) := Ee^{i(\vec{t}, \vec{X}_n)}$, $n \in N$, are pointwise convergent to φ if and only if

$$\lim_{n\to\infty} \mathrm{E}\{Y_n(\vec{t})\exp i(\vec{t}, \vec{X}_n)\} = 1, \qquad \vec{t} \in R^d.$$

Proof. The above follows at once from the inequalities

$$|\varphi_n(\vec{t})-\varphi(\vec{t})| \leqslant |\mathrm{E}\exp i(\vec{t},\vec{X}_n)-\varphi(\vec{t})\mathrm{E}\{Y_n(\vec{t})\exp i(\vec{t},\vec{X}_n)\}|+ \\ +|\varphi(\vec{t})\mathrm{E}\{Y_n(\vec{t})\exp i(\vec{t},\vec{X}_n)\}-\varphi(\vec{t})| \\ \leqslant |\varphi(\vec{t})|\cdot \mathrm{E}\left|\frac{1}{\varphi(\vec{t})}-Y_n(\vec{t})\right|+|\varphi(\vec{t})|\cdot|\mathrm{E}\{Y_n(\vec{t})\exp i(\vec{t}),\vec{X}_n)\}-1|,$$

$$|\mathrm{E}\{Y_n(\vec{t})\exp i(\vec{t},\vec{X}_n)\}-1| \\ \leqslant \left|\mathrm{E}\{Y_n(\vec{t})\exp i(\vec{t},\vec{X}_n)\}-\mathrm{E}\left\{\frac{1}{\varphi(\vec{t})}\exp i(\vec{t},\vec{X}_n)\right\}\right|+ \\ +\left|\frac{1}{\varphi(\vec{t})}\mathrm{E}\exp i(\vec{t},\vec{X}_n)-1\right| \\ \leqslant \mathrm{E}\left|Y_n(\vec{t})-\frac{1}{\varphi(\vec{t})}\right|+\frac{1}{|\varphi(\vec{t})|}\cdot|\varphi_n(\vec{t})-\varphi(\vec{t})|. \blacksquare$$

Now we shall find conditions equivalent to convergence in law to $\mathscr{L}(\vec{a},A,\mu)$ for $\mathscr{D}$-systems not necessarily satisfying (3.7).

If $V_{nk}:=\sum_{j=1}^{k}E_{n,j-1}\left(\frac{\|\vec{Y}_{nj}\|^2}{1+\|\vec{Y}_{nj}\|^2}\right)$, $1\leqslant k\leqslant k_n$, $n\in\boldsymbol{N}$, then, by (3.5),

$$\lim_{n\to\infty}P(V_{n,k_n}>C)=0 \tag{3.23}$$

for every $C>\mathrm{Tr}\,\mathbf{A}+\mu(\boldsymbol{R}^d)$. For arbitrary $C>\mathrm{Tr}\,\mathbf{A}+\mu(\boldsymbol{R}^d)$ we define

$$\vec{X}^c_{nk}:=\vec{X}_{nk}I(V_{nk}\leqslant C),\qquad 1\leqslant k\leqslant k_n,\ n\in\boldsymbol{N}, \tag{3.24}$$

and, as before,

$$\vec{A}^c_{nk}:=E_{n,k-1}\big(\vec{X}^c_{nk}I(\vec{X}^c_{nk}\in V)\big)=\vec{A}_{nk}I(V_{nk}\leqslant C), \tag{3.25}$$

$$\vec{Y}^c_{nk}:=\vec{X}^c_{nk}-\vec{A}^c_{nk}=\vec{Y}_{nk}I(V_{nk}\leqslant C), \tag{3.26}$$

$$V^c_{nk}=\sum_{j=1}^{k}E_{n,j-1}\left(\frac{\|\vec{Y}^c_{nj}\|^2}{1+\|\vec{Y}^c_{nj}\|^2}\right)=\sum_{j=1}^{k}E_{n,j-1}\left(\frac{\|\vec{Y}_{nj}\|^2}{1+\|\vec{Y}_{nj}\|^2}\right)I(V_{nj}\leqslant C), \tag{3.27}$$

$$\vec{S}^c_n:=\sum_{k=1}^{k_n}\vec{X}^c_{nk}, \tag{3.28}$$

for $1\leqslant k\leqslant k_n$, $n\in\boldsymbol{N}$. System (3.24) satisfies (3.4)–(3.6) with respect to the σ-fields (3.1) and $V\in O$, i.e., it is a $\mathscr{D}$-system. Moreover, one has $\lim_{n\to\infty}P[\vec{S}_n\neq\vec{S}^c_n]=0$, i.e., the limit laws of the $\mathscr{D}$-systems (2.1) and (3.24) are equal provided that either of them exists. By (3.27) we get $V^c_{n,k_n} \leqslant C$, $n \in \boldsymbol{N}$; hence the $\mathscr{D}$-system (3.24) satisfies (3.7). Thus we have proved

THEOREM 3.2. *For every $\mathscr{D}$-system (2.1) the following conditions are equivalent:*

(1) *the $\mathscr{D}$-system (2.1) converges in law to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$,*

(2) *for every $C > \operatorname{Tr}\mathbf{A} + \mu(\boldsymbol{R}^d)$ the $\mathscr{D}$-system (3.24) satisfies (3.8),*

(3) *there exists a $C > \operatorname{Tr}\mathbf{A} + \mu(\boldsymbol{R}^d)$ such that the $\mathscr{D}$-system (3.24) fulfils* (3.8). ■

The remaining part of this chapter is devoted to giving some sufficient conditions for the convergence in law of the $\mathscr{D}$-system to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$.

THEOREM 3.3. *If the $\mathscr{D}$-system (2.1) satisfies*

$$\exp\Big\{i \sum_{k=1}^{k_n} (\vec{t}, \vec{Y}_{nk}) - \sum_{k=1}^{k_n} E_{n,k-1}(e^{i(\vec{t}, \vec{Y}_{nk})} - 1)\Big\} \xrightarrow{P} 1, \quad \vec{t} \in \boldsymbol{R}^d, \tag{3.29}$$

then it converges in law to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$.

Proof. For every $C > \operatorname{Tr}\mathbf{A} + \mu(\boldsymbol{R}^d)$, system (3.24) satisfies (3.29) and (3.7). By (3.7) the sequence

$$B_n := \exp\Big\{i \sum_{k=1}^{k_n} (\vec{t}, \vec{Y}^c_{nk}) - \sum_{k=1}^{k_n} E_{n,k-1}(e^{i(\vec{t}, \vec{Y}^c_{nk})} - 1)\Big\}, \quad n \in \boldsymbol{N},$$

is bounded. Then (3.29) implies $\lim_{n\to\infty} \mathrm{E} B_n = 1$, i.e. the $\mathscr{D}$-system (3.24) fulfils (3.8) and the conclusion follows from Theorem 3.2. ■

THEOREM 3.4. *If the $\mathscr{D}$-system (2.1) satisfies*

$$\sum_{k=1}^{k_n} |E_{n,k-1}(e^{i(\vec{t}, \vec{Y}_{nk})} - 1)|^2 \xrightarrow{P} 0, \quad t \in \boldsymbol{R}^d, \tag{3.30}$$

then it converges in law to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$.

Proof. By the inequality

$$\begin{aligned} |E_{n,k-1}(e^{i(\vec{t}, \vec{Y}^c_{nk})} - 1)|^2 &= I(V_{nk} \leqslant C) |E_{n,k-1}(e^{i(t, \vec{Y}_{nk})} - 1)|^2 \\ &\leqslant |E_{n,k-1}(e^{i(\vec{t}, \vec{Y}_{nk})} - 1)|^2, \quad \vec{t} \in \boldsymbol{R}^d,\ 1 \leqslant k \leqslant k_n,\ n \in \boldsymbol{N}, \end{aligned}$$

we can assume that $\mathscr{D}$-system (2.1) satisfies (3.7). But in this case the sequence $\sum_{k=1}^{k_n} |E_{n,k-1}(e^{i(\vec{t}, \vec{Y}_{nk})} - 1)|^2$, $n \in \boldsymbol{N}$, is bounded and hence (3.30) is equivalent to

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathrm{E} |E_{n,k-1}(e^{i(\vec{t}, \vec{Y}_{nk})} - 1)|^2 = 0, \quad \vec{t} \in \boldsymbol{R}^d. \tag{3.31}$$

(In the proof of the boundedness of the above sequence we use the expansion

$$E_{n,k-1}(e^{i(\vec{t},\vec{Y}_{nk})}-1) = E_{n,k-1}\left(\frac{i(\vec{t},\vec{Y}_{nk})}{1+\|\vec{Y}_{nk}\|^2}\right) - E_{n,k-1}\left(\frac{(\vec{t},\vec{Y}_{nk})^2}{2(1+\|\vec{Y}_{nk}\|^2)}\right) + $$

$$+E_{n,k-1}\left(h(\vec{Y}_{nk},\vec{t})\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right), \quad 1\leqslant k\leqslant k_n,\ n\in \boldsymbol{N},\ \vec{t}\in \boldsymbol{R}^d,$$

and moreover, inequalities (3.20), (3.21) and

$$\sum_{k=1}^{k_n}\left|E_{n,k-1}\left(\frac{i(\vec{t},\vec{Y}_{nk})}{1+\|\vec{Y}_{nk}\|^2}\right)\right|^2 \leqslant \|\vec{t}\|^2\sum_{k=1}^{k_n}E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right), \quad n\in\boldsymbol{N},\ \vec{t}\in\boldsymbol{R}^d.)$$

For an arbitrary fixed $\vec{t}\in\boldsymbol{R}^d$ we define

$$W_{nk} := \exp\left\{i\left(\vec{t},\sum_{j=1}^{k}\vec{Y}_{nj}\right)-\sum_{j=1}^{k}E_{n,j-1}(e^{i(\vec{t},\vec{Y}_{nj})}-1)\right\} -$$

$$-\exp\left\{i\left(\vec{t},\sum_{j=1}^{k-1}\vec{Y}_{nj}\right)-\sum_{j=1}^{k-1}E_{n,j-1}(e^{i(\vec{t},\vec{Y}_{nj})}-1)\right\}$$

$$= [\exp i(\vec{t},\vec{Y}_{nk}) - \exp E_{n,k-1}(e^{i(\vec{t},\vec{Y}_{nk})}-1)]\times$$

$$\times\exp\left\{i\left(\vec{t},\sum_{j=1}^{k-1}\vec{Y}_{nj}\right)-\sum_{j=1}^{k}E_{n,j-1}(e^{i(\vec{t},\vec{Y}_{nj})}-1)\right\},\ 1\leqslant k\leqslant k_n,\ n\in\boldsymbol{N}.$$

Thus $B_n-1=\sum_{k=1}^{k_n}W_{nk}$, $n\in\boldsymbol{N}$, and $\mathbf{E}B_n-1=\sum_{k=1}^{k_n}\mathbf{E}E_{n,k-1}W_{nk}$, $n\in\boldsymbol{N}$. Next,

$$E_{n,k-1}W_{nk} = \left[E_{n,k-1}\left(\exp i(\vec{t},\vec{Y}_{nk})\right)-\exp E_{n,k-1}(e^{i(\vec{t},\vec{Y}_{nj})}-1)\right]\times$$

$$\times\exp\left\{i\left(\vec{t},\sum_{j=1}^{k-1}\vec{Y}_{nj}\right)-\sum_{j=1}^{k}E_{n,j-1}(e^{i(\vec{t},\vec{Y}_{nj})}-1)\right\}$$

$$= Q\left(E_{n,k-1}(e^{i(\vec{t},\vec{Y}_{nk})}-1)\right)\exp\left\{i\left(\vec{t},\sum_{j=1}^{k-1}\vec{Y}_{nj}\right)-\sum_{j=1}^{k}E_{n,j-1}(e^{i(\vec{t},\vec{Y}_{nj})}-1)\right\},$$

where $Q(z):=1+z-\exp z$, $z\in\boldsymbol{C}$.

