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On an integro-differential equation

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Introduction. The paper deals with the following integro-differential equation

$$\left[\int_a^b u(t, y)dy\right]u_t'(t, x) = -\lambda(t, x)u(t, x)\,, \tag{1}$$

where a, b and $\lambda(t, x) \geqslant 0$ are given and $u(t, x)$ is the unknown function. An equation of this type was derived by J. Bodziony for the description of the process of screening of granular bodies [1]. The unknown quantity $\int_a^b u(t, y)dy$ has the physical meaning of the total volume of the material contained in the screen at the moment t.

A function $u(t, x)$ defined for $0 \leqslant t < t_0$ and almost every x in (a, b) will be called the *solution* of equation (1) if it satisfies (1) almost everywhere and has the following properties:

1° *$u(t, x)$ is absolutely continuous in t for almost every x,*

2° *$u(t, x)$ is Lebesgue integrable with respect to x,*

3° *$\left[\int_a^b u(t, y)dy\right]^{-1}$ is integrable with respect to t in every compact interval contained in $0 \leqslant t < t_0$* (¹).

The following initial value problem for equation (1) is considered (see [1]).

PROBLEM P_1. *To find a solution of* (1) *satisfying the initial condition*

$$u(0, x) = v(x) \quad \textit{for almost every } x\,, \tag{2}$$

where $v(x)$ is a given function in (a, b).

The purpose of this note is to prove, under suitable assumptions on the coefficient $\lambda(t, x)$ and on the initial function $v(x)$, the uniqueness, the existence and continuous dependence on λ and v of the solution of problem P_1 and to discuss some of its important properties. The method

(¹) We restrict ourselves to solutions satisfying 3° since without this requirement the solution of problem P_1 would not be uniquely determined (see § 2, Example 1).

used is that of successive approximations. In the case when $\lambda(t,x)$ does not depend on t and has at most one point of discontinuity the existence and uniqueness of the solution as well as some of its properties were proved by J. Bodziony and S. Gołąb [2]. Their method is quite different from ours and cannot be applied if λ depends on t.

§ 1. Reduction of problem P_1.

ASSUMPTION H_1. *We suppose that $v(x)$ is non-negative, Lebesgue integrable in (a,b) and*

$$\int_a^b v(x)dx > 0 . \tag{3}$$

ASSUMPTION H_2. *We assume that $\lambda(t,x)$ is defined and non-negative almost everywhere in $0<t$, $a<x<b$. In every finite rectangle $0<t<t_1$, $a<x<b$ it is bounded and Lebesgue integrable (as function of two variables).*

PROPOSITION 1. *Under the assumption H_2 problem P_1 is equivalent to the following one.*

PROBLEM P_2. *To find a solution of the equation*

$$u(t,x) = v(x)\exp\left\{-\int_0^t \left[\lambda(s,x)\Big/\int_a^b u(s,y)dy\right]ds\right\}, \tag{4}$$

which is defined for $0\leqslant t<t_0$ and almost every x in (a,b) and satisfies 2° and 3°.

Proof. Let $u(t,x)$ be a solution of problem P_1. Then, for almost every fixed x in (a,b), $u(t,x)$, as a function of t satisfies almost everywhere in $0<t<t_0$ the linear differential equation

$$u_t' = a(t,x)u \tag{5}$$

with the initial condition (2), where

$$a(t,x) = -\lambda(t,x)\Big/\int_a^b u(t,y)dy$$

is, by 2°, 3° (see the Introduction) and by assumption H_2, integrable in t for almost every x. Therefore, owing to 1° (see the Introduction) $u(t,x)$ satisfies equation (4) for almost every x and hence is a solution of problem P_2.

Now, suppose $u(t,x)$ to be a solution of problem P_2. From (4) it follows that 1° holds true and that equation (1) is satisfied almost everywhere in $0<t<t_0$, $a<x<b$. Hence $u(t,x)$ is a solution of problem P_1.

