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COMBINATIONS OF RATIONAL SINGULARITIES ON PLANE SEXTIC CURVES WITH THE SUM OF MILNOR NUMBERS LESS THAN SIXTEEN

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0. Introduction

Recently I am interested in combinations of singularities on projective surfaces and curves. I believe that there exists a systematic simple law describing the appearance of singularities on them (cf. Urabe [12], [13], [14]).

In this paper I would like to explain the following result on plane sextic curves, which is obtained by checking case by case.

MAIN THEOREM. *Let $B \subset P^2$ denote a reduced (possibly reducible) plane sextic curve over the complex number field C. Let $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$ be a combination of rational singularities. Assume that the sum of Milnor numbers $\sum a_k k + \sum b_l l + \sum c_m m$ is less than sixteen. Then there exists a curve B with G if and only if $G \neq A_4 + 11A_1$, $2A_2 + 11A_1$, $A_2 + 13A_1$.*

The rational singularity is defined by the following local defining equation (cf. Durfee [6]).

$$A_k\colon\ x^2 + y^{k+1} = 0 \quad (k \geqslant 1),$$

$$D_l\colon\ x^2 y + y^{l-1} = 0 \quad (l \geqslant 4),$$

$$E_6\colon\ x^3 + y^4 = 0, \qquad E_7\colon\ x^3 + xy^3 = 0, \qquad E_8\colon\ x^3 + y^5 = 0.$$

We use (1) the surjectivity of the period mapping for K3 surfaces and (2) the lattice embedding theory due to Nikulin (Nikulin [8]) as the main tools for the proof. But they are *not* enough to complete the proof. We need several *devices* to complete it. In particular the notion of elementary transformation of Dynkin graphs, which I proposed to describe singularities on plane quartic curves (cf. Urabe [12]), is effective even for sextic curves. It gives a sufficient condition for a combination G to exist on B (see Section 4).

Here I give a remark about the case where the sum of Milnor numbers is greater than fifteen. If there is a sextic curve B with a combination G, then the sum is less than twenty. Even in this case it is possible to make the list of possible combination of rational singularities on B, if we use the original exact version of the Nikulin's lattice embedding theorem. But it is very tiresome to count up isotropic subgroups to determine overlattices of the given root lattice for complecated cases. Therefore I think it is not very interesting to carry out the calculation when $\sum \mu_x > 15$. I think rather that we should try to find out more conceptual proof.

1. Theory of K3 surfaces

We assume that every variety is defined over the complex number field C.

Let $B \subset P^2$ be a reduced sextic curve in the 2-dimensional projective space. We assume that B has only rational singularities. Let $F(z_0, z_1, z_2)$ be the homogeneous defining polynomial of B. We give weight one to each variable z_0, z_1, z_2. Let z_3 be another variable with weight 3. By X we denote the irreducible surface defined by $z_3^2 - F(z_0, z_1, z_2) = 0$ in the weighted projective space $P(1, 1, 1, 3)$ (cf. Dolgachev [5]). The projection π: $P(1, 1, 1, 3) - \{(0, 0, 0, 1)\} \to P^2; (z_0, z_1, z_2, z_3) \to (z_0, z_1, z_2)$ defines a surjective morphism π: $X \to P^2$ of degree 2. The branching locus of π is B. The following is well known (cf. Arnold [2]).

PROPOSITION 1.1. *A point $x \in X$ is singular if and only if $\pi(x) \in B$ and $\pi(x)$ is a singular point of B. Moreover the isomorphism class of the surface singularity (x, X) and that of the curve singularity $(\pi(x), B)$ determines each other uniquely.*

Thus the study of singularities on B is reduced to that on X. Let ϱ: $Z \to X$ be the minimal resolution of singularities of X. The following is an easy consequence of standard technics in the surface theory, if we note that $R^1 \varrho_* \mathcal{O}_Z = 0$ under the assumption (cf. Barth et al. [3], Durfee [6]).

PROPOSITION 1.2. *Z is a K3 surface, that is, $H^1(\mathcal{O}_Z) = 0$ and the canonical line bundle K_Z is trivial. The line bundle $L = \varrho^* \pi^* \mathcal{O}_{P^2}(1)$ is numerically effective and of degree 2.*

Recall that L is *numerically effective* if for every algebraic curve C on Z the intersection $L \cdot C$ is non-negative.

Here obviously the morphism $\pi\varrho$ coincides with the one φ_L associated with the line bundle L. But conversely, when does the given numerically effective line bundle L of degree 2 on a K3 surface Z define a surjective morphism φ_L: $Z \to P^2$ of degree 2? In the following we consider this question. Let Z denote a K3 surface. We owe ideas in the proof to Saint-

Donat [10]. We give a precise proof to the case where L is numerically effective and of degree 2, since he treated the case where L is ample in [10].

PROPOSITION 1.3 (Saint-Donat [10]). *Let M be a line bundle on Z. Assume that the complete linear system $|M|$ is not empty and that $|M|$ has no fixed components. Then one of the following* (i), (ii) *holds.*

(i) $M^2 > 0$ *and any general member of $|M|$ is an irreducible curve with arithmetic genus* $P_a = (M^2/2)+1$. *In this case* $h^1(M) = 0$.

(ii) $M^2 = 0$ *and there exists a smooth elliptic curve E and a positive integer k with* $M \cong \mathcal{O}_Z(kE)$. *In this case* $h^1(M) = k-1$ *and every member in $|M|$ can be written in the form* $E_1 + E_2 + \dots + E_k$ *with* $E_i \in |E|$ *for* $1 \leqslant i \leqslant k$.

LEMMA 1.4. (1) *For any non-zero effective divisor D,* $H^2(\mathcal{O}_Z(D)) = 0$.

(2) *For any non-zero effective reduced connected divisor D,* $H^1(\mathcal{O}_Z(D)) = 0$.

(3) *If* $h^i(\mathcal{O}_Z(A)) = h^i(\mathcal{O}_Z(B))$ *for divisors A, B and for* $i = 0, 1, 2$, *then* $A^2 = B^2$.

Proof. By the Serre duality, cohomology exact sequences and the Riemann–Roch theorem it is obvious. □

PROPOSITION 1.5. *Let D be an effective divisor on Z. We set* $D \sim D' + \Delta$ *where Δ is the fixed components of the complete linear system $|D|$. Let* $\Delta_1, \Delta_2, \dots, \Delta_N$ *be the connected reduced components of Δ.*

(1) *Any irreducible component of Δ is a smooth rational curve with self-intersection number -2. Furthermore if* $\Delta = \Delta' + \Delta''$ *for some effective divisors Δ' and Δ'' with* $\Delta' \neq 0$, *then* $\Delta'^2 = -2h^0(\mathcal{O}_{\Delta'}) \leqslant -2$.

(2) *Assume moreover that D' is reducible. We have a smooth elliptic curve E and an integer k with $k \geqslant 2$ such that* $D \sim kE$. *If* $D' \cdot \Delta_i > 0$, *then* $E \cdot \Delta_i = 1$.

Proof. (1) Assume that $\Delta = \Delta' + \Delta''$. ($\Delta'$ and Δ'' are effective divisors with $\Delta' \neq 0$.) Since Δ' is fixed components, $h^0(\mathcal{O}_Z(\Delta')) = 1$. By Lemma 1.4(1), $h^2(\mathcal{O}_Z(\Delta')) = 0$. Therefore by the Riemann–Roch theorem

$$1 - h^1(\mathcal{O}_Z(\Delta')) = (\Delta'^2/2) + 2.$$

We have

$$\Delta'^2 = -2\big(h^1(\mathcal{O}_Z(\Delta')) + 1\big).$$

Next consider the exact sequence

$$0 \to \mathcal{O}_Z(-\Delta') \to \mathcal{O}_Z \to \mathcal{O}_{\Delta'} \to 0.$$

We have $h^1(\mathcal{O}_Z(\Delta')) + 1 = h^0(\mathcal{O}_{\Delta'})$ since $h^0(\mathcal{O}_Z) = 1$, $h^1(\mathcal{O}_Z) = h^0(\mathcal{O}_Z(-\Delta')) = 0$, and $h^1(\mathcal{O}_Z(-\Delta')) = h^1(\mathcal{O}_Z(\Delta'))$ by Serre's duality.

If Δ' is an irreducible component of Δ, then we have $\Delta'^2 = -2$. By the adjunction formula for the arithmetic genus $p_a(\Delta')$ of Δ', $p_a(\Delta') = (\Delta'^2/2) + 1 = 0$ holds. It implies that Δ' is a smooth rational curve.

(2) The former half follows from Proposition 1.3. Set $A = D' + \Delta_i$. We have $h^0(\mathcal{O}_Z(D')) = h^0(\mathcal{O}_Z(A))$ since Δ_i is fixed components. By Lemma 1.4(1),

$$h^2(\mathcal{O}_Z(D')) = h^2(\mathcal{O}_Z(A)) = 0.$$

On the other hand $h^1(\mathcal{O}_Z(D')) = k-1$ by Proposition 1.3 and $h^1(\mathcal{O}_Z(A)) = 0$ by Lemma 1.4(2). Thus

$$\chi(\mathcal{O}_Z(D')) + k - 1 = \chi(\mathcal{O}_Z(A)).$$

By the Riemann–Roch theorem we have

$$(D'^2/2) + 2 + k - 1 = (D' + \Delta_i)^2/2 + 2.$$

Thus we have

$$kE \cdot \Delta_i = D' \cdot \Delta_i = k$$

since $\Delta_i^2 = -2$ by (1). □

The next theorem is due to Saint-Donat, too.

THEOREM 1.6 (Saint-Donat [10]). *Any complete linear system on a projective K3 surface has no fixed points outside its fixed components.*

PROPOSITION 1.7. *Let L be a numerically effective line bundle of degree 2 on Z. The following conditions are equivalent.*

(1) *The line bundle L does not define a surjective morphism φ_L: $Z \to \mathbf{P}^2$ of degree 2.*

(2) *The complete linear system $|L|$ has a fixed component.*

(3) *There is a smooth elliptic curve E and a smooth rational curve Γ with $E \cdot \Gamma = 1$ such that any member in $|L|$ is linearly equivalent to $2E + \Gamma$.*

(4) *There exists an element $U \in \mathrm{Pic}(Z)$ with $U^2 = 0$ and $U \cdot L = 1$.*

If one of the above conditions holds, then Γ in (3) is the fixed component of $|L|$.

Proof. Since $L^2 = \deg L > 0$, Z is projective (Theorem IV. 5. 2, Barth et al. [3]). Applying Kawamata's vanishing theorem (Kawamata [7]) and the Riemann–Roch theorem one has $h^1(L) = h^2(L) = 0$ and $h^0(L) = (L^2/2) + 2 = 3$.

(1) ⇒ (2) Assume that (2) does not hold. By Theorem 1.6 (1) does not hold.