By the inequality $|Q(z)|\leqslant\frac{1}{2}|z|^2\exp|z|$, $z\in\boldsymbol{C}$, we get

$$\left|Q\left(E_{n,k-1}(e^{i(\vec{t},\vec{Y}_{nk})}-1)\right)\right|\leqslant 5|E_{n,k-1}(e^{i(\vec{t},\vec{Y}_{nk})}-1)|^2, \quad 1\leqslant k\leqslant k_n,\ n\in\boldsymbol{N}.$$

Using (3.20), (3.21), we can prove that for $1\leqslant k\leqslant k_n$, $n\in\boldsymbol{N}$:

$$\left|\exp\left\{i\left(\vec{t},\sum_{j=1}^{k-1}\vec{Y}_{nj}\right)-\sum_{j=1}^{k}E_{n,j-1}(e^{i(\vec{t},\vec{Y}_{nj})}-1)\right|\leqslant C_0=C_0(\vec{t})< +\infty.$$

Therefore, we have obtained for $n \in N$

$$|\mathrm{E}B_n - 1| \leqslant \sum_{k=1}^{k_n} \mathrm{E}|E_{n,k-1}W_{nk}| \leqslant 5C_0(\vec{t}) \cdot \sum_{k=1}^{k_n} \mathrm{E}|E_{n,k-1}(e^{i(\vec{t}, \vec{Y}_{nk})} - 1)|^2.$$

By (3.31) Theorem 3.1 yields the required result. ■

System (2.1) is said to be *conditionally infinitesimal* (*with respect to the σ-fields* (3.1)) if and only if

$$\max_{1 \leqslant k \leqslant k_n} P_{n,k-1}(\|\vec{X}_{nk}\| > \varepsilon) \xrightarrow{P} 0, \quad \varepsilon > 0. \tag{3.32}$$

By the inequalities

$$\frac{\varepsilon^2}{1+\varepsilon^2} P_{n,k-1}(\|\vec{X}_{nk}\| > \varepsilon) \leqslant E_{n,k-1}\left(\frac{\|\vec{X}_{nk}\|^2}{1+\|\vec{X}_{nk}\|^2} I(\|\vec{X}_{nk}\| > \varepsilon)\right)$$

$$\leqslant E_{n,k-1}\left(\frac{\|\vec{X}_{nk}\|^2}{1+\|\vec{X}_{nk}\|^2}\right) \leqslant \frac{\varepsilon^2}{1+\varepsilon^2} + P_{n,k-1}(\|\vec{X}_{nk}\| > \varepsilon), \quad \varepsilon > 0,$$

property (3.32) is equivalent to

$$\max_{1 \leqslant k \leqslant k_n} E_{n,k-1}\left(\frac{\|\vec{X}_{nk}\|^2}{1+\|\vec{X}_{nk}\|^2}\right) \xrightarrow{P} 0. \tag{3.33}$$

If (2.1) is conditionally infinitesimal, then by the inequality

$$0 \leqslant \max_{1 \leqslant k \leqslant k_n} P(\|\vec{X}_{nk}\| > \varepsilon) = \max_{1 \leqslant k \leqslant k_n} \mathrm{E}P_{n,k-1}(\|\vec{X}_{nk}\| > \varepsilon)$$

$$\leqslant \mathrm{E} \max_{1 \leqslant k \leqslant k_n} P_{n,k-1}(\|\vec{X}_{nk}\| > \varepsilon), \quad \varepsilon > 0,$$

it is also infinitesimal, but the converse implication is not necessarily true. If (2.1) is infinitesimal and the system of σ-fields (3.1) is such that every $\vec{X}_{nk}$ is independent of $\mathscr{F}_{n,k-1}$, then (2.1) is conditionally infinitesimal with respect to the σ-fields under consideration.

LEMMA 3.3. *Let system* (2.1) *be conditionally infinitesimal. Then*

$$\max_{1 \leqslant k \leqslant k_n} \|\vec{A}_{nk}\| \xrightarrow{P} 0, \tag{3.34}$$

$$\max_{1 \leqslant k \leqslant k_n} P_{n,k-1}(\|\vec{Y}_{nk}\| > \varepsilon) \xrightarrow{P} 0, \quad \varepsilon > 0, \tag{3.35}$$

where $\vec{A}_{nk}$ *and* $\vec{Y}_{nk}$ *are defined by* (3.2) *and* (3.3), *respectively.*

Proof. If $V \subset K(\vec{O}, r)$, then for every $\varepsilon > 0$

$$\|\vec{A}_{nk}\| \leqslant \varepsilon \cdot \sqrt{d} + r\sqrt{d} P_{n,k-1}(\|\vec{X}_{nk}\| > \varepsilon)$$

and by Lemma 3.1 we get (3.34).

Next, $\{\|\vec{Y}_{nk}\| > \varepsilon\} \subset \{\|\vec{X}_{nk}\| > \varepsilon/2\} \cup \{\|\vec{A}_{nk}\| > \varepsilon/2\}$ for every $\varepsilon > 0$ and then the by $\mathscr{F}_{n,k-1}$-measurability of $\|\vec{A}_{nk}\|$

$$0 \leqslant \max_{1 \leqslant k \leqslant k_n} P_{n,k-1}(\|\vec{Y}_{nk}\| > \varepsilon)$$
$$\leqslant \max_{1 \leqslant k \leqslant k_n} P_{n,k-1}(\|\vec{X}_{nk}\| > \varepsilon/2) + \max_{1 \leqslant k \leqslant k_n} I(\|\vec{A}_{nk}\| > \varepsilon/2), \quad n \in N.$$

But for $\omega \in \Omega$

$$[\max_{1 \leqslant k \leqslant k_n} I(\|\vec{A}_{nk}\| > \varepsilon/2)](\omega) > 0$$

if and only if

$$[\max_{1 \leqslant k \leqslant k_n} \|\vec{A}_{nk}\|](\omega) > \varepsilon/2,$$

which, by (3.34), implies

$$\max_{1 \leqslant k \leqslant k_n} I(\|\vec{A}_{nk}\| > \varepsilon/2) \xrightarrow{P} 0. \ \blacksquare$$

THEOREM 3.5. *Every conditionally infinitesimal $\mathscr{D}$-system* (2.1) *converges in law to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$.*

Proof. Let us write $r(V) := \sup\{r > 0;\ K(\vec{O}, r) \subset V\}$. It follows from (3.34) that

$$\lim_{n \to \infty} P\big(\max_{1 \leqslant k \leqslant k_n} \|\vec{A}_{nk}\| > r(V)/2\big) = 0. \tag{3.36}$$

We define a new system

$$\tilde{X}_{nk} := \vec{X}_{nk} I\big(\|\vec{A}_{nk}\| \leqslant r(V)/2\big), \quad 1 \leqslant k \leqslant k_n,\ n \in N. \tag{3.37}$$

As for system (3.24), we can prove that (3.37) is a conditionally infinitesimal $\mathscr{D}$-system. By (3.36) this system is convergent in law if and only if so is (2.1). Therefore we can assume that our $\mathscr{D}$-system satisfies additionally

$$\max_{1 \leqslant k \leqslant k_n} \|\vec{A}_{nk}\| \leqslant r(V)/2, \quad n \in N. \tag{3.38}$$

Condition (3.35) is equivalent to

$$\max_{1 \leqslant k \leqslant k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1 + \|\vec{Y}_{nk}\|^2}\right) \xrightarrow{P} 0, \tag{3.39}$$

and thus, by (3.5), it suffices to show that for every $\vec{t} \in \boldsymbol{R}^d$ there exists a constant $\eta(\vec{t}, V) > 0$, independent of k and n, such that

$$(3.40) \qquad |E_{n,k-1}(e^{i(\vec{t}, \vec{Y}_{nk})} - 1)| \leqslant \eta(\vec{t}, V) E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right)$$

for every $1 \leqslant k \leqslant k_n$ and $n \in \boldsymbol{N}$.

If $\vec{X}_{nk} \notin V$, then $\|\vec{X}_{nk}\| > r(V)$ and next $\|\vec{Y}_{nk}\| > r(V)/2$. Hence

$$(3.41) \qquad |E_{n,k-1}\big((e^{i(\vec{t}, \vec{Y}_{nk})} - 1) I(\vec{X}_{nk} \notin V)\big)| \leqslant 2P_{n,k-1}(\vec{X}_{nk} \notin V)$$
$$\leqslant s(V) E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{X}_{nk} \in V)\right) \leqslant s(V) E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right),$$

where $S(V)$ is a constant depending only on $r(V)$. Next

$$(3.42) \qquad \big|E_{n,k-1}\big((e^{i(\vec{t}, \vec{Y}_{nk})} - 1) I(\vec{X}_{nk} \in V)\big)\big|$$
$$\leqslant \big|E_{n,k-1}\big(i(\vec{t}, \vec{Y}_{nk}) I(\vec{X}_{nk} \in V)\big)\big| + \tfrac{1}{2} E_{n,k-1}\big((\vec{t}, \vec{Y}_{nk})^2 I(\vec{X}_{nk} \in V)\big)$$
$$\leqslant |(\vec{t}, \vec{A}_{nk})| P_{n,k-1}(\vec{X}_{nk} \notin V) + \tfrac{1}{2}\|\vec{t}\|^2 E_{n,k-1}\big(\|\vec{Y}_{nk}\|^2 I(\vec{X}_{nk} \in V)\big)$$
$$\leqslant \tfrac{1}{2}\|\vec{t}\| d(V) P_{n,k-1}(\vec{X}_{nk} \notin V) + \tfrac{1}{2}\|\vec{t}\|^2 \big[1 + \big(d(V) + r(V)/2\big)^2\big] \times$$
$$\times E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{X}_{nk} \in V)\right) \leqslant s(V, \vec{t}) E_{n,k-1}\left(\frac{\|Y_{nk}\|^2}{1+\|Y_{nk}\|^2}\right),$$

where $s(V, \vec{t})$ depends on $\|\vec{t}\|$, $r(V)$ and the diameter $d(V)$ of the set V. By (3.41) and (3.42) we get (3.40). ■

Observe that under assumption (3.5) condition (3.35) is equivalent to

$$(3.43) \qquad \sum_{k=1}^{k_n} \left[E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right)\right]^2 \xrightarrow{P} 0.$$

Dvoretzky has stated in [2] that (in the one-dimensional case) (3.43) is sufficient for the $\mathscr{D}$-system (2.1) to converge in law to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$. We cannot prove this result with no additional assumptions. For example, an immediate consequence of the proof of Theorem 3.5 is

COROLLARY 3.1. *If the $\mathscr{D}$-system* (2.1) *fulfils* (3.35) *and there exists a constant a, $0 < a < r(V)$, such that*

$$(3.44) \qquad \lim_{n\to\infty} P[\max_{1\leqslant k\leqslant k_n} \|\vec{A}_{nk}\| > a] = 0,$$

then it converges in law to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$. ■

THEOREM 3.6. *If the $\mathscr{D}$-system* (2.1) *fulfils* (3.35) *and*

$$\sum_{k=1}^{k_n}\left[E_{n,k-1}\left(\frac{(\vec{t},\vec{Y}_{nk})}{1+\|\vec{Y}_{nk}\|^2}\right)\right]^2\xrightarrow{P}0,\quad \vec{t}\in\vec{R}^d,\tag{3.45}$$

or, slightly stronger,

$$\sum_{k=1}^{k_n}\left[E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|}{1+\|\vec{Y}_{nk}\|^2}\right)\right]^2\xrightarrow{P}0,\tag{3.46}$$

then it converges in law to $\mathscr{L}(\vec{a},\mathbf{A},\mu)$.

Proof. One has, for an arbitrary $\varepsilon>0$,

$$E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|}{1+\|\vec{Y}_{nk}\|^2}\right)\leqslant\frac{\varepsilon}{1+\varepsilon^2}+P_{n,k-1}(\|\vec{Y}_{nk}\|>\varepsilon),$$

$$\begin{aligned}|E_{n,k-1}(e^{i(\vec{t},\vec{Y}_{nk})}-1)|&\leqslant E_{n,k-1}\big(|(\vec{t},\vec{Y}_{nk})|I(\|\vec{Y}_{nk}\|\leqslant\varepsilon)\big)+2P_{n,k-1}(\|\vec{Y}_{nk}\|>\varepsilon)\\&\leqslant\varepsilon\|\vec{t}\|+2P_{n,k-1}(\|\vec{Y}_{nk}\|>\varepsilon),\quad 1\leqslant k\leqslant k_n,\ n\in\boldsymbol{N},\end{aligned}$$

and thus by Lemma 3.1 we obtain

$$\max_{1\leqslant k\leqslant k_n}E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|}{1+\|\vec{Y}_{nk}\|^2}\right)\xrightarrow{P}0,\tag{3.47}$$

$$\max_{1\leqslant k\leqslant k_n}|E_{n,k-1}(e^{i(\vec{t},\vec{Y}_{nk})}-1)|\xrightarrow{P}0,\quad t\in\boldsymbol{R}^d.\tag{3.48}$$

Moreover, (3.47) implies

$$\max_{1\leqslant k\leqslant k_n}\left|E_{n,k-1}\left(\frac{(\vec{t},\vec{Y}_{nk})}{1+\|\vec{Y}_{nk}\|^2}\right)\right|\xrightarrow{P}0,\quad \vec{t}\in\boldsymbol{R}^d.\tag{3.49}$$

Let us introduce, for a fixed $\vec{t}\in\vec{R}^d$, the sequence of random variables

$$\begin{aligned}T_n(\vec{t}):&=\sum_{k=1}^{k_n}\left\{|E_{n,k-1}(e^{i(\vec{t},\vec{Y}_{nk})}-1)|^2-\left|E_{n,k-1}\left(\frac{i(\vec{t},\vec{Y}_{nk})}{1+\|\vec{Y}_{nk}\|^2}\right)\right|^2\right\}\\&=\sum_{k=1}^{k_n}\left\{|E_{n,k-1}(e^{i(\vec{t},\vec{Y}_{nk})}-1)|+\left|E_{n,k-1}\left(\frac{i(\vec{t},\vec{Y}_{nk})}{1+\|\vec{Y}_{nk}\|^2}\right)\right|\right\}\times\\&\quad\times\left\{|E_{n,k-1}(e^{i(\vec{t},\vec{Y}_{nk})}-1)|-\left|E_{n,k-1}\left(\frac{i(\vec{t},\vec{Y}_{nk})}{1+\|\vec{Y}_{nk}\|^2}\right)\right|\right\},\quad n\in\boldsymbol{N}.\end{aligned}$$

Observe that

$$\left| |E_{n,k-1}(e^{i(\vec{t}, \vec{Y}_{nk})}-1)| - \left| E_{n,k-1}\left(\frac{i(\vec{t}, \vec{Y}_{nk})}{1+\|\vec{Y}_{nk}\|^2}\right)\right|\right|$$

$$\leqslant \left| E_{n,k-1}\left(e^{i(\vec{t}, \vec{Y}_{nk})}-1-\frac{i(\vec{t}, \vec{Y}_{nk})}{1+\|\vec{Y}_{nk}\|^2}\right)\right| \leqslant (2+\tfrac{1}{2}\|\vec{t}\|^2) E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right)$$

and thus, by

$$|T_n(\vec{t})| \leqslant (2+\tfrac{1}{2}\|\vec{t}\|^2)\Bigg[\|\vec{t}\| \max_{1\leqslant k\leqslant k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|}{1+\|\vec{Y}_{nk}\|^2}\right)+$$

$$+\max_{1\leqslant k\leqslant k_n} |E_{n,k-1}(e^{i(\vec{t}, \vec{Y}_{nk})}-1)|\Bigg]\cdot \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right), \quad n \in \boldsymbol{N},$$

we obtain $T_n(\vec{t}) \xrightarrow{P} 0$, $\vec{t} \in \boldsymbol{R}^d$. Theorem 3.4 implies the required convergence in law to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$. ∎