PROPOSITION 2. *Under the assumptions* H_1 *and* H_2 *we have for the solution* $u(t, x)$ *of the problem* P_2

4° $u(t, x) \geqslant 0, \int_a^b u(t, y)dy > 0,$

5° $\int_a^b u(t, y)dy$ *is non-increasing with respect to* t,

6° $\int_a^b u(t, y)dy$ *is absolutely continuous in every compact interval contained in* $0 \leqslant t < t_0$.

Proof. From assumption H_1 and from equation (4) follows 4°. Therefore, $u(t, x)$ satisfying almost everywhere equation (1), we have by 4° and assumption H_2 $u_t'(t, x) \leqslant 0$ almost everywhere. Hence, by 1° $u(t, x)$ is for almost every x a non-increasing function of t and consequently the same is true for $\int_a^b u(t, y)dy$. From this and 4° we get for any $t_1 < t_0$

$$\int_a^b u(t, y)dy \geqslant \int_a^b u(t_1, y)dy = K_1 > 0 \quad \text{for} \quad 0 \leqslant t \leqslant t_1 . \tag{6}$$

By assumption H_2 we have for some constant A_1

$$0 \leqslant \lambda(t, x) \leqslant A_1 \tag{7}$$

almost everywhere in $0 < t < t_1$, $a < x < b$. Put

$$\int_a^b v(x)dx = A . \tag{8}$$

Now, from (4), (6) and (7) it follows that for $t', t'' \in [0, t_1]$

$$\begin{aligned}
&\Big|\int_a^b u(t', x)dx - \int_a^b u(t'', x)dx\Big| \\
&= \Big|\int_a^b v(x)\Big[\exp\Big\{-\int_0^{t'} \Big[\lambda(s, x)\Big/\int_a^b u(s, y)dy\Big]ds\Big\} - \\
&\qquad\qquad -\exp\Big\{-\int_0^{t''} \Big[\lambda(s, x)\Big/\int_a^b u(s, y)dy\Big]ds\Big\}\Big]dx\Big| \\
&\leqslant \int_a^b v(x)\Big|\int_{t'}^{t''} \Big[\lambda(s, x)\Big/\int_a^b u(s, y)dy\Big]ds\Big|dx \\
&\leqslant \frac{AA_1}{K_1}|t' - t''| ,
\end{aligned}$$

which completes the proof of 6°.

§ 2. Uniqueness of the solution of problem P_2.

THEOREM 1. *Under the assumptions* H_1 *and* H_2 *two solutions of problem* P_2 *in* $0 \leqslant t < t_0$, $a < x < b$ *are equal for almost every* x.

Proof. Let $u_1(t, x)$ and $u_2(t, x)$ be two solutions of problem P_2. It is sufficient to prove that for any $0 < t_1 < t_0$ the statement of Theorem 1 holds true in $0 \leqslant t \leqslant t_1$, $a < x < b$. By the same argument which led us to conclusion (6) there is a constant K such that

$$\begin{aligned} &\int_a^b u_1(t, y)\,dy \geqslant K > 0\,, \\ &\int_a^b u_2(t, y)\,dy \geqslant K > 0\,, \end{aligned} \qquad \text{for} \quad 0 \leqslant t \leqslant t_1\,. \tag{9}$$

Since u_1 and u_2 satisfy equation (4), it follows that for almost every x

$$|u_1(t, x) - u_2(t, x)| = v(x)\Big|\exp\Big\{-\int_0^t \Big[\lambda(s, x)\Big/\int_a^b u_1(s, y)\,dy\Big]ds\Big\} - \exp\Big\{-\int_0^t \Big[\lambda(s, x)\Big/\int_a^b u_2(t, y)\,dy\Big]ds\Big\}\Big|\,.$$

Hence, by (7) and (9)

$$\begin{aligned} &|u_1(t, x) - u_2(t, x)| \\ &\leqslant v(x)\int_0^t \Big[\lambda(s, x)\int_a^b |u_1(s, y) - u_2(s, y)|\,dy\Big/\int_a^b u_1(s, y)\,dy\int_a^b u_2(s, y)\,dy\Big]ds \\ &\leqslant v(x)\frac{\Lambda_1}{K^2}\int_0^t \Big[\int_a^b |u_1(s, y) - u_2(s, y)|\,dy\Big]ds\,. \end{aligned}$$