(2) ⇒ (3) Let $D' + \Delta$ denote the general member in $|L|$, where Δ is the fixed components of $|L|$. Note that $D' \neq 0$ since $\dim|L| = 2$. First assume that D' is irreducible. We have $h^1(\mathcal{O}_Z(D')) = h^1(L) = 0$ by Proposition 1.3. On the other hand $h^2(\mathcal{O}_Z(D')) = h^2(L) = 0$ by Lemma 1.4(1) and $h^0(\mathcal{O}_Z(D')) = h^0(L)$ by definition. Thus we have $D'^2 = (D' + \Delta)^2$ by Lemma 1.4(3) and it implies that $2D' \cdot \Delta + \Delta^2 = 0$. Here recall that L and D' are numerically effective. We have

$$0 \leqslant L \cdot \Delta = (D' + \Delta) \cdot \Delta = -D' \cdot \Delta \leqslant 0.$$

Thus

$$D' \cdot \Delta = \Delta^2 = 0.$$

Proposition 1.5(1) implies that $\Delta = 0$, which contradicts the assumption. Therefore D' is reducible. We have a smooth elliptic curve E and an integer $k \geqslant 2$ such that $D' \sim kE$. If $E \cdot \Delta = 0$, we have $0 \leqslant L \cdot \Delta = (kE+\Delta) \cdot \Delta = \Delta^2 \leqslant -2$, which is a contradiction. Thus $E \cdot \Delta > 0$. By Proposition 1.5(2) one sees that there is a component Γ of Δ with $\Gamma \cdot E = 1$. Γ is a smooth rational curve with $\Gamma^2 = -2$ by Proposition 1.5(1). Set $\Delta = \Gamma + \Delta'$ and $D = kE + \Gamma$. By definition $D + \Delta' \in |L|$. It follows from Lemma 1.4(2) that $h^1(\mathcal{O}_Z(D)) = 0$, since D is linearly equivalent to a reduced connected effective non-zero divisor. We have $D^2 = (D+\Delta)^2$ and $2D \cdot \Delta' + \Delta'^2 = 0$. Thus $0 \leqslant L \cdot \Delta' = (D + \Delta') \cdot \Delta' = -D \cdot \Delta'$. However, setting $\Delta' = m\Gamma + \Delta''$ ($m \in \mathbf{Z}$, Δ'' contains no Γ), we have

$$D \cdot \Delta' = (kE+\Gamma) \cdot (m\Gamma + \Delta'') = m(k-2) + kE \cdot \Delta'' + \Gamma \cdot \Delta'' \geqslant 0.$$

Thus $D \cdot \Delta' = \Delta'^2 = 0$. We can conclude that $\Delta' = 0$, and $kE + \Gamma \in |L|$. Note that by the adjunction formula $p_a(E) = (E^2/2) + 1 = 1$, $E^2 = 0$ holds. We have $k = 2$, since $2 = (kE+\Gamma)^2 = 2k-2$.

(3) $\Rightarrow$ (4) The line bundle $U = \mathcal{O}_Z(E)$ satisfies the condition.

(4) $\Rightarrow$ (1) Assume that (4) holds but (1) does not hold. We will deduce a contradiction. By assumption we have a surjective morphism $\varphi_L\colon Z \to \mathbf{P}^2$. Now $h^0(U^*) = 0$ for the dual line bundle U^* of U, since $L \cdot U^* = -L \cdot U = -1$. By the Riemann–Roch theorem we have $h^0(U) \geqslant 2$. Let $D + \Delta$ be a general member of $|U|$, where Δ is the fixed component of $|U|$. Note that $D \neq 0$. Obviously $D^2 \geqslant 0$. If $L \cdot D = 0$, then $D^2 < 0$ by the Hodge index theorem. Thus $L \cdot D > 0$, since L is numerically effective. Furthermore $L \cdot \Delta \geqslant 0$ holds. We have $L \cdot D = 1$ since $1 = L \cdot U = L \cdot D + L \cdot \Delta$. It implies that the restriction $\varphi_L | D$ is an isomorphism and $\varphi_L(D)$ is a line in $\mathbf{P}^2$. Thus D is irreducible and $p_a(D) = 0$. However, in this case $D^2 = 2p_a(D) - 2 = -2$, which is a contradiction. □

PROPOSITION 1.8. (1) *Let $M \in \operatorname{Pic}(Z)$ be a line bundle with $M^2 = -2$. Then M or its dual M^*, and only one of them is effective.*

(2) *Let L be a numerically effective line bundle on Z with* $\deg L = L^2 > 0$. *The set $R = \{M \in \operatorname{Pic}(Z) \mid M^2 = -2,\ L \cdot M = 0\}$ is a root system whose fundamental system of roots is $\Delta = \{\mathcal{O}_Z(C) \in \operatorname{Pic}(Z) \mid C$ is an irreducible smooth rational curve with $C^2 = -2,\ C \cdot L = 0\}$* (see Bourbaki [4] for the notion of root systems).

Proof. (1) It is easily obtained by the Riemann–Roch theorem.

(2) This follows from the Hodge index theorem and the adjunction formula. □

Since exceptional curves in the minimal resolution of a rational double point on a surface constitute a fundamental system of roots, we have the next corollary (cf. Durfee [6]).

COROLLARY 1.9. *Let* E_L *be the union of curves* C *with* $\mathcal{O}_Z(C)\in\Delta$. *Every connected component of* E_L *coincides with the exceptional divisor in the minimal resolution of a rational double point. Let* $\varrho\colon Z\to X$ *be the contraction morphism of connected components of* E_L *to a normal surface* X. *The surface* X *has only rational double points as singularities and their combination is described by the numbers of irreducible root systems of each type* A_k, D_l, E_6, E_7, E_8 *which appear in the irreducible decomposition of the root system* $R = \bigoplus_{i=1}^{m} R_i$.

PROPOSITION 1.10. *Let* L *be a numerically effective line bundle of degree* 2 *on* Z. *Moreover we assume that* L *does not satisfy any equivalent condition in Proposition* 1.7. *Then the morphism* $\varphi_L\colon Z\to \boldsymbol{P}^2$ *factors through* $\varrho\colon Z\to X$ *and the induced morphism* $\pi\colon X\to\boldsymbol{P}^2$ *defines a branched double covering branching along a reduced sextic curve* B *with only rational singularities. The combination of singularities on* B *is described by the irreducible decomposition of the root system* $R=\{M\in\operatorname{Pic}(Z)\mid M^2=-2,\ M\cdot L=0\}$.

Proof. We consider the graded algebra $\bigoplus_{m=0}^{\infty} H^0(Z, L^{\otimes m})$. It follows from vanishing theorem and Riemann–Roch theorem that $h^0(Z, L^{\otimes m}) = m^2+2$, if $m>0$. Let u_0, u_1, u_2 be a basis of $H^0(Z, L)$. Let $S_m\subset H^0(Z, L^{\otimes m})$ be the subspace generated by monomials in u_i's. If $P(u_0, u_1, u_2)=0$ for a homogeneous polynomial $P(z_0, z_1, z_2)$, then $P=0$ since φ_L is surjective. We have $\dim S_m =$ the number of monomials in u_i's of degree $m = (m+1)(m+2)/2$. Thus $S_m = H^0(Z, L^{\otimes m})$ for $m = 1$, 2 and there is an element $v\in H^0(Z, L^{\otimes 3})$ such that $H^0(Z, L^{\otimes 3}) = S_3 + \boldsymbol{C}v$ (direct sum).

Let $\Phi\colon Z\to\boldsymbol{P}(1, 1, 1, 3)$ be the morphism defined by $\Phi(z) = (u_0(z), u_1(z), u_2(z), v(z))$ for $z\in Z$. Let Y be the image of Φ. Note that $(0, 0, 0, 1)\notin Y$ since u_i's do not simultaneously vanish on Z. Let $\pi\colon Y\to\boldsymbol{P}^2$ be the morphism defined by $(z_0, z_1, z_2, z_3)\to(z_0, z_1, z_2)$. The composition $\pi\Phi$ coincides with φ_L. Since $\pi(Y)=\varphi_L(Z)=\boldsymbol{P}^2$, $\dim Y = 2$. In view of following Lemma 1.11, one sees that $H^0(Z, L^{\otimes m}) = S_m + vS_{m-3}$ for $m\geqslant 4$ by comparing the dimension. Thus there are homogeneous polynomials $F(z_0, z_1, z_2)$ and $G(z_0, z_1, z_2)$ with $\deg F = 6$ and $\deg G = 3$ such that $v^2 = vG(u_0, u_1, u_2)+F(u_0, u_1, u_2)$. By exchanging v by $v+G(u_0, u_1, u_2)/2$ we can assume that $G=0$. Then one sees that

$$\bigoplus_{m=0}^{\infty} H^0(Z, L^{\otimes m}) \cong \boldsymbol{C}[z_0, z_1, z_2, z_3]/(z_3^2 - F(z_0, z_1, z_2)).$$

Y is the branched double covering over $\boldsymbol{P}^2$ branching along the sextic curve $B\colon F(z_0, z_1, z_2) = 0$.

Assume that B has a multiple component. Let $l \subset \boldsymbol{P}^2$ be a general line. The curve $C = \pi^{-1}(l)$ is a singular one with $p_a(C) = 2$, since l intersects with the multiple component. On the other hand $\tilde{C} = \varphi_L^{-1}(l) \in |L|$ is smooth by the Bertini theorem. Thus $\tilde{C}$ is a smooth model of C and $p_a(\tilde{C}) < 2$. However $p_a(\tilde{C}) = (L^2/2) + 1 = 2$ by the adjunction formula, which is a contradiction. One sees that B is reduced and the singular locus of Y is 0-dimensional by Proposition 1.1. Note that we can conclude that Y is normal by Serre's criterion for normality since Y is a hypersurface in a smooth variety $\boldsymbol{P}(1, 1, 1, 3) - \{(0, 0, 0, 1)\}$ (cf. Altman–Kleiman [1]). Now by the Hartogs extension theorem one sees that there is a morphism $\bar{\Phi}\colon X \to \boldsymbol{P}(1, 1, 1, 3)$ with $\bar{\Phi}\varrho = \Phi$ since X is normal. In particular $\varphi_L = \pi\bar{\Phi}\varrho$ factors through ϱ. By definition $\bar{\Phi}$ is a finite birational morphism to a normal variety Y and thus it is an isomorphism. In view of Proposition 1.1, it is obvious that singularities on B is described by R. □

LEMMA 1.11. *If $P(u_0, u_1, u_2) + vQ(u_0, u_1, u_2) = 0$ for homogeneous polynomials P, Q with $\deg P = \deg Q + 3$, then $P = Q = 0$.*

Proof. Obviously we can assume moreover that P and Q has no non-constant common divisor. Then the variety $Y' \subset \boldsymbol{P}(1, 1, 1, 3)$ defined by $P(z_0, z_1, z_2) + z_3 Q(z_0, z_1, z_2) = 0$ is irreducible and it contains a surface Y. Thus $Y = Y'$. If $\deg Q > 0$, then $(0, 0, 0, 1) \in Y' = Y$, which is a contradiction. If Q is a non-zero constant, then we have $v \in S_3$, a contradiction. □

We show further several propositions for later use.

Note that the *positive cone* $\Sigma = \{x \in H^{1,1}(Z, \boldsymbol{R}) \mid x^2 > 0\}$ in $H^{1,1}(Z, \boldsymbol{R}) = H^2(Z, \boldsymbol{R}) \cap H^1(Z, \Omega_Z^1)$ has two connected components since the signature of the intersection form on $H^{1,1}(Z, \boldsymbol{R})$ is $(1, 19)$. Let Σ_+ denote the connected component of Σ containing the Kähler class $\varkappa$ (cf. Siu [11]). Another component is $\Sigma_- = -\Sigma_+$.

PROPOSITION 1.12. *Let L be a line bundle on Z with $\deg L > 0$. Assume that $L \in \operatorname{Pic}(Z) \subset H^{1,1}(Z, \boldsymbol{R})$ belongs to Σ_+. Then the following two conditions are equivalent.*

(1) *L is numerically effective.*

(2) *For every smooth rational curve C on Z, $L \cdot C \geqslant 0$.*

Proof. (1) ⇒ (2) It is obvious.