From the above proof it follows that under assumption (3.35) properties (3.30) and (3.45) are equivalent. Let us remark that (3.45) is equivalent to

$$\text{(3.50)} \qquad \sum_{k=1}^{k_n} \left\| E_{n,k-1}\left(\frac{\vec{Y}_{nk}}{1+\|\vec{Y}_{nk}\|^2}\right)\right\|^2 \xrightarrow{P} 0.$$

In fact, if $\vec{e}_i := (0, \ldots, \underset{i}{1}, \ldots, 0) \in \boldsymbol{R}^d$, $1 \leqslant i \leqslant d$, then

$$\sum_{k=1}^{k_n} \left\| E_{n,k-1}\left(\frac{\vec{Y}_{nk}}{1+\|Y_{nk}\|^2}\right)\right\|^2 = \sum_{k=1}^{k_n}\sum_{i=1}^{d}\left(E_{n,k-1}\left(\frac{Y_{nki}}{1+\|Y_{nk}\|^2}\right)\right)^2$$

$$= \sum_{i=1}^{d}\sum_{k=1}^{k_n}\left(E_{n,k-1}\left(\frac{(\vec{e}_i, \vec{Y}_{nk})}{1+\|\vec{Y}_{nk}\|^2}\right)\right)^2 \xrightarrow{P} 0.$$

Conversely, by (3.50) and

$$\sum_{k=1}^{k_n}\left[E_{n,k-1}\left(\frac{(\vec{t}, \vec{Y}_{nk})}{1+\|\vec{Y}_{nk}\|^2}\right)\right]^2 = \sum_{k=1}^{k_n}\left[\sum_{i=1}^{d} E_{n,k-1}\left(\frac{t_i Y_{nki}}{1+\|\vec{Y}_{nk}\|^2}\right)\right]^2$$

$$\leqslant 2\sum_{k=1}^{k_n}\sum_{i=1}^{d}\left[E_{n,k-1}\left(\frac{t_i Y_{nki}}{1+\|\vec{Y}_{nk}\|^2}\right)\right]^2 \leqslant 2\cdot\|\vec{t}\|^2 \sum_{k=1}^{k_n}\left\|E_{n,k-1}\left(\frac{\vec{Y}_{nk}}{1+\|\vec{Y}_{nk}\|^2}\right)\right\|^2,$$

we obtain (3.45).

Now assume that all vectors (2.1) have finite expectations. In this case we can define

$$\vec{B}_{nk} := E_{n,k-1}\vec{X}_{nk}, \tag{3.51}$$

$$\vec{Z}_{nk} := \vec{X}_{nk} - \vec{B}_{nk'}, \quad 1 \leqslant k \leqslant k_{n'}, \ n \in \boldsymbol{N}. \tag{3.52}$$

Repeating previously used arguments, we can verify that Theorems 3.1–3.4, 3.6, remain true if we replace in all conditions $\vec{Y}_{nk}$, $\vec{A}_{nk}$ by $\vec{Z}_{nk}$, $\vec{B}_{nk}$, respectively. Moreover, if system (3.52) is conditionally infinitesimal, then each of the properties

$$\sum_{k=1}^{k_n} |E_{n,k-1}(e^{i(\vec{t},\vec{Z}_{nk})} - 1)| \leqslant C = C(\vec{t}) < +\infty, \quad \vec{t} \in \boldsymbol{R}^d, \ n \in \boldsymbol{N}, \tag{3.53}$$

$$\lim_{n\to\infty} P\Big[\sum_{k=1}^{k_n} E_{n,k-1}(\|\vec{Z}_{nk}\|) > C\Big] = 0, \quad \text{for some } C > 0, \tag{3.54}$$

is sufficient for the convergence in law of (2.1) to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$. Assume that all vectors (2.1) have additionally finite variances. If there exists a constant $C > 0$ such that

$$\lim_{n\to\infty} P\Big[\sum_{k=1}^{k_n} E_{n,k-1}(\|Z_{nk}\|^2) > C\Big] = 0 \tag{3.55}$$

and system (3.53) is conditionally infinitesimal, or stronger,

$$\max_{1\leqslant k\leqslant k_n} E_{n,k-1}(\|Z_{nk}\|^2) \xrightarrow{P} 0, \tag{3.56}$$

then system (2.1) converges in law to $\mathscr{L}(\vec{a}, \mathbf{A}, \mu)$.

IV. Convergence in law to $l(\vec{a}, \mathbf{A}, \nu)$ for sums of dependent random vectors

Let $l(\vec{a}, \mathbf{A}, \nu)$ be a fixed infinitely divisible probability measure on $\boldsymbol{R}^d$. If $l(\vec{a}, \mathbf{A}, \nu) = \mathscr{L}(\vec{a}, \mathbf{A}, \mu)$, then the relationship between ν and μ is given by (1.7), (1.8) and (1.11). Hence condition (3.6) is equivalent to

$$\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in E)\right) \xrightarrow{P} \int_E \frac{\|\vec{x}\|^2}{1+\|\vec{x}\|^2}\, d\nu, \tag{4.1}$$

$$E \in \mathrm{Cont}\,\nu, \quad \vec{O} \notin \bar{E}.$$

Similarly, (3.5) is equivalent to

$$(4.2)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki}Y_{nkj}}{1+\|\vec{Y}_{nk}\|^2}\right) \xrightarrow{P} a_{ij} + \int_{R^d} \frac{x_i x_j\, d\nu}{1+\|\vec{x}\|^2}, \quad 1 \leqslant i,\ j \leqslant d.$$

In this way all the theorems of the previous chapter can be reformulated to give sufficient conditions for weak convergence to $l(\vec{a}, \mathbf{A}, \nu)$. A more interesting result is given by

THEOREM 4.1. *The random vectors* (2.1) *form a* $\mathscr{D}(\vec{a}, \mathbf{A}, \mu)$*-system if and only if they satisfy* (3.4), (4.2) *and*

$$(4.3)\qquad \sum_{k=1}^{k_n} P_{n,k-1}(\vec{Y}_{nk} \in E) \xrightarrow{P} \nu(E), \qquad E \in \mathrm{Cont}\,\nu,\ \vec{O} \notin \bar{E},$$

the measure ν *being defined by* (1.7) *and* (1.8).

An analogous result is true for the systems of random vectors with finite expectations with $\vec{Y}_{nk}$, $\vec{A}_{nk}$ *replaced by* $\vec{Z}_{nk}$ *and* $\vec{B}_{nk}$.

Proof. It suffices to prove the equivalence of (3.6) and (4.3). Assume (3.6) and fix an arbitrary $E \in \mathrm{Cont}\,\nu$, $\vec{O} \notin \bar{E}$. Since $\vec{O} \notin \bar{E}$, there exists a $\beta > 0$ such that $K(\vec{O}, \beta) \cap \bar{E} = \emptyset$. Choose an arbitrary $\varepsilon > 0$. There exists a bounded set $E_0 \in \mathrm{Cont}\,\nu$ such that $E_0 \subset E$ and

$$(4.4)\qquad |\mu(E) - \mu(E_0)| < \frac{\varepsilon}{2} \cdot \frac{\beta^2}{1+\beta^2},$$

which implies

$$(4.5)\qquad |\nu(E) - \nu(E_0)| < \varepsilon/2.$$

Pick a $\delta > 0$ such that

$$\left| \frac{1+\|\vec{x}\|^2}{\|\vec{x}\|^2} - \frac{1+\|\vec{y}\|^2}{\|\vec{y}\|^2} \right| < \frac{\varepsilon}{2\mu(E)} \quad \text{if} \quad \|\vec{x} - \vec{y}\| < \delta,\ \vec{x}, \vec{y} \in E_0.$$

Choose a finite partition of the set E_0 into disjoint subsets $E_i \in \mathrm{Cont}\,\mu$, $1 \leqslant i \leqslant S$, with diameters less than δ and a selection $x_i \in E_i$, $1 \leqslant i \leqslant s$, such that

$$(4.6)\qquad \left| \nu(E_0) - \sum_{i=1}^{s} \frac{1+\|\vec{x}_i\|^2}{\|\vec{x}_i\|^2}\, \mu(E_i) \right| < \frac{\varepsilon}{2}.$$

Then by (3.6)

$$(4.7)\quad \Big|\sum_{k=1}^{k_n}\sum_{i=1}^{s}\frac{1+\|\vec{x}_i\|^2}{\|\vec{x}_i\|^2}E_{n,k-1}\Big(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}I(\vec{Y}_{nk}\in E_i)\Big)-$$

$$-\sum_{k=1}^{k_n}\sum_{i=1}^{s}P_{n,k-1}(\vec{Y}_{nk}\in E_i)\Big|$$

$$\leqslant\frac{\varepsilon}{2\mu(E)}\sum_{i=1}^{s}\sum_{k=1}^{k_n}E_{n,k-1}\Big(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}I(Y_{nk}\in E_i)\Big)$$

$$\leqslant\frac{\varepsilon}{2\mu(E)}\sum_{k=1}^{k_n}E_{n,k-1}\Big(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}I(\vec{Y}_{nk}\in E_i)\Big)\xrightarrow{P}\varepsilon/2,$$

and

$$(4.8)\quad \sum_{i=1}^{s}\sum_{k=1}^{k_n}\frac{1+\|\vec{x}_i\|^2}{\|\vec{x}_i\|^2}E_{n,k-1}\Big(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}I(\vec{Y}_{nk}\in E_i)\Big)-$$

$$-\sum_{i=1}^{s}\frac{1+\|\vec{x}_i\|^2}{\|\vec{x}_i\|^2}\mu(E_i)\xrightarrow{P}0.$$

Finally, by (3.6) and (4.4)

$$(4.9)\quad \Big|\sum_{k=1}^{k_n}P_{n,k-1}(\vec{Y}_{nk}\in E)-\sum_{i=1}^{s}\sum_{k=1}^{k_n}P_{n,k-1}(\vec{Y}_{nk}\in E_i)\Big|$$

$$=\sum_{k=1}^{k_n}P_{n,k-1}(\vec{Y}_{nk}\in E\setminus E_0)$$

$$\leqslant\frac{1+\beta^2}{\beta^2}\sum_{k=1}^{k_n}E_{n,k-1}\Big(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}I(\vec{Y}_{nk}\in E\setminus E_0)\Big)\xrightarrow{P}\frac{1+\beta^2}{\beta^2}\mu(E\setminus E_0)\leqslant\frac{\varepsilon}{2}.$$

Therefore by (4.5)–(4.9) Lemma 3.1 yields (4.3).

Now assume (4.3) and as before fix an $E\in\operatorname{Cont}\mu$, $\vec{O}\notin\bar{E}$ and an $\varepsilon>0$. Choose a bounded set $E_0\in\operatorname{Cont}\nu$ such that $E_0\subset E$ and $|\nu(E)-\nu(E_0)|<\varepsilon/2$, which implies

$$(4.10)\quad |\mu(E)-\mu(E_0)|<\varepsilon/2.$$

By the uniform continuity of the function $g(\vec{x}):=\|\vec{x}\|^2/(1+\|\vec{x}\|^2)$ on $\boldsymbol{R}^d$ there exists a $\delta>0$ such that $|g(\vec{x})-g(\vec{y})|<\varepsilon/2\nu(E)$ if $\|\vec{x}-\vec{y}\|<\delta$. Divide the set E_0 into non-empty disjoint sets $E_i \in \operatorname{Cont}\nu$, $1 \leqslant i \leqslant s$, with diameters less than δ and choose $\vec{x}_i \in E_i$, $1 \leqslant i \leqslant s$, such that

$$(4.11)\qquad \left|\mu(E_0) - \sum_{i=1}^{s} \frac{\|\vec{x}_i\|^2}{1+\|\vec{x}_i\|^2}\nu(E_i)\right| < \frac{\varepsilon}{2}.$$

By (4.3)

$$(4.12)\qquad \sum_{i=1}^{s}\sum_{k=1}^{k_n} \frac{\|\vec{x}_i\|^2}{1+\|\vec{x}_i\|^2} P_{n,k-1}(\vec{Y}_{nk} \in E_i) - \sum_{i=1}^{s} \frac{\|\vec{x}_i\|^2}{1+\|\vec{x}_i\|^2}\nu(E_i) \xrightarrow{P} 0.$$

Next,

$$(4.13)\qquad \left|\sum_{i=1}^{s}\sum_{k=1}^{k_n} \frac{\|x_i\|^2}{1+\|x_i\|^2} P_{n,k-1}(\vec{Y}_{nk} \in E_i) - \right.$$
$$\left. - \sum_{i=1}^{s}\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in E_i)\right)\right|$$
$$\leqslant \frac{\varepsilon}{2\nu(E)} \sum_{k=1}^{k_n} P_{n,k-1}(\vec{Y}_{n,k} \in E) \xrightarrow{P} \frac{\varepsilon}{2\nu(E)} \cdot \nu(E) = \frac{\varepsilon}{2},$$

and

(4.14)

$$\left|\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in E_0)\right) - \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in E)\right)\right|$$
$$\leqslant \sum_{k-1}^{k_n} P_{n,k-1}(\vec{Y}_{nk} \in E \setminus E_0) \xrightarrow{P} \nu(E \setminus E_0) < \frac{\varepsilon}{2}.$$

Thus (4.10)–(4.14) and Lemma 3.1 imply (3.6). ∎

Now we are going to obtain a generalization of the multidimensional Lévy–Takano criterion (cf. Theorem 2.4).