From the last inequality we get by (8)

$$\int_a^b |u_1(t, x) - u_2(t, x)|\,dx \leqslant \frac{A\Lambda_1}{K^2}\int_0^t \Big[\int_a^b |u_1(s, y) - u_2(s, y)|\,dy\Big]ds\,. \tag{10}$$

Hence, it follows by a standard argument that

$$\int_a^b |u_1(t, x) - u_2(t, x)|\,dx = 0 \qquad \text{for} \qquad 0 \leqslant t \leqslant t_1\,, \tag{11}$$

whence $u_1(t, x) = u_2(t, x)$ for almost every x.

Remark 1. By 4° and 6° (see Proposition 2) it is evident that if $v(x)$ is continuous in a certain subinterval $\Delta \subset (a, b)$ and $\lambda(t, x)$ is continuous for $0 \leqslant t < t_0$, $x \in \Delta$, then the solution of problem P_2 is continuous in the same set. In that case, by Theorem 1, any two solutions of problem P_2 are identically equal for $0 \leqslant t < t_0$, $x \in \Delta$.

EXAMPLE 1. The following example shows that if we drop the requirement that the solution of problem P_1 satisfy 3° (see Introduction), then it may happen not to be uniquely determined.

Let $\lambda(t,x) = 1$, $a = 0$, $b = 1$, $v(x) = 1$. Then

$$u_1(t,x) = 1-t \quad \text{for} \quad t \geqslant 0\,,$$

and

$$u_2(t,x) = \begin{cases} 1-t & \text{for} \quad 0 \leqslant t \leqslant 1\,, \\ 0 & \text{for} \quad t > 1 \end{cases}$$

are two different solutions of equation (1) satisfying the initial condition (2). Of course, in accordance with Theorem 1, they are equal in the interval $0 \leqslant t < 1$.

§ 3. Existence of the solution of problem P_2.

THEOREM 2. *Under assumptions* H_1 *and* H_2, *for* t_0 *sufficiently small, there exists a solution of problem* P_2 *in*

$$(12) \qquad 0 \leqslant t < t_0\,, \quad a < x < b\,.$$

Proof. Choose $t_0 > 0$ so small that

$$(13) \qquad \exp\left(-\frac{2t_0 A_0}{A}\right) > \frac{1}{2}\,, \quad t_0 < 1\,,$$

where A is defined by (8) and A_0 is the upper bound of $\lambda(t,x)$ in $0 \leqslant t \leqslant 1$, $a < x < b$. We now define the sequence of successive approximations

$$(14) \quad u_0(t,x) = v(x)\,, \quad u_{\nu+1}(t,x) = v(x)\exp\left\{-\int_0^t \left[\lambda(s,x)\Big/\int_a^b u_\nu(s,y)\,dy\right]ds\right\}.$$

First we prove by induction that formulas (14) define in (12) for almost every x functions satisfying 1°, 2° (see Introduction) and that

$$(15) \qquad \int_a^b u_\nu(t,y)\,dy > \frac{A}{2} \quad \text{and is continuous in } 0 \leqslant t < t_0\,,$$

$$(16) \qquad u_\nu(t,x) \leqslant v(x)\,.$$

This is obvious for $\nu = 0$ by assumption H_1 and (8). Suppose it to be true for $\nu = k$. Then, $\int_a^b u_k(t,y)\,dy$ being continuous, (15) holding true for $\nu = k$ and $\lambda(t,x)$ being integrable and bounded in (12), the function $\lambda(t,x)/\int_a^b u_k(t,y)\,dy$ is also integrable and bounded in (12). Hence it follows that

$$\exp\left\{-\int_0^t \left[\lambda(s,x)\Big/\int_a^b u_k(s,y)\,dy\right]ds\right\}$$