(2) ⇒ (1) It is enough to show that for every irreducible curve C on Z, inequality $L \cdot C \geqslant 0$ holds. First assume that $C^2 < 0$. Then by the adjunction formula C is smooth and rational. By the assumption the inequality holds. Next we consider the case $C^2 \geqslant 0$. In this case the orthogonal complement $[C]^\perp$ to C in $H^{1,1}(Z, \boldsymbol{R})$ does not contain any element in Σ since the signature is $(1, 19)$. Thus Σ_+ and Σ_- are in the opposite side with respect to $[C]^\perp$. One sees that $L \cdot C > 0$ since $\varkappa \cdot C > 0$ and $L, \varkappa \in \Sigma_+$. □

For every element $M \in H^2(Z, \mathbf{Z})$ with $M^2 = -2$, we can define an isomorphism s_M: $H^2(Z, \mathbf{Z}) \to H^2(Z, \mathbf{Z})$ by $s_M(P) = P + (P \cdot M) M$. It is easily checked that s_M preserves the intersection form and s_M^2 is the identity. s_M is called the *reflection* with respect to M. Indeed the induced mapping s_M: $H^2(Z, \mathbf{R}) \to H^2(Z, \mathbf{R})$ on the real coefficient cohomology is the reflection with respect to the hyperplane orthogonal to M. Note that when we regard $\mathrm{Pic}(Z)$ as a subgroup of $H^2(Z, \mathbf{R})$, if $M \in \mathrm{Pic}(Z)$, then s_M induces an isomorphism s_M: $\mathrm{Pic}(Z) \to \mathrm{Pic}(Z)$.

PROPOSITION 1.13. *Let L be a line bundle on Z with* $\deg L = L^2 > 0$. *Assume that* $L \in \mathrm{Pic}(Z) \subset H^{1,1}(Z, \mathbf{R})$ *belongs to* Σ_+. *Then there exist finite number of elements* $M_1, \ldots, M_r \in \mathrm{Pic}(Z)$ *with* $M_i^2 = -2$ $(1 \leqslant i \leqslant r)$ *such that* $s_{M_1} s_{M_2} \ldots s_{M_r}(L)$ *is numerically effective.*

Proof. Set $\tilde{R} = \{M \in \mathrm{Pic}(Z) \mid M^2 = -2\}$. We denote the orthogonal hyperplane to $M \in \tilde{R}$ by $H_M = \{x \in H^2(Z, \mathbf{R}) \mid x \cdot M = 0\}$. Note that if $x \in \Sigma$, then $\{M \in \tilde{R} \mid H_M \ni x\} = \{M \in \tilde{R} \mid M \cdot x = 0\}$ is a finite set, since the intersection form on $\{N \in \mathrm{Pic}(Z) \mid N \cdot x = 0\}$ is negatively definite. Therefore $(\bigcup_{M \in \tilde{R}} H_M) \cap \Sigma$ is closed set in Σ and $\Sigma^0 = \Sigma - \bigcup_{M \in \tilde{R}} H_M$ is a union of finite or countably many disjoint connected open sets. Let T_0 be the connected component of Σ^0 containing the Kähler class $\varkappa$ and T' be the connected component whose closure contains L. Let x' be an interior point in T'. We fix such an x'. We can connect $\varkappa$ and x' with a piecewise smooth path γ in Σ_+ with the following properties (a) and (b).

(a) The path γ does not pass through any point on $H_M \cap H_{M'}$ for any M, $M' \in \tilde{R}$ with $M \neq M'$.

(b) If γ intersects with H_M for some $M \in \tilde{R}$, then γ is smooth at any intersection point P and it is tranversal to H_M at P.

We denote connected components of Σ^0 through which γ passes by $x' \in T' = T_r, T_{r-1}, \ldots, T_1, T_0 \ni \varkappa$ in order. The number of them is finite since γ is compact. We use induction an r. If $r = 0$, then $L \in \bar{T}_0$ and the condition (2) in Proposition 1.12 is satisfied. Thus we obtain the conclusion. Assume $r > 0$. Let H_{M_r} be the wall between T_r and T_{r-1}. One sees that $L' = s_{M_r}(L)$ belongs to the closure of T_{r-1}. Thus we can apply the induction hypothesis to L' and we obtain the conclusion. □

Here we explain the theory of periods for K3 surfaces.

Let Λ be an even unimodular lattice with signature (3, 19). It is known that such a lattice is unique up to isomorphisms and thus it is isomorphic to $Q(2E_8) \oplus H \oplus H \oplus H$ (orthogonal direct sum). ($Q(2E_8)$ is the negatively definite root lattice of type $2E_8$ and $H = \mathbf{Z}u + \mathbf{Z}v$ is a lattice of rank 2 with $u^2 = v^2 = 0$ and $u \cdot v = v \cdot u = 1$. H is called a *hyberbolic plane*.) Λ is isomorphic to the second cohomologh group $H^2(Z, \mathbf{Z})$ of any K3 surface Z, if we define the bilinear form on $H^2(Z, \mathbf{Z})$ by the intersection form. A pair (Z, α) where Z is a K3 surface and an isomorphism of lattices α: $H^2(Z, \mathbf{Z}) \to \Lambda$ is called a *marked K3 surface.*

For any marked K3 surface (Z, α), $H^2(Z, \mathbf{Z}) \otimes \mathbf{C} = H^2(Z, \mathbf{C})$ has the Hodge decomposition $H^2(Z, \mathbf{C}) = H^2(Z, \mathcal{O}_Z) \oplus H^1(Z, \Omega_Z^1) \oplus H^0(Z, K_Z)$. We have a nowhere vanishing holomorphic 2-form $\psi \in H^0(Z, K_Z)$, since the canonical line bundle K_Z is trivial. The 2-form ψ is unique up to the multiple of non-zero complex number. Thus the point $[\alpha(\psi)] = \alpha(\psi) \bmod \mathbf{C}^* \in \mathbf{P}(\Lambda \otimes \mathbf{C}) = \Lambda \otimes \mathbf{C} - \{0\}/\mathbf{C}^*$ is uniquely determined depending on the pair (Z, α). The point $[\alpha(\psi)]$ is called the *period* of (Z, α). Set

$$\Delta = \{[\omega] \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid 0 \neq \omega \in \Lambda \otimes \mathbf{C}, \omega \cdot \omega = 0, \omega \cdot \bar{\omega} > 0\}.$$

The point $[\alpha(\psi)]$ belongs to Δ since $\psi \cdot \psi = 0$ and $\psi \cdot \bar{\psi} > 0$. The 20-dimensional complex manifold Δ is called the *period domain.*

Now we can state the next remarkable theorem (cf. Barth et al. [3]).

THEOREM 1.14 (Horikawa–Shah–Kulikov–Persson–Pinkham–Todrov–Looijenga). *For every point* $[\omega]$ *on* Δ, *there exists a marked K3 surface* (Z, α) *whose period agrees with* $[\omega]$.

LEMMA 1.15. $\operatorname{Pic}(Z) = \{x \in H^2(Z, \mathbf{Z}) \mid x \cdot \psi = 0\}$.

Summing up above results, we have the next one.

THEOREM 1.16. *Let Q be the negatively definite root lattice* (i.e., *the lattice generated by roots*) *corresponding to a Dynkin graph* $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$. *Assume that the lattice* $S = \mathbf{Z}\lambda \oplus Q$ ($\lambda^2 = 2$, *orthogonal direct sum*) *has an embedding* $S \subset \Lambda$ *as lattices with the following properties* (a) *and* (b). *Then there exists a reduced sextic curve* $B \subset \mathbf{P}^2$ *whose combination of singularities agrees with G. Now let* $\hat{S} = \{x \in \Lambda \mid$ *for some non-zero integer m, mx belongs to* $S\}$ *denote the primitive hull of S in Λ.*

(a) *If* $\eta \in \hat{S}$, $\lambda \cdot \eta = 0$, *and* $\eta^2 = -2$, *then* $\eta \in Q$.

(b) $\hat{S}$ *does not contain any element u with* $u^2 = 0$ *and* $u \cdot \lambda = 1$.

Proof. Let T be the orthogonal complement of S in Λ. T is a lattice with signature $(2, t)$ for some $t \geqslant 0$. Let μ be an element in $T \otimes \mathbf{R}$ such that $\mu^2 > 0$ and $\hat{S} = \{x \in \Lambda \mid x \cdot \mu = 0\}$. Such a μ exists since $\mathbf{R}$ is an infinite-dimensional linear space over $\mathbf{Q}$. Set $T' = \{\lambda \in T \otimes \mathbf{R} \mid \lambda \cdot \mu = 0\}$. T' is a linear space over $\mathbf{R}$ equipped with a bilinear form with values in $\mathbf{R}$ with signature $(1, t)$. Pick an element $\nu \in T'$ with $\nu^2 = \mu^2$. Set $\omega = \mu + \sqrt{-1}\nu \in \Lambda \otimes \mathbf{C}$. The point $[\omega]$ belongs to Δ since $\omega \cdot \omega = \mu^2 - \nu^2 + 2\sqrt{-1}\mu \cdot \nu = 0$ and $\omega \cdot \bar{\omega} = \mu^2 + \nu^2 = 2\mu^2 > 0$. Here note that $\hat{S} = \{x \in \Lambda \mid x \cdot \omega = 0\}$. Let (Z, α) be the marked K3 surface whose period is $[\omega]$. In view of Lemma 1.15, one sees that α induces an isomorphism α: $\operatorname{Pic}(Z) \xrightarrow{\sim} \hat{S}$. We consider the line bundle L

$= \alpha^{-1}(\lambda)$. By considering $(Z, -\alpha)$ instead of (Z, α) if necessary, we can assume that L and the Kähler class $\varkappa$ belongs to the same connected component of the positive cone in $H^{1,1}(Z, \mathbf{R})$, because (Z, α) and $(Z, -\alpha)$ defines the same period. Then by Proposition 1.13, there are finite number of $M_1, \ldots, M_r \in \mathrm{Pic}(Z)$ with $M_i^2 = -2$ for $1 \leqslant i \leqslant r$ such that $L' = s_{M_1} \cdots s_{M_r}(L)$ is numerically effective. Now if $M \in \mathrm{Pic}(Z)$ and $M^2 = -2$, then (Z, β) and $(Z, \beta s_M)$ defines the same period, since $M \cdot \psi = 0$ for $\psi \in H^0(Z, K_Z)$. Thus considering $(Z, \alpha s_{M_r} \cdots s_{M_1})$ instead of (Z, α), we can assume that $L = \alpha^{-1}(\lambda)$ is numerically effective. Then by assumption (b) we can apply Proposition 1.10. By condition (a) the root system $R = \{M \in \mathrm{Pic}(Z) \mid M^2 = -2,\ M \cdot L = 0\}$ is of type G. □

Remark. The converse of the above theorem is obviously true.

2. Theory of lattices

By the last theorem in the previous section, it is very important to determine whether the lattice S can be embedded into Λ or not. In this section we explain Nikulin's lattice embedding theorem which gives us a criterion. We explain it in a simplified form compared with the original form in [8]. It is enough for our purpose in this article.

We begin with this section by a chain of definitions.