First we introduce an auxiliary notion. Let there be given for every $\varepsilon > 0$ three sequences of random variables $Y_n^\varepsilon \leqslant X_n^\varepsilon \leqslant Z_n^\varepsilon$, $n \in \boldsymbol{N}$. Assume that for all but a countably many $\varepsilon > 0$ we have $Y_n^\varepsilon \xrightarrow{P} \alpha(\varepsilon)$, $Z_n^\varepsilon \xrightarrow{P} \beta(\varepsilon)$; $\alpha(\varepsilon), \beta(\varepsilon) \in \boldsymbol{R}$ and $\lim_{\varepsilon\to 0}\alpha(\varepsilon) = \lim_{\varepsilon\to 0}\beta(\varepsilon) = a$, ε being allowed to take values off a countable subset of positive numbers (the same convention applies to similar formulae in the subsequent theorem). Then this common limit a will be denoted by $\operatorname*{Lim}_{\substack{n\to\infty\\ \varepsilon\to 0}} X_n^\varepsilon$, or shortly, by $\operatorname*{Lim}_{\varepsilon,n} X_n^\varepsilon$. It can easily be proved that the limit $\operatorname*{Lim}_{\varepsilon,n} X_n^\varepsilon$ is well defined and the following lemma is true.

LEMMA 4.1. *Let X_n^ε, W_n^ε, $n \in \boldsymbol{N}$, be sequences of random variables such that* $\operatorname{Lim}_{\varepsilon,n} X_n^\varepsilon = \alpha$. *Assume that for every $\varepsilon > 0$ there exists a sequence of random variables T_n^ε; $n \in \boldsymbol{N}$, such that $T_n^\varepsilon \to \gamma(\varepsilon) \in \boldsymbol{R}$, $\lim_{\varepsilon\to 0}\gamma(\varepsilon) = 0$ and $|X_n^\varepsilon - W_n^\varepsilon| \leqslant T_n^\varepsilon$, $n \in \boldsymbol{N}$. Then* $\operatorname{Lim}_{\varepsilon,n} W_n^\varepsilon = \alpha$. ■

THEOREM 4.2. *Let system* (2.1) *be conditionally infinitesimal with respect to the σ-fields* (3.1). *Moreover, if it satisfies*

$$\sum_{k=1}^{k_n} P_{n,k-1}(\vec{X}_{nk} \in E) \xrightarrow{P} \nu(E), \qquad E \in \operatorname{Cont}\nu,\ \vec{O} \notin \bar{E}, \tag{4.15}$$

$$\begin{aligned}\operatorname{Lim}_{\varepsilon,n}\Big\{ &\sum_{k=1}^{k_n} E_{n,k-1}\big(X_{nki}X_{nkj}I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big) - \\ &- \sum_{k=1}^{k_n} E_{n,k-1}\big(X_{nki}I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big)E_{n,k-1}\big(X_{nkj}I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big)\Big\} \\ &= a_{ij}; \quad 1 \leqslant i,\ j \leqslant d,\end{aligned} \tag{4.16}$$

and for some $V \in O \cap \operatorname{Cont}\nu$

$$\sum_{k=1}^{k_n} E_{n,k-1}\big(\vec{X}_{nk}I(\vec{X}_{nk} \in V)\big) \xrightarrow{P} \vec{a} + \int_V \frac{\vec{x}\|\vec{x}\|^2}{1+\|\vec{x}\|^2}\,d\nu - \int_{\boldsymbol{R}^d\setminus V} \frac{\vec{x}}{1+\|\vec{x}\|^2}\,d\nu, \tag{4.17}$$

or, more generally,

$$\begin{aligned}\sum_{k=1}^{k_n}\big\{\vec{A}_{nk} + E_{n,k-1}\big(\vec{Y}_{nk}I(\vec{Y}_{nk} \in U)\big)\big\}& \\ \xrightarrow{P} \vec{a} + \int_U \frac{\vec{x}\|\vec{x}\|^2}{1+\|\vec{x}\|^2}\,d\nu &- \int_{\boldsymbol{R}^d\setminus U} \frac{\vec{x}}{1+\|\vec{x}\|^2}\,d\nu\end{aligned} \tag{4.18}$$

for some $U \in O \cap \operatorname{Cont}\nu$ and for $\vec{A}_{nk}$, $\vec{Y}_{nk}$ defined by (3.2), (3.3) *for some $V \in O$, then it converges in law to $l(\vec{a}, \mathbf{A}, \nu)$.*

Proof. Condition (4.15) is equivalent to (4.3) for every σ-finite measure ν on $\boldsymbol{R}^d$ satisfying (1.9) and (1.10). Consequently it suffices to prove that

$$V_n^E \xrightarrow{P} 0, \qquad E \in \operatorname{Cont}\nu,\ \vec{O} \notin \bar{E}, \tag{4.19}$$

where

$$V_n^E := \sum_{k=1}^{k_n}\{P_{n,k-1}(\vec{X}_{nk} \in E) - P_{n,k-1}(\vec{Y}_{nk} \in E)\}, \qquad n \in \boldsymbol{N}.$$

For arbitrary $\varepsilon > 0$ we have

$$|V_n^E| \leqslant \sum_{k=1}^{k_n} \{P_{n,k-1}(\vec{X}_{nk} \in E,\ \vec{Y}_{nk} \notin E) + P_{n,k-1}(\vec{Y}_{nk} \in E,\ \vec{X}_{nk} \notin E)\}$$
$$\leqslant 2 \sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{A}_{nk}\| > \varepsilon) +$$
$$+ \sum_{k=1}^{k_n} \{P_{n,k-1}(\vec{X}_{nk} \in E, \vec{Y}_{nk} \notin E, \|\vec{A}_{nk}\| \leqslant \varepsilon) +$$
$$+ P_{n,k-1}(\vec{X}_{nk} \notin E,\ \vec{Y}_{nk} \in E,\ \|A_{nk}\| \leqslant \varepsilon)\}.$$

Let us define for $E \in \mathscr{B}^d$ and $\varepsilon > 0$

$$E_\varepsilon := \{\vec{x} \in E;\ d(\vec{x}, \mathbf{R}^d \setminus E) \leqslant \varepsilon\}, \tag{4.20}$$

$$E^\varepsilon := \{\vec{x} \notin E;\ d(\vec{x}, E) \leqslant \varepsilon\}, \tag{4.21}$$

where $d(\vec{x}, A) := \inf\{\|\vec{x} - \vec{y}\|;\ \vec{y} \in A\}$.

If ε_i, $i \in \mathbf{N}$, monotonically decreases to zero, then

$$E_{\varepsilon_i} \supseteq E_{\varepsilon_{i+1}}, \quad E^{\varepsilon_i} \supseteq E^{\varepsilon_{i+1}},\ i \in \mathbf{N}, \quad \text{and}$$
$$\bigcap_{i=1}^{\infty} E_{\varepsilon_i} = E \cap \overline{\mathbf{R}^d \setminus E} \subset \partial E, \quad \bigcap_{i=1}^{\infty} E^{\varepsilon_i} = (\mathbf{R}^d \setminus E) \cap \bar{E} \subset \partial E.$$

Observe that for every $1 \leqslant k \leqslant k_n$, $n \in \mathbf{N}$ and $\varepsilon > 0$,

$$\{\vec{X}_{nk} \in E,\ \vec{Y}_{nk} \notin E,\ \|\vec{A}_{nk}\| \leqslant \varepsilon\} \subset \{\vec{X}_{nk} \in E_\varepsilon\},$$
$$\{\vec{X}_{nk} \notin E,\ \vec{Y}_{nk} \in E,\ \|\vec{A}_{nk}\| \leqslant \varepsilon\} \subset \{\vec{X}_{nk} \in E^\varepsilon\}.$$

We can choose ε_i, $i \in \mathbf{N}$, such that all E_{ε_i}, E^{ε_i} are continuity sets of ν. Then for a fixed $i \in \mathbf{N}$

$$|V_n^E| \leqslant 2 \sum_{k=1}^{k_n} I(\|\vec{A}_{nk}\| > \varepsilon_i) + \sum_{k=1}^{k_n} P_{n,k-1}(\vec{X}_{nk} \in E_{\varepsilon_i}) +$$
$$+ \sum_{k=1}^{k_n} P_{n,k-1}(\vec{X}_{nk} \in E^{\varepsilon_i}) \xrightarrow{P} \nu(E_{\varepsilon_i}) + \nu(E^{\varepsilon_i}).$$

But $\lim\limits_{i\to\infty} [\nu(E_{\varepsilon_i}) + \nu(E^{\varepsilon_i})] \leqslant 2\nu(\partial E) = 0$ and then by Lemma 3.1 we obtain (4.19).

Now we shall prove that under assumption (4.15) conditions (4.16) and (4.2) are equivalent for every non-negative definite $d \times d$-matrix $\mathbf{A} = [a_{ij}]$. First we shall prove that for every $\varepsilon > 0$, $K(\vec{O}, \varepsilon) \in \mathrm{Cont}\,\nu$,

$$\sum_{k=1} E_{n,k-1}\left(\frac{Y_{nki}\,Y_{nkj}}{1 + \|\vec{Y}_{nk}\|^2}\, I(\|\vec{Y}_{nk}\| > \varepsilon)\right) \xrightarrow{P} \int\limits_{\|\vec{x}\| > \varepsilon} \frac{x_i x_j}{1 + \|\vec{x}\|^2}\, d\nu, \tag{4.22}$$
$$1 \leqslant i,\ j \leqslant d.$$

Choose an arbitrary $\delta > 0$. Then there exists an $\eta > 0$ such that $K(\vec{O}, \eta) \in \operatorname{Cont}\nu$, $\nu(\|\vec{x}\| > \eta) < \delta/2$ and

$$(4.23)\qquad \left| \int\limits_{\varepsilon < \|\vec{x}\| \leqslant \eta} \frac{x_i x_j}{1 + \|\vec{x}\|^2}\, d\nu - \int\limits_{\|\vec{x}\| > \varepsilon} \frac{x_i x_j}{1 + \|\vec{x}\|^2}\, d\nu \right| < \frac{\delta}{2}.$$

The function $f(x) := x_i x_j/(1 + \|\vec{x}\|^2)$ is uniformly continuous on $\boldsymbol{R}^d$ and hence there exists a $\beta > 0$ such that, for $\|\vec{x} - \vec{y}\| < \delta$,

$$|f(\vec{x}) - f(\vec{y})| < \frac{\delta}{2\nu(\|\vec{x}\| > \varepsilon)}.$$

Divide the set $\{\varepsilon < \|\vec{x}\| \leqslant \eta\}$ into non-empty disjoint sets $E_l \in \operatorname{Cont}\nu$, $1 \leqslant l \leqslant s$, with diameters less than β and choose $\vec{x}_l \in E$, $1 \leqslant l \leqslant s$, such that

$$(4.24)\qquad \left| \int\limits_{\varepsilon < \|\vec{x}\| \leqslant \eta} \frac{x_i x_j}{1 + \|\vec{x}\|^2}\, d\nu - \sum_{l=1}^{s} \frac{x_{li} x_{lj}}{1 + \|\vec{x}_l\|^2}\, \nu(E_l) \right| < \frac{\delta}{2}.$$

By (4.3)

$$(4.25)\qquad \sum_{l=1}^{s} \frac{x_{li} x_{lj}}{1 + \|\vec{x}_l\|^2} \sum_{k=1}^{k_n} P_{n,k-1}(\vec{Y}_{nk} \in E_l) - \sum_{l=1}^{s} \frac{x_{li} x_{lj}}{1 + \|\vec{x}_l\|^2}\, \nu(E_l) \xrightarrow{P} 0,$$

$$(4.26)\qquad \Bigg| \sum_{l=1}^{s} \frac{x_{li} x_{lj}}{1 + \|\vec{x}_l\|^2} \sum_{k=1}^{k_n} P_{n,k-1}(\vec{Y}_{nk} \in E_l) - \sum_{l=1}^{s} \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki} Y_{nkj}}{1 + \|\vec{Y}_{nk}\|^2}\, I(\vec{Y}_{nk} \in E_l) \right) \Bigg| \leqslant \frac{\delta}{2\nu(\|\vec{x}\| > \varepsilon)} \cdot \sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Y}_{nk}\| > \varepsilon) \xrightarrow{P} \frac{\delta}{2},$$

and

$$(4.27)\qquad \Bigg| \sum_{l=1}^{s} \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki} Y_{nkj}}{1 + \|\vec{Y}_{nk}\|^2}\, I(\vec{Y}_{nk} \in E_l) \right) - \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki} Y_{nkj}}{1 + \|\vec{Y}_{nk}\|^2}\, I(\|\vec{Y}_{nk}\| > \varepsilon) \right) \Bigg| \leqslant \sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Y}_{nk}\| > \eta) \xrightarrow{P} \nu(\|\vec{x}\| > \eta) < \delta/2.$$

Letting δ tend to zero and applying Lemma 3.1, we obtain (4.22). Therefore (4.2) is equivalent to

$$(4.28)\quad \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{(\vec{t},\vec{Y}_{nk})^2}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\|\leqslant\varepsilon)\right)$$

$$\xrightarrow{P} (\vec{t}, A\vec{t}) + \int_{\|\vec{x}\|\leqslant\varepsilon} \frac{(\vec{t},\vec{x})}{1+\|\vec{x}\|^2}\,d\nu, \quad \vec{t}\in\boldsymbol{R}^d,\ \varepsilon>0,\ K(\vec{O},\varepsilon)\in\operatorname{Cont}\nu.$$

Now we shall prove that the last condition is equivalent to

$$(4.29)\quad \operatorname*{Lim}_{\varepsilon,n} \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{(\vec{t},\vec{Y}_{nk})^2}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\|\leqslant\varepsilon)\right) = (\vec{t}, \mathbf{A}\vec{t}), \qquad \vec{t}\in\boldsymbol{R}^d.$$