is defined in (12) for almost every x and is integrable with respect to x; consequently, the same is true for $u_{k+1}(t, x)$ and inequalities (15) and (16) hold true for $\nu = k+1$ by (13) and (14). Since 1° is obviously satisfied by u_{k+1}, it remains to prove that $\int_a^b u_{k+1}(t, y)dy$ is continuous. Now, this follows by Lebesgue's theorem from the continuity of $u_{k+1}(t, x)$ in t and from (16). Next we will prove that sequence (14) is convergent for all t and for almost every x in (12). Indeed, by (14) and (15) we have for $\nu \geqslant 1$

$$\begin{aligned}
&|u_{\nu+1}(t, x) - u_\nu(t, x)| \\
&\leqslant v(x)\Big|\exp\Big\{-\int_0^t \Big[\lambda(s, x)\Big/\int_a^b u_\nu(s, y)dy\Big]ds\Big\} - \\
&\qquad\qquad -\exp\Big\{-\int_0^t \Big[\lambda(s, x)\Big/\int_a^b u_{\nu-1}(s, y)dy\Big]ds\Big\}\Big| \\
&\leqslant v(x)\Big|\int_0^t \lambda(s, x)\Big[\int_a^b \big(u_\nu(s, y) - u_{\nu-1}(s, y)\big)dy\Big/\int_a^b u_\nu(s, y)dy\int_a^b u_{\nu-1}(s, y)dy\Big]ds \\
&\leqslant \frac{4A_0}{A^2}v(x)\int_0^t\Big[\int_a^b |u_\nu(s, y) - u_{\nu-1}(s, y)|dy\Big]ds\,.
\end{aligned}$$

Hence, using the obvious inequality

$$|u_1(t, x) - u_0(t, x)| \leqslant v(x)$$

we get by induction

$$|u_{\nu+1}(t, x) - u_\nu(t, x)| \leqslant v(x)\left[\frac{4A_0 t}{A}\right]^\nu \frac{1}{\nu!}, \qquad \nu = 0, 1, \ldots$$

The last inequalities imply our assertion. By Lebesgue's theorem and by (16) we infer that

$$u(t, x) = \lim_{\nu\to+\infty} u_\nu(t, x)$$

is integrable with respect to x and

$$\lim_{\nu\to+\infty}\int_a^b u_\nu(t, y)dy = \int_a^b u(t, y)dy \qquad \text{for} \qquad 0 \leqslant t < t_0\,.$$

By (15) we have moreover

$$\int_a^b u(t, y)dy \geqslant \frac{A}{2} > 0\,.$$

Therefore

$$\lim_{\nu\to+\infty} \lambda(t, x)\Big/\int_a^b u_\nu(t, y)dy = \lambda(t, x)\Big/\int_a^b u(t, y)dy$$

almost everywhere in (12) and since by (15)

$$\Big|\lambda(t, x)\Big/\int_a^b u_\nu(t, y)dy\Big| \leqslant 2\lambda(t, x)/A ,$$

we infer by assumption H_2 and by Lebesgue's theorem that

$$\lim_{\nu\to+\infty} \int_0^t \Big[\lambda(s, x)\Big/\int_a^b u_\nu(s, y)dy\Big]ds = \int_0^t \Big[\lambda(s, x)\Big/\int_a^b u(t, y)dy\Big]ds$$

for all $t \in [0, t_0)$ and almost every $x \in (a, b)$. Hence, letting ν tend to ∞ in (14) we find that $u(t, x)$ satisfies equation (4) in (12) for almost every x and consequently is a solution of problem P_2 in (12).

Remark 2. By Remark 1, if $v(x)$ is continuous in a certain subinterval $\Delta \subset (a, b)$ and $\lambda(t, x)$ is continuous for $0 \leqslant t < t_0$, $x \in \Delta$, then the solution $u(t, x)$, just obtained, is continuous in the same set.