A free $\mathbf{Z}$-module S of finite rank equipped with a symmetric non-degenerate bilinear form with values in $\mathbf{Z}$ is called a *lattice*. A lattice S is said to be *even* if $x \cdot x = x^2$ is an even number for every $x \in S$. Note that we can define a canonical morphism $S \to S^* = \mathrm{Hom}(S, \mathbf{Z})$ by associating an element $x \in S$ with a morphism $y \to x \cdot y$. S is *unimodular* if this morphism $S \to S^*$ is an isomorphism. The following theorem is well known (cf. Milnor–Husemoller [9]). By $Q(G)$ we denote the negatively definite root lattice associated with the Dynkin graph G (cf. Bourbaki [4]).

THEOREM 2.1. (1) *An even unimodular lattice with signature (s, t) (s and t are non-negative integers) exists if and only if $s - t$ is a multiple of 8.*

(2) *Any indefinite even unimodular lattice is uniquely determined by its signature.*

(3) *There are two isomorphism classes of negatively definite even unimodular lattices of rank 16. One is $Q(2E_8)$. The other is Γ_{16}, which is an overlattice of $Q(D_{16})$ with index 2.*

Here for a sublattice S of a lattice S', S' is said to be an *overlattice* of S, if S'/S is a finite group. The number $\#(S'/S)$ is called its *index*.

Now let A denote a finite abelian group. A symmetric bilinear form $b\colon A \times A \to \mathbf{Q}/\mathbf{Z}$ over A is called a *finite symmetric bilinear form*. A mapping

$q: A \to Q/2Z$ satisfying the following conditions (1) and (2) is called a *finite quadratic form*.

(1) $q(na) = n^2 q(a)$ $(n \in Z, a \in A)$;

(2) $q(a+a') - q(a) - q(a') \equiv 2b(a, a') \pmod{2Z}$ $(a, a' \in A)$. Here $b: A \times A \to Q/Z$ is a finite symmetric bilinear form; b is called the *bilinear form of* q. Note that we can define the notion of orthogonality among subsets of A by using b.

Let S be an even lattice. Its dual $S^* = \mathrm{Hom}(S, Z)$ contains S as the image of the canonical morphism $S \to S^*$ and S^* has a bilinear form with values in Q which is the extension of the one on S.

We can define a finite symmetric bilinear form b_S and a finite quadratic form q_S on $A_S = S^*/S$ as follows

$$b_S(t_1+S, t_2+S) = t_1 \cdot t_2 + Z \quad (t_1, t_2 \in S^*),$$

$$q_S(t+S) = t^2 + 2Z \quad (t \in S^*).$$

b_S is the bilinear form of q_S and it is non-degenerate. b_S is called the *discriminant bilinear form* of S and q_S is called the *discriminant quadratic form* of S.

Now by $q_\theta^{(2)}(2)$: $Z/2 \to Q/2Z$ $(\theta = \pm 1)$ we denote a finite quadratic form over $Z/2$ defined by $q_\theta^{(2)}(2)(a \bmod 2) \equiv \theta a^2/2 \pmod{2Z}$.

An embedding $S \subset A$ of lattices is said to be *primitive* if A/S is free.

By $l(A)$ we denote the minimum number of generators of a finite abelian group A. A_p is the p-Sylow subgroup of A for a prime number p.

Under the above definitions we can state Nikulin's theorem.

THEOREM 2.2 (Nikulin [8]). *If the following conditions* (1), (2), (3), (4) *are satisfied, then the even lattice S with signature* (t_+, t_-) *and the discriminant quadratic form q has a primitive embedding into an even unimodular lattice with signature* (l_+, l_-).

(1) $l_+ - l_- \equiv 0 \pmod 8$.

(2) $l_+ - t_+ \geqslant 0$, $l_- - t_- \geqslant 0$, $l_+ + l_- - t_+ - t_- \geqslant l(S^*/S)$.

(3) *For every odd prime number p,* $l((S^*/S)_p) < l_+ + l_- - t_+ - t_-$.

(4) *If* $l((S^*/S)_2) = l_+ + l_- - t_+ - t_-$, *then* $q \cong q_\theta^{(2)}(2) \oplus q'$ *for some* q' *and for some* $\theta = \pm 1$.

When we consider non-primitive embeddings, the following proposition is very useful.

Let S' be an even overlattice of an even lattice S. A chain of embeddings $S \subset S' \subset S'^* \subset S^*$ is defined. Set $H_{S'} = S'/S$. We say that a subgroup $H \subset S^*/S$ is *isotropic* if $q_S | H \equiv 0$.

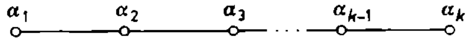
PROPOSITION 2.3 (Nikulin [8]). (1) *The correspondence* $S' \mapsto H_{S'}$ *gives one-to-one correspondence between even overlattices of S and isotropic subgroups in* S^*/S.

(2) *The orthogonal complement $H_{S'}^{\perp}$ of $H_{S'}$ in S^*/S equals S'^*/S and* $(q_S|(H_{S'})^{\perp})/H_{S'} = q_{S'}$.

In the sequel we give the discriminant quadratic form for an irreducible root lattice of type A, D, E. By the tables in the appendix in Bourbaki [4] we can calculate it easily. We assume that every root lattice is negatively definite.

1. $Q = Q(A_k)$ $(k \geqslant 1)$.

Let $\alpha_1, \ldots, \alpha_k$ be the basis of Q associated with the Dynkin graph (the fundamental system of roots).



Let $\omega_1, \omega_2, \ldots, \omega_k$ be the dual basis of $\alpha_1, \alpha_2, \ldots, \alpha_k$. They are basis of $Q^* = P(A_k)$. (The weight lattice of type A_k).

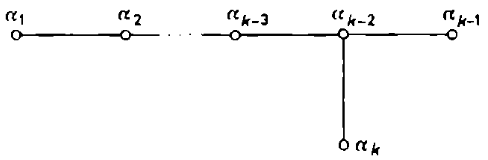
$$A_Q = Q^*/Q \cong \mathbf{Z}/(k+1).$$

A_Q is generated by $g = \omega_1 + Q$; $\omega_1 = -\{k\alpha_1 + (k-1)\alpha_2 + \ldots + 2\alpha_2 + \alpha_1\}/(k+1)$. For the discriminant quadratic form $q = q_S$: $A_S \to \mathbf{Q}/2\mathbf{Z}$

$$q(g) \equiv -k/(k+1) \bmod 2\mathbf{Z}.$$

2. $Q = Q(D_k)$ $(k \geqslant 4)$.

Let $\alpha_1, \ldots, \alpha_k$ be the basis of Q.



Let $\omega_1, \omega_2, \ldots, \omega_k$ be the dual basis of $\alpha_1, \alpha_2, \ldots, \alpha_k$. We denote $\bar{\omega}_i = \omega_i + Q$ $(1 \leqslant i \leqslant k)$, $\# A_Q = 4$ and $A_Q = Q^*/Q = \{0, \bar{\omega}_1, \bar{\omega}_{k-1}, \bar{\omega}_k\}$; $\bar{\omega}_k = \bar{\omega}_1 + \bar{\omega}_{k-1}$;

$$\begin{aligned}
\omega_1 &= -(\alpha_1 + \alpha_2 + \ldots + \alpha_{k-2} + \tfrac{1}{2}\alpha_{k-1} + \tfrac{1}{2}a_k),\\
\omega_{k-1} &= -\{\alpha_1 + 2\alpha_2 + \ldots + (k-2)\alpha_{k-2} + \tfrac{1}{2}k\alpha_{k-1} + \tfrac{1}{2}(k-2)\alpha_k\}/2,\\
\omega_k &= -\{\alpha_1 + 2\alpha_2 + \ldots + (k-2)\alpha_{k-2} + \tfrac{1}{2}(k-2)\alpha_{k-1} + \tfrac{1}{2}k\alpha_k\}/2.
\end{aligned}$$

Case 1. k is even.

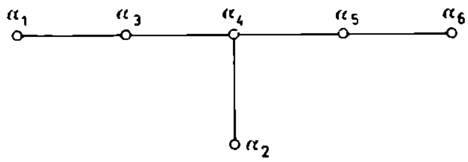
$$A_Q = Q^*/Q \cong (\mathbf{Z}/2) \times (\mathbf{Z}/2),$$

$$q(a\bar{\omega}_{k-1} + b\bar{\omega}_k) \equiv -\frac{k}{4}a^2 - \frac{k-2}{2}ab - \frac{k}{4}b^2 \pmod{2\mathbf{Z}}.$$

Case 2. k is odd.

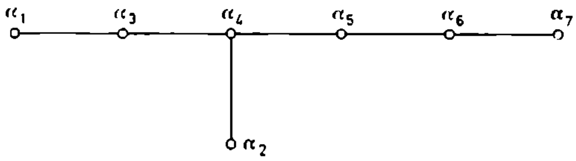
$$A_Q = Q^*/Q \cong Z/4,$$
$$\bar{\omega}_{k-1} = 3\bar{\omega}_k, \quad \bar{\omega}_1 = 2\bar{\omega}_k,$$
$$q(\bar{\omega}_k) \equiv -k/4 \pmod{2Z}.$$

3. $Q = Q(E_6)$.
Let $\omega_1, \omega_2, \ldots, \omega_6$ be the dual basis of the next basis $\alpha_1, \alpha_2, \ldots, \alpha_6$.



$$A_Q = Q^*/Q \cong Z/3, \quad A_Q \text{ is generated by } g = \omega_6 + Q,$$
$$\omega_6 = -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6)/3,$$
$$q(g) \equiv -4/3 \pmod{2Z}.$$

4. $Q = Q(E_7)$.
Let $\omega_1, \omega_2, \ldots, \omega_7$ denote the dual basis of the following basis $\alpha_1, \alpha_2, \ldots, \alpha_7$.



$$A_Q = Q^*/Q \cong Z/2, \; A_Q \text{ is generated by } g = \omega_7 + Q,$$
$$\omega_7 = -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7)/2,$$
$$q(g) \equiv -3/2 \pmod{2Z}.$$

5. $Q = Q(E_8)$..
Q is an unimodular even lattice.

$$A_Q = Q^*/Q = 0.$$

The next lemma holds, since it holds for irreducible ones.

LEMMA 2.4. *Let $Q = Q(G)$ be the root lattice of type G.*
(1) *For any odd prime number p, $(p-1)\,l((Q^*/Q)_p) \leqslant \operatorname{rank} Q$.*
(2) $\operatorname{rank} Q \equiv l((Q^*/Q)_2) \pmod 2$.

3. Reduction of the problem

In this section by $Q = Q(G)$ we denote the negatively definite root lattice of type $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$. We assume that

$$\operatorname{rank} Q = \sum a_k k + \sum b_l l + \sum c_m m \leqslant 15$$

throughout this section.

LEMMA 3.1. *If $l((Q^*/Q)_p) > 20 - \operatorname{rank} Q$ for some odd prime number p, then $Q = Q(7A_2)$, $Q(A_3+6A_2)$, $Q(7A_2+A_1)$, $Q(6A_2+3A_1)$ or $Q(A_5+5A_2)$.*

Proof. By Lemma 2.4,

$$\operatorname{rank} Q \geqslant 21(p-1) - (p-1)\operatorname{rank} Q.$$

Thus we have $15p \geqslant 21(p-1)$ and $p = 3$. Note that for an irreducible root lattice $\bar{Q}$, $(\bar{Q}^*/\bar{Q})_3 \neq 0$ if and only if $\bar{Q}$ is type A_{3k+2} or E_6. Since $\operatorname{rank} Q \leqslant 15$ and $l((Q^*/Q)_3) \geqslant 6$, we have the conclusion. □

LEMMA 3.2. *There exists an even overlattice $\hat{Q}'$ of $Q' = Q(6A_2)$ with index 3 with the following property* (a*).