It is obvious that the latter condition is implied by the former one. Conversely, let us fix a $\vec{t}\in\boldsymbol{R}^d$ and let us write

$$X_n^\delta := \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{(\vec{t},\vec{Y}_{nk})^2}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\|\leqslant\delta)\right) - (\vec{t}, \mathbf{A}\vec{t}), \qquad n\in\boldsymbol{N},\ \delta>0.$$

By (4.29) for every $\delta>0$ there exist sequences Y_n^δ, Z_n^δ, $n\in\boldsymbol{N}$, such that $Y_n^\delta\leqslant X_n^\delta\leqslant Z_n^\delta$, $n\in\boldsymbol{N}$, $Y_n^\delta\xrightarrow{P}\alpha(\delta)$, $Z_n^\delta\xrightarrow{P}\beta(\delta)$, $\lim\limits_{\delta\to0}\alpha(\delta)=\lim\limits_{\delta\to0}\beta(\delta)=0$. Thus for every $\delta>0$

$$0\leqslant X_n^\delta - Y_n^\delta = |X_n^\delta - Y_n^\delta| \leqslant Z_n^\delta - Y_n^\delta \xrightarrow{P} \beta(\delta)-\alpha(\delta),$$

$$|X_n^\delta| \leqslant |X_n^\delta - Y_n^\delta| + |Y_n^\delta - \alpha(\delta)| + |\alpha(\delta)|, \qquad n\in\boldsymbol{N}.$$

Let us fix an $\varepsilon>0$ such that $K(\vec{O},\varepsilon)\in\operatorname{Cont}\nu$. Then for $0<\delta<\varepsilon$ such that $K(0,\delta)\in\operatorname{Cont}\nu$ by (4.22) it follows that

$$V_n^\delta := \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{(\vec{t},\vec{Y}_{nk})^2}{1+\|\vec{Y}_{nk}\|^2} I(\delta<\|\vec{Y}_{nk}\|\leqslant\varepsilon)\right) - \int_{\delta<\|\vec{x}\|\leqslant\varepsilon} \frac{(\vec{t},\vec{x})^2}{1+\|\vec{x}\|^2}\,d\nu \xrightarrow{P} 0.$$

Therefore for $0<\delta<\varepsilon$

$$\left|X_n^\varepsilon - \int_{\|\vec{x}\|\leqslant\varepsilon} \frac{(\vec{t},\vec{x})^2}{1+\|\vec{x}\|^2}\,d\nu\right| \leqslant |X_n^\delta| + |V_n^\delta| + \int_{\|\vec{x}\|\leqslant\delta} \frac{(\vec{t},\vec{x})^2}{1+\|\vec{x}\|^2}\,d\nu$$

$$\leqslant |X_n^\delta - Y_n^\delta| + |Y_n^\delta - \alpha(\delta)| + |V_n^\delta| + |\alpha(\delta)| + \int_{\|\vec{x}\|\leqslant\delta} \frac{(\vec{t},\vec{x})^2}{1+\|\vec{x}\|^2}\,d\nu, \qquad n\in\boldsymbol{N},$$

and Lemma 3.1 yields (4.28).

The inequalities

$$\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{(\vec{t},\vec{Y}_{nk})^2}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\|\leqslant\varepsilon)\right)\leqslant \sum_{k=1}^{k_n} E_{n,k-1}\left((\vec{t},\vec{Y}_{nk})^2 I(\|\vec{Y}_{nk}\|\leqslant\varepsilon)\right)$$
$$\leqslant (1+\varepsilon^2)\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{(\vec{t},\vec{Y}_{nk})^2}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\|\leqslant\varepsilon)\right),\quad \varepsilon>0,\ n\in \boldsymbol{N},$$

imply that (4.29) is equivalent to

$$\text{(4.29)}\qquad \operatorname*{Lim}_{\varepsilon,n}\sum_{k=1}^{k_n} E_{n,k-1}\left((\vec{t},\vec{Y}_{nk})^2 I(\|\vec{Y}_{nk}\|\leqslant\varepsilon)\right) = (\vec{t},\mathbf{A}\vec{t}),\qquad \vec{t}\in\boldsymbol{R}^d,$$

and next to

$$\text{(4.30)}\qquad \operatorname*{Lim}_{\varepsilon,n}\sum_{k=1}^{k_n} E_{n,k-1}\left(Y_{nki}Y_{nkj} I(\|\vec{Y}_{nk}\|\leqslant\varepsilon)\right) = a_{ij},\qquad 1\leqslant i,\ j\leqslant d.$$

Observe that for arbitrary $1\leqslant i,j\leqslant d$ and every $\varepsilon>0$ such that $K(\vec{O},\varepsilon)\in$ Contν we have

$$\left|\sum_{k=1}^{k_n} E_{n,k-1}\left(Y_{nki}Y_{nkj}I(\|\vec{Y}_{nk}\|\leqslant\varepsilon)\right)-\sum_{k=1}^{k_n} E_{n,k-1}\left(Y_{nki}Y_{nkj}I(\|\vec{X}_{nk}\|\leqslant\varepsilon)\right)\right|$$
$$\leqslant\left|\sum_{k=1}^{k_n} E_{n,k-1}\left(Y_{nki}Y_{nkj}I(\|\vec{X}_{nk}\|\leqslant\varepsilon)I(\|\vec{Y}_{nk}\|>\varepsilon)\right)\right|+$$
$$+\left|\sum_{k=1}^{k_n} E_{n,k-1}\left(Y_{nki}Y_{nkj}I(\|\vec{X}_{nk}\|>\varepsilon)I(\|\vec{Y}_{nk}\|\leqslant\varepsilon)\right)\right|$$
$$\leqslant \varepsilon^2\sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{X}_{nk}\|>\varepsilon)+\varepsilon^2\sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Y}_{nk}\|>\varepsilon)+$$
$$+\varepsilon\sum_{k=1}^{k_n}(|A_{nki}|+|A_{nkj}|)P_{n,k-1}(\|\vec{Y}_{nk}\|>\varepsilon)+$$
$$+\sum_{k=1}^{k_n}|A_{nki}|\cdot|A_{nkj}|P_{n,k-1}(\|\vec{Y}_{nk}\|>\varepsilon)$$
$$\leqslant \varepsilon^2\sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{X}_{nk}\|>\varepsilon)+\varepsilon^2\sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Y}_{nk}\|>\varepsilon)+$$
$$+2\varepsilon\max_{1\leqslant k\leqslant k_n}\|\vec{A}_{nk}\|\cdot\sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Y}_{nk}\|>\varepsilon)+$$
$$+(\max_{1\leqslant k\leqslant k_n}\|\vec{A}_{nk}\|)^2\sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Y}_{nk}\|>\varepsilon)$$
$$\xrightarrow{P} 2\varepsilon^2\nu(\|\vec{x}\|>\varepsilon).$$

The finiteness of $\int\limits_{\|\vec{x}\|\leqslant 1} \|\vec{x}\|^2 d\nu$ implies $\lim\limits_{\varepsilon\to 0}\varepsilon^2\nu(\|\vec{x}\|>\varepsilon)=0$. and thus by Lemma 3.1 property (4.30) is equivalent to

$$(4.31)\quad \operatorname*{Lim}_{\varepsilon,n}\sum_{k=1}^{k_n} E_{n,k-1}\big(Y_{nki}Y_{nkj}I(\|\vec{X}_{nk}\|\leqslant\varepsilon)\big)=a_{ij},\qquad 1\leqslant i,\,j\leqslant d.$$

If $\varepsilon>0$ is so small that $K(\vec{O},\varepsilon)\subset V$, $K(\vec{O},\varepsilon)\in \mathrm{Cont}\,\nu$, then

$$\sum_{k=1}^{k_n} E_{n,k-1}\big(Y_{nki}Y_{nkj}I(\|\vec{X}_{nk}\|\leqslant\varepsilon)\big)-\Big\{\sum_{k=1}^{k_n} E_{n,k-1}\big(X_{nki}X_{nkj}I(\|\vec{X}_{nk}\|\leqslant\varepsilon)\big)-$$
$$-\sum_{k=1}^{k_n} E_{n,k-1}\big(X_{nki}I(\|\vec{X}_{nk}\|\leqslant\varepsilon)\big)E_{n,k-1}\big(X_{nkj}I(\|\vec{X}_{nk}\|\leqslant\varepsilon)\big)\Big\}$$
$$=\sum_{k=1}^{k_n}\big\{E_{n,k-1}\big(X_{nki}I(\|\vec{X}_{nk}\|>\varepsilon)I(\vec{X}_{nk}\in V)\big)E_{n,k-1}\big(X_{nkj}I(\|\vec{X}_{nk}\|$$
$$>\varepsilon)I(\vec{X}_{nk}\in V)\big)\big\}-\sum_{k=1}^{k_n}A_{nki}A_{nkj}P_{n,k-1}(\|\vec{X}_{nk}\|>\varepsilon),\ 1\leqslant i,\,j\leqslant d,\ n\in N.$$

But both these summands converge in probability to zero and by Lemma 4.1 we obtain the equivalence of (4.31) and (4.16). Hence (4.2) and (4.16) are equivalent.

Finally we shall prove that under assumptions (4.15) and (4.16) condition (4.18) is equivalent to (3.4) for every $U\in O\cap\mathrm{Cont}\,\nu$. By the equality

$$E_{n,k-1}\left(\frac{\vec{Y}_{nk}}{1+\|\vec{Y}_{nk}\|^2}\right)=E_{n,k-1}\big(\vec{Y}_{nk}I(\vec{Y}_{nk}\in U)\big)+$$
$$+E_{n,k-1}\left(\frac{\vec{Y}_{nk}}{1+\|\vec{Y}_{nk}\|^2}I(\vec{Y}_{nk}\notin U)\right)-E_{n,k-1}\left(\vec{Y}_{nk}\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}I(\vec{Y}_{nk}\in U)\right),$$

the required equivalence will follow if we prove

$$(4.32)\quad \sum_{k=1}^{k_n}E_{n,k-1}\left(Y_{nki}\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}I(\vec{Y}_{nk}\in U)\right)\xrightarrow{P}\int\limits_U x_i\,d\mu,$$
$$U\in\mathrm{Cont}\,\nu\cap O,\ 1\leqslant i\leqslant d,$$

$$(4.33)\quad \sum_{k=1}^{k_n}E_{n,k-1}\left(\frac{Y_{nki}}{1+\|\vec{Y}_{nk}\|^2}I(\vec{Y}_{nk}\notin U)\right)\xrightarrow{P}\int\limits_{R^d\setminus U}\frac{x_i}{\|\vec{x}\|^2}\,d\mu,$$
$$U\in O\cap\mathrm{Cont}\,\nu,\ 1\leqslant i\leqslant d,$$

where the measure μ is defined by (1.11).

Choose an arbitrary $\varepsilon > 0$. There exists an $\eta > 0$ such that $\operatorname{Cont}\mu \ni K(\vec{O}, \eta) \subset U$ and

$$\Big| \int_{\|\vec{x}\| < \eta} x_i \, d\mu \Big| < \varepsilon/2. \tag{4.34}$$

We may assume that $\eta < \varepsilon/2\big(\operatorname{Tr}\mathbf{A} + \mu(\boldsymbol{R}^d)\big)$. Divide the set $U \setminus K(\vec{O}, \eta)$ into non-empty disjoint sets $E_l \in \operatorname{Cont}\mu$, $1 \leqslant l \leqslant s$, with diameter less than η and choose $\vec{x}_l \in E_l$, $1 \leqslant l \leqslant s$, such that

$$\Big| \int_{U \setminus K(\vec{O}, \eta)} x_i \, d\mu - \sum_{l=1}^{s} x_{li} \mu(E_l) \Big| < \varepsilon/2. \tag{4.35}$$

By Theorem 4.1 condition (3.6) is satisfied and thus

$$\sum_{l=1}^{s} x_{li} \sum_{k=1}^{k_n} E_{n,k-1}\Big(\frac{\|\vec{Y}_{nk}\|^2}{1 + \|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in E_l)\Big) - \sum_{l=1}^{s} x_{li} \mu(E_l) \xrightarrow{P} 0. \tag{4.36}$$

Moreover, by (3.5),

$$\begin{aligned} &\Big| \sum_{l=1}^{s} \sum_{k=1}^{k_n} x_{li} E_{n,k-1}\Big(\frac{\|\vec{Y}_{nk}\|^2}{1 + \|\vec{Y}_{nk}\|^2} I(Y_{nk} \in E_l)\Big) - \\ &\qquad - \sum_{l=1}^{s} \sum_{k=1}^{k_n} E_{n,k-1}\Big(Y_{nki} \frac{\|\vec{Y}_{nk}\|^2}{1 + \|\vec{Y}_{nk}\|^2} I(Y_{nk} \in E_l)\Big) \Big| \\ &\qquad \leqslant \frac{\varepsilon}{2\big(\operatorname{Tr}\mathbf{A} + \mu(\boldsymbol{R}^d)\big)} \sum_{k=1}^{k_n} E_{n,k-1}\Big(\frac{\|\vec{Y}_{nk}\|^2}{1 + \|\vec{Y}_{nk}\|^2}\Big) \xrightarrow{P} \frac{\varepsilon}{2}, \end{aligned} \tag{4.37}$$

$$\begin{aligned} &\Big| \sum_{k=1}^{k_n} E_{n,k-1}\Big(Y_{nki} \frac{\|\vec{Y}_{nk}\|^2}{1 + \|\vec{Y}_{nk}\|^2} I\big(\vec{Y}_{nk} \in U \setminus K(\vec{O}, \eta)\big)\Big) - \\ &\qquad - \sum_{k=1}^{k_n} E_{n,k-1}\Big(Y_{nki} \frac{\|\vec{Y}_{nk}\|^2}{1 + \|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in U)\Big) \Big| \\ &\qquad = \Big| \sum_{k=1}^{k_n} E_{n,k-1}\Big(Y_{nki} \frac{\|\vec{Y}_{nk}\|^2}{1 + \|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\| \leqslant \eta)\Big) \Big| \\ &\qquad \leqslant \frac{\varepsilon}{2\big(\operatorname{Tr}\mathbf{A} + \mu(\boldsymbol{R}^d)\big)} \sum_{k=1}^{k_n} E_{n,k-1}\Big(\frac{\|\vec{Y}_{nk}\|^2}{1 + \|\vec{Y}_{nk}\|^2}\Big) \xrightarrow{P} \frac{\varepsilon}{2}. \end{aligned} \tag{4.38}$$