Let us denote by $T > 0$ the largest positive number such that the solution of problem P_2 exists in

$$0 \leqslant t < T , \quad a < x < b . \tag{17}$$

By Theorems 1 and 2 such a number is uniquely determined and obviously there are two possible cases: I) $T < +\infty$, II) $T = +\infty$.

§ 4. Some properties of the solution of problem P_2. We suppose throughout this section that assumptions H_1 and H_2 hold true.

THEOREM 3. *For the solution $u(t, x)$ of problem P_2 the limits*

$$\lim_{t\to T} u(t, x) , \quad \lim_{t\to T} \int_a^b u(t, y)dy \tag{18}$$

exist and are finite and non-negative. If we put

$$v^*(x) = \lim_{t\to T} u(t, x) , \tag{19}$$

then $v^*(x)$ is integrable and

$$\int_a^b v^*(x)dx = \lim_{t\to T} \int_a^b u(t, y)dy . \tag{20}$$

Proof. Since $u(t, x)$ and $\int_a^b u(t, y)dy$ are non-negative and non-increasing with respect to t (see § 1), limits (19) exist and are finite and non-negative. Now, from the obvious inequality

$$0 \leqslant u(t, x) \leqslant v(x) \tag{21}$$

and by Lebesgue's theorem, we conclude that $v^*(x)$ is integrable and that (20) holds true.

THEOREM 4. *For the solution $u(t, x)$ of problem P_2 we have*

$$\frac{d}{dt}\int_a^b u(t, y)dy = \int_a^b u_t'(t, y)dy \quad \text{for almost every } t \in [0, T). \tag{22}$$

Proof. Since $u(t, x)$ satisfies almost everywhere equation (1) (see Proposition 1, § 1), it follows by (6), (21) and H_2 that

$$|u_t'(t, x)| \leqslant \Lambda_1 u(t, x) \Big/ \int_a^b u(t, y)dy \leqslant \frac{\Lambda_1}{K_1} v(x)$$

almost everywhere in every finite rectangle $0 \leqslant t \leqslant t_1$, $a < x < b$ contained in (17) with suitable constants Λ_1 and K_1. The right-hand member of the last inequality being integrable, relation (22) is an immediate consequence of Lebesgue's theorem and of the absolute continuity of $u(t, x)$ in t.

§ 5. Discussion of cases I and II.

We assume in this section H_1 and H_2 to hold true.

THEOREM 5. *In case $T < +\infty$ we have*

$$\lim_{t\to T}\int_a^b u(t, y)dy = 0. \tag{23}$$

Proof. By Theorem 3 the limit (23) exists and is non-negative. Now, suppose that

$$\lim_{t\to T}\int_a^b u(t, y)dy > 0.$$

Then, from (4) and (20) it would follow that

$$v^*(x) = \lim_{t\to T} u(t, x) = v(x)\exp\left\{-\int_0^T \left[\lambda(s, x) \Big/ \int_a^b u(s, y)dy\right] ds\right\} \tag{24}$$

and

$$\int_a^b v^*(x)dx = \lim_{t\to T}\int_a^b u(t, y)dy > 0. \tag{25}$$

In virtue of (25) we might apply Theorem 2, replacing $v(x)$ by $v^*(x)$ and 0 by T, and hence for a suitable $T_0 > T$ there would exist in $T \leqslant t < T_0$, $a < x < b$ a solution $u^*(t, x)$ of the problem P_2 for the equation

$$(26) \qquad u^*(t, x) = v^*(x)\exp\Big\{-\int_T^t \Big[\lambda(s, x)\Big/\int_a^b u^*(s, y)dy\Big]ds\Big\}.$$

Writing

$$(27) \qquad u^{**}(t, x) = \begin{cases} u(t, x) & \text{in} \quad 0 \leqslant t < T, \ a < x < b, \\ u^*(t, x) & \text{in} \quad T \leqslant t < T_0, \ a < x < b, \end{cases}$$

we would have by (4), (24), and (27)

$$u^{**}(t, x) = v(x)\exp\Big\{-\int_0^t \Big[\lambda(s, x)\Big/\int_a^b u^{**}(s, y)dy\Big]ds\Big\}$$

almost everywhere in

$$(28) \qquad 0 \leqslant t < T_0, \quad a < x < b.$$

Therefore, $u^{**}(t, x)$ would be the solution of problem P_2 in (28) and thus we would obtain a contradiction because of the definition of T and of the fact that $T_0 > T$.