(a*) *If $\eta \in \hat{Q}'$ and $\eta^2 = -2$, then $\eta \in Q'$.*

Proof. Let $\alpha_1, \ldots, \alpha_6, \beta_1, \ldots, \beta_6$ be the basis of Q' associated with the next Dynkin graph.

Set $\omega = \sum_{i=1}^{6} \frac{1}{3}(2\alpha_i + \beta_i)$, and $\hat{Q}' = Q' + Z\omega$. It is easily checked that $\hat{Q}'$ is an even overlattice of Q' with index 3. We show (a*). Assume that there is an element $\eta \in \hat{Q}' - Q'$ with $\eta^2 = -2$. Set

$$\eta = \sum_{i=1}^{6} a_i \alpha_i + \sum_{i=1}^{6} b_i \beta_i + \varepsilon\omega \qquad (a_i, b_i \in Z, \varepsilon = \pm 1).$$

By considering $-\eta$ instead of η, we can assume that $\varepsilon = 1$. Note that $\omega^2 = -4$, $\omega \cdot \alpha_i = -1$ and $\omega \cdot \beta_i = 0$. We have

$$-2 = \eta^2 = -4 - 2\sum_{i=1}^{6} (a_i^2 + a_i + b_i^2 - a_i b_i).$$

Thus $\sum_{i=1}^{6} X_i + 1 = 0$. Here

$$X_i = a_i^2 + a_i + b_i^2 - a_i b_i = \{a_i - \tfrac{1}{2}b_i - 1)\}^2 + \tfrac{3}{4}[\{b_i + \tfrac{1}{3}\}^2 - \tfrac{4}{9}].$$

By the last expression $X_i \geqslant 0$ if $b_i \neq 0$. In the case $b_i = 0$, $X_i = a_i(a_i+1) \geqslant 0$. Consequently $\sum_{i=1}^{6} X_i \geqslant 0$, which is a contradiction. □

LEMMA 3.3. *There exists a plane reduced sextic curve with* $7A_2$, A_3+6A_2, $7A_2+A_1$ *or* $6A_2+3A_1$.

Proof. First we consider the case $Q = Q(7A_2)$. Set $S = Z\lambda \oplus Q$ ($\lambda^2 = 2$). Note that $Q = Q' \oplus Q''$ ($Q' = Q(6A_2)$, $Q'' = Q(A_2)$). Take the overlattice $\hat{Q}'$ in Lemma 3.2. Set $\hat{S} = Z\lambda \oplus \hat{Q}' \oplus Q''$. Let I be the cyclic subgroup of $Q'^*/Q' \cong (Z/3)^{\oplus 6}$ generated by $\omega + Q'$. By Proposition 2.3, $\hat{Q}'^*/\hat{Q}' \cong I^{\perp}/I$. Since $Z/3$ is a field, $l((\hat{Q}'^*/\hat{Q}')_p) = 4$ if $p = 3$, and it is zero if $p \neq 3$. Thus

$$l((\hat{S}^*/\hat{S})_p) \leqslant 1 < \operatorname{rank} \Lambda - \operatorname{rank} S = 22 - 15 = 7, \quad \text{if } p \neq 3,$$

and

$$l((\hat{S}^*/\hat{S})_3) = 5 < \operatorname{rank} \Lambda - \operatorname{rank} S.$$

Thus by Theorem 2.2 there is a primitive embedding $\hat{S} \subset \Lambda$. The condition (a*) in Lemma 3.2 implies condition (a) in Theorem 1.16. It is easy to check condition (b) in Theorem 1.16. Indeed if $u \cdot \lambda = 1$ for $u \in \hat{S}$, then $u = \frac{1}{2}\lambda + \alpha$ for some $\alpha \in \hat{Q}' \oplus Q''$. But $\frac{1}{2} \notin Z$.

Secondly assume $Q = Q(A_3 + 6A_2)$. Set $Q' = Q(6A_2)$ and $Q'' = Q(A_3)$. Taking the overlattice $\hat{Q}'$ in Lemma 3.2, set $\hat{S} = Z\lambda \oplus \hat{Q}' \oplus Q''$.

$$l((\hat{S}^*/\hat{S})_p) \leqslant 2 < \operatorname{rank} \Lambda - \operatorname{rank} \hat{S} = 22 - 16 = 6, \quad \text{if } p \neq 3$$

and

$$l((\hat{S}^*/\hat{S})_3) = 4 < \operatorname{rank} \Lambda - \operatorname{rank} \hat{S}.$$

Thus by the same reason as above we obtain the conclusion.

Thirdly we consider $Q = Q(7A_2 + A_1)$. Set $Q' = Q(6A_2)$, $Q'' = Q(A_2 + A_1)$ and $\hat{S} = Z\lambda \oplus \hat{Q}' \oplus Q''$, where $\hat{Q}'$ is the overlattice in Lemma 3.2.

$$l((\hat{S}^*/\hat{S})_p) \leqslant 2 < \operatorname{rank} \Lambda - \operatorname{rank} \hat{S} = 6, \quad \text{if } p \neq 3$$

and

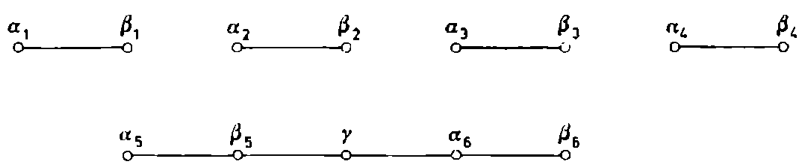
$$l((\hat{S}^*/\hat{S})_3) = 5 < \operatorname{rank} \Lambda - \operatorname{rank} \hat{S}.$$

The fourth case $Q(6A_2 + 3A_1)$ is similar. □

LEMMA 3.4. *The root lattice* $Q' = Q(A_5 + 4A_2)$ *has an even overlattice* $\hat{Q}'$ *with index* 3 *and with the next property* (a*).

(a*) *If* $\eta \in \hat{Q}'$ *and* $\eta^2 = -2$, *then* $\eta \in Q'$.

Proof. Let $\alpha_1, \ldots, \alpha_6, \beta_1, \ldots, \beta_6, \gamma$ be the basis of Q' associated with the next Dynkin graph.



Set $\omega = \frac{1}{3}\sum_{i=1}^{6}(2\alpha_i+\beta_i)$ and $\hat{Q}' = Q' + \boldsymbol{Z}\omega$. It is easily checked that $\hat{Q}'$ is an even overlattice of Q' with index 3. Note that $\omega^2 = -4$, $\omega\cdot\alpha_i = -1$, $\omega\cdot\beta_i = 0$, and $\omega\cdot\gamma = 1$. Next assume that there is an element $\eta\in\hat{Q}'-Q$ with $\eta^2 = -2$. We can write it in the next form.

$$\eta = \sum_{i=1}^{6} a_i\alpha_i + \sum_{i=1}^{6} b_i\beta_i + c\gamma + \varepsilon\omega \qquad (a_i,\, b_i,\, c\in\boldsymbol{Z},\ \varepsilon = \pm 1).$$

By considering $-\eta$ instead of η if necessary, we can assume that $\varepsilon = 1$. We have

$$-2 = \eta^2 = -4 - 2\sum_{i=1}^{6}(a_i^2 + a_i - a_i b_i + b_i^2) - 2c^2 + 2c + 2a_6 c + 2b_5 c.$$

Setting

$$X_i = a_i^2 + a_i - a_i b_i + b_i^2 = \{a_i - \tfrac{1}{2}(b_i-1)\}^2 + \tfrac{3}{4}[\{b_i + \tfrac{1}{3}\}^2 - \tfrac{4}{9}]$$

and

$$\begin{aligned} Y &= a_5^2 + a_5 + b_5^2 - a_5 b_5 + a_6^2 + a_6 + b_6^2 - a_6 b_6 + c^2 - c - b_5 c - a_6 c \\ &= \{c - \tfrac{1}{2}(a_6 + b_5 + 1)\}^2 + \{a_5 - \tfrac{1}{2}(b_5 - 1)\}^2 + \{b_6 - \tfrac{1}{2}a_6\}^2 + \tfrac{1}{2}\{a_6 - \tfrac{1}{2}(b_5-1)\}^2 \\ &\qquad + \tfrac{3}{8}\{b_5 + \tfrac{1}{3}\}^2 - \tfrac{2}{3}, \end{aligned}$$

one knows

$$\sum_{i=1}^{4} X_i + Y + 1 = 0.$$

Since $X_i \geqslant 0$ and $Y+1 > 0$, it is a contradiction. □

By Lemma 3.4 we can show the next one.

LEMMA 3.5. *There is a plane reduced sextic curve with* $A_5 + 5A_2$.

PROPOSITION 3.6. *Let* $Q = Q(G)$ *be the negatively definite root lattice associated with the Dynkin graph G. We assume that*

(1) $\operatorname{rank} Q \leqslant 15$;

(2) $l((Q^*/Q)_2) \leqslant 20 - \operatorname{rank} Q$.

Then there is a plane reduced sextic curve with the combination of singularities G.

Proof. We can assume moreover that $G \neq 7A_2$, $A_3 + 6A_2$, $7A_2 + A_1$, $6A_2 + 3A_1$ and $A_5 + 5A_2$ by Lemma 3.3 and 3.5. Set $S = \boldsymbol{Z}\lambda \oplus Q$ ($\lambda^2 = 2$). By Lemma 3.1 for every odd prime number p

$$l((S^*/S)_p) = l((Q^*/Q)_p) \leqslant 20 - \operatorname{rank} Q = \operatorname{rank} \Lambda - \operatorname{rank} S - 1.$$

On the other hand by assumption (2)

$$l((S^*/S)_2) = 1 + l((Q^*/Q)_2) \leqslant \operatorname{rank} \Lambda - \operatorname{rank} S.$$

Note that

$$l(S^*/S) = \max_p l((S^*/S)_p).$$

If $l((S^*/S)_2) < \operatorname{rank} \Lambda - \operatorname{rank} S$, then by Theorem 2.2 there is a primitive embedding $S \subset \Lambda$. Assume that $l((S^*/S)_2) = \operatorname{rank} \Lambda - \operatorname{rank} S$. In this case the discriminant quadratic form q on $S^*/S \cong Z/2 \oplus Q^*/Q$ has the form $q \cong q_1^{(2)}(2) \oplus q'$ because $\lambda^2 = 2$. Thus by Theorem 2.2 there is a primitive embedding $S \subset \Lambda$. Obviously conditions (a) and (b) in Theorem 1.16 are satisfied. We have the conclusion by Theorem 1.16. □

PROPOSITION 3.7. *Let $Q = Q(G)$ be the negatively definite root lattice of type G. We assume that*

(1) $\operatorname{rank} Q \leqslant 15$;

(2) $l((Q^*/Q)_2) > 20 - \operatorname{rank} Q$.

Then G is one in the next list. (We write the values of $r = \operatorname{rank} Q$ and $l = l((Q^*/Q)_2)$ in the list.)