By (4.34)–(4.38) and Lemma 3.1 we obtain (4.32). For the proof of (4.33) choose a $\beta > 2(\operatorname{Tr}\mathbf{A}+\mu(\mathbf{R}^d))/\varepsilon$ so large that $U \subset K(\vec{O}, \beta) \in \operatorname{Cont}\mu$ and

$$(4.39) \qquad \left| \int\limits_{\mathbf{R}^d\setminus U} \frac{x_i}{\|\vec{x}\|^2}\, d\mu - \int\limits_{K(\vec{O},\beta)\setminus U} \frac{x_i}{\|\vec{x}\|^2}\, d\mu \right| < \frac{\varepsilon}{2}.$$

The function $f(\vec{x}) := x_i/\|\vec{x}\|^2$ is uniformly continuous on the set $K(\vec{O}, \beta)\setminus U$ and hence there exists a $\delta > 0$ such that

$$|f(\vec{x})-f(\vec{y})| < \frac{\varepsilon}{2(\operatorname{Tr}\mathbf{A}+\mu(\mathbf{R}^d))} \quad \text{if} \quad \|\vec{x}-\vec{y}\| < \delta, \quad \vec{x}, \vec{y} \in K(\vec{O}, \beta)\setminus U.$$

Divide the set $K(\vec{O}, \beta)\setminus U$ into disjoint sets $E_l \in \operatorname{Cont}\mu$, $1 \leqslant l \leqslant s$, with diameters less than δ and choose $\vec{x}_l \in E_l$, $1 \leqslant l \leqslant s$, such that

$$(4.40) \qquad \left| \sum_{l=1}^{s} \frac{x_{li}}{\|\vec{x}\|^2}\mu(E_l) - \int\limits_{K(\vec{O},\beta)\setminus U} \frac{x_i}{\|\vec{x}\|^2}\, d\mu \right| < \frac{\varepsilon}{2}.$$

Property (3.6) implies

(4.41)

$$\sum_{l=1}^{s} \frac{x_{li}}{\|\vec{x}_l\|^2} \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(Y_{nk} \in E_l)\right) - \sum_{l=1}^{s} \frac{x_{li}}{\|\vec{x}_l\|^2}\mu(E_l) \xrightarrow{P} 0.$$

By (3.5) we obtain

$$(4.42) \qquad \left| \sum_{l=1}^{s} \sum_{k=1}^{k_n} \frac{x_{li}}{\|\vec{x}_l\|^2} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in E_l)\right) - \right.$$
$$\left. - \sum_{l=1}^{s} \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki}}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in E_l)\right) \right|$$
$$\leqslant \frac{\varepsilon}{2(\operatorname{Tr}\mathbf{A}+\mu(\mathbf{R}^d))} \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right) \xrightarrow{P} \frac{\varepsilon}{2},$$

$$(4.43) \qquad \left| \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki}}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \in K(\vec{O}, \beta)\setminus U)\right) - \right.$$
$$\left. - \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki}}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk} \notin U)\right) \right|$$

$$= \left| \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki}}{\|\vec{Y}_{nk}\|^2} \cdot \frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\| > \beta) \right) \right|$$

$$\leqslant \frac{1}{\beta} \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} \right) \xrightarrow{P} \varepsilon_0 < \frac{\varepsilon}{2}.$$

Therefore (4.39)–(4.43) imply (4.33). Now assume $V \in \operatorname{Cont}\nu$. For (4.17) it suffices to prove

$$\sum_{k=1}^{k_n} E_{n,k-1}\big(\vec{Y}_{nk} I(\vec{Y}_{nk} \in V)\big) \xrightarrow{P} \vec{O}, \tag{4.44}$$

or, equivalently,

$$\sum_{k=1}^{k_n} E_{n,k-1}\big(Y_{nki} I(\vec{Y}_{nk} \in V)\big) \xrightarrow{P} 0, \quad 1 \leqslant i \leqslant d. \tag{4.45}$$

By (3.34) and the estimate

$$\left| \sum_{k=1}^{k_n} E_{n,k-1}\big(Y_{nki} I(\vec{Y}_{nk} \in V)\big) \right| \leqslant \left| \sum_{k=1}^{k_n} E_{n,k-1}\big(Y_{nki} I(\vec{X}_{nk} \in V)\big) \right| +$$
$$+ \left| \sum_{k=1}^{k_n} E_{n,k-1}\big(Y_{nki} I(\vec{Y}_{nk} \in V)\big) - \sum_{k=1}^{k_n} E_{n,k-1}\big(Y_{nki} I(\vec{X}_{nk} \in V)\big) \right|$$
$$\leqslant \max_{1 \leqslant k \leqslant k_n} \|A_{nk}\| \sum_{k=1}^{k_n} P_{n,k-1}(\vec{X}_{nk} \notin V) + d(V) \sum_{k=1}^{k_n} P_{n,k-1}(\vec{Y}_{nk} \in V, \vec{X}_{nk} \notin V) +$$
$$+ d(V) \sum_{k=1}^{k_n} P_{n,k-1}(\vec{Y}_{nk} \notin V, \vec{X}_{nk} \in V),$$

it remains to prove that the last two summands converge in probability to zero. Choose a monotonically decreasing sequence ε_l, $l \in N$, such that V_{ε_l}, $V^{\varepsilon_l} \in \operatorname{Cont}\nu$, $l \in N$ (cf. definitions (4.20), (4.21)). For every $l \in N$ one has $\vec{O} \notin \bar{V}^{\varepsilon_l}$; moreover, if ε_l is small enough, then $\vec{O} \notin \bar{V}_{\varepsilon_l}$. Hence for such an ε_l the above-mentioned summands are bounded from above by

$$2 \sum_{k=1}^{k_n} I(\|\vec{A}_{nk}\| > \varepsilon_l) + \sum_{k=1}^{k_n} P_{n,k-1}(\vec{X}_{nk} \in V_{\varepsilon_l}) + \sum_{k=1}^{k_n} P_{n,k-1}(\vec{X}_{nk} \in V^{\varepsilon_l})$$
$$\xrightarrow{P} \nu(V_{\varepsilon_l}) + \nu(V^{\varepsilon_l}).$$

Letting l tend to $+\infty$ and applying Lemma 3.1, we obtain (4.45). Now, by Theorem 4.1 and 3.5 we obtain the required convergence in law to $l(\vec{a}, \mathbf{A}, \nu)$. ■

Let us remark that (4.18) is equivalent to

$$(4.46)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\vec{X}_{nk} I(\vec{Y}_{nk} \in U)\big) \xrightarrow{P} \vec{a} + \int_U \vec{x}\, d\mu - \int_{R^d \setminus U} \frac{\vec{x}}{\|\vec{x}\|^2}\, d\mu,$$

$$U \in O \cap \operatorname{Cont}\nu.$$

V. Convergence in law to $\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa)$ for sums of dependent random vectors with finite variances

Let $\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa) \in \mathscr{P}(\boldsymbol{R}^d)$ be an infinitely divisible distribution with finite variance. In this section we consider the convergence in law to $\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa)$ only for those systems (2.1) in which every $\vec{X}_{nk}$ has finite variance $v(\vec{X}_{nk}) = \mathrm{E}\|\vec{X}_{nk} - \mathrm{E}\vec{X}_{nk}\|^2$, $1 \leqslant k \leqslant k_n$, $n \in N$. This system will be called a $\mathscr{B}(\vec{m}, \mathbf{A}, \varkappa)$*-system* (or shortly a $\mathscr{B}$*-system*) if there exists a double array of σ-fields (3.1) such that every $\vec{X}_{nk}$ is $\mathscr{F}_{nk}$-measurable and the following properties hold:

$$(5.1)\qquad \sum_{k=1}^{k_n} \vec{B}_{nk} \xrightarrow{P} \vec{m},$$

$$(5.2)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(Z_{nki} Z_{nkj} I(\vec{Z}_{nk} \in V)\big) \xrightarrow{P} a_{ij} + \int_V \frac{x_i x_j}{\|\vec{x}\|^2}\, d\varkappa, \qquad 1 \leqslant i < j \leqslant d,$$

for some $V \in O \cap \operatorname{Cont}\varkappa$,

$$(5.3)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\|\vec{Z}_{nk}\|^2 I(\vec{Z}_{nk} \in E)\big) \xrightarrow{P} \varkappa(E)$$

for every bounded $E \in \operatorname{Cont}\varkappa$, $\vec{O} \notin \bar{E}$,

where $\vec{B}_{nk}$ and $\vec{Z}_{nk}$ are defined by (3.51) and (3.52), respectively.

Repeating slightly modified arguments of Chapters III and IV, we can prove the following results:

THEOREM 5.1. *If system* (2.1) *satisfies*

$$(3.55)\qquad \lim_{n\to\infty} P\Big[\sum_{k=1}^{k_n} E_{n,k-1}(\|\vec{Z}_{nk}\|^2) > C\Big] = 0$$

for some $C > 0$, *then* (5.3) *is equivalent to*

$$(5.4)\qquad \sum_{k=1}^{k_n} P_{n,k-1}(\vec{Z}_{nk} \in E) \xrightarrow{P} \int_E \frac{d\varkappa}{\|\vec{x}\|^2}, \qquad E \in \operatorname{Cont}\varkappa,\ \vec{O} \notin \bar{E}. \ \blacksquare$$

THEOREM 5.2. *If the $\mathscr{B}$-system* (2.1) *has additionally the property*

$$(5.5)\qquad \sum_{k=1}^{k_n} E_{n,k-1}(\|\vec{Z}_{nk}\|^2) \leqslant C, \qquad n \in \boldsymbol{N},$$

for some $C > 0$, then it converges in law to $\mathscr{K}(\vec{m}, \mathbf{A}, \varkappa)$ if and only if

$$(5.6)\qquad \lim_{n\to\infty} \mathrm{E}\exp\Big\{i\sum_{k=1}^{k_n}(\vec{t}, \vec{Z}_{nk}) - \sum_{k=1}^{k_n} E_{n,k-1}\,(e^{i(\vec{t}, \vec{Z}_{nk})}-1)\Big\} = 1. \ \blacksquare$$

THEOREM 5.3. *If the $\mathscr{B}$-system* (2.1) *fulfils* (3.55) *for some $C > 0$, then it converges in law to $\mathscr{K}(\vec{m}, \mathbf{A}, k)$ if and only if for this $C > 0$ the system*

$$(5.7)\qquad \vec{X}^C_{nk} := \vec{X}_{nk} I\Big(\sum_{i=1}^{k} E_{n,k-1}(\|\vec{Z}_{nk}\|^2) \leqslant C\Big), \qquad 1 \leqslant k < k_n,\ n \in \boldsymbol{N},$$

satisfies (5.6). $\blacksquare$

THEOREM 5.4. *If the $\mathscr{B}$-system* (2.1) *satisfies* (3.55) *for some $C > 0$ and at least one of the following properties holds*:

$$(5.8)\qquad \exp\Big\{i\Big(\vec{t}, \sum_{k=1}^{k_n}\vec{Z}_{nk}\Big) - \sum_{k=1}^{k_n} E_{n,k-1}(e^{i(\vec{t}, \vec{Z}_{nk}}-1)\Big\} \xrightarrow{P} 1, \qquad \vec{t} \in \boldsymbol{R}^d,$$

$$(5.9)\qquad \sum_{k=1}^{k_n} |E_{n,k-1}(e^{i(\vec{t}, \vec{Z}_{nk})}-1)|^2 \xrightarrow{P} 0, \qquad \vec{t} \in \boldsymbol{R}^d,$$

$$(5.10)\qquad \max_{1\leqslant k\leqslant k_n} P_{n,k-1}(\|\vec{Z}_{nk}\| > \varepsilon) \to 0, \qquad \varepsilon > 0,$$

$$(3.56)\qquad \max_{1\leqslant k\leqslant k_n} E_{n,k-1}(\|\vec{Z}_{nk}\|^2) \xrightarrow{P} 0,$$

then it converges in law to $\mathscr{K}(m, \mathbf{A}, \varkappa)$. $\blacksquare$

THEOREM 5.5. *If the random vectors* (2.1) *with finite variances satisfy* (5.1), (5.3) *for every $E \in \operatorname{Cont}\varkappa$, $\vec{O} \notin \bar{E}$ and*

$$(5.11)\qquad \sum_{k=1}^{k_n} E_{n,k-1}(Z_{nki}Z_{nkj}) \xrightarrow{P} a_{ij} + \int_{\boldsymbol{R}^d} \frac{x_i x_j}{\|\vec{x}\|^2}\,d\varkappa, \qquad 1 \leqslant i, j \leqslant d,$$

with respect to some σ-fields (3.1) *($\vec{X}_{nk}$ is $\mathscr{F}_{n,k}$-measurable) then they form a $\mathscr{B}$-system.* $\blacksquare$

Remark. It can be proved that if the random vectors (2.1) form a $\mathscr{B}$-system, then (5.2) is satisfied for every $V \in O \cap \operatorname{Cont}\varkappa$.