THEOREM 6. *If for almost every t and almost every x in a certain subset $E \subset (a, b)$ of positive measure we have $\lambda(t, x) = 0$ and $\int_E v(x)dx > 0$, then $T = +\infty$.*

Proof. Under the assumptions of our theorem equation (4) gives $u(t, x) = v(x)$ for almost every $x \in E$ and for all t. Hence, by 4° (see § 1)

$$\int_a^b u(t, y)dy \geqslant \int_E u(t, y)dy = \int_E v(y)dy > 0 \qquad \text{for} \quad 0 \leqslant t < T,$$

and consequently by Theorem 5 we have $T = +\infty$.

The following example shows that the conditions $\lambda(t, x) > 0$ and $v(x) > 0$ everywhere, do not imply $T < +\infty$.

EXAMPLE 2. Let $\lambda(t, x) = \exp(-t)$, $v(x) = 1$, $a = 0$, $b = 1$. Then $u(t, x) = \exp(-t)$ is the solution of problem P_2 defined in $0 \leqslant t$, $0 \leqslant x \leqslant 1$. Here we have $T = +\infty$.

However, the next theorem shows that $T < +\infty$ whenever $\lambda(t, x)$ has a positive lower bound.

THEOREM 7. *If almost everywhere in $0 < t$, $a < x < b$*

$$(29) \qquad \lambda(t, x) \geqslant \lambda_0 > 0,$$

then $T < +\infty$.

Proof. From (1), (29) and $u(t,x) \geqslant 0$ (see § 1) it follows that almost everywhere

$$\Big(\int_a^b u(t,y)\,dy\Big)\,u_t'(t,x) \leqslant -\lambda_0 u(t,x)\,,$$

whence integrating with respect to x and dividing by $\int_a^b u(t,y)\,dy$ we get for almost every t

$$\int_a^b u_t'(t,y)\,dy \leqslant -\lambda_0\,.$$

Hence, by Theorem 4 we have for almost every $t \in (0, T)$

$$\frac{d}{dt}\int_a^b u(t,y)\,dy \leqslant -\lambda_0$$

and, putting $a_0 = \int_a^b u(0,y)\,dy$, we get by 6° (see § 1)

$$\int_a^b u(t,y)\,dy \leqslant a_0 - \lambda_0 t$$

for all $t \in [0, T)$. Put $t_0 = a_0/\lambda_0$; then, if $t_0 < T$, we would have by the last inequality

$$\int_a^b u(t_0,y)\,dy \leqslant 0\,,$$

which is impossible by 4°. Hence, we must have $T \leqslant t_0$ and consequently T is finite.

§ 6. Continuous dependence on the initial values and on λ of the solution of problem P_2.

THEOREM 8. *Let $v(x)$, $v_n(x)$ and $\lambda(t,x)$, $\lambda_n(t,x)$ $(n = 1, 2, \ldots)$ satisfy assumptions H_1 and H_2 respectively. Denote by $u(t,x)$ the solution of problem P_2 corresponding to the initial function $v(x)$ and to the coefficient $\lambda(t,x)$ and let*

$$(30) \qquad 0 \leqslant t < T\,, \quad a < x < b$$

be its largest existence domain. Suppose that

$$(31) \qquad \lim_{n\to\infty}\int_a^b |v_n(x) - v(x)|\,dx = 0\,,$$

$$(32) \qquad \lim_{n\to\infty}\lambda_n(t,x) = \lambda(t,x) \quad \textit{almost everywhere in } (30)\,,$$

and that

(33) *$\lambda_n(t,x)$, $\lambda(t,x)$ have a common upper bound in every finite rectangle contained in* (30).