$r = 11$

[1]	$11A_1$	$l = 11$			

$r = 12$

[2]	A_3+9A_1	$l = 10$	[4]	$8A_1+D_4$	$l = 10$
[3]	A_2+10A_1	$l = 10$	[5]	$12A_1$	$l = 12$

$r = 13$

[6]	A_5+8A_1	$l = 9$	[13]	$8A_1+D_5$	$l = 9$
[7]	A_4+9A_1	$l = 9$	[14]	$7A_1+D_6$	$l = 9$
[8]	$2A_3+7A_1$	$l = 9$	[15]	$5A_1+2D_4$	$l = 9$
[9]	$A_3+A_2+8A_1$	$l = 9$	[16]	A_3+10A_1	$l = 11$
[10]	$A_3+6A_1+D_4$	$l = 9$	[17]	$9A_1+D_4$	$l = 11$
[11]	$2A_2+9A_1$	$l = 9$	[18]	A_2+11A_1	$l = 11$
[12]	$A_2+7A_1+D_4$	$l = 9$	[19]	$13A_1$	$l = 13$

$r = 14$

[20]	A_7+7A_1	$l = 8$	[41]	$A_3+A_2+5A_1+D_4$	$l = 8$
[21]	A_6+8A_1	$l = 8$	[42]	$A_3+5A_1+D_6$	$l = 8$
[22]	$A_5+A_3+6A_1$	$l = 8$	[43]	$A_3+3A_1+2D_4$	$l = 8$
[23]	$A_5+A_2+7A_1$	$l = 8$	[44]	$A_2+4A_1+2D_4$	$l = 8$
[24]	$A_5+5A_1+D_4$	$l = 8$	[45]	$5A_1+D_5+D_4$	$l = 8$
[25]	$A_4+A_3+7A_1$	$l = 8$	[46]	$4A_1+D_6+D_4$	$l = 8$
[26]	$A_4+A_2+8A_1$	$l = 8$	[47]	$2A_1+3D_4$	$l = 8$
[27]	$A_4+6A_1+D_4$	$l = 8$	[48]	$2A_3+8A_1$	$l = 10$
[28]	$2A_3+A_2+6A_1$	$l = 8$	[49]	$A_3+7A_1+D_4$	$l = 10$
[29]	$A_3+2A_2+7A_1$	$l = 8$	[50]	$8A_1+D_6$	$l = 10$
[30]	$A_3+6A_1+D_5$	$l = 8$	[51]	$6A_1+2D_4$	$l = 10$

[31] $3A_2+8A_1$ $l=8$
[32] $2A_2+6A_1+D_4$ $l=8$
[33] $A_2+7A_1+D_5$ $l=8$
[34] $A_2+6A_1+D_6$ $l=8$
[35] $8A_1+E_6$ $l=8$
[36] $7A_1+D_7$ $l=8$
[37] $7A_1+E_7$ $l=8$
[38] $6A_1+D_8$ $l=8$
[39] $3A_3+5A_1$ $l=8$
[40] $2A_3+4A_1+D_4$ $l=8$

[52] A_5+9A_1 $l=10$
[53] A_4+10A_1 $l=10$
[54] $A_3+A_2+9A_1$ $l=10$
[55] $2A_2+10A_1$ $l=10$
[56] $\mathrm{A}_2+8A_1+D_4$ $l=10$
[57] $9A_1+D_5$ $l=10$
[58] A_3+11A_1 $l=12$
[59] A_2+12A_1 $l=12$
[60] $10A_1+D_4$ $l=12$
[61] $14A_1$ $l=14$

$r=15$, $l=7$

[62] A_9+6A_1
[63] A_8+7A_1
[64] $A_7+A_2+6A_1$
[65] $A_6+A_3+6A_1$
[66] $A_6+A_2+7A_1$
[67] $2A_5+5A_1$
[68] $A_5+A_4+6A_1$
[69] $A_5+A_3+A_2+5A_1$
[70] $A_5+2A_2+6A_1$
[71] $A_5+5A_1+D_5$
[72] $2A_4+7A_1$
[73] $A_4+A_3+A_2+6A_1$
[74] $A_4+2A_2+7A_1$
[75] $A_4+6A_1+D_5$
[76] $A_3+3A_2+6A_1$
[77] $A_3+6A_1+E_6$
[78] $A_3+5A_1+E_7$
[79] $4A_2+7A_1$
[80] $2A_2+6A_1+D_5$
[81] $A_2+7A_1+E_6$
[82] $A_2+6A_1+D_7$
[83] $A_2+6A_1+E_7$
[84] $7A_1+E_8$
[85] $6A_1+D_9$
[86] $A_7+A_3+5A_1$
[87] $A_7+4A_1+D_4$
[88] $A_5+2A_3+4A_1$
[89] $A_5+A_2+4A_1+D_4$
[90] $A_5+4A_1+D_6$
[91] $A_5+2A_1+2D_4$
[92] $4A_3+3A_1$

[98] $A_3+A_2+4A_1+D_6$
[99] $A_3+A_2+2A_1+2D_4$
[100] $A_3+3A_1+D_5+D_4$
[101] $A_3+2A_1+D_6+D_4$
[102] A_3+3D_4
[103] $4A_1+D_6+D_5$
[104] $4A_1+D_4+E_7$
[105] $3A_1+D_8+D_4$
[106] $3A_1+2D_6$
[107] $2A_1+D_5+2D_4$
[108] $A_4+A_3+4A_1+D_4$
[109] $A_3+4A_1+D_8$
[110] $A_2+4A_1+D_5+D_4$
[111] $5A_1+D_{10}$
[112] $5A_1+2D_5$
[113] $4A_1+D_7+D_4$
[114] $A_1+D_6+2D_4$
[115] $A_4+2A_3+5A_1$
[116] $A_4+5A_1+D_6$
[117] $2A_3+2A_2+5A_1$
[118] $A_3+2A_2+4A_1+D_4$
[119] $A_3+A_2+5A_1+D_5$
[120] $A_3+5A_1+D_7$
[121] $2A_2+5A_1+D_6$
[122] $A_2+3A_1+D_6+D_4$
[123] $A_6+5A_1+D_4$
[124] $A_5+A_3+3A_1+D_4$
[125] $A_4+A_2+5A_1+D_4$
[126] $3A_2+5A_1+D_4$
[127] $5A_1+D_4+E_6$
[128] $2A_3+A_2+3A_1+D_4$

[93]	$3A_3+A_2+4A_1$	[129]	$A_2+5A_1+D_8$
[94]	$3A_3+2A_1+D_4$	[130]	$A_4+3A_1+2D_4$
[95]	$2A_3+4A_1+D_5$	[131]	$2A_2+3A_1+2D_4$
[96]	$2A_3+3A_1+D_6$	[132]	$A_2+A_1+3D_4$
[97]	$2A_3+A_1+2D_4$		

$r=15,\ l=9$

[133]	$2A_3+5A_1+D_4$	[147]	$2A_3+A_2+7A_1$
[134]	$A_3+4A_1+2D_4$	[148]	$A_3+2A_2+8A_1$
[135]	$5A_1+D_6+D_4$	[149]	$A_3+A_2+6A_1+D_4$
[136]	A_7+8A_1	[150]	$A_3+7A_1+D_5$
[137]	$A_5+6A_1+D_4$	[151]	$A_3+6A_1+D_6$
[138]	$7A_1+D_8$	[152]	$3A_2+9A_1$
[139]	$3A_1+3D_4$	[153]	$2A_2+7A_1+D_4$
[140]	A_6+9A_1	[154]	$A_2+8A_1+D_5$
[141]	$A_5+A_3+7A_1$	[155]	$A_2+7A_1+D_6$
[142]	$A_5+A_2+8A_1$	[156]	$A_2+5A_1+2D_4$
[143]	$A_4+A_3+8A_1$	[157]	$9A_1+E_6$
[144]	$A_4+A_2+9A_1$	[158]	$8A_1+D_7$
[145]	$A_4+7A_1+D_4$	[159]	$8A_1+E_7$
[146]	$3A_3+6A_1$	[160]	$6A_1+D_5+D_4$

$r=15,\ l=11$

[161]	A_5+10A_1	[166]	$10A_1+D_5$
[162]	$2A_3+9A_1$	[167]	$9A_1+D_6$
[163]	$A_3+A_2+10A_1$	[168]	$7A_1+2D_4$
[164]	$A_3+8A_1+D_4$	[169]	A_4+11A_1
[165]	$A_2+9A_1+D_4$	[170]	$2A_2+11A_1$

$r=15,\ l=13$

[171]	A_3+12A_1	[173]	A_2+13A_1
[172]	$11A_1+D_4$		

$r=15,\ l=15$

[174]	$15A_1$

Sketch of the proof. Since $l \leqslant r$ and $l > 20-r$, $11 \leqslant r \leqslant 15$. Let $Q = \bigoplus_{i=1}^{m} Q_i$ denote the decomposition into irreducible root lattice. Let a be the number of components Q_i which are of type A_{2k-1}, D_{2l+1} $(k \geqslant 1,\ l \geqslant 2)$ or E_7, and let b be the number of components Q_i of type D_{2l} $(l \geqslant 2)$. Then

$$21-r \leqslant a+2b = l \leqslant a+4b \leqslant r.$$

By Lemma 2.4, $r \equiv a \pmod 2$.

Case $r=11$. If $b \geqslant 1$, we have $l \leqslant r-2b \leqslant 9 < 21-r$, a contradiction. Thus $b=0$ and $10 \leqslant a \leqslant 11$. Since a is odd $a = 11 = r$. Thus $Q = Q(11A_1)$.

Case $r = 12$. If $b \geqslant 2$. We have $l \leqslant r-2b \leqslant 8 < 21-r$, a contradiction. First assume $b = 0$. Then $9 \leqslant a \leqslant 12$ and thus $a = 10$ or 12. If $a = 10$, it is easy to see that $Q = Q(10A_1 + A_2)$ or $Q(9A_1 + A_3)$. If $a = 12$, $Q = Q(12A_1)$. Next assume $b = 1$. We have $7 = 21-r-2b \leqslant a \leqslant r-4b = 8$ and thus $a = 8$. Therefore $Q = Q(8A_1 + D_4)$.

In the case $r = 13$, 14, or 15, the argument is more complicated but by the same method we can show the proposition. □

LEMMA 3.8. *Set $Q' = Q(5A_1)$ and $S' = Z\lambda \oplus Q'$ ($\lambda^2 = 2$). S' has an even overlattice $\hat{S}'$ with index 2 and with the next properties* (a) *and* (b) *for any negatively definite root lattice Q''.*

(a) *If $\eta \in \hat{S}' \oplus Q''$, $\eta^2 = -2$ and $\eta \cdot \lambda = 0$, then $\eta \in Q' \oplus Q''$.*

(b) *$\hat{S}' \oplus Q''$ contains no element u with $u^2 = 0$ and $u \cdot \lambda = 1$.*

Proof. The next Dynkin graph shows a basis of Q'.

$$\overset{\alpha_1}{\circ} \qquad \overset{\alpha_2}{\circ} \qquad \overset{\alpha_3}{\circ} \qquad \overset{\alpha_4}{\circ} \qquad \overset{\alpha_5}{\circ}$$

Set $\omega = \frac{1}{2}(\lambda + \sum_{i=1}^{5} \alpha_i)$ and $\hat{S}' = S' + Z\omega$. $\hat{S}'$ is an even overlattice with index 2 of S'. Note that $\omega^2 = -2$, $\omega \cdot \lambda = 1$ and $\omega \cdot \alpha_1 = -1$. Since the orthogonal complement of λ in $\hat{S}' \oplus Q''$ is $Q' \oplus Q''$, (a) holds. Next consider (b). Assume that there is an element $u \in \hat{S}' \oplus Q''$ with $u^2 = 0$, $u \cdot \lambda = 1$. We can write it in the form

$$u = \omega + \sum_{i=1}^{5} a_i \alpha_i + \beta \qquad (a_i \in Z,\ \beta \in Q'').$$

Set $\beta^2 = -2m$, where m is an integer with $m \geqslant 0$. We have $0 = -2-2\sum_{i=1}^{5} (a_i^2 + a_i) - 2m$, and thus $\sum_{i=1}^{5} a_i(a_i+1) + m + 1 = 0$. But the left-hand side of the last equality is positive, which is a contradiction. □

By Lemma 3.8 we can deduce the next lemma.