VI. Particular cases of limit distributions

A. The normal distribution in $\boldsymbol{R}^d$. The probability measure $P \in \mathscr{P}(\boldsymbol{R}^d)$ is said to be *normal* if its characteristic function φ_p has the form

$$\varphi_p(t) = \exp\{i(\vec{t}, \vec{a}) - \tfrac{1}{2}(t, \mathbf{A}\vec{t})\}, \qquad \vec{t} \in \boldsymbol{R}^d,$$

where $\vec{a} \in \boldsymbol{R}^d$ and $\mathbf{A}$ is a non-negative definite $d \times d$-matrix.

It is easily seen that $P := \mathscr{N}(\vec{a}, \mathbf{A}) = \mathscr{L}(\vec{a}, \mathbf{A}, 0) = l(\vec{a}, \mathbf{A}, 0) = \mathscr{K}(\vec{a}, \mathbf{A}, 0)$. Hence in this case conditions (3.4)–(3.6) (or (3,4), (4.2) and (4.3)) have the following form:

$$(6.1)\qquad \sum_{k=1}^{k_n} \left\{\vec{A}_{nk} + E_{n,k-1}\left(\frac{\vec{Y}_{nk}}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\| \leqslant \varepsilon)\right)\right\} \xrightarrow{P} \vec{a}, \qquad \varepsilon > 0,$$

$$(6.2)\qquad \sum_{n=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki}\, Y_{nkj}}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\| \leqslant \varepsilon)\right) \xrightarrow{P} a_{ij}; \qquad 1 \leqslant i,\ j \leqslant d,\ \varepsilon > 0.$$

$$(6.3)\qquad \sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Y}_{nk}\| > \varepsilon) \xrightarrow{P} 0, \qquad \varepsilon > 0.$$

One can easily reformulate the theorems of Chapter III for systems (2.1) satisfying (6.1)–(6.3). Similarly we obtain the criteria of asymptotic normality for the random vectors with finite moments replacing $\vec{Y}_{nk}$, $\vec{A}_{nk}$ by $\vec{Z}_{nk}$, $\vec{B}_{nk}$, respectively.

Now assume that system (2.1) is conditionally infinitesimal. In this case (6.3) is equivalent to

$$(6.4)\qquad \sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{X}_{nk}\| > \varepsilon) \xrightarrow{P_n} 0, \qquad \varepsilon > 0.$$

Under assumption (6.3) condition (6.2) is equivalent to

$$(6.5)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(X_{nki} X_{nkj} I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big) -$$
$$- \sum_{k=1}^{k_n} E_{n,k-1}\big(X_{nki} I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big) E_{n,k-1}\big(X_{nkj} I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big) \xrightarrow{P} a_{ij},$$
$$1 \leqslant i,\ j \leqslant d, \qquad \varepsilon > 0.$$

Finally, assuming (6.2) and (6.3), we have the equivalence of (6.1) and

$$(6.6)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\vec{X}_{nk} I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big) \xrightarrow{P} \vec{a}, \qquad \varepsilon > 0.$$

COROLLARY 6.1. *If system* (2.1) *satisfies* (6.4)–(6.6) (*with respect to some σ-fields* (3.1)), *then it is asymptotically normal* $\mathcal{N}(\vec{a}, \mathbf{A})$. ■

The theorems of Chapter V yield

COROLLARY 6.2. *If system* (2.1) *of random vectors with finite variances satisfies* (3.55), (5.1),

$$(6.7)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(Z_{nki}Z_{nkj}I(\|\vec{Z}_{nk}\| \leqslant \varepsilon)\big) \xrightarrow{P} a_{ij}, \quad 1 \leqslant i,\ j \leqslant d, \quad \varepsilon > 0.$$

and

$$(6.8)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\|\vec{Z}_{nk}\|^2 I(\varepsilon < \|\vec{Z}_{nk}\| \leqslant \delta)\big) \xrightarrow{P} 0, \quad \varepsilon > 0,\ \delta > 0,$$

or, equivalently,

$$(6.9)\qquad \sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Z}_{nk}\| > \varepsilon) \xrightarrow{P} 0, \quad \varepsilon > 0,$$

then it is asymptotically normal $\mathcal{N}(\vec{a}, \mathbf{A})$. ■

COROLLARY 6.3. *If the random vectors* (2.1) *with finite variances satisfy* (5.1), (6.7) *and*

$$(6.10)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\|\vec{Z}_{nk}\|^2 I(\|\vec{Z}_{nk}\| > \varepsilon)\big) \xrightarrow{P} 0, \quad \varepsilon > 0,$$

then this system converges in law to $\mathcal{N}(\vec{a}, \mathbf{A})$. *Under the assumption of* (6.10) *condition* (6.7) *is equivalent to*

$$(6.11)\qquad \sum_{k=1}^{k_n} E_{n,k-1}(Z_{nki}Z_{nkj}) \xrightarrow{P} a_{ij}, \quad 1 \leqslant i,\ j \leqslant d.\ ■$$

COROLLARY 6.4. *If system* (2.1) *of random vectors with finite variances satisfies* (5.1), (6.11) *and*

$$(6.12)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\|\vec{X}_{nk}\|^2 I(\|\vec{X}_{nk}\| > \varepsilon)\big) \xrightarrow{P} 0, \quad \varepsilon > 0,$$

then it is asymptotically normal $\mathcal{N}(\vec{a}, \mathbf{A})$. ■

This corollary can be proved in two different ways:

(a) By using the following lemma of Dvoretzky ([3], Lemma 3.3): Let X be an integrable random variable and $\mathcal{F}_0 \subset \mathcal{F}$ a σ-subfield in the probability space and let $\mu := \mathrm{E}(X|\mathcal{F}_0)$. Then we have for every $\varepsilon > 0$

$$4E\big(X^2 I(|X| > \varepsilon)|\mathcal{F}_0\big) \geqslant E\big((X-\mu)^2 I(|X-\mu| > 2\varepsilon)|\mathcal{F}_0\big).$$

(b) By showing that such a system satisfies (6.4)–(6.6). ■

In [3] Dvoretzky proved one theorem on asymptotic normality without restrictions about the existence of moments of random variables. By an analogous method we can prove a result of the following kind:

THEOREM 6.1. *If system* (2.1) *satisfies* (6.4) *and*

$$\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{X_{nki}}{1+X_{nki}^2}\right) \xrightarrow{P} a_i, \qquad 1 \leqslant i \leqslant d, \tag{6.13}$$

$$\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{X_{nki}^2}{1+X_{nki}^2}\right) \xrightarrow{P} a_{ii}, \qquad 1 \leqslant i \leqslant d, \tag{6.14}$$

$$\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{X_{nki}X_{nkj}}{1+X_{nki}^2 X_{nkj}^2}\right) \xrightarrow{P} a_{ij}, \qquad 1 \leqslant i \neq j \leqslant d, \tag{6.15}$$

$$\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{X_{nki}}{1+X_{nki}^2}\right) E_{n,k-1}\left(\frac{X_{nkj}}{1+X_{nkj}^2}\right) \xrightarrow{P} 0, \qquad 1 \leqslant i,\ j \leqslant d, \tag{6.16}$$

then it converges in law to $\mathcal{N}(\vec{a}, \mathbf{A})$. ∎

Let us remark that in [4] we proved for the one-dimensional case that system (2.1) has the above properties if and only if it satisfies (6.4)–(6.6) and, in addition,

$$\sum_{k=1}^{k_n} E_{n,k-1}[X_{nk}^2 I(|X_{nk}| \leqslant \varepsilon)] \xrightarrow{P} a_{11}, \qquad \varepsilon > 0. \tag{6.17}$$

B. The weak law of large numbers. The convergence in law of system (2.1) to the measure $\delta_{\vec{a}}$, concentrated at $\vec{a} \in \boldsymbol{R}^d$, is often called as the weak law of large numbers.

$\delta_{\vec{a}}$ can be considered as the normal distribution with the zero covariance matrix and then we obtain from the above theorems sufficent conditions for the weak law. It is easy to verify that

COROLLARY 6.5. *If the random vectors* (2.1) *satisfy* (6.1) *and*

$$\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}\right) \xrightarrow{P} 0 \tag{6.18}$$

then the weak law of large numbers holds.

If the random vectors (2.1) *with finite variances satisfy* (5.1) *and*

$$\sum_{k=1}^{k_n} E_{n,k-1}(\|\vec{Z}_{nk}\|^2) \xrightarrow{P} 0, \tag{6.19}$$

then the weak law of large numbers holds. ∎

C. The Poisson distribution in $\boldsymbol{R}^d$. Let us fix an arbitrary $\vec{O} \neq \vec{a} \in \boldsymbol{R}^d$ and let $X\colon \Omega \to \boldsymbol{R}$ be a Poisson random variable with the expectation λ. Then $\vec{a}X\colon \Omega \to \boldsymbol{R}^d$, $\vec{a}X := (a_1X, \ldots, a_dX)$, is a random vector with the characteristic function

$$\varphi(\vec{t}) = \int_\Omega a^{i(\vec{t}, \vec{a}X)} dP = \int_\Omega e^{i(\vec{t}, \vec{a})X} dP = \exp[\lambda(e^{i(\vec{t}, \vec{a})} - 1)]$$

$$= \exp\{i(\lambda\vec{a}, \vec{t}) + \lambda(e^{i(\vec{t}, \vec{a})} - 1 - i(\vec{t}, \vec{a}))\}$$

$$= \exp\left\{i(\lambda\vec{a}, \vec{t}) + (e^{i(\vec{t}, \vec{x})} - 1 - i(\vec{t}, \vec{x}))\frac{dk(\vec{x})}{\|\vec{x}\|^2}\right\}, \qquad \vec{t} \in \boldsymbol{R}^d,$$

where $k := \lambda\|\vec{a}\|^2\delta_{\vec{a}}$.

Let us denote its d-dimensional distribution by $\mathscr{P}(\vec{a}, \lambda)$. The expectation of $\mathscr{P}(\vec{a}, \lambda)$ is equal to $\lambda\vec{a}$ and its covariance matrix is $[\lambda a_i a_j]$; moreover,

$$\mathscr{P}(\vec{a}, \lambda) = \mathscr{K}(\lambda\vec{a}, 0, \lambda\|\vec{a}\|^2\delta_{\vec{a}}) = \mathscr{L}\left(\frac{\lambda\vec{a}}{1 + \|\vec{a}\|^2}, 0, \frac{\lambda\|\overleftarrow{a}\|^2}{1 + \|\vec{a}\|^2}\delta_{\vec{a}}\right)$$

$$= l\left(\frac{\lambda\vec{a}}{1 + \|\vec{a}\|^2}, 0, \lambda\delta_{\vec{a}}\right).$$

Let us consider a system (2.1) of random vectors with finite variances. Conditions (5.1)–(5.3) with $\vec{m} = \lambda\vec{a}$, $K = \lambda\|\vec{a}\|^2\delta_{\vec{a}}$, define a $\mathscr{BP}(\vec{a}, \lambda)$-*system*. Assume that this system satisfies (3.55) for some $C > 0$. Then condition (5.3) is equivalent to

$$(6.2) \qquad \sum^{k_n} P_{n,k-1}(\|\vec{Z}_{nk} - \vec{a}\| \leqslant \varepsilon) \xrightarrow{P} \lambda, \qquad 0 < \varepsilon < \|\vec{a}\|,$$

and

$$(6.21) \qquad \sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Z}_{nk} - \vec{a}\| > \varepsilon,\ \|\vec{Z}_{nk}\| > \delta) \xrightarrow{P} 0, \qquad \varepsilon > 0,\ \delta > 0.$$

Condition (5.2) is equivalent to

$$(6.22) \qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\|\vec{Z}_{nk}\|^2 I(\|\vec{Z}_{nk}\| \leqslant \varepsilon)\big) \xrightarrow{P} 0, \qquad 0 < \varepsilon < \|\vec{a}\|,$$

$$(6.23) \qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(Z_{nki}Z_{nkj} I(\vec{Z}_{nk} \in E)\big) \xrightarrow{P} \lambda a_i a_j \delta_{\vec{a}}(E)$$

for every bounded $E \in \operatorname{Cont}\varkappa$, $\vec{O} \notin \bar{E}$.

It can be proved that (5.3) implies (6.23). Thus we have obtained:

COROLLARY 6.6. *Under assumption* (3.55) *the random vectors* (2.1) *with finite variances form the* $\mathscr{BP}(\vec{a}, \lambda)$*-system if and only if they satisfy* (5.1), (6.20)–(6.22). ■

COROLLARY 6.7. *Under assumption* (3.55) *the random vectors* (2.1), *with finite variances form the* $\mathscr{BP}(\vec{a}, \lambda)$*-system if and only if they satisfy* (5.1), (6.20) *and*

$$(6.24)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\|\vec{Z}_{nk}\|^2 I(\|\vec{Z}_{nk}-\vec{a}_{nk}\| > \varepsilon) I(\|\vec{Z}_{nk}\| \leqslant a)\big) \xrightarrow{P} 0, \qquad \varepsilon > 0,\ a > 0.$$

Assume in addition that

$$(6.25)\qquad \sum_{k=1}^{k_n} E_{n,k-1}(Z_{nki}Z_{nkj}) \xrightarrow{P} \lambda a_i a_j, \qquad 1 \leqslant i,\ j \leqslant d.$$

Then the random vectors (2.1) *form a* $\mathscr{BP}(\vec{a}, \lambda)$*-system if and only if* (5.1) *and*

$$(6.26)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\|\vec{Z}_{nk}\|^2 I(\vec{Z}_{nk}-\vec{a}\| > \varepsilon)\big) \xrightarrow{P} 0, \qquad \varepsilon > 0,$$

hold.