Under these assumptions, for every $0 < t_1 < T$ the solutions $u_n(t, x)$ of problem P_2, corresponding to the initial functions $v_n(x)$ and coefficients $\lambda_n(t, x)$, exist in $0 \leqslant t \leqslant t_1$, $a < x < b$ for n sufficiently large, and

$$\lim_{n\to\infty} \int_a^b |u_n(t, x) - u(t, x)|\,dx = 0 \tag{34}$$

uniformly with respect to $t \in [0, t_1]$.

Proof. By (31) we have for all indices n sufficiently large

$$2A + 1 \geqslant \int_a^b v_n(x)\,dx \geqslant A > 0\,, \tag{35}$$

where

$$2A = \int_a^b v(x)\,dx\,.$$

By (32) and (33) and by Lebesgue's theorem we obtain

$$\lim_{n\to\infty} \int_a^b\int_0^t |\lambda_n(s, x) - \lambda(s, x)|\,v(x)\,ds\,dx = 0 \quad \text{for} \quad 0 \leqslant t < T\,. \tag{36}$$

Let t_0 satisfy the inequality

$$0 < t_0 < \min\left(1, -\frac{A}{2\Lambda_0}\ln\frac{1}{2}\right),$$

where Λ_0 is the common upper bound of $\lambda(t, x)$ and $\lambda_n(t, x)$ in $0 \leqslant t \leqslant 1$, $a < x < b$. Then, from the proof of Theorem 2 it follows by (35) that $u(t, x)$ and $u_n(t, x)$, for n sufficiently large, exist in

$$0 \leqslant t \leqslant t_0\,, \quad a < x < b\,, \tag{37}$$

and satisfy

$$\int_a^b u(t, y)\,dy \geqslant \frac{A}{2}\,, \quad \int_a^b u_n(t, y)\,dy \geqslant \frac{A}{2}\,. \tag{38}$$

We will now prove that (34) holds for $t \in [0, t_0]$. Indeed, from equation (4) it follows that for almost every x in (37)

$$|u_n(t, x) - u(t, x)| = \Big|v_n(x)\exp\Big\{-\int_0^t \Big[\lambda_n(s, x)\Big/\int_a^b u_n(s, y)\,dy\Big]ds\Big\} - $$
$$- v(x)\exp\Big\{-\int_0^t \Big[\lambda(s, x)\Big/\int_a^b u(s, y)\,dy\Big]ds\Big\}\Big|\,.$$

Hence, by (38) and by the definition of A and Λ_0 we obtain

$$|u_n(t,x)-u(t,x)|$$

$$\leqslant |v_n(x)-v(x)|\exp\Big\{-\int_0^t\Big[\lambda_n(s,x)\Big/\int_a^b u(s,y)\,dy\Big]ds\Big\}+$$

$$+v_n(x)\Big|\exp\Big\{-\int_0^t\Big[\lambda_n(s,x)\Big/\int_a^b u_n(s,y)\,dy\Big]ds\Big\}-$$

$$-\exp\Big\{-\int_0^t\Big[\lambda_n(s,x)\Big/\int_a^b u(s,y)\,dy\Big]ds\Big\}\Big|+$$

$$+v(x)\Big|\exp\Big\{-\int_0^t\Big[\lambda_n(s,x)\Big/\int_a^b u(s,y)\,dy\Big]ds\Big\}-$$

$$-\exp\Big\{-\int_0^t\Big[\lambda(s,x)\Big/\int_a^b u(s,y)\,dy\Big]ds\Big\}\Big|$$

$$\leqslant |v_n(x)-v(x)|+$$

$$+v_n(x)\int_0^t\Big[\lambda_n(s,x)\int_a^b|u_n(s,y)-u(s,y)|\,dy\Big/\int_a^b u_n(s,y)\,dy\int_a^b u(s,y)\,dy\Big]ds+$$

$$+v(x)\int_0^t\Big[|\lambda_n(s,x)-\lambda(s,x)|\Big/\int_a^b u(s,y)\,dy\Big]ds$$