LEMMA 3.9. *Let Q be the root lattice associated with the Dynkin graph G. Assume the following three conditions*:

(1) $\operatorname{rank} Q \leqslant 15$;

(2) $l((Q^*/Q)_2) + \operatorname{rank} Q = 22$;

(3) *G is a sum of a Dynkin graph G' of type $5A_1$ and another Dynkin graph G'' containing A_1, A_5 or E_7 as its component.*

Then there is a reduced plane sextic curve with the combination of singularities G.

COROLLARY 3.10. *Dynkin graphs* [1]–[4], [6]–[14], [20]–[38], [62]–[85] *can be realized on reduced sextic curves as their combinations of singularities.*

4. Elementary transformations of Dynkin graphs

The notion of elementary transformation is useful to manipulate our problem. This notion is the one which I first proposed to describe singularities on quartic curves (Urabe [12]).

DEFINITION 4.1. The following procedure by which we can make a root subsystem R' from a given root system R is called an *elementary transformation* of the root system R.

(1) Decompose $R = \bigoplus_{i=1}^{m} R_i$ into irreducible root systems.

(2) Choose a fundamental system of roots $\Delta_i \subset R_i$ for $1 \leqslant i \leqslant m$. Set $\tilde{\Delta}_i = \Delta_i \cup \{-\eta_i\}$ for $1 \leqslant i \leqslant m$, where η_i is the maximal root associated with Δ_i.

(3) Choose a proper subset $\Delta_i' \subset \tilde{\Delta}_i$ for $1 \leqslant i \leqslant m$.

(4) Set $R' = \bigoplus_{i=1}^{m} R_i'$, where R_i' is the root system generated by Δ_i'.

Now for a given root lattice Q, its root system $R = \{\eta \in Q \mid \eta^2 = -2\}$ is uniquely defined. Conversely to a given root system R, its root lattice $Q = \sum_{\alpha \in R} Z\alpha$ corresponds. Thus the above definition also gives a procedure by which we can make a root sublattice Q' from a given root lattice Q.

Let M denote an even lattice and Q be a negatively definite root lattice. Assume that an embedding $Q \subset M$ is defined. Let $\hat{Q} = \{\mu \in M \mid$ For some non-zero integer m, $m\mu \in Q\}$ be the primitive hull of Q in M. We say that Q is *full* in M if the following condition $(*)$ is satisfied.

$$(*) \qquad \text{If } \eta \in \hat{Q} \text{ and } \eta^2 = -2, \text{ then } \eta \in Q.$$

PROPOSITION 4.2. *Let M be an even lattice and Q be its full root sublattice. For any root lattice Q' which is obtains from Q by one elementary transformation, there is a full embedding $Q' \subset M \oplus H$, where H is a hyperbolic plane.*

Proof. Let R be the root system of Q. We use the notation in Definition 4.1. Then $\bigcup_{i=1}^{m} \Delta_i'$ is a free basis of Q'. We define an embedding $\varphi\colon Q' \to M \oplus H$ by setting for $\alpha \in \Delta_i'$,

$$\begin{aligned} \varphi(\alpha) &= \alpha \oplus 0 \quad (\text{if } \alpha \neq -\eta_i) \\ &= \alpha \oplus u \quad (\text{if } \alpha = -\eta_i). \end{aligned}$$

Here u, v is a basis of $H = Zu + Zv$ with $u^2 = v^2 = 0$, $u \cdot v = 1$. Obviously φ preserves bilinear forms. Thus we have only to show the fullness. Let $\hat{Q}'$ denote the primitive hull of $\varphi(Q')$ in $M \oplus H$. Assume that $\eta \in \hat{Q}'$ and $\eta^2 = -2$. We can write it in the form

$$\eta = \sum_{i=1}^{m} \sum_{\alpha \in \Delta_i'} a_\alpha \varphi(\alpha) \in M \oplus H \qquad (a_\alpha \in \boldsymbol{Q}).$$

We would like to show that $a_\alpha \in \boldsymbol{Z}$ for every α.

Now set $A = \{i \mid 1 \leqslant i \leqslant m,\ -\eta_i \notin \Delta_i'\}$, $B = \{i \mid 1 \leqslant i \leqslant m,\ -\eta_i \in \Delta_i'\}$ and $\Delta_i'' = \Delta_i' - \{-\eta_i\}$. If $i \in B$, Δ_i'' is a proper subset of Δ_i. Set $-\eta_i = \sum_{\alpha \in \Delta_i} c_{i\alpha} \alpha$ $(c_{i\alpha} \in \boldsymbol{Z})$. Then we have

$$\eta = \sum_{i \in A} \sum_{\alpha \in \Delta_i'} a_\alpha (\alpha \oplus 0) + \sum_{i \in B} \Big\{ \sum_{\alpha \in \Delta_i''} (a_\alpha + a_i c_{i\alpha})(\alpha \oplus 0) + \sum_{\alpha \in \Delta_i - \Delta_i''} a_i c_{i\alpha} (\alpha \oplus 0) \Big\} + \Big\{ \sum_{i \in B} a_i \Big\} (0 \oplus u).$$

Here we set $a_{-\eta_i} = a_i$. Let π: $M \oplus H \to M$ denote the projection. $\pi(\eta)$ belongs to the primitive hull of Q in M. By the fullness of Q we have $\pi(\eta) \in R \subset Q$, since $\pi(\eta)^2 = -2$. There is a number k with $1 \leqslant k \leqslant m$ such that $\pi(\eta) \in R_k$.

Case 1. $k \in A$.

First we have $a_\alpha \in \boldsymbol{Z}$ for $\alpha \in \Delta_k'$ since $\pi(\eta) \in R_k$. For $i \in A$ with $i \neq k$, $a_\alpha = 0$ for every $\alpha \in \Delta_i'$. Next fix an arbitrary $i \in B$. $a_i c_{i\alpha} = 0$ for $\alpha \in \Delta_i - \Delta_i'' \neq \phi$. We have $a_i = 0$, since $c_{i\alpha} \neq 0$ (cf. Bourbaki [4]). Thus for every $\alpha \in \Delta_i''$, $a_\alpha = a_\alpha + a_i c_{i\alpha} = 0$.

Case 2. $k \in B$.

We have $a_\alpha = 0$ for $i \in A$, $\alpha \in \Delta_i'$. We consider $i \in B$ with $i \neq k$. We have $a_i c_{i\beta} = 0$ for $\beta \in \Delta_i - \Delta_i'' \neq \phi$ and thus $a_i = 0$. It implies that $a_\alpha = a_\alpha + a_i c_{i\alpha} = 0$ for every $\alpha \in \Delta_i''$. Lastly we consider $k \in B$. We have $a_k = \sum_{i \in B} a_i = \eta \cdot v \in \boldsymbol{Z}$. By the fullness of Q, $a_\alpha + a_k c_{k\alpha} \in \boldsymbol{Z}$ for every $\alpha \in \Delta_k''$. We have $a_\alpha \in \boldsymbol{Z}$ for $\alpha \in \Delta_k''$, since $c_{k\alpha} \in \boldsymbol{Z}$.

Consequently we have

$$\eta = \sum_{\alpha \in \Delta_k'} a_\alpha \varphi(\alpha) \qquad (a_\alpha \in \boldsymbol{Z}). \qquad \square$$

Definition 4.3. A disjoint finite union of connected Dynkin graphs is called a *Dynkin graph*. For a Dynkin graph, the following procedure is called an *elementary transformation* of it.

(1) Replace each component by the extended Dynkin graph of the corresponding type.

(2) Choose in an arbitrary manner at least one vertex from each component (of the extended Dynkin graph) and then remove these vertices together with the edges issuing from them.

We give some explanation. We use the notation in Definition 4.1. We fix an integer i. We can obtain the Dynkin graph of the corresponding type from Δ_i by the following rule: (a) Draw a vertex o corresponding to each element α in Δ_i. (b) If $\alpha \cdot \beta = 1$, then connect the correspnding vertices o—o with an edge for α, $\beta \in \Delta_i$ with $\alpha \neq \beta$. If $\alpha \cdot \beta = 0$, then we do not connect the corresponding vertices o o. If we apply the same rule to $\tilde{\Delta}_i$, then the obtained graph is the extended Dynkin graph the corresponding type. Therefore choosing a subset $\Delta_i' \subset \tilde{\Delta}_i$ corresponds to choosing a proper subgraph of the extended Dynkin graph.

By these facts one sees the following. Let R be the root system corresponding to a Dynkin graph G. Let R' be a root subsystem of R which is obtained by one elementary transformation from R. If the type of R' is described by a Dynkin graph G', then G' is the one obtained from G by an elementary transformation of Dynkin graphs.

THEOREM 4.4. *Any Dynkin graph which is obtained by elementary transformations repeated twice from the one of type $2E_8$ or D_{16} can be realized as a combination of singularities on a reduced plane sextic curve.*

Proof. We consider $Q = Q(2E_8)$. Assume that Q'' is a root lattice which is obtained from Q by elementary transformations repeated twice. By Proposition 4.2 there is a full embedding $Q'' \subset Q \oplus H \oplus H$. Let $\hat{Q}''$ denote the primitive hull of Q'' in $Q \oplus H \oplus H$. Set $\hat{S} = \hat{Q}'' \oplus \mathbf{Z}(u+v)$, when u and v are basis of the third hyperbolic plane H. The lattice $\hat{S}$ has a natural primitive embedding $\hat{S} \subset Q \oplus H \oplus H \oplus H$, which has the following properties (0), (a), (b).

(0) For $\lambda = u+v$, $\lambda^2 = 2$.

(a) If $\eta \in \hat{S}$, $\eta \cdot \lambda = 0$ and $\eta^2 = -2$, then $\eta \in Q''$.

(b) $\hat{S}$ does not contain any element u with $u^2 = 0$ and $u \cdot \lambda = 1$.

By Theorem 1.16 we obtain the conclusion.

Next consider another even unimodular lattice Γ_{16} with signature (0, 16). It is known that $Q = Q(D_{16})$ is a full sublattice of Γ_{16}. Noting that $\Gamma_{16} \oplus H \oplus H \oplus H \cong Q(2E_8) \oplus H \oplus H \oplus H$, we have the conclusion by Proposition 4.2 and Theorem 1.16. □

COROLLARY 4.5. *Dynkin graphs* [5], [15]–[17], [39]–[52], [86]–[114], [133]–[139] *in Proposition* 3.7 *can be realized on a reduced plane sextic curve as a combination of singularities.*

Indeed, graphs [52], [108]–[114], [136]–[139] are obtained from D_{16}. Other ones are obtained from $2E_8$ by elementary transformations repeated twice.

5. Concrete construction

For Dynkin graphs [18], [19], [53]–[61], [115]–[122], [140]–[168], [171], [172], [174] it is not difficult to find out a reduced reducible plane sextic curve whose combination of singularities agrees with the Dynkin graph. In the following we give several typical examples. (As for singularities on plane quartic curves, see Urabe [13].)