Proof. For sufficiently small $\delta > 0$ we have the inequality

$$\sum_{k=1}^{k_n} E_{n,k-1}\big(\|\vec{Z}_{nk}\|^2 I(\|\vec{Z}_{nk}-\vec{a}_{nk}\| > \varepsilon) I(\|\vec{Z}_{nk}\| \leqslant a)\big)$$
$$\leqslant \sum_{k=1}^{k_n} E_{n,k-1}\big(\|\vec{Z}_{nk}\|^2 I(\|\vec{Z}_{nk}\| \leqslant \delta)\big) + a^2 \sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Z}_{nk}-\vec{a}\| > \varepsilon,\ \|\vec{Z}_{nk}\| > \delta)$$

and then (6.21) and (6.22) imply (6.24).

Conversely, (6.24) implies at once (6.22). Moreover, arguments similar to those used in the proof of Theorem 4.1 show that (6.24) implies (6.21).

Assume (6.25). Then

$$(6.27)\qquad \sum_{k=1}^{k_n} E_{n,k-1}(\|\vec{Z}_{nk}\|^2) \xrightarrow{P_n} \lambda \|\vec{a}\|^2$$

and in this case condition (3.55) is satisfied. It can be proved (as Theorem 5.1) that (6.20) is equivalent to

$$(6.28)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\|\vec{Z}_{nk}\|^2 I(\|\vec{Z}_{nk}-\vec{a}\| \leqslant \varepsilon)\big) \xrightarrow{P} \lambda \|\vec{a}\|^2, \qquad 0 < \varepsilon < \|\vec{a}\|,$$

and then, by (6.27), to (6.26).

Conversely, (6.26) implies (6.24). ■

One can easily reformulate the results of Chapter V for the systems satisfying the properties described in the last two corollaries.

Other properties, which quarantee the convergence in law to $P(\vec{a}, \lambda)$ are obtained from the theorems of Chapters III and IV.

The conditions

$$(6.29)\qquad \sum_{k=1}^{k_n}\left\{\vec{A}_{nk}+E_{n,k-1}\left(\frac{\vec{Y}_{nk}}{1+\|\vec{Y}_{nk}\|^2}\right)\right\}\xrightarrow{P}\frac{\lambda\vec{a}}{1+\|\vec{a}\|^2},$$

$$(6.30)\qquad \sum_{k=1}^{k_n}E_{n,k-1}\left(\frac{Y_{nki}Y_{nkj}}{1+\|\vec{Y}_{nk}\|^2}\right)\xrightarrow{P}\frac{\lambda a_i a_j}{1+\|\vec{a}\|^2},\qquad 1\leqslant i,\ j\leqslant d,$$

$$(6.31)\qquad \sum_{k=1}^{k_n}E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}I(\vec{Y}_{nk}\in E)\right)\xrightarrow{P}\begin{cases}\dfrac{\lambda\|\vec{a}\|^2}{1+\|\vec{a}\|^2} & \vec{a}\in\operatorname{Int}E,\ \vec{O}\notin\bar{E},\\ 0 & \vec{a}\notin\bar{E},\ \vec{O}\notin\bar{E}.\end{cases}$$

define a $\mathscr{DP}(\vec{a}, \lambda)$-*system*.

COROLLARY 6.8. *The random vectors* (2.1) *form a* $\mathscr{DP}(\vec{a}, \lambda)$-*system if and only if they satisfy* (6.29), (6.30) *and*

$$(6.32)\qquad \sum_{k=1}^{k_n}E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}I(\|\vec{Y}_{nk}-\vec{a}\|>\varepsilon)\right)\xrightarrow{P}0,\qquad \varepsilon>0.$$

Proof. Condition (6.31) is equivalent to

$$(6.33)\qquad \sum_{k=1}^{k_n}E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}I(\|\vec{Y}_{nk}-\vec{a}\|\leqslant\varepsilon)\right)\xrightarrow{P}\frac{\lambda\|\vec{a}\|^2}{1+\|\vec{a}\|^2},\qquad 0<\varepsilon<\|\vec{a}\|,$$

and

$$(6.34)\qquad \sum_{k=1}^{k_n}E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2}I(\|\vec{Y}_{nk}-\vec{a}\|>\varepsilon)I(\|\vec{Y}_{nk}\|>\delta)\right)\xrightarrow{P}0,\qquad \varepsilon,\delta>0.$$

By (6.30) and (6.33) we obtain (6.32). Conversely, (6.30) and (6.32) give (6.33), (6.32) implies (6.34) and thus we obtain (6.31). ■

By Theorem 5.1 condition (6.31) is equivalent to

$$(6.35)\qquad \sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Y}_{nk}-\vec{a}\| \leqslant \varepsilon) \xrightarrow{P} \lambda, \qquad \varepsilon < \|\vec{a}\|,$$

and

$$(6.36)\qquad \sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{Y}_{nk}-\vec{a}\| > \varepsilon,\ \|\vec{Y}_{nk}\| > \delta) \xrightarrow{P} 0, \qquad \varepsilon, \delta > 0.$$

Observe that (6.30) and (6.33) imply

$$(6.37)\qquad \sum_{k=1} E_{n,k-1}\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\| \leqslant \varepsilon)\right) \xrightarrow{P} 0, \qquad \varepsilon < \|\vec{a}\|,$$

and this implies

$$(6.38)\qquad \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki}Y_{nkj}}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\| \leqslant \varepsilon)\right) \xrightarrow{P} 0, \qquad \varepsilon < \|\vec{a}\|.$$

Conditions (6.35) and (6.36) yield

(6.39)

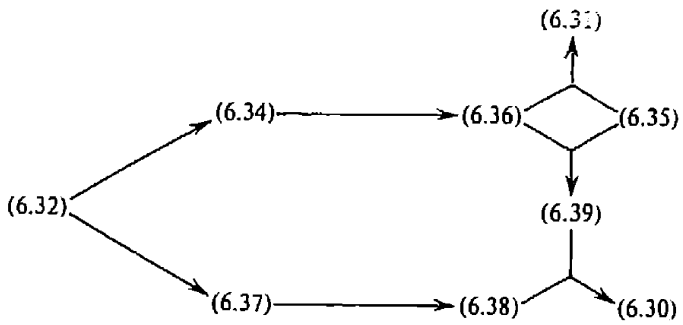
$$\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nki}Y_{nkj}}{1+\|\vec{Y}_{nk}\|^2} I(\|\vec{Y}_{nk}\| > \varepsilon)\right) \xrightarrow{P} \frac{\lambda a_i a_j}{1+\|\vec{a}\|^2}, \qquad 1 \leqslant i, j \leqslant d \qquad \varepsilon < \|\vec{a}\|,$$

and by (6.38) we obtain (6.30).

Thus we have proved

COROLLARY 6.9. *Conditions* (6.29), (6.35)–(6.37) *are equivalent to* (6.29)–(6.31). ∎

The implications



lead to the following

COROLLARY 6.10. *The random vectors* (2.1) *form a* $\mathcal{DP}(\vec{a}, \lambda)$*–system if and only if they satisfy* (6.29), (6.32) *and* (6.35). ∎

Assume that the $\mathscr{DP}(\vec{a}, \lambda)$-system (2.1) is conditionaly infinitesimal. Then (6.31) is equivalent to

$$(6.40) \qquad \sum_{k=1}^{k_n} P_{n,k-1}(\|\vec{X}_{nk} - \vec{a}\| \leqslant \varepsilon) \xrightarrow{P} \lambda, \qquad \varepsilon < \|\vec{a}\|,$$

and

$$(6.41) \qquad \sum_{k=1}^{k_n} P_{n,k-1}(\|X_{nk}\| > \delta,\ \|\vec{X}_{nk} - \vec{a}\| > \varepsilon) \xrightarrow{P} 0; \qquad \varepsilon, \delta > 0.$$

Under assumption (6.31) condition (6.30) is equivalent to

$$(6.42) \qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(X_{nki}X_{nkj}I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big) - \\ - \sum_{k=1}^{k_n} \big\{E_{n,k-1}\big(X_{nki}I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big)E_{n,k-1}\big(X_{nkj}I(\|\vec{X}_{nk}\| \leqslant \varepsilon)\big)\big\} \xrightarrow{P} 0, \\ 1 \leqslant i, j \leqslant d, \qquad \varepsilon < \|\vec{a}\|.$$

Assuming (6.31) and (6.30), condition (6.29) is equivalent to

$$(6.43) \qquad \sum_{k=1}^{k_n} \big\{\vec{A}_{nk} + E_{n,k-1}\big(\vec{Y}_{nk}I(\|\vec{Y}_{nk}\| < \|\vec{a}\| + \varepsilon)\big)\big\} \xrightarrow{P} \lambda\vec{a}, \qquad \varepsilon > 0,$$

and, simultaneously, to

$$(6.44) \qquad \sum_{k=1}^{k_n} \big\{\vec{A}_{nk} + E_{n,k-1}\big(\vec{Y}_{nk}I(\|\vec{Y}_{nk}\| < \varepsilon)\big)\big\} \xrightarrow{P} 0, \qquad \varepsilon < \|\vec{a}\|.$$

Assume $\vec{a} \notin \partial V$ (the set V being used in the definition of $\vec{A}_{nk}$). Then, in the case of $\vec{a} \in \operatorname{Int} V$, (6.29) implies

$$(6.45) \qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\vec{X}_{nk}I(\vec{X}_{nk} \in V)\big) \xrightarrow{P} \lambda\vec{a}$$

and in the contrary case

$$(6.46) \qquad \sum_{k=1}^{k_n} E_{n,k-1}\big(\vec{X}_{nk}I(\vec{X}_{nk} \in V)\big) \xrightarrow{P} \vec{0}.$$

Conversely, if (6.45) is satisfied for some $V \in O$, $\vec{a} \in \operatorname{Int} V$, or (6.46) is satisfied for $V \in O$, $\vec{a} \notin \bar{V}$, then under assumption (6.40)–(6.42) we obtain (6.29) with $\vec{A}_{nk}$, $\vec{Y}_{nk}$ determined by that V. Thus we have proved

COROLLARY 6.11. *If the conditionally infinitesimal system* (2.1) *satisfies* (6.40)–(6.42) *and one of the properties* (6.43)–(6.46), *then it converges in law to* $\mathscr{P}(\vec{a}, \lambda)$. ∎

VII. Another method of conditioning

So far we have considered all the conditional quantities with respect to the row-wise increasing double arrays. Now, for the random vectors (2.1) we define

$$\mathscr{H}_{nk} := \mathscr{B}(\vec{S}_{nk}), \quad 1 \leqslant k \leqslant k_n, \; n \in \boldsymbol{N}, \tag{7.1}$$

where $\vec{S}_{nk} := \sum_{j=1}^{k} \vec{X}_{nj}$ and $\mathscr{B}(\vec{X}) \subset \mathscr{F}$ denotes the σ-field generated by the random vector $\vec{X}$.

We can consider some groups of previously studied conditions with $\mathscr{F}_{nk}$ replaced by $\mathscr{H}_{nk}$, for example, (5.1)–(5.3), and ask if they are sufficient for adequate convergence in law.

Some previous proofs are based essentially on the fact that the σ-fields $\mathscr{F}_{nk}$ form row-wise increasing array. Trying omit this inconvenience, Dvoretzky proved in [3] the following

LEMMA 7.1. *Let $X_1, \ldots, X_n$ be random variables defined on some probability space $(\Omega, \mathscr{F}, P)$. Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ and random variables $\tilde{X}_1, \ldots, \tilde{X}_n$ defined on it such that*

$$P(X_1 \leqslant u) = \tilde{P}(\tilde{X}_1 \leqslant u), \quad u \in \boldsymbol{R}, \tag{7.2}$$

$$P\Big(X_k \leqslant u \,\Big|\, \sum_{i=1}^{k-1} X_i = v\Big) = \tilde{P}\Big(\tilde{X}_k \leqslant u \,\Big|\, \sum_{i=1}^{k-1} \tilde{X}_i = v\Big), \quad u, v \in \boldsymbol{R}, \tag{7.3}$$

the conditional probabilities in (7.3) *are regular and*

$$\tilde{P}(\tilde{X}_k \leqslant u | \tilde{\mathscr{F}}_{k-1}) = \tilde{P}(\tilde{X}_k \leqslant u | \tilde{\mathscr{H}}_{k-1}), \quad u \in \boldsymbol{R}, \; k = 1, 2, \ldots, n, \tag{7.4}$$

where

$$\tilde{\mathscr{F}}_k := \mathscr{B}(\tilde{X}_1, \ldots, \tilde{X}_k), \quad \tilde{\mathscr{H}}_k := \mathscr{B}\Big(\sum_{i=1}^{k} \tilde{X}_i\Big).$$

The generalization of this lemma for the multidimensional case is easy.

Let us apply this procedure to random vectors (2.1). This lemma guarantees that the probability distributions of sums $\sum_{k=1}^{k_n} \vec{X}_{nk}$ and $\sum_{k=1}^{k_n} \vec{\tilde{X}}_{nk}$ are equal. Moreover, the same is true for $\mathrm{E}(\vec{X}_{nk}|\mathscr{H}_{n,k-1})$ and $\mathrm{E}(\vec{\tilde{X}}_{nk}|\tilde{\mathscr{H}}_{n,k-1})$, but it is not clear why $\sum_{k=1}^{k_n} \mathrm{E}(\vec{\tilde{X}}_{nk}|\tilde{\mathscr{H}}_{n,k-1})$ has the same probability distribution as $\sum_{k=1}^{k_n} \mathrm{E}(\vec{X}_{nk}\mathscr{H}_{n,k-1})$, i.e., condition (5.1) is preserved by this procedure. Similar questions arise for conditions (5.2) and (5.3). If the answer is affirmative, then all the theorems of this paper hold true for conditioning (7.1).

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