$$\leqslant |v_n(x)-v(x)|+v_n(x)\frac{4\Lambda_0}{A^2}\int_0^t\int_a^b|u_n(s,y)-u(s,y)|\,dy\,ds+$$

$$+\frac{2}{A}v(x)\int_0^t|\lambda_n(s,x)-\lambda(s,x)|\,ds\,.$$

Putting

$$A_n=\int_a^b|v_n(x)-v(x)|\,dx\,,\qquad B_n=\int_a^b\int_0^{t_0}|\lambda_n(s,x)-\lambda(s,x)|\,v(x)\,ds\,dx\,,$$

we get from the last inequality by (35)

$$(39)\qquad \int_a^b|u_n(t,x)-u(t,x)|\,dx$$

$$\leqslant A_n+\frac{2}{A}B_n+\frac{(2A+1)4\Lambda_0}{A^2}\int_0^t\Big[\int_a^b|u_n(s,y)-u(s,y)|\,dy\Big]ds\,.$$

It follows from (39) that ([3]) putting $C=(2A+1)4\Lambda_0/A^2$

$$\int_a^b|u_n(t,x)-u(t,x)|\,dx\leqslant\Big(A_n+\frac{2}{A}B_n\Big)\exp(Ct)\qquad\text{for}\qquad 0\leqslant t\leqslant t_0\,.$$

Hence, by (31) and (36) we get (34) uniformly in $0 \leqslant t \leqslant t_0$. Thus we have proved that the set Z of numbers $t_0 > 0$, such that, for indices n sufficiently large, $u_n(t, x)$ exist in (37) and (34) holds uniformly in $0 \leqslant t \leqslant t_0$, is non-void. Now, denote by t^* the least upper bound of the set Z (or $+\infty$ if Z is unbounded). To complete the proof of our theorem it is sufficient to show that $t^* > t_1$. For this purpose, suppose that the contrary $t^* \leqslant t_1$ holds true. Then $t^* < T$ and $u(t, x)$ exists in $0 \leqslant t \leqslant t^*$, $a < x < b$. Therefore, we have by 5° (see Proposition 2)

$$\int_a^b u(t, x)dx \geqslant 2A^* > 0 \quad \text{for} \quad 0 \leqslant t \leqslant t^*, \tag{40}$$

where $2A^* = \int_a^b u(t^*, x)dx$. Choose $0 < t_2 < t^*$ so that

$$t^* - t_2 < \min\left(1, -\frac{A^*}{2\Lambda^*}\ln\frac{1}{2}\right), \tag{41}$$

where Λ^* is the common upper bound of $\lambda(t, x)$ and $\lambda_n(t, x)$ in $0 \leqslant t \leqslant t_2+1$, $a < x < b$. Since $t_2 < t^*$, $t_2 \in Z$ and the solutions $u_n(t, x)$, for indices n sufficiently large, exist in $0 \leqslant t \leqslant t_2$, $a < x < b$ and (34) holds true for $0 \leqslant t \leqslant t_2$. Hence, by (40) we have

$$\int_a^b u_n(t_2, x)dx \geqslant A^* > 0 \tag{42}$$

for n sufficiently large. Now choose t_0^* so that

$$t^* - t_2 < t_0^* < \min\left(1, -\frac{A^*}{2\Lambda^*}\ln\frac{1}{2}\right). \tag{43}$$

From (42) and (43) it follows, by the argument used in the first part of our proof (replacing $v_n(x)$ by $v_n^*(x) = u_n(t_2, x)$, 0 by t_2, A by A^* and Λ_0 by Λ^*) and by the argument applied in the proof of Theorem 5, that for n sufficiently large $u_n(t, x)$ exist in $0 \leqslant t \leqslant t_2 + t_0^*$, $a < x < b$ and satisfy (34) in $0 \leqslant t \leqslant t_2 + t_0^*$. Thus we have obtained a contradiction because of the definition of t^* since by (43) $t^* < t_2 + t_0^*$. This completes the proof.

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