[57] $9A_1+D_5$: a cuspidal cubic + a general line passing through its singular point + a general conic;

[60] $10A_1+D_4$: 2 general conics + 2 general lines passing through one point on a conic;

[122] $A_2+3A_1+D_6+D_4$: a quartic curve C with A_2+A_1 + a general tangent line at a general point P to C + the line passing through P and the A_1-point on C;

[140] A_6+9A_1: a quartic curve with A_6 + 2 general lines;

[141] $A_5+A_3+7A_1$: 3 conics C_1, C_2, C_3 with the following properties C_1 and C_2 have intersection number 3 at one point, C_2 and C_3 have intersection number 2 at one point and intersect transversally at the other 2 points, C_3 and C_1 intersect transversally at 4 points not on C_2;

[145] $A_4+7A_1+D_4$: a quartic curve with A_4+A_1 + a general line passing through the A_1-point + a general line;

[147] $2A_3+A_2+7A_1$: a cuspidal cubic A + a conic B tangent to A at 2 different points + a general line

$$(A\colon x^2z=y^3,\quad B\colon 4x^2-8y^2+5yz-z^2=0);$$

[157] $9A_1+E_6$: a quartic curve with E_6 + 2 general lines;

[158] $8A_1+D_7$: a quartic curve with A_4+A_1 + a general line passing through the A_4-point + a general line;

[159] $8A_1+E_7$: a quartic curve with E_7 + a general conic;

[161] A_5+10A_1: 2 conics with intersection number 3 at one point + 2 general lines;

[163] $A_3+A_2+10A_1$: a cuspidal cubic + its general tangent line + 2 general lines;

[174] $15A_1$: 6 general lines.

6. The remaining cases

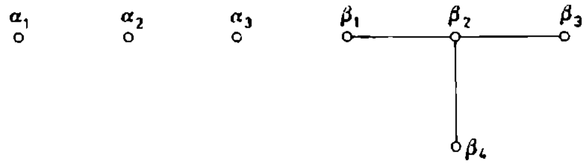
The remaining items in Proposition 3.7 are [123]–[132], [169], [170] and [173]. For [123]–[132] there are corresponding sextic curves. On the contrary we can show that there is no plane sextic curve with [169] A_4+11A_1, [170] $2A_2+11A_1$ and [173] A_2+13A_1.

LEMMA 6.1. *Set* $S'=\mathbf{Z}\lambda\oplus Q'$ *with* $\lambda^2=2$ *and with* $Q'=Q(3A_1+D_4)$. *There is an even overlattice* $\hat{S}'$ *of* S' *with index* 2 *and with the next properties* (a) *and* (b) *for any negatively definite root lattice* Q''.

(a) *If* $\eta\in\hat{S}'\oplus Q''$, $\lambda\cdot\eta=0$ *and* $\eta^2=-2$, *then* $\eta\in Q'\oplus Q''$.

(b) $\hat{S}'\oplus Q''$ *does not contain any element* u *with* $u^2=0$ *and* $u\cdot\lambda=1$.

Proof. The following is a basis of Q'.



Set $\omega_4 = (\beta_1 + 2\beta_2 + \beta_3 + 2\beta_4)/2$. We have $\omega_4^2 = -1$, $\omega_4 \cdot \beta_i = 0$ $(1 \leqslant i \leqslant 3)$, $\omega_4 \cdot \beta_4 = -1$. Set $\theta = (\lambda + \alpha_1 + \alpha_2 + \alpha_3)/2 + \omega_4$ and $\hat{S}' = S' + Z\theta$. It is easy to check that $\hat{S}'$ is an even overlattice with index 2. Obviously (a) holds. We show (b). Assume that we have such an element u. We can write it in the form

$$u = \tfrac{1}{2}\lambda + \sum_{i=1}^{3} \{a_i + \tfrac{1}{2}\} \alpha_i + (\beta + \omega_4) + \gamma \qquad (a_i \in Z,\ \beta \in Q(D_4),\ \gamma \in Q'').$$

We have

$$0 = u^2 = -2a_1(a_1+1) - 2(a_2 + \tfrac{1}{2})^2 - 2(a_3 + \tfrac{1}{2})^2 + (\beta + \omega_4)^2 + \gamma^2.$$

Since a_i's are integers, $a_1(a_1+1) \geqslant 0$, $(a_2 + \frac{1}{2})^2 > 0$ and $(a_3 + \frac{1}{2})^2 > 0$. Moreover $(\beta + \omega_4)^2 \leqslant 0$ and $\gamma^2 \leqslant 0$. Thus the right-hand side of the above equality is negative, which is a contradiction. □

By using the above lemma we can show the next one.

LEMMA 6.2. *Let G' be the Dynkin graph of type $3A_1 + D_4$, let G'' be a Dynkin graph with the following properties*

(1) *G'' has 8 vertices;*

(2) *G'' contains A_1 or A_5 as its component.*

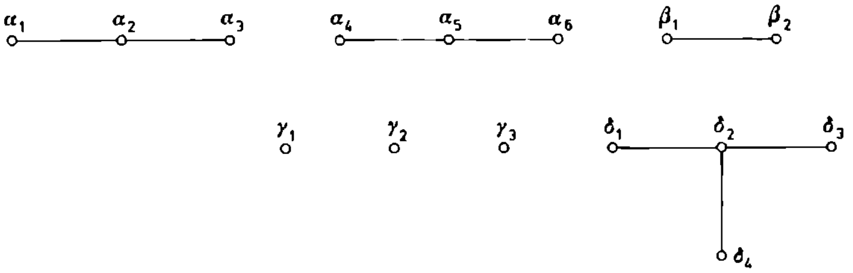
Then there is a plane reduced sextic curve corresponding to $G = G' + G''$.

COROLLARY 6.3. *There are corresponding reduced plane sextic curves for* [123]-[127].

PROPOSITION 6.4. *There are corresponding curves for* [128]-[132].

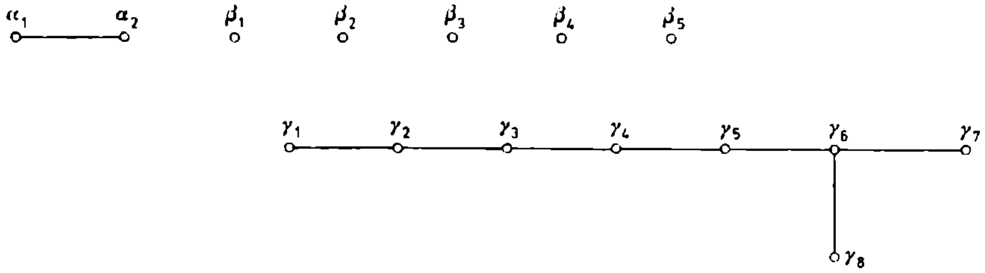
Proof. We indicate an even overlattice $\hat{S}$ which has an primitive embedding into Λ with the properties (a), (b) in Theorem 1.16. We set $Q = Q(G)$ and $S = Z\lambda \oplus Q$ $(\lambda^2 = 2)$.

[128] $2A_3 + A_2 + 3A_1 + D_4$. The basis of Q is the following.



$$\hat{S} = S + Z\theta, \quad \theta = (\lambda + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_6 + \gamma_1)/2.$$

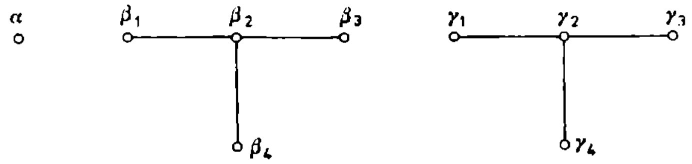
[129] $A_2 + 5A_1 + D_8$. The following is the basis of $Q = Q(A_2 + 5A_1 + D_8)$.



Set $\omega = (\gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 6\gamma_6 + 3\gamma_7 + 4\gamma_8)/2$.

$$\hat{S} = S + Z\theta, \quad \theta = (\lambda + \beta_5)/2 + \omega.$$

[130] $A_4 + 3A_1 + 2D_4$.
[131] $2A_2 + 3A_1 + 2D_4$.
[132] $A_2 + A_1 + 3D_4$.
The following is the basis of $Q' = Q(A_1 + 2D_4)$.



$$\hat{S} = S + Z\theta, \quad \theta = (\lambda + \alpha + \beta_1 + \beta_4 + \gamma_1 + \gamma_4)/2.$$

Set $\hat{S}' = (Z\lambda \oplus Q') + Z\theta$. We can check that the discriminant form on $\hat{S}'^*/\hat{S}'$ is $q_1^{(2)}(2)^2 \oplus q_{-1}^{(2)}(2)^2$. □

PROPOSITION 6.5. *There is not a reduced plane sextic curve with a combination of singularities $A_4 + 11A_1$, $2A_2 + 11A_1$ and $A_2 + 13A_1$.*

Proof. Let $p_n = (n-1)(n-2)/2$ denote the arithmetic genus of a plane curve of degree n. Note that $p_6 = 10$, $p_5 = 6$, $p_4 = 3$, $p_3 = 1$, $p_2 = p_1 = 0$. For any irreducible reduced curve D of degree n and for its smooth model $\tilde{D}$, the following equality holds.

$$g + \sum \delta_p = p_n.$$

Here g is the genus of $\tilde{D}$, δ_p is a positive number defined for every singular point $p \in D$, and the sum is taken over all singular points on D. The above equality is called the *Plücker formula*.

Assume that there is a plane reduced sextic curve C with $A_4 + 11A_1$. Note that $\delta_p = 2$ for A_4 and $\delta_p = 1$ for A_1. C cannot be irreducible, since

$2+11\times 1 = 13 > p_6$. Let C_1 be the irreducible component of C with the A_4-singularity. Since the A_4-point has only one branch, C_1 is uniquely defined. Now $\deg C_1 = 4$ or 5, since C_1 has the A_4-point.

Assume that $\deg C_1 = 5$. We can write C in the form $C = C_1 \cup C_2$, and C_2 is a line. C_1 and C_2 intersect at at most 5 points and at every intersection point C has a singularity. In our case the singularity is of type A_1. Thus C_1 and C_2 intersect at 5 points transversally. Then C_1 has to have $11-5=6$ A_1-singularities, since C_2 is smooth. The sum of δ_p over singularities on C_1 is $2+6=8$. But 8 is greater than p_5, which is a contradiction.

Assume that $\deg C_1 = 4$. If we write C in the form $C = C_1 \cup C_2$, then C_2 is a conic. C_1 and C_2 intersect transversally at 8 points. C_2 has at most one A_1-singularity. Thus C_1 has at least $11-8-1 = 3$ A_1-singularities besides the A_4-point. Thus the sum of δ_p over singularities on C_1 is at least $2+3=5$. But 5 is greater than p_4, which is a contradiction.

Noting that an A_2-singular point has only one branch and that $\delta_p = 1$ for an A_2-singularity, we can give similar proofs for the case $2A_2 + 11A_1$ and the case $A_2 + 13A_1$. □

Remark. It is interesting to ask whether we can determine all possible combinations of singularities on plane sextic curves only by the Plücker formula. The answer is negative. Consider the A_{20}-singularity. Since $\delta_p = 10$ for A_{20}, the existence of an irreducible reduced rational sextic curve whose singularities consist of an A_{20}-point cannot be prohibited by the Plücker formula. But such a sextic curve never exists, since rank $\mathrm{Pic}(Z) \leqslant 20$ for any K3 surface Z.